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$$\begin{cases} u_t = \nabla\cdot(D(u)\nabla u) - \chi\nabla\cdot(u\nabla v) + \xi\nabla\cdot(u\nabla w), & x\in\Omega,\ t>0,\\ \tau_1 v_t = \Delta v + \alpha u - \beta v, & x\in\Omega,\ t>0,\\ \tau_2 w_t = \Delta w + \gamma u - \delta w, & x\in\Omega,\ t>0,\\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & x\in\partial\Omega,\ t>0,\\ u(x,0)=u_0(x),\ \tau_1 v(x,0)=\tau_1 v_0(x), \tau_2 w(x,0)=\tau_2 w_0(x), & x\in\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n (n\geq 2)$ with smooth boundary $\partial\Omega$, the parameters $\chi,\xi,\alpha,\beta,\gamma$ and δ are positive, $\tau_1,\tau_2\geq 0$ and, ν denotes the outward normal vector of $\partial\Omega$. The unknown $u(x,t)$ denotes the concentration of cell, $v(x,t)$ and $w(x,t)$ describe the concentrations of chemoattractant and chemorepellent, respectively. The model (1.1) was also introduced in [34] to model the quorum sensing effect in the chemotactic movement. Throughout this text, the C^2-smooth nonlinear diffusion $D(u)$ is assumed to satisfy

$$D(u)>0 \text{ for } u\geq 0 \text{ and } D(u)\geq du^\theta \text{ for some } d>0, \theta\in\mathbb{R} \text{ and for all } u>0. \tag{1.2}$$

Intuitively, large values of θ seem to enhance boundedness of solutions. Neglect of the repulsion effect, the model (1.1) reduces to the widely known (attractive) Keller-Segel chemotaxis model with nonlinear diffusion:

$$\begin{cases} u_t = \nabla\cdot(D(u)\nabla u) - \chi\nabla\cdot(u\nabla v), & x\in\Omega,\ t>0,\\ \tau_1 v_t = \Delta v + \alpha u - \beta v, & x\in\Omega,\ t>0, \end{cases} \tag{1.3}$$

whose solution behavior has been extensively studied in the past four decades in various perspectives (see the survey articles [12, 4] and references therein). More often than not, when $D(u)$ is a constant, the system (1.3) is known as the classical Keller-Segel model. A striking feature of the classical model (1.3) is the blow-up of solutions in two or higher dimensions [31, 41, 42]. While, when $D(u)$ is a nonlinear function, there is a critical exponent $\theta^* = 1-\frac{2}{n}$ which distinguishes between occurrence and impossibility of blow-up. More precisely, the solution of system (1.3) will globally exist [14, 15, 36, 39] if $\theta>\theta^*$ and blow up in finite time [5, 7, 8, 13, 30, 32, 16] if $\theta<\theta^*$ with $n\geq 3$; when $\theta=\theta^*$, based on the availability of Lyapunov functional, it has been proved there exists radially symmetric initial data such that the solution of (1.3) will blow up in finite time [17, 22].

Another important subsystem of (1.1) is the following repulsive Keller-Segel model:

$$\begin{cases} u_t = \nabla\cdot(D(u)\nabla u) + \xi\nabla\cdot(u\nabla w), & x\in\Omega,\ t>0,\\ \tau_2 w_t = \Delta w + \gamma u - \delta w, & x\in\Omega,\ t>0. \end{cases} \tag{1.4}$$

Based on a Lyapunov functional different from that of the attractive Keller-Segel model (1.3), the global existence of classical solutions in two dimensions and weak solutions in three and four dimensions were established in [6] with $D(u)=1$. Further results on a repulsive Keller-Segel model with nonlinear chemosensitivity can be found in [37].

The attraction-repulsion model (1.1) is a mixed combination of the attractive KS model (1.3) and the repulsive one (1.4). Hence, the mathematical analysis on the boundedness and blow-up of solutions offers great challenges due to the complex interactions between the three components u,v and w, and the difficulty of constructing a Lyapunov functional. To motivate our study, we summarize the known results on boundedness versus blow-up for (1.1) in the literature as follows:

- $\tau_1 = \tau_2 = \tau$ and $D(u) = 1$: In one dimensional space, the global existence of classical solutions, non-trivial stationary state, asymptotic behavior and pattern formation of the system (1.1) have been studied in [20, 28, 27] with $\tau = 1$. In higher dimensions ($n \geq 2$), Tao and Wang [38] studied the global solvability, boundedness, blow-up, existence of steady states by introducing the transformation $s = \xi w - \chi v$. It has been proved that when repulsion dominates or cancels attraction in the sense of $\xi\gamma \geq \chi\alpha$, the global classical solution will exists for both $\tau = 0$ [38] and $\tau = 1$ [18, 26]. Whereas, if attraction dominates in the sense of $\xi\gamma < \chi\alpha$, then the solution of system (1.1) with $\tau = 0$ will blow up in finite time for large initial mass and exist globally with small initial mass in two dimensional spaces [9, 23, 43]. The large time behavior of solution with small initial data was established in [24]. Part

Employing the same Lyapunov functional as in [25] in a close way, when the repulsion dominates or cancels attraction in the sense

$$\text{either} \quad \{\xi\gamma > \chi\alpha\} \quad \text{or} \quad \{\xi\gamma = \chi\alpha, \beta \geq \delta\}, \tag{1.6}$$

we establish the refined $L^{1+\theta}$-boundedness of u compared to the L^1-boundedness of u as used in [25] as the starting point toward our further bootstrap type argument and then the remaining procedure leading to its L^∞-boundedness is standard and developed since [39]. Namely, we first use the gained $L^{1+\theta}$-bound for u and the v-equation to improve the regularity of v and then, with an elliptic Agmon-Douglis-Nirenberg L^p-estimate applied to the w-equation, the restriction $\theta > 1-\frac{4}{n+2}$ enables us to derive a Grownwall type inequality for the coupled quantity $\int_\Omega u$

By the blowup criterion (2.1) of Lemma 2.1, it suffices to derive $\|u(\cdot,t)\|_{L^\infty} < \infty$ for all $t > 0$ to obtain the global-in-time solutions. We first notice that L^1-norm of the solution triple of (1.5) is bounded by integrating equations in (1.5) over Ω.

Lemma 2.2. *The solution* (u, v, w) *of system* (1.1) *satisfies the following properties*

$$\|u(\cdot,t)\|_{L^1} = \|u_0\|_{L^1}, \tag{2.2}$$

$$\|v(\cdot,t)\|_{L^1} = \frac{\alpha}{\beta}\|u_0\|_{L^1} - \left(\frac{\alpha}{\beta}\|u_0\|_{L^1} - \|v_0\|_{L^1}\right)e^{-\beta t}, \tag{2.3}$$

$$\|w(\cdot,t)\|_{L^1} = \frac{\gamma}{\delta}\|u_0\|_{L^1}. \tag{2.4}$$

The following version of Gagliardo-Nirenberg inequality will be used in several places in our upcoming discussions.

Lemma 2.3 (Gagliardo-Nirenberg inequality [10])**.** *Let* Ω *be a bounded domain in* $\mathbb{R}^n$ *with smooth boundary. Let* l *and* k *be any integers satisfying* $0 \le l < k$, *and let* $1 \le q, r \le \infty$, *and* $p \in \mathbb{R}^+$, $\frac{l}{k} \le a \le 1$ *such that*

$$\frac{1}{p} - \frac{l}{n} = a\left(\frac{1}{q} - \frac{k}{n}\right) + (1-a)\frac{1}{r}.$$

Then, for any $f \in W^{k,q}(\Omega) \cap L^r(\Omega)$, *there exist a constant* c *depending only on* Ω, q, k, r *and* n *such that:*

$$\|D^l f\|_{L^p} \le c(\|D^k f\|_{L^q}^a \|f\|_{L^r}^{1-a} + \|f\|_{L^r}), \tag{2.5}$$

with the following exception: if $1 < q < \infty$ *and* $k - l - \frac{n}{q}$ *is a nonnegative integer, then* (2.5) *holds only for* a *satisfying* $\frac{l}{k} \le a < 1$.

We should mention that the original Gagliardo-Nirenberg inequality (e.g. see [33]) is stated only for $r \ge 1$, but this condition can be easily relaxed to $r \in (0, p)$

$$
\begin{aligned}
&\int_\Omega u_t(G(u) - \chi v + \xi w) = \int_\Omega \nabla\cdot(D(u)\nabla u - \chi u\nabla v + \xi u\nabla w)(G(u) - \chi v + \xi w)\\
&= -\int_\Omega (D(u)\nabla u - \chi u \nabla v + \xi u \nabla w)\left(\frac{D(u)}{u}\nabla u - \chi\nabla v + \xi\nabla w\right)\\
&= -\int_\Omega u\left|\frac{D(u)}{u}\nabla u - \chi\nabla v + \xi\nabla w\right|^2.
\end{aligned}
\tag{2.8}
$$

A use of integration by part yields

$$\int_\Omega u_t(G(u) - \chi v + \xi w) = \frac{d}{dt}\int_\Omega G(u) - \chi\frac{d}{dt}\int_\Omega uv + \chi\int_\Omega uv_t + \xi\int_\Omega u_t w. \tag{2.9}$$

It follows from the second equation of (1.5) that $u = \frac{1}{\alpha}v_t - \frac{1}{\alpha}\Delta v + \frac{\beta}{\alpha}v$, which gives

$$\int_\Omega uv_t = \frac{1}{\alpha}\int_\Omega v_t^2 + \frac{1}{2\alpha}\frac{d}{dt}\int_\Omega|\nabla v|^2 + \frac{\beta}{2\alpha}\frac{d}{dt}\int_\Omega v^2. \tag{2.10}$$

Similarly, a substitution of the third equation of (1.1) shows

$$u = \frac{\delta}{\gamma}w - \frac{1}{\gamma}\Delta w, \tag{2.11}$$

which gives rise to

$$\int_\Omega u_t w = \frac{\delta}{2\gamma}\frac{d}{dt}\int_\Omega w^2 + \frac{1}{2\gamma}\frac{d}{dt}\int_\Omega|\nabla w|^2. \tag{2.12}$$

The combination of (2.9), (2.10) and (2.12) leads to

$$
\begin{aligned}
&\int_\Omega u_t(G(u) - \chi v + \xi w)\\
&= \frac{d}{dt}\int_\Omega\left(G(u) - \chi uv + \frac{\beta\chi}{2\alpha}v^2 + \frac{\chi}{2\alpha}|\nabla v|^2 + \frac{\xi\delta}{2\gamma}w^2 + \frac{\xi}{2\gamma}|\nabla w|^2\right) + \frac{\chi}{\alpha}\int_\Omega v_t^2,
\end{aligned}
$$

which together with (2.8) implies (2.7). The proof of Lemma 2.4 is completed. □

Employing the energy identity obtained in Lemma 2.4, we next derive $L^{1+\theta}$-boundedness of the u-component provided the repulsion dominates or cancels attraction as specified in (1.6), which serves a key starting point towards our further analysis.

Lemma 2.5. *Let* $\Omega\subset\mathbb{R}^n (n \geq 3)$ *and* $D(u)$ *satisfy* (1.2). *If* (1.6) *holds, then there exists a constant* $C > 0$ *independent of* t *such that the solution of* (1.5) *satisfies*

$$\|u(\cdot, t)\|_{L^{1+\theta}} \leq C. \tag{2.13}$$

Proof. Thanks to the mass conservation of u in (2.2), we shall proceed with $\theta > 0$. Then using (2.11) and the Cauchy-Schwarz inequality, one can readily derive that

$$
\begin{aligned}
\chi\int_\Omega uv &= \frac{\chi\delta}{\gamma}\int_\Omega vw + \frac{\chi}{\gamma}\int_\Omega \nabla w\cdot\nabla v\\
&\leq \frac{\chi\delta}{\gamma}\left(\frac{\xi}{2\chi}\int_\Omega w^2 + \frac{\chi}{2\xi}\int_\Omega v^2\right) + \frac{\chi}{\gamma}\left(\frac{\xi}{2\chi}\int_\Omega|\nabla w|^2 + \frac{\chi}{2\xi}\int_\Omega|\nabla v|^2\right)\\
&= \frac{\xi\delta}{2\gamma}\int_\Omega w^2 + \frac{\chi^2\delta}{2\xi\gamma}\int_\Omega v^2 + \frac{\xi}{2\gamma}\int_\Omega|\nabla w|^2 + \frac{\chi^2}{2\xi\gamma}\int_\Omega|\nabla v|^2.
\end{aligned}
\tag{2.14}
$$

Substituting (2.14) into (2.7), we obtain

$$\begin{aligned} F(u,v,w) &\geq \int_\Omega G(u) + \left(\frac{\beta\chi}{2\alpha} - \frac{\chi^2\delta}{2\xi\gamma}\right)\int_\Omega v^2 + \left(\frac{\chi}{2\alpha} - \frac{\chi^2}{2\xi\gamma}\right)\int_\Omega |\nabla v|^2 \\ &= \int_\Omega G(u) + \frac{\chi(\xi\gamma\beta - \chi\alpha\delta)}{2\xi\gamma\alpha}\int_\Omega v^2 + \frac{\chi(\xi\gamma - \chi\alpha)}{2\xi\gamma\alpha}\int_\Omega |\nabla v|^2. \end{aligned} \tag{2.15}$$

Integrating (2.6) with respect to t and using (2.15), we have

$$\begin{aligned} &\int_\Omega G(u) + \frac{\chi}{\alpha}\int_0^t\int_\Omega v_t^2 + \int_0^t\int_\Omega u\left|\frac{D(u)}{u}\nabla u - \chi\nabla v + \xi\nabla w\right|^2 \\ &\leq F(u_0, v_0) - \frac{\chi(\xi\gamma - \chi\alpha)}{2\xi\gamma\alpha}\int_\Omega |\nabla v|^2 - \frac{\chi(\xi\gamma\beta - \chi\alpha\delta)}{2\alpha\xi\gamma}\int_\Omega v^2. \end{aligned} \tag{2.16}$$

In the first case of (1.6), i.e., $\xi\gamma > \chi\alpha$, using the Gagliardo-Nirenberg inequality (2.3) and the boundedness of $\|v(\cdot,t)\|_{L^1}$ in (2.3), we have

$$\begin{aligned} -\frac{\chi(\xi\gamma\beta - \chi\alpha\delta)}{2\alpha\xi\gamma}\|v\|_{L^2}^2 &\leq c_1\|\nabla v\|_{L^2}^{\frac{n}{n+1}}\|v\|_{L^1}^{\frac{2}{n+2}} + c_2\|v\|_{L^1}^2 \\ &\leq \frac{\chi(\xi\gamma - \chi\alpha)}{2\xi\gamma\alpha}\|\nabla v\|_{L^2}^2 + c_3. \end{aligned} \tag{2.17}$$

While, in the second case of (1.6), i.e., $\xi\gamma = \chi\alpha$ and $\beta \geq \delta$, one trivially has

$$-\frac{\chi(\xi\gamma - \chi\alpha)}{2\xi\gamma\alpha}\int_\Omega |\nabla v|^2 - \frac{\chi(\xi\gamma\beta - \chi\alpha\delta)}{2\alpha\xi\gamma}\int_\Omega v^2 = \frac{(\delta-\beta)\chi}{2\alpha}\int_\Omega v^2 \leq 0. \tag{2.18}$$

In summary, under (1.6), the combination of (2.16), (2.17) and (2.18) implies that

$$\int_\Omega G(u) \leq F(u_0, v_0) + c_3. \tag{2.19}$$

Since $D(u)$ fulfills (1.2) and $G(z) := \int_{z_0}^z\int_{s_0}^\sigma \frac{D(\tau)}{\tau}d\tau d\sigma$, one can easily infer that $G(u) \geq c_4 u^{\theta+1} - c_5$ for some $c_4, c_5 > 0$. Then the estimate (2.13) follows directly from (2.19). □

which together with the $L^{1+\theta}$-boundeness of u in (2.13) gives

$$\|v(\cdot,t)\|_{L^s} \le e^{-\beta t}\|e^{t\Delta}v_0\|_{L^s} + \alpha\int_0^t e^{-\beta(t-\tau)}\|e^{(t-\tau)\Delta}u(\cdot,\tau)\|_{L^s}d\tau$$

$$\le c_1e^{-\beta t}\|v_0\|_{W^{1,\infty}} + c_2\alpha\int_0^t e^{-\beta(t-\tau)}\cdot(1+(t-\tau)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{1+\theta}-\frac{1}{s})})\|u(\cdot,t)\|_{L^{1+\theta}}d\tau$$

$$\le c_1e^{-\beta t}\|v_0\|_{W^{1,\infty}} + c_2c_3\alpha\int_0^\infty e^{-\beta\sigma}\cdot(1+\sigma^{-\frac{1}{2}}$$

Remark 3. The larger the exponent θ is, the easier the boundedeness of u is. Indeed, if $\theta + 1 > n$, then Lemma 2.6 tells us that $\|\nabla v(\cdot,t)\|_{L^\infty}$ is bounded. Then (3.1) of Lemma 3.1 quickly shows that $\|u(\cdot,t)\|_{L^p}$ is bounded for any $p > 2$. Thus, we directly jump to the proof of Theorem 1.1 on P. 13 to obtain that $\|u(\cdot,t)\|_{L^\infty}$ is bounded. Accordingly, we shall proceed henceforth with $\theta \le n-1$.

The condition $\theta > 1 - \frac{4}{n+2}$ (along with $\theta \le n-1$ as noted in Remark 3) indeed enables us to select certain parameters in an appropriate way such that we have the license to apply Gagliardo-Nirenberg inequality in Lemma 3.2.

Lemma 2.8. *Let $n \ge 2$, $\theta > 1 - \frac{4}{n+2}$, $\bar{p} \ge 1 + 2\theta$ and $\bar{q} \ge \frac{n-2}{2}$. Then there exist numbers $p \ge \bar{p}, q \ge \bar{q}$, $s \in [1, \frac{n(\theta+1)}{n-1-\theta})$, $\lambda \in (1, \frac{n}{n-2})$ and $\mu > \max\{1, \frac{n}{2}\}$ such that*

$$p > 1 + 2\theta, \tag{2.26}$$

$$\max\left\{1 - \frac{2}{s}, \quad \frac{n-2}{n}\cdot\frac{p-\theta}{p+\theta}\right\} < \frac{1}{\lambda} < 1 - \frac{n-2}{n}\cdot\frac{1}{q}, \tag{2.27}$$

$$\max\left\{1 - \frac{2(q-1)}{s}, \quad \frac{n-2}{n}\frac{2}{p+\theta}\right\} < \frac{1}{\mu} < \min\left\{\frac{2}{1+\theta}, \quad \frac{2}{n} + \frac{n-2}{n}\cdot\frac{1}{q}\right\}, \tag{2.28}$$

$$\frac{\frac{p-\theta}{2(1+\theta)} - \frac{1}{2\lambda}}{\frac{p+\theta}{2(1+\theta)} + \frac{1}{n} - \frac{1}{2}} + \frac{\frac{1}{s} + \frac{1}{2\lambda} - \frac{1}{2}}{\frac{q}{s} + \frac{1}{n} - \frac{1}{2}} < 1 \tag{2.29}$$

and

$$\frac{\frac{1}{1+\theta} - \frac{1}{2\mu}}{\frac{p+\theta}{2(1+\theta)} + \frac{1}{n} - \frac{1}{2}} + \frac{\frac{q-1}{s} + \frac{1}{2\mu} - \frac{1}{2}}{\frac{q}{s} + \frac{1}{n} - \frac{1}{2}} < 1. \tag{2.30}$$

Proof. Using the similar argument as in [39, Lemma 2.1], after various honest computations, we obtain the respective equalities directly. □

3. Proof of Theorem 1.1.

In this section, we show the proof of Theorem 1.1; the idea for the proof starts with the following coupled energy estimates.

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^n (n \ge 2)$ be a bounded domain with smooth boundary. Suppose that $D(u)$ satisfies (1.2). Assume that $p > 2$ and $q > 2$. Then there exist two constants $C_1 > 0$ and $C_2 > 0$ such that for all $t \in (0, T_{max})$, the solution of (1.5) satisfies*

$$\begin{aligned}
&\frac{d}{dt}\left(\int_\Omega u^p + \int_\Omega |\nabla v|^{2q}\right) + \frac{2p(p-1)d}{(p+\theta)^2}\int_\Omega |\nabla u^{\frac{p+\theta}{2}}|^2 \\
&+ \frac{2(q-1)}{q}\int_\Omega |\nabla|\nabla v|^q|^2 + 2q\beta\int_\Omega |\nabla v|^{2q} + \frac{(p-1)\xi\gamma}{2}\int_\Omega u^{p+1} \\
&\le \frac{p(p-1)\chi^2}{2d}\int_\Omega u^{p-\theta}|\nabla v|^2 + C_1\int_\Omega u^2|\nabla v|^{2q-2} + C_2.
\end{aligned} \tag{3.1}$$

Proof. We multiply the first equation of (1.5) by pu^{p-1}, and integrate the equation with respect to x over Ω to obtain

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega u^p + p(p-1)d\int_\Omega u^{p-2+\theta}|\nabla u|^2 \\
&\le p(p-1)\chi\int_\Omega u^{p-1}\nabla u\cdot\nabla v - p(p-1)\xi\int_\Omega u^{p-1}\nabla u\cdot\nabla w
\end{aligned}$$

$$
\begin{aligned}
&= p(p-1)\chi \int_\Omega u^{p-1} \nabla u \cdot \nabla v + (p-1)\xi \int_\Omega u^p(\delta w - \gamma u) \\
&\le \frac{p(p-1)d}{2} \int_\Omega u^{p-2+\theta} |\nabla u|^2 + \frac{p(p-1)\chi^2}{2d} \int_\Omega u^{p-\theta} |\nabla v|^2 \\
&\quad + (p-1)\xi\delta \int_\Omega u^p w - (p-1)\xi\gamma \int_\Omega u^{p+1},
\end{aligned}
$$

which yields

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega u^p + \frac{p(p-1)d}{2} \int_\Omega u^{p-2+\theta} |\nabla u|^2 + (p-1)\xi\gamma \int_\Omega u^{p+1} \\
&\le \frac{p(p-1)\chi^2}{2d} \int_\Omega u^{p-\theta} |\nabla v|^2 + (p-1)\xi\delta \int_\Omega u^p w.
\end{aligned} \tag{3.2}
$$

Now, we use Young's inequality and Lemma 2.7 to estimate the last term in (3.2) as

$$
\begin{aligned}
(p-1)\xi\delta \int_\Omega u^p w &\le \frac{(p-1)\xi\gamma}{4} \int_\Omega u^{p+1} + c_1 \int_\Omega w^{p+1} \\
&\le \frac{(p-1)\xi\gamma}{2} \int_\Omega u^{p+1} + c_2.
\end{aligned} \tag{3.3}
$$

Substituting (3.3) into (3.2), one has

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega u^p + \frac{2p(p-1)d}{(p+\theta)^2} \int_\Omega |\nabla u^{\frac{p+\theta}{2}}|^2 + \frac{(p-1)\xi\gamma}{2} \int_\Omega u^{p+1} \\
&\le \frac{p(p-1)\chi^2}{2d} \int_\Omega u^{p-\theta} |\nabla v|^2 + c_2.
\end{aligned} \tag{3.4}
$$

Even with the lower regularity of $\|\nabla v\|_{L^s} (s \in [1, \frac{n}{n-1}))$ compared to that of Lemma 2.[illegible] and the standard technique to control the resulting boundary integral [14], the following estimate associated with the v-equation in (1.5) is quite known, cf. [25, Lemma 3.4]:

$$
\frac{d}{dt}\int_\Omega |\nabla v|^{2q} + 2q\beta \int_\Omega |\nabla v|^{2q}
$$

q

In the sequel, we shall utilize the parameters chosen in Lemma 2.8 to bound the two coupling integrals on the right-hand side in terms of the dissipation terms on its left-hand side. To start off, the Hölder inequality gives that

$$\int_\Omega u^{p-\theta}|\nabla v|^2 \le \left(\int_\Omega u^{(p-\theta)\lambda}\right)^{\frac{1}{\lambda}} \cdot \left(\int_\Omega |\nabla v|^{2\lambda'}\right)^{\frac{1}{\lambda'}} \tag{3.9}$$

and

$$\int_\Omega u^2|\nabla v|^{2q-2} \le \left(\int_\Omega u^{2\mu}\right)^{\frac{1}{\mu}} \cdot \left(\int_\Omega |\nabla v|^{2(q-1)\mu'}\right)^{\frac{1}{\mu'}}, \tag{3.10}$$

where $\lambda \in (1, \frac{n}{n-2})$ and $\mu > \max\{1, \frac{n}{2}\}$ such that

$$\frac{1}{\lambda} + \frac{1}{\lambda'} = 1, \quad \frac{1}{\mu} + \frac{1}{\mu'} = 1 \qquad \lambda' = \frac{\lambda}{\lambda - 1}, \quad \mu' = \frac{\mu}{\mu - 1}. \tag{3.11}$$

The facts $\lambda > 1$ and $p > 1 + 2\theta$ as in (2.26) entail that

$$\frac{p+\theta}{2(1+\theta)} > \frac{p+\theta}{2(p-\theta)\lambda}.$$

On the other hand, from the first inequality of (2.27), we have

$$\frac{p+\theta}{2(p-\theta)\lambda} > \frac{n-2}{2n}.$$

These two inequalities enable us to apply the Gagliardo-Nirenberg inequality to bound

$$\begin{aligned}\left(\int_\Omega u^{(p-\theta)\lambda}\right)^{\frac{1}{\lambda}} &= \|u^{\frac{p+\theta}{2}}\|^{\frac{2(p-\theta)}{p+\theta}}_{L^{\frac{2\lambda(p-\theta)}{p+\theta}}} \\ &\le c_3 \|\nabla u^{\frac{p+\theta}{2}}\|^{\frac{2k_1(p-\theta)}{p+\theta}}_{L^2} \|u^{\frac{p+\theta}{2}}\|^{\frac{2(1-k_1)(p-\theta)}{p+\theta}}_{L^{\frac{2(1+\theta)}{p+\theta}}} + c_3 \|u^{\frac{p+\theta}{2}}\|^{\frac{2(p-\theta)}{p+\theta}}_{L^{\frac{2(1+\theta)}{p+\theta}}}\end{aligned} \tag{3.12}$$

with

$$k_1 = \frac{\frac{p+\theta}{2(1+\theta)} - \frac{p+\theta}{2(p-\theta)\lambda}}{\frac{p+\theta}{2(1+\theta)} + \frac{1}{n} - \frac{1}{2}} = \frac{\frac{p+\theta}{2(1+\theta)} - \frac{p+\theta}{2(p-\theta)\lambda}}{\frac{p+\theta}{2(1+\theta)} - \frac{n-2}{2n}} \in (0,1).$$

By noting the $L^{1+\theta}$-boundedness of u by (2.13) and setting

$$a_1 := \frac{(p-\theta)}{p+\theta} \cdot k_1 = \frac{\frac{p-\theta}{2(1+\theta)} - \frac{1}{2\lambda}}{\frac{p+\theta}{2(1+\theta)} - \frac{n-2}{2n}} \in (0,1) \tag{3.13}$$

due to $\lambda \in (1, \frac{n}{n-2})$, we then get from (3.12) that

$$\left(\int_\Omega u^{(p-\theta)\lambda}\right)^{\frac{1}{\lambda}} \le c_4 \|\nabla u^{\frac{p+\theta}{2}}\|^{2a_1}_{L^2} + c_4. \tag{3.14}$$

Fixing $s \in [1, \frac{n(\theta+1)}{(n-1-\theta)^+})$, using the second inequality in (2.27), (3.11) and the Gagliaro-Nirenberg inequality again, we find a $k_2 = \frac{\frac{q}{s} - \frac{q}{2\lambda'}}{\frac{q}{s} + \frac{1}{n} - \frac{1}{2}} \in (0,1)$ such that

$$\left(\int_\Omega |\nabla v|^{2\lambda'}\right)^{\frac{1}{\lambda'}} = \Big\||\nabla v|^q\Big\|^{\frac{2}{q}}_{L^{\frac{2\lambda'}{q}}} \le c_6 \Big\|\nabla|\nabla v|^q\Big\|^{\frac{2k_2}{q}}_{L^2} \cdot \Big\||\nabla v|^q\Big\|^{\frac{2(1-k_2)}{q}}_{L^{\frac{s}{q}}} + c_6 \Big\||\nabla v|^q\Big\|^{\frac{2}{q}}_{L^{\frac{s}{q}}}.$$

This along with the boundedness of $\|\nabla v\|_{L^s}$ as shown in (2.20) of Lemma 2.6 entails

$$\left(\int_\Omega |\nabla v|^{2\lambda'}\right)^{\frac{1}{\lambda'}} \le c_7 \Big\|\nabla|\nabla v|^q\Big\|^{2b_1}_{L^2} + c_7 \tag{3.15}$$

with

$$b_1 := \frac{1}{q} \cdot \frac{\frac{q}{s} - \frac{q}{2\lambda}}{\frac{q}{s} + \frac{1}{n} - \frac{1}{2}} = \frac{\frac{1}{s} - \frac{1}{2\lambda}}{\frac{q}{s} + \frac{1}{n} - \frac{1}{2}} \quad (0,1) \tag{3.16}$$

implied by (2.29). Similarly, in light of (2.28) and the boundedness of $u^{\frac{p+\theta}{2}}{}_{L^{\frac{2(1+\theta)}{p+\theta}}}$, one can find $k_3 = \frac{\frac{p+\theta}{2(1+\theta)} - \frac{p+\theta}{4\mu}}{\frac{p+\theta}{2(1+\theta)} + \frac{1}{n} - \frac{1}{2}} \quad (0,1)$ and then use G-N inequality to estimate

$$\begin{aligned}\left(\int_\Omega u^{2\mu}\right)^{\frac{1}{\mu}} = u^{\frac{p+\theta}{2}}{}^{\frac{4}{p+\theta}}_{L^{\frac{4\mu}{p+\theta}}} &\leq c_8 \quad u^{\frac{p+\theta}{2}}{}^{\frac{4k_3}{p+\theta}}_{L^2} \; u^{\frac{p+\theta}{2}}{}^{\frac{4(1-k_3)}{p+\theta}}_{L^{\frac{2(1+\theta)}{p+\theta}}} + c_8 \; u^{\frac{p+\theta}{2}}{}^{\frac{4}{p+\theta}}_{L^{\frac{2(1+\theta)}{p+\theta}}} \\ &\leq c_9 \quad u^{\frac{p+\theta}{2}}{}^{\frac{4k_3}{p+\theta}}_{L^2} + c_9 \\ &= c_9 \quad u^{\frac{p+\theta}{2}}{}^{2a_2}_{L^2} + c_9,\end{aligned} \tag{3.17}$$

where

$$a_2 := \frac{2k_3}{p+\theta} = \frac{\frac{1}{(1+\theta)} - \frac{1}{2\mu}}{\frac{p+\theta}{2(1+\theta)} + \frac{1}{n} - \frac{1}{2}} \quad (0,1). \tag{3.18}$$

By (2.28), (2.30) and (3.11), it follows that $k_4 = \frac{\frac{q}{s} - \frac{q}{2(q-1)\mu}}{\frac{q}{s} + \frac{1}{n} - \frac{1}{2}} \quad (0,1)$. Then the Gagliardo-Nirenberg inequality together with the boundedness of $\quad v \;{}_{L^s}$ implies

$$\begin{aligned}\left(\int_\Omega | \quad v|^{2(q-1)\mu}\right)^{\frac{1}{\mu}} &= \Big\|| \quad v|^q\Big\|^{\frac{2(q-1)}{q}}_{L^{\frac{2(q-1)\mu}{q}}} \\ &\leq c_{10}\Big\| \quad | \quad v|^q\Big\|^{\frac{2(q-1)k_4}{q}}_{L^2} \cdot \Big\|| \quad v|^q\Big\|^{\frac{2(q-1)(1-k_4)}{q}}_{L^{\frac{s}{q}}} + c_{10}\Big\|| \quad v|^q\Big\|^{\frac{2(q-1)}{q}}_{L^{\frac{s}{q}}} \\ &\leq c_{11}\Big\| \quad | \quad v|^q\Big\|^{2b_2}_{L^2} + c_{11},\end{aligned} \tag{3.19}$$

where

$$b_2 := \frac{q-1}{q} \cdot k_4 = \frac{\frac{q-1}{s} - \frac{1}{2\mu}}{\frac{q}{s} + \frac{1}{n} - \frac{1}{2}} \quad (0,1). \tag{3.20}$$

A collection of (3.9), (3.10), (3.14), (3.15), (3.17) and (3.19) shows that

$$c_1 \int_\Omega u^p | \quad v|^2 + c_1 \int_\Omega u^2 | \quad v|^{2q-2}$$

$$\leq c_{12} \quad u$$

For $a > 0$ and $b > 0$ with $a + b < 1$, a couple of simple uses of Young's inequality with epsilon, for any $\varepsilon > 0$ and $X, Y \geq 0$, there exists a constant $C_\varepsilon > 0$ such that

$$X^a + Y^b + X^a Y^b \leq \varepsilon(X + Y) + C_\varepsilon. \tag{3.24}$$

The facts (3.22) and (3.23) enable us to apply (3.24) to (3.21) to deduce

$$c_1 \int_\Omega u^p |\nabla v|^2 + c_1 \int_\Omega u^2 |\nabla v|^{2q-2} \leq \varepsilon \left(\int_\Omega |\nabla u^{\frac{p}{2}}|^2 + \int_\Omega |\nabla |\nabla v|^q|^2 dx \right) + c_{13}. \tag{3.25}$$

Substituting (3.25) into (3.8) and choosing $\varepsilon = \min\left\{\frac{q-1}{q}, \frac{(p-1)d}{p}\right\}$, one has

$$\begin{aligned} \frac{d}{dt}\left(\int_\Omega u^p + \int_\Omega |\nabla v|^{2q}\right) &\leq -\frac{(p-1)\xi\gamma}{2}\int_\Omega u^{p+1} - 2q\beta \int_\Omega |\nabla v|^{2q} + c_{14} \\ &\leq -2q\beta\left(\int_\Omega u^p + \int_\Omega |\nabla v|^{2q}\right) + c_{15}, \end{aligned} \tag{3.26}$$

where we have used the Young's inequality to get

$$-\frac{(p-1)\xi\gamma}{2}\int_\Omega u^{p+1} \leq -2q\beta \int_\Omega u^p + c_{16}.$$

Solving the Grownwall's inequality (3.26) gives the existence of a constant c_{17} such that

$$\int_\Omega u^p + \int_\Omega |\nabla v|^{2q} \leq c_{17},$$

which yields the desired estimates (3.6) and (3.7). This completes the proof. □

Proof of Theorem 1.1. Lemma 3.2 provides a constant $c_1 > 0$ such that, for all $p > n$,

$$\|u(\cdot, t)\|_{L^p} \leq c_1 \text{ for all } t \in (0, T_{max}),$$

which together with the parabolic and elliptic regularity applied to the second and third equations in (1.5) shows

$$\|v(\cdot, t)\|_{L^\infty} + \|w(\cdot, t)\|_{L^\infty} \leq c_2 \text{ for all } t \in (0, T_{max}). \tag{3.27}$$

Then using (3.27) and Alikakos-Moser type iteration (cf. [3] or [39, Lemma A.1]), one can easily obtain that

$$\|u(\cdot, t)\|_{L^\infty} \leq c_3 \text{ for all } t \in (0, T_{max}). \tag{3.28}$$

Finally, Theorem 1.1 is an immediate consequence of (3.28) and Lemma 2.1. □

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