

*Research article***Predator-prey systems with defense switching and density-suppressed dispersal strategy**Jiawei Chu<sup>1</sup> and Hai-Yang Jin<sup>2,\*</sup><sup>1</sup> Department of Applied Mathematics, The Hong Kong Polytechnic University, Hung Hom, Hong Kong, China<sup>2</sup> School of Mathematics, South China University of Technology, Guangzhou 510640, China\* **Correspondence:** Email: mahyjin@scut.edu.cn.**Abstract:** In this paper, we consider the following predator-prey system with defense switching mechanism and density-suppressed dispersal strategy

$$\begin{cases} u_t = \Delta(d_1(w)u) + \frac{\beta_1 uvw}{u+v} - \alpha_1 u, & x \in \Omega, \quad t > 0, \\ v_t = \Delta(d_2(w)v) + \frac{\beta_2 uvw}{u+v} - \alpha_2 v, & x \in \Omega, \quad t > 0, \\ w_t = \Delta w - \frac{\beta_3 uvw}{u+v} + \sigma w \left(1 - \frac{w}{K}\right), & x \in \Omega, \quad t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega, \quad t > 0, \\ (u, v, w)(x, 0) = (u_0, v_0, w_0)(x), & x \in \Omega, \end{cases}$$

where  $\Omega \subset \mathbb{R}^2$  is a bounded domain with smooth boundary. Based on the method of energy estimates and Moser iteration, we establish the existence of global classical solutions with uniform-in-time boundedness. We further prove the global stability of co-existence equilibrium by using the Lyapunov functionals and LaSalle's invariant principle. Finally we conduct linear stability analysis and perform numerical simulations to illustrate that the density-suppressed dispersal may trigger the pattern formation.

**Keywords:** prey-predator system; defense switching; density-suppressed diffusion; global stability; pattern formation

**1. Introduction**

Defense switching means that prey species pay more attention on guarding against the relatively more abundant population predator [1], certain fish species in Lake Tanganyika against two

phenotypes (dextral and sinistral) of cichlid *Perissodus microlepis* [2] is a typical example. The dextral and sinistral phenotypes attack the prey fishes from the left-side and right-side, respectively. Pretend that the population of dextral individuals is more abundant, then prey fishes tend to be more defensive against the attacks from left-side, which leads to greater hunting success for sinistral individuals (relatively rare population). Based on a simple Lotka-Volterra equations of a two-predator one-prey system, Saleem et al. proposed a defensive switching model [1]

$$\begin{cases} u_t = u(-\alpha_1 + f_1 w), \\ v_t = v(-\alpha_2 + f_2 w), \\ w_t = w(\sigma - f_1 u - f_2 v), \end{cases} \quad (1.1)$$

with the predatory rates functions

$$\begin{cases} f_1(u, v) = \frac{\beta_1}{1 + \frac{u}{v}}, \\ f_2(u, v) = \beta_2(1 - \frac{f_1}{\beta_1}), \end{cases} \quad (1.2)$$

where  $u := u(x, t)$  and  $v := v(x, t)$  denote the density of two predators while  $w := w(x, t)$  is the prey density. The parameters  $\alpha_j$  ( $j = 1, 2$ ) account the death rate,  $\beta_j > 0$  ( $j = 1, 2$ ) are the predation coefficients and  $\sigma > 0$  accounts for the growth rate of the prey species. Moreover, the predatory rates  $f_1$  and  $f_2$ , also called “defensive switching functions”, possess a characteristic property that the rate of the prey attacked by a predator will decrease if this predator population becomes much more abundant than the population of another predator. Specifically, when a predator population becomes large, the prey species guards against it more vigilantly and switches to another predator, which is in relatively small population, to keep its individual from being hunted too much. Such prey behaviors result in less successful hunting for abundant population predator species and more successful hunting for relatively rare one [1].

In light of the defensive switching model Eqs (1.1) and (1.2) in [1], Pang and Wang [3] considered the following reaction-diffusion system by introducing the random movements of species and the intra-specific interaction between the prey in Eq (1.1)

$$\begin{cases} u_t = d_1 \Delta u + \frac{\beta_1 u v w}{u+v} - \alpha_1 u, \\ v_t = d_2 \Delta v + \frac{\beta_2 u v w}{u+v} - \alpha_2 v, \\ w_t = d_3 \Delta w - \frac{(\beta_1 + \beta_2) u v w}{u+v} + \sigma w \left(1 - \frac{w}{K}\right), \end{cases} \quad (1.3)$$

where constants  $d_j > 0$  ( $j = 1, 2, 3$ ) account for diffusion coefficients and the positive parameter  $K$  represents the environmental carrying capacity to the prey species. We note that the interaction mechanism between the predators and prey in the defensive switching model is substantially different from that in the ratio-dependent predator-prey system [4] though they seem to have some similar structures.

In [1], Saleem et al. proved that the co-existence steady state is globally asymptotically stable except the case where the two predators have the same mortality rates; Otherwise, the system has a periodic solution. Pang and Wang [3] proved the co-existence steady states is globally asymptotically stable no matter  $\alpha_1$  and  $\alpha_2$  are equal or not and hence no pattern formation will arise from the system (1.3). If the term  $d_1 \Delta u$  is replaced by a cross diffusion term  $\Delta(d_1 u + \frac{k u}{\epsilon + v^2})$  ( $k, \epsilon > 0$ ), however, they showed that the cross diffusion between predators can drive stationary patterns by using

Leray-Schauder degree theory. Recently, the system (1.3) with prey-taxis (the cross diffusion between the predators and prey) was considered by Wang and Guo [5]. They established the existence of globally bounded solutions and global stability of constant steady states for small prey-tactic coefficient and numerically demonstrated that strong prey-taxis can induce the pattern formation if the two predators have different mortality rates. Subsequently, the existence of nonconstant steady states was obtained for some range of repulsive prey-taxis coefficient with more general functional response functions in [6]. From the above results, we find that the dispersal strategies (like cross-diffusion or prey-taxis) play important roles in determining the population distribution profiles.

In this paper, we shall consider a different dispersal strategy - density-suppressed diffusion, which was first used in [7] to describe the directed movement of predators in the predator-prey systems to fit the experimental observations. This type of diffusion assumes that the predator's diffusion decreasingly depends on the density distribution of the prey. As shown in [8], the density-suppressed diffusion can explain the heterogenous population distribution observed in the field experiment of [7] while random diffusion can not. Hence density-suppressed diffusion is a dispersal strategy employed in the predator-prey system. By taking into account the density-suppressed diffusion, the defensive switching model reads

$$\begin{cases} u_t = \Delta(d_1(w)u) + \frac{\beta_1 uvw}{u+v} - \alpha_1 u, \\ v_t = \Delta(d_2(w)v) + \frac{\beta_2 uvw}{u+v} - \alpha_2 v, \\ w_t = d_3 \Delta w - \frac{(\beta_1 + \beta_2)uvw}{u+v} + \sigma w \left(1 - \frac{w}{K}\right), \end{cases} \quad (1.4)$$

where  $d_j(w) > 0$  and  $d_j^0(w) < 0$  ( $j = 1, 2$ ). Note that the property  $d_j^0(w) < 0$  means that predators will decrease their random diffusion rates at higher density of the prey species in order for predation. If we expand the density-suppressed diffusion

$$\Delta(d_j(w)\phi) = \nabla \cdot (d_j(w) \nabla \phi + \phi \nabla d_j(w)), \quad \phi = u, v,$$

we find that the density-suppressed diffusion indeed intrinsically includes both random diffusion and advection (prey-taxis) components. The differences from the spatial models considered in [3,5,6] is that here both diffusion and advection coefficient are not constant but functions of the prey density. It was shown that the density-suppressed diffusion in the predator-prey systems may generate spatially non-homogeneous patterns that the random diffusion can not do [8] and may bring substantially different dynamics [9].

Beyond the predator-prey systems, the density-suppressed diffusion has already been commonly used in the modeling of other biological processes such as the chemotaxis [10, 11], bacterial movement [12, 13] and so on. Since the possible degeneracy caused by the density-suppressed diffusion brings considerable challenges for analysis, the studies of these biological models with density-suppressed diffusion have been increasingly attracting attentions and produced many interesting analytical results [14–28] alongside rich numerical simulations demonstrating complex dynamics and patterns [29–31].

The purpose of this paper is to study the global dynamics of the defensive switching model with density-suppressed diffusion including global existence and asymptotic behavior of solutions as well

as pattern formations. Specifically we consider the following problem

$$\begin{cases} u_t = \Delta(d_1(w)u) + \frac{\beta_1 uvw}{u+v} - \alpha_1 u, & x \in \Omega, \quad t > 0, \\ v_t = \Delta(d_2(w)v) + \frac{\beta_2 uvw}{u+v} - \alpha_2 v, & x \in \Omega, \quad t > 0, \\ w_t = \Delta w - \frac{\beta_3 uvw}{u+v} + \sigma w \left(1 - \frac{w}{K}\right), & x \in \Omega, \quad t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega, \quad t > 0, \\ (u, v, w)(x, 0) = (u_0, v_0, w_0)(x), & x \in \Omega, \end{cases} \quad (1.5)$$

where  $\Omega \subset \mathbb{R}^2$  is a bounded domain with smooth boundary. The parameters  $\alpha_j$  ( $j = 1, 2$ ),  $\beta_j$  ( $j = 1, 2, 3$ ),  $\sigma$  and  $K$  are all positive constants. Note that we have assumed  $d_3 = 1$  without loss of generality and consider more general constant  $\beta_3$  compared to Eq (1.4). We suppose the density-suppressed diffusion coefficients  $d_j(w)$  satisfy the following conditions:

$(H_0)$   $d_j(w) \in C^2([0, 1])$  with  $d_j(w) > 0$  and  $d_j'(w) \leq 0$  for all  $w \geq 0$ ,  $j = 1, 2$ .

We further suppose the initial data satisfy

$$(u_0, v_0, w_0) \in [W^{1,1}(\Omega)]^3 \text{ with } u_0, v_0 > 0, w_0 \geq 0. \quad (1.6)$$

Then our first result on the global existence of solutions is stated in the following theorem.

**Theorem 1.1** (Global boundedness). *Let  $\Omega \subset \mathbb{R}^2$  be a bounded domain with smooth boundary. Assume the assumption  $(H_0)$  holds and the initial data  $(u_0, v_0, w_0)$  satisfy Eq (1.6). Then there exists a uniquely determined triple  $(u, v, w)$  of nonnegative functions which solves Eq (1.5) classically in  $\Omega \times (0, 1)$  and satisfies*

$$ku(\cdot, t)k_{L^1} + kv(\cdot, t)k_{L^1} + kw(\cdot, t)k_{W^{1,1}} \leq M_0 \text{ for all } t > 0,$$

where  $M_0 > 0$  is a constant independent of  $t$ . Particularly, one has

$$0 < w(x, t) \leq K := \max\{w_0, K\} \text{ for all } (x, t) \in \Omega \times (0, 1).$$

**Remark 1.2.** Note that in works [5, 6] considering the defensive switching model with prey-taxis, the smallness of prey-taxis coefficient was required to ensure the global boundedness of solutions. Here we obtain the existence of global classical solutions with uniform-in-time boundedness without any smallness assumptions on the parameters.

Next, we shall show the global stability of constant steady states. In fact it is straightforward to find that a positive constant steady state  $(u, v, w)$  exists if and only if  $K > \frac{\alpha_1}{\beta_1} + \frac{\alpha_2}{\beta_2}$ , where  $u, v, w > 0$ , satisfying

$$\alpha_1 = \frac{\beta_1 v w}{u + v}, \quad \alpha_2 = \frac{\beta_2 u w}{u + v}, \quad \sigma \left(1 - \frac{w}{K}\right) = \frac{\beta_3 u v}{u + v}, \quad (1.7)$$

which can be solved to obtain

$$u = \frac{\sigma}{\beta_3} \left(1 - \frac{w}{K}\right) \left(1 + \frac{\alpha_2 \beta_1}{\alpha_1 \beta_2}\right), \quad v = \frac{\sigma}{\beta_3} \left(1 - \frac{w}{K}\right) \left(1 + \frac{\alpha_1 \beta_2}{\alpha_2 \beta_1}\right), \quad w = \frac{\alpha_1}{\beta_1} + \frac{\alpha_2}{\beta_2}. \quad (1.8)$$

Then we can show that  $(u, v, w)$  is globally asymptotically stable under certain conditions, as stated in the following theorem.

**Theorem 1.3** (Global stabilization). *Suppose the conditions in Theorem 1.1 hold and let  $(u, v, w)$  be the solution obtained in Theorem 1.1. If  $K > \frac{\alpha_1}{\beta_1} + \frac{\alpha_2}{\beta_2}$  and*

$$\frac{1}{\sigma} > \max_{0 < w \leq K} \left( \frac{|d_1^0(w)|^2 w^2}{\beta_1 d_1(w)} + \frac{|d_2^0(w)|^2 w^2}{\beta_2 d_2(w)} \right) \frac{K - w}{4Kw}$$

*Proof.* From the third equation of (1.5), we have

$$w_t = \Delta w - w + g(u, v, w) \text{ in } \Omega, \quad \frac{\partial w}{\partial \nu} = 0,$$

with  $g(u, v, w) := w - \frac{\beta_3 uvw}{u+v} + \sigma w \left(1 - \frac{w}{K}\right)$ . Since  $0 < w \leq K$  (see Eq (2.1)) and  $u, v > 0$ , we have

$$\begin{aligned} |g(u, v, w)| &\leq K + \beta_3 K u + \sigma K + \frac{\sigma K^2}{K} \\ &= K \left(1 + \beta_3 u + \sigma + \frac{\sigma K}{K}\right) \\ &\leq K \left(1 + \beta_3 + \sigma + \frac{\sigma K}{K}\right)(u + 1), \end{aligned}$$

which, together with Eq (2.3), gives

$$\|g(u, v, w)\|_{L^r} \leq c_3. \quad (2.6)$$

With Eq (2.6), we use the results in [39, Lemma 1] to obtain Eq (2.4) with Eq (2.5) directly.  $\square$

Now we will show some basic boundedness properties of the solution  $(u, v, w)$  obtained in Lemma 2.1.

**Lemma 2.4.** *Let  $(u, v, w)$  be the classical solution of (1.5) obtained in Lemma 2.1. Then it holds that*

$$\int_{\Omega} u + \int_{\Omega} v \leq K_1 \text{ for all } t \in (0, T_{\max}), \quad (2.7)$$

and

$$\int_t^{t+\tau} \int_{\Omega} u^2 + \int_t^{t+\tau} \int_{\Omega} v^2 \leq K_2 \text{ for all } t \in (0, T_{\max} - \tau), \quad (2.8)$$

where the constants  $K_1, K_2 > 0$  are independent of  $t$  and  $\tau := \min\{1, T_{\max}/2\}$ .

*Proof.* Multiplying the first and second equations of (1.5) by  $\beta_3$ , and multiplying the third equation of (1.5) by  $(\beta_1 + \beta_2)$ , then adding the resulting equations, one obtains

$$\begin{aligned} [\beta_3(u + v) + (\beta_1 + \beta_2)w]_t &= \Delta[\beta_3(d_1(w)u + d_2(w)v) + (\beta_1 + \beta_2)w] \\ &\quad - \beta_3(\alpha_1 u + \alpha_2 v) + \sigma(\beta_1 + \beta_2)w \left(1 - \frac{w}{K}\right). \end{aligned} \quad (2.9)$$

Thus, integrating Eq (2.9) with respect to  $x$  over  $\Omega$ , we have

$$\frac{d}{dt} \int_{\Omega} [\beta_3(u + v) + (\beta_1 + \beta_2)w] + \beta_3 \int_{\Omega} (\alpha_1 u + \alpha_2 v) + \frac{\sigma(\beta_1 + \beta_2)}{K} \int_{\Omega} w^2 = \sigma(\beta_1 + \beta_2) \int_{\Omega} w. \quad (2.10)$$

Using Young's inequality, one has

$$(\sigma + 1)(\beta_1 + \beta_2) \int_{\Omega} w \leq \frac{\sigma(\beta_1 + \beta_2)}{K} \int_{\Omega} w^2 + \frac{(\sigma + 1)^2(\beta_1 + \beta_2)K|\Omega|}{4\sigma},$$

which, substituted into Eq (2.10), gives

$$\frac{d}{dt} \int_{\Omega} [\beta_3 u + \beta_3 v + (\beta_1 + \beta_2)w] + \gamma_0 \int_{\Omega} [\beta_3 u + \beta_3 v + (\beta_1 + \beta_2)w] \leq c_1, \quad (2.11)$$

where  $\gamma_0 := \min\{\alpha_1, \alpha_2, 1\}$  and  $c_1 := \frac{(\sigma+1)^2(\beta_1+\beta_2)K[\Omega]}{4\sigma}$ . Then, applying the Grönwall's inequality to Eq (2.11) gives

$$\int_{\Omega} (u + v) \leq \int_{\Omega} (u_0 + v_0) + \frac{(\beta_1 + \beta_2)}{\beta_3} \int_{\Omega} w_0 + \frac{c_1}{\gamma_0 \beta_3},$$

which yields Eq (2.7).

Next, we shall show Eq (2.8) holds based on some ideas in [8]. Under the homogeneous Neumann boundary conditions, we define a shifted Laplacian operator  $B = -\Delta + \gamma_1$  with  $\gamma_1 := \min\left\{\frac{\alpha_1}{2d_1(0)}, \frac{\alpha_2}{2d_2(0)}\right\} > 0$ . Then  $B$  is a sectorial operator in  $L^p(\Omega)$  for all  $p \in (1, \infty)$  [32], and one can easily show that its inverse  $B^{-1}$  satisfies

$$\|B^{-1}\phi\|_{L^2} \leq c_2 \|\phi\|_{L^2} \text{ for all } \phi \in L^2(\Omega), \quad (2.12)$$

and

$$\|B^{-\frac{1}{2}}\phi\|_{L^2}^2 \leq c_2 \|\phi\|_{L^2}^2 \text{ for all } \phi \in L^2(\Omega) \quad (2.13)$$

for some constant  $c_2 > 0$ . Using the assumptions in  $(H_0)$  and Eq (2.1), we can derive that there exist two positive constants  $\delta_1$  and  $\delta_2$  independent of  $t$  such for all  $j = 1, 2$  one has

$$0 < \delta_1 \leq |d_j^0(w)| \leq \delta_2, \quad (2.14)$$

and

$$0 < d_j(K) \leq d_j(w) \leq d_j(0). \quad (2.15)$$

Using the definition of  $B$ , we can rewrite Eq (2.9) as

$$+ \int_{\Omega} w$$

which combined with Eq (2.15) enables us to find a positive constant  $\gamma_2 := \min\{d_1(K), d_2(K)\}$ ,  $1/g$  such that

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} \left| B^{-\frac{1}{2}} [\beta_3(u+v) + (\beta_1 + \beta_2)w] \right|^2 + 2\gamma_2 \int_{\Omega} [\beta_3(u+v) + (\beta_1 + \beta_2)w]^2 \\ & \leq 2c_3 \int_{\Omega} B^{-1} [\beta_3(u+v) + (\beta_1 + \beta_2)w]. \end{aligned} \quad (2.17)$$

On one hand, using Eq (2.12), the Hölder inequality and Young's inequality, one has

$$\begin{aligned} 2c_3 \int_{\Omega} B^{-1} [\beta_3(u+v) + (\beta_1 + \beta_2)w] & \leq 2c_3 j \Omega^{\frac{1}{2}} \|B^{-1} [\beta_3(u+v) + (\beta_1 + \beta_2)w]\|_{L^2} \\ & \leq 2c_2 c_3 j \Omega^{\frac{1}{2}} \|\beta_3(u+v) + (\beta_1 + \beta_2)w\|_{L^2} \\ & \leq \frac{\gamma_2}{2} \int_{\Omega} [\beta_3(u+v) + (\beta_1 + \beta_2)w]^2 + \frac{2c_2^2 c_3^2 j \Omega}{\gamma_2}. \end{aligned} \quad (2.18)$$

On the other hand, using Eq (2.13), we get

$$\frac{\gamma_2}{2c_2} \int_{\Omega} j B^{-\frac{1}{2}} [\beta_3(u+v) + (\beta_1 + \beta_2)w]^2 \leq \frac{\gamma_2}{2} \int_{\Omega} [\beta_3(u+v) + (\beta_1 + \beta_2)w]^2. \quad (2.19)$$

Defining  $z(t) := \int_{\Omega} j B^{-\frac{1}{2}} [\beta_3(u+v) + (\beta_1 + \beta_2)w]^2$ , and substituting Eqs (2.18) and (2.19) into Eq (2.17), we obtain

$$z'(t) + \frac{\gamma_2}{2c_2} z(t) + \gamma_2 \int_{\Omega} [\beta_3(u+v) + (\beta_1 + \beta_2)w]^2 \leq \frac{2c_2^2 c_3^2 j \Omega}{\gamma_2}. \quad (2.20)$$

Then applying the Grönwall's inequality to Eq (2.20) alongside Eq (2.13), one has

$$\begin{aligned} z(t) & \leq \frac{4c_2^3 c_3^2 j \Omega}{\gamma_2^2} + \int_{\Omega} \left| B^{-\frac{1}{2}} [\beta_3(u_0 + v_0) + (\beta_1 + \beta_2)w_0] \right|^2 \\ & \leq \frac{4c_2^3 c_3^2 j \Omega}{\gamma_2^2} + c_2 \int_{\Omega} j [\beta_3(u_0 + v_0) + (\beta_1 + \beta_2)w_0]^2 \leq c_4. \end{aligned} \quad (2.21)$$

Integrating Eq (2.20) over  $(t, t + \tau)$  for  $\tau = \min\{1, T_{\max}/2g\}$  and applying Eq (2.21), we get

$$\begin{aligned} \gamma_2 \beta_3^2 \int_t^{t+\tau} \int_{\Omega} (u^2 + v^2) & \leq \gamma_2 \int_t^{t+\tau} \int_{\Omega} [\beta_3(u+v) + (\beta_1 + \beta_2)w]^2 \\ & \leq \frac{2c_2^2 c_3^2 j \Omega}{\gamma_2} \tau + z(t) \\ & \leq \frac{2c_2^2 c_3^2 j \Omega}{\gamma_2} + c_4, \end{aligned}$$

which gives Eq (2.8). Then we complete the proof of Lemma 2.4.  $\square$

**Lemma 2.5.** *Let  $(u, v, w)$  be the classical solution of (1.5) and  $\tau = \min\{1, T_{\max}/2g\}$ . Then it holds that*

$$\int_t^{t+\tau} \int_{\Omega} j r w^2 \leq K_3 \quad \text{for all } t \in (0, T_{\max} - \tau), \quad (2.22)$$

and

$$\|r w\|_{L^2} \leq K_4 \quad \text{for all } t \in (0, T_{\max}), \quad (2.23)$$

as well as

$$\int_t^{t+\tau} \int_{\Omega} |j \Delta w|^2 \leq K_5 \quad \text{for all } t \in (0, T_{\max} - \tau), \quad (2.24)$$

where the positive constants  $K_3$ ,  $K_4$  and  $K_5$  are independent of  $t$ .

*Proof.* Multiplying the third equation of (1.5) by  $w$ , integrating the resulting equation by parts and using  $0 < w \leq K$  in Eq (2.1), we obtain

$$\frac{d}{dt} \int_{\Omega} w^2 + 2 \int_{\Omega} |r w|^2 + 2\beta_3 \int_{\Omega} \frac{uvw^2}{u+v} + \frac{2\sigma}{K} \int_{\Omega} w^3 = 2\sigma \int_{\Omega} w^2 \leq 2\sigma K^2 j \Omega j. \quad (2.25)$$

Integrating Eq (2.25) over  $(t, t + \tau)$  and using the facts  $0 < \tau \leq 1$  and Eq (2.1), one has Eq (2.22) directly.

Next, we multiply the third equation of (1.5) by  $-\Delta w$ , use the integration by parts to the resulting equation, and apply Young's inequality as well as Eq (2.1) to get

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} |r w|^2 + 2 \int_{\Omega} |j \Delta w|^2 &= 2\beta_3 \int_{\Omega} \frac{uvw}{u+v} \Delta w + \frac{2\sigma}{K} \int_{\Omega} w^2 \Delta w - 2\sigma \int_{\Omega} w \Delta w \\ &\leq \int_{\Omega} |j \Delta w|^2 + 3\beta_3^2 \int_{\Omega} \left( \frac{uvw}{u+v} \right)^2 + \frac{3\sigma^2}{K^2} \int_{\Omega} w^4 + 3\sigma^2 \int_{\Omega} w^2 \\ &\leq \int_{\Omega} |j \Delta w|^2 + 3\beta_3^2 K^2 \int_{\Omega} u^2 + \frac{3\sigma^2 K^4 j \Omega j}{K^2} + 3\sigma^2 K^2 j \Omega j, \end{aligned}$$

which entails

$$\frac{d}{dt} \int_{\Omega} |r w|^2 + \int_{\Omega} |j \Delta w|^2 \leq 3\beta_3^2 K^2 \int_{\Omega} u^2 + c_1, \quad (2.26)$$

where  $c_1 := \frac{3\sigma^2 K^4 j \Omega j}{K^2} + 3\sigma^2 K^2 j \Omega j$ . Applying a Gagliardo-Nirenberg type inequality derived in [16, Lemma 2.5] and noting the fact  $\|w\|_{L^2} \leq K \|j \Omega j\|^{\frac{1}{2}}$ , we have

$$\begin{aligned} \int_{\Omega} |r w|^2 &\leq c_2 (\| \Delta w \|_{L^2} \|w\|_{L^2} + \|w\|_{L^2}^2) \\ &\leq c_2 K \|j \Omega j\|^{\frac{1}{2}} (\| \Delta w \|_{L^2} + K \|j \Omega j\|^{\frac{1}{2}}) \\ &\leq \frac{1}{2} \int_{\Omega} |j \Delta w|^2 + \frac{(2 + c_2)c_2 K^2 j \Omega j}{2}, \end{aligned}$$

which substituted into Eq (2.26) gives

$$\frac{d}{dt} \int_{\Omega} |r w|^2 + \int_{\Omega} |r w|^2 + \frac{1}{2} \int_{\Omega} |j \Delta w|^2 \leq c_3 \left( \int_{\Omega} u^2 + 1 \right), \quad (2.27)$$

with  $c_3 := 3\beta_3^2 K^2 + c_1 + \frac{(2+c_2)c_2 K^2 j \Omega j}{2}$ . For any  $t \in (0, T_{\max})$ , by Eq (2.22) there exists a  $t_0 := t_0(t) \in ((t - \tau)_+, t)$  such that  $t_0 \geq 0$  and

$$\int_{\Omega} |r w(x, t_0)|^2 dx \leq c_4. \quad (2.28)$$

Moreover, from Eq (2.8), we know that

$$\int_{t_0}^{t_0+\tau} \int_{\Omega} u^2(x, s) dx ds \leq K_2 \text{ for all } t_0 \in (0, T_{\max} - \tau). \quad (2.29)$$

Multiplying Eq (2.27) by  $e^t$  and integrating the results with respect to  $t$  over  $(t_0, t)$ , and using the facts Eqs (2.28) and (2.29) and  $t_0 < t \leq t_0 + \tau \leq t_0 + 1$ , one has

$$\begin{aligned} \|r w(\cdot, t)\|_{L^2}^2 &\leq e^{t_0-t} \|r w(\cdot, t_0)\|_{L^2}^2 + e^{-t} c_3 \int_{t_0}^t e^s (k u(\cdot, s)\|_{L^2}^2 + 1) \\ &\leq c_4 + c_3 \int_{t_0}^t (k u(\cdot, s)\|_{L^2}^2 + 1) \\ &\leq c_4 + c_3 \int_{t_0}^{t_0+\tau} (k u(\cdot, s)\|_{L^2}^2 + 1) \\ &\leq c_4 + c_3 K_2 + c_3. \end{aligned}$$

Hence Eq (2.23) holds. On the other hand, integrating Eq (2.27) along with Eqs (2.8) and (2.23), it holds that

$$\begin{aligned} \int_t^{t+\tau} \int_{\Omega} |j \Delta w|^2 &\leq 2c_3 \int_t^{t+\tau} \int_{\Omega} u^2 + 2c_3 \tau + 2 \int_{\Omega} |j r w(\cdot, t)|^2 \\ &\leq 2c_3(K_2 + 1) + 2K_4^2, \end{aligned}$$

which gives Eq (2.24). Then we complete the proof of Lemma 2.5.  $\square$

### 3. Global existence: Proof of Theorem 1.1

In this section, we shall show the existence of global classical solution based on some ideas in [8]. To this end, we first establish the *a priori*  $L^2$ -estimates of  $u$  and  $v$ .

**Lemma 3.1** ( $L^2$ -Boundedness). *Suppose that the assumptions in Lemma 2.1 hold and let  $(u, v, w)$  be the classical solution of (1.5) which is defined on its maximal existence time interval  $[0, T_{\max})$ . Then the solution of (1.5) satisfies*

$$k u(\cdot, t)\|_{L^2} + k v(\cdot, t)\|_{L^2} \leq K_6 \text{ for all } t \in (0, T_{\max}), \quad (3.1)$$

where the constant  $K_6 > 0$  is independent of  $t$ .

*Proof.* Multiplying the first equation of (1.5) by  $u$ , integrating the results with respect to  $x$  over  $\Omega$  by parts and using the fact  $u, v > 0$  for all  $t \in (0, T_{\max})$ , we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\Omega} u^2 + \alpha_1 \int_{\Omega} u^2 + \int_{\Omega} d_1(w) |j r u|^2 &= \beta_1 \int_{\Omega} \frac{u v w}{u + v} u - \int_{\Omega} d_1^0(w) u r w \cdot r u \\ &\leq \beta_1 K \int_{\Omega} u^2 + \int_{\Omega} u |j d_1^0(w) |j r w |j r u|, \end{aligned}$$

which along with Eqs (2.14) and (2.15) indicates

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} u^2 + \alpha_1 \int_{\Omega} u^2 + d_1(K) \int_{\Omega} |j r u|^2 \leq \beta_1 K \int_{\Omega} u^2 + \delta_2 \int_{\Omega} u |j r w |j r u|. \quad (3.2)$$

Applying Young's inequality and the Hölder inequality, one gets

$$\begin{aligned} \delta_2 \int_{\Omega} u_j r w_j r u_j &\leq \frac{d_1(K)}{2} \int_{\Omega} j r u_j^2 + \frac{\delta_2^2}{2d_1(K)} \int_{\Omega} u_j^2 j r w_j^2 \\ &\leq \frac{d_1(K)}{2} \int_{\Omega} j r u_j^2 + \frac{\delta_2^2}{2d_1(K)} k u k_{L^4}^2 k r w k_{L^4}^2, \end{aligned}$$

which updates Eq (3.2) as

$$\frac{d}{dt} k u k_{L^2}^2 + 2\alpha_1 k u k_{L^2}^2 + d_1(K) k r u k_{L^2}^2 \leq 2\beta_1 K k u k_{L^2}^2 + \frac{\delta_2^2}{d_1(K)} k u k_{L^4}^2 k r w k_{L^4}^2. \quad (3.3)$$

Using the Gagliardo-Nirenberg inequality, the following inequality [16, Lemma 2.5]

$$k r w k_{L^4}^2 \leq c_1 (k \Delta w k_{L^2} k r w k_{L^2} + k r w k_{L^2}^2),$$

and Eq (2.23), we obtain

$$\begin{aligned} \frac{\delta_2^2}{d_1(K)} k u k_{L^4}^2 k r w k_{L^4}^2 &\leq c_2 (k r u k_{L^2} k u k_{L^2} + k u k_{L^2}^2) (k \Delta w k_{L^2} K_4 + K_4^2) \\ &\leq c_2 K_4 k r u k_{L^2} k u k_{L^2} k \Delta w k_{L^2} + c_2 K_4 k u k_{L^2}^2 k \Delta w k_{L^2} \\ &\quad + c_2 K_4^2 k r u k_{L^2} k u k_{L^2} + c_2 K_4^2 k u k_{L^2}^2 \\ &\leq d_1(K) k r u k_{L^2}^2 + c_3 k u k_{L^2}^2 k \Delta w k_{L^2}^2 + c_4 k u k_{L^2}^2, \end{aligned} \quad (3.4)$$

where  $c_3 := \frac{c_2^2 K_4^2}{d_1(K)}$  and  $c_4 := \frac{c_2^2 K_4^4}{2d_1(K)} + \frac{d_1(K)}{2} + c_2 K_4^2$ .

Then substituting Eq (3.4) into Eq (3.3), we obtain

$$\begin{aligned} \frac{d}{dt} k u k_{L^2}^2 &\leq (2\beta_1 K + c_4) k u k_{L^2}^2 + c_3 k u k_{L^2}^2 k \Delta w k_{L^2}^2 \\ &\leq c_5 k u k_{L^2}^2 (1 + k \Delta w k_{L^2}^2), \end{aligned} \quad (3.5)$$

where  $c_5 := 2\beta_1 K + c_4 + c_3$ . For any  $t \in (0, T_{\max})$ , by Eq (2.8) there exists a  $t_0 := t_0(t) \in ((t - \tau)_+, t)$  such that  $t_0 \geq 0$  and

$$\int_{\Omega} u^2(x, t_0) dx \leq c_6. \quad (3.6)$$

On the other hand, from Eq (2.24), we know that

$$\int_{t_0}^{t_0+\tau} \int_{\Omega} |j \Delta w(x, s)|^2 dx ds \leq K_5, \text{ for all } t \in (0, T_{\max} - \tau). \quad (3.7)$$

Then integrating Eq (3.5) with respect to  $t$  over  $(t_0, t)$ , and using the Eqs (3.6) and (3.7) and  $t \leq t_0 + \tau \leq t_0 + 1$ , one has

$$\begin{aligned} k u(\cdot, t) k_{L^2}^2 &\leq k u(\cdot, t_0) k_{L^2}^2 \cdot e^{c_5 \int_{t_0}^t (1 + k \Delta w(\cdot, s) k_{L^2}^2) ds} \\ &\leq k u(\cdot, t_0) k_{L^2}^2 \cdot e^{c_5 \int_{t_0}^{t_0+\tau} (1 + k \Delta w(\cdot, s) k_{L^2}^2) ds} \\ &\leq c_6 e^{c_5(1+K_5)}. \end{aligned} \quad (3.8)$$

Similarly, we can show that

$$k v(\cdot, t) k_{L^2}^2 \leq c_7. \quad (3.9)$$

Then the combination of Eqs (3.8) and (3.9) gives Eq (3.1). The proof of Lemma 3.1 is completed.  $\square$

**Lemma 3.2** ( $L^1$ -Boundedness). Suppose that the assumptions in Lemma 2.1 hold and let  $(u, v, w)$  be the classical solution of (1.5) defined on its maximal existence time interval  $[0, T_{\max})$ . Then it holds that

$$ku(\cdot, t)k_{L^1} + kv(\cdot, t)k_{L^1} \leq K_7 \text{ for all } t \in (0, T_{\max}), \quad (3.10)$$

where the constant  $K_7 > 0$  is independent of  $t$ .

*Proof.* Multiplying the first equation of (1.5) by  $u^{p-1}$  with  $p > 2$  and integrating the resulting equation over  $\Omega$  by parts, one has

$$\begin{aligned} & \frac{1}{p} \frac{d}{dt} \int_{\Omega} u^p + (p-1) \int_{\Omega} u^{p-2} d_1(w) j r u j^2 + \alpha_1 \int_{\Omega} u^p \\ &= -(p-1) \int_{\Omega} u^{p-1} d_1^0(w) r u \cdot r w + \beta_1 \int_{\Omega} \frac{vw}{u+v} u^p. \end{aligned} \quad (3.11)$$

Using Eqs (2.14) and (2.15) and  $0 < \frac{vw}{u+v} \leq w \leq K$  which is satisfied by the fact  $u, v > 0$  for all  $t \in (0, T_{\max})$ , from Eq (3.11) we get

$$\begin{aligned} & \frac{1}{p} \frac{d}{dt} \int_{\Omega} u^p + (p-1) d_1(K) \int_{\Omega} u^{p-2} j r u j^2 + \alpha_1 \int_{\Omega} u^p \\ & \leq (p-1) \delta_2 \int_{\Omega} u^{p-1} j r u j j r w j + \beta_1 K \int_{\Omega} u^p \\ & \leq \frac{(p-1) d_1(K)}{2} \int_{\Omega} u^{p-2} j r u j^2 + \frac{\delta_2^2 (p-1)}{2 d_1(K)} \int_{\Omega} u^p j r w j^2 + \beta_1 K \int_{\Omega} u^p, \end{aligned}$$

which together with the fact

$$\frac{2(p-1) d_1(K)}{p} \int_{\Omega} j r u^{\frac{p}{2}} j^2 = \frac{p(p-1) d_1(K)}{2} \int_{\Omega} u^{p-2} j r u j^2,$$

gives

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} u^p + \frac{2(p-1) d_1(K)}{p} \int_{\Omega} j r u^{\frac{p}{2}} j^2 + \alpha_1 p \int_{\Omega} u^p \\ & \leq \frac{p(p-1) \delta_2^2}{2 d_1(K)} \int_{\Omega} u^p j r w j^2 + p \beta_1 K \int_{\Omega} u^p. \end{aligned} \quad (3.12)$$

Noting the fact  $ku k_{L^2} \leq K_6$  in Eq (3.1), and using Lemma 2.3, we can find a constant  $c_1 > 0$  such that

$$k r w k_{L^4} \leq c_1. \quad (3.13)$$

Then we can use the Hölder inequality, Gagliardo-Nirenberg inequality and Young's inequality as well

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as Eq (3.13) to obtain

where the constant  $c_6 > 0$  is independent of  $t$  and  $p$ . Then using the Moser iteration process [33], there exists a constant  $c_7 > 0$  independent of  $t$  such that

$$ku(\cdot, t)k_{L^1} \leq c_7. \quad (3.19)$$

Similarly, we can find a constant  $c_8 > 0$  such that

$$kv(\cdot, t)k_{L^1} \leq c_8. \quad (3.20)$$

Then the combination of Eqs (3.19) and (3.20) gives Eq (3.10). The proof of Lemma 3.2 is completed.  $\square$

*Proof of Theorem 1.1.* From Lemma 3.2, we know there exists a constant  $c_1 > 0$  independent of  $t$  such that

$$ku(\cdot, t)k_{L^1} \leq c_1 \left( \frac{1}{t} + \frac{1}{t^2} + \frac{1}{t^3} + \frac{1}{t^4} + \frac{1}{t^5} + \frac{1}{t^6} + \frac{1}{t^7} + \frac{1}{t^8} + \frac{1}{t^9} + \frac{1}{t^{10}} + \frac{1}{t^{11}} + \frac{1}{t^{12}} + \frac{1}{t^{13}} + \frac{1}{t^{14}} + \frac{1}{t^{15}} + \frac{1}{t^{16}} + \frac{1}{t^{17}} + \frac{1}{t^{18}} + \frac{1}{t^{19}} + \frac{1}{t^{20}} \right)$$

and  $J_u(t) = 0$  iff  $u = u$ . Using the similar way, one has  $J_v(t)$ ,  $J_w(t) \geq 0$  and “=” holds iff  $v = v$ ,  $w = w$ . Hence according to the definition of  $E(t)$  in Eq (4.1), we obtain that  $E(u, v, w) > 0$  for all  $(u, v, w) \neq (u, v, w)$  and moreover  $E(u, v, w) = 0$ .

Next, we shall show that  $\frac{d}{dt}E(t) \leq 0$  under the conditions Eq (4.2) and  $K > \frac{\alpha_1}{\beta_1} + \frac{\alpha_2}{\beta_2}$ . In fact, using the equations in Eq (1.5), one has

$$\begin{aligned} \frac{dE(t)}{dt} &= \beta_2 v \int_{\Omega} \left( \frac{u - u}{u} \right) u_t + \beta_1 u \int_{\Omega} \left( \frac{v - v}{v} \right) v_t + \frac{\beta_1 \beta_2 (u + v)}{\beta_3} \int_{\Omega} \left( \frac{w - w}{w} \right) w_t \\ &= -\beta_2 v u \int_{\Omega} \frac{d_1(w) j r u j^2}{u^2} - \beta_2 v u \int_{\Omega} \frac{d_1^0(w) r u \cdot r w}{u} + \beta_2 v \int_{\Omega} (u - u) \left( \frac{\beta_1 v w}{u + v} - \alpha_1 \right) \\ &\quad - \beta_1 u v \int_{\Omega} \frac{d_2(w) j r v j^2}{v^2} - \beta_1 u v \int_{\Omega} \frac{d_2^0(w) r v \cdot r w}{v} + \beta_1 u \int_{\Omega} (v - v) \left( \frac{\beta_2 u w}{u + v} - \alpha_2 \right) \\ &\quad - \frac{\beta_1 \beta_2 (u + v) w}{\beta_3} \int_{\Omega} \frac{j r w j^2}{w^2} + \frac{\beta_1 \beta_2 (u + v)}{\beta_3} \int_{\Omega} (w - w) \left( \sigma - \frac{\sigma w}{K} - \frac{\beta_3 u v}{u + v} \right), \end{aligned}$$

which can be rewritten as

$$\begin{aligned} \frac{dE(t)}{dt} &= - \int_{\Omega} X^T B X + \beta_2 v \int_{\Omega} (u - u) \left( \frac{\beta_1 v w}{u + v} - \alpha_1 \right) \\ &\quad + \beta_1 u \int_{\Omega} (v - v) \left( \frac{\beta_2 u w}{u + v} - \alpha_2 \right) \\ &\quad + \frac{\beta_1 \beta_2 (u + v)}{\beta_3} \int_{\Omega} (w - w) \left( \sigma - \frac{\sigma w}{K} - \frac{\beta_3 u v}{u + v} \right) \\ &=: - \int_{\Omega} X^T B X + J_1 + J_2 + J_3, \end{aligned} \tag{4.4}$$

where

$$X = \begin{pmatrix} \frac{r u}{u} \\ \frac{r v}{v} \\ \frac{r w}{w} \end{pmatrix} \text{ and } B = \begin{pmatrix} v u d_1(w) \beta_2 & 0 & \frac{\beta_2 v u d_1^0(w) w}{2} \\ 0 & \beta_1 u v d_2(w) & \frac{\beta_1 u v d_2^0(w) w}{2} \\ \frac{\beta_2 v u d_1^0(w) w}{2} & \frac{\beta_1 u v d_2^0(w) w}{2} & \frac{\beta_1 \beta_2 (u + v) w}{\beta_3} \end{pmatrix}.$$

After some calculations, we can check that the matrix  $B$  is positive definite provided

$$\frac{4(u + v) w}{\beta_3 u v} > \frac{j d_1^0(w) j^2 w^2}{\beta_1 d_1(w)} + \frac{j d_2^0(w) j^2 w^2}{\beta_2 d_2(w)},$$

which, together with the fact  $\frac{\beta_3 u v}{u + v} = \sigma \left( 1 - \frac{w}{K} \right)$  (see Eq (1.7)), gives

$$\frac{1}{\sigma} > \frac{K - w}{4Kw} F(w), \tag{4.5}$$

where the function

$$F(w) := \frac{j d_1^0(w) j^2 w^2}{\beta_1 d_1(w)} + \frac{j d_2^0(w) j^2 w^2}{\beta_2 d_2(w)}.$$

If  $kw_0k_{L^1} \leq K$ , one can easily check that Eq (4.5) holds provided Eq (4.2) is true. Now, we consider the cases  $kw_0k_{L^1} \geq K$ . In fact if Eq (4.2) holds, then there exists a small  $\varepsilon_0 > 0$  such that

$$\frac{1}{\sigma} > \frac{K-w}{4Kw} \max_{0 < w \leq K} F(w) + \varepsilon_0. \quad (4.6)$$

From the assumptions  $(H_0)$ , we know that  $F(w)$  belongs to  $C^1([0, 1])$ , which combined with the fact Eq (2.2), implies that

$$\limsup_{t \rightarrow 1} \frac{K-w}{4Kw} F(w) = \frac{K-w}{4Kw} \limsup_{t \rightarrow 1} F(w) \leq \frac{K-w}{4Kw} \max_{0 < w \leq K} F(w),$$

and thus for the chosen  $\varepsilon_0$  in Eq (4.6), there exists a  $T > 0$  such that

$$\frac{K-w}{4Kw} F(w) \leq \frac{K-w}{4Kw} \max_{0 < w \leq K} F(w) + \varepsilon_0 \text{ for all } (x, t) \in \bar{\Omega} \times [T, 1]. \quad (4.7)$$

Then the combination of Eqs (4.6) and (4.7) gives

$$\frac{1}{\sigma} > \frac{K-w}{4Kw} F(w), \text{ for all } (x, t) \in \bar{\Omega} \times [T, 1].$$

Hence the matrix  $B$  is positive and there is a constant  $c_1 > 0$  such that

$$-\int_{\Omega} X^T B X \leq -c_1 \int_{\Omega} \left( \frac{|r u|^2}{u^2} + \frac{|r v|^2}{v^2} + \frac{|r w|^2}{w^2} \right) \text{ for all } t \geq T. \quad (4.8)$$

On the other hand, using the fact  $\frac{\alpha_1}{\beta_1} = \frac{v w}{u + v}$  (see Eq (1.7)), one gets

$$\begin{aligned} J_1 &:= \beta_2 v \int_{\Omega} (u - u) \left( \frac{\beta_1 v w}{u + v} - \alpha_1 \right) \\ &= \beta_1 \beta_2 v \int_{\Omega} (u - u) \left( \frac{v w}{u + v} - \frac{\alpha_1}{\beta_1} \right) \\ &= \beta_1 \beta_2 v \int_{\Omega} (u - u) \left( \frac{v w}{u + v} - \frac{v w}{u + v} \right) \\ &= \frac{\beta_1 \beta_2 v}{u + v} \int_{\Omega} (u - u) \frac{v w u + v w v - u v w - v v w}{u + v}, \end{aligned}$$

which together with the fact

$$\begin{aligned} v w u + v w v - u v w - v v w &= v v (w - w) + v w u - u v w \\ &= v v (w - w) + v u (w - w) + v w u - u v w \\ &= v(u + v)(w - w) + v w u - v w u + v w u - u v w \\ &= v(u + v)(w - w) - v w (u - u) + u w (v - v), \end{aligned}$$

gives

$$\begin{aligned} J_1 &= \beta_1 \beta_2 v \int_{\Omega} \frac{v(u - u)(w - w)}{u + v} + \frac{\beta_1 \beta_2 v w}{u + v} \int_{\Omega} \frac{u(u - u)(v - v)}{u + v} \\ &\quad - \frac{\beta_1 \beta_2 v w}{u + v} \int_{\Omega} \frac{v(u - u)^2}{u + v}. \end{aligned} \quad (4.9)$$

Similarly, we can use the fact  $\frac{\alpha_2}{\beta_2} = \frac{u w}{u+v}$  (see Eq (1.7)) to obtain

$$\begin{aligned} J_2 &:= \beta_1 u \int_{\Omega} (v - v) \left( \frac{\beta_2 u w}{u+v} - \alpha_2 \right) \\ &= \beta_1 \beta_2 u \int_{\Omega} \frac{u(v-v)(w-w)}{u+v} + \frac{\beta_1 \beta_2 u w}{u+v} \int_{\Omega} \frac{v(u-u)(v-v)}{u+v} \\ &\quad - \frac{\beta_1 \beta_2 u w}{u+v} \int_{\Omega} \frac{u(v-v)^2}{u+v}. \end{aligned} \quad (4.10)$$

At last, using the fact  $\sigma = \frac{\beta_3 u v}{u+v} + \frac{\sigma w}{K}$  (see Eq (1.7)), we have

$$\begin{aligned} J_3 &:= \frac{\beta_1 \beta_2 (u+v)}{\beta_3} \int_{\Omega} (w-w) \left( \sigma - \frac{\sigma w}{K} - \frac{\beta_3 u v}{u+v} \right) \\ &= \frac{\beta_1 \beta_2 (u+v)}{\beta_3} \int_{\Omega} (w-w) \left( \frac{\beta_3 u v}{u+v} - \frac{\beta_3 u v}{u+v} \right) - \frac{\beta_1 \beta_2 (u+v) \sigma}{\beta_3 K} \int_{\Omega} (w-w)^2 \\ &= \beta_1 \beta_2 (u+v) \int_{\Omega} (w-w) \frac{u v u + u v v - u v u - u v v}{(u+v)(u+v)} \\ &\quad - \frac{\beta_1 \beta_2 (u+v) \sigma}{\beta_3 K} \int_{\Omega} (w-w)^2 \\ &= -\beta_1 \beta_2 u \int_{\Omega} \frac{u(v-v)(w-w)}{u+v} - \beta_1 \beta_2 v \int_{\Omega} \frac{v(u-u)(w-w)}{u+v} \\ &\quad - \frac{\beta_1 \beta_2 (u+v) \sigma}{\beta_3 K} \int_{\Omega} (w-w)^2. \end{aligned} \quad (4.11)$$

Combining Eqs (4.9) and (4.10) with Eq (4.11), we have

$$\sum_{j=1}^3$$

Then we can update Eq (4.12) as

$$\sum_{j=1}^3 J_j = -\frac{\beta_1 \beta_2 w}{u+v} \int_{\Omega} \frac{(v u - u v)^2}{u+v} - \frac{\beta_1 \beta_2 (u+v) \sigma}{\beta_3 K} \int_{\Omega} (w - w)^2. \quad (4.13)$$

Substituting Eqs (4.8) and (4.13) into Eq (4.4) gives for all  $t \geq T$

$$\begin{aligned} \frac{dE(t)}{dt} \leq & -\frac{\beta_1 \beta_2 w}{u+v} \int_{\Omega} \frac{(v u - u v)^2}{u+v} - \frac{\beta_1 \beta_2 (u+v) \sigma}{\beta_3 K} \int_{\Omega} (w - w)^2 \\ & - c_1 \int_{\Omega} \left( \frac{j r u j^2}{u^2} + \frac{j r v j^2}{v^2} + \frac{j r w j^2}{w} \right), \end{aligned}$$

which indicates  $\frac{dE(t)}{dt} \leq 0$  for all  $u, v, w$ . Moreover, one can check that if  $\frac{dE(t)}{dt} = 0$ , then  $r u = r v = 0$  and  $w = w$ , which implies that

$$u = \tilde{u}, \quad v = \tilde{v}$$

with  $\tilde{u}$  and  $\tilde{v}$  are some positive constants. Since  $(u, v, w) = (\tilde{u}, \tilde{v}, w)$  is a solution of (1.5), then one has

$$\begin{cases} \frac{\beta_1 \tilde{v} w}{\tilde{u} + \tilde{v}} - \alpha_1 = 0, \\ \frac{\beta_2 \tilde{u} w}{\tilde{u} + \tilde{v}} - \alpha_2 = 0, \\ \frac{\beta_3 \tilde{u} \tilde{v}}{\tilde{u} + \tilde{v}} - \sigma + \frac{\sigma w}{K} = 0, \end{cases}$$

which gives

$$\tilde{u} = \frac{\sigma}{\beta_3} \left( 1 - \frac{w}{K} \right) \left( 1 + \frac{\alpha_2 \beta_1}{\alpha_1 \beta_2} \right) = u, \quad \tilde{v} = \frac{\sigma}{\beta_3} \left( 1 - \frac{w}{K} \right) \left( 1 + \frac{\alpha_1 \beta_2}{\alpha_2 \beta_1} \right) = v,$$

Thus  $\frac{dE(t)}{dt} = 0$  if and only if  $(u, v, w) = (u, v, w)$ . By using the LaSalle's invariant principle (cf. [41, Theorem 5.24]), we get that the co-existence steady state  $(u, v, w)$  is globally asymptotically stable.  $\square$

*Proof of Theorem 1.3.* Theorem 1.3 is a consequence of Lemma 4.1.  $\square$

## 5. Spatio-temporal patterns

In this section, we assume  $K > \frac{\alpha_1}{\beta_1} + \frac{\alpha_2}{\beta_2}$  and will study the effect of density-suppressed dispersals on the dynamics of the system (1.5).

### 5.1. Linear instability analysis

Denote  $d_j := d_j(w)$ ,  $d_j^0 := d_j^0(w)$  ( $j = 1, 2$ ) and linearize (1.5) at  $(u, v, w)$  to obtain

$$\begin{cases} \Phi_t = A \Delta \Phi + B \Phi, & x \in \Omega, \quad t > 0, \\ (v \cdot r) \Phi = 0, & x \in \partial \Omega, \quad t > 0, \\ \Phi(x, 0) = (u_0 - u, v_0 - v, w_0 - w)^T, & x \in \Omega, \end{cases} \quad (5.1)$$

where  $T$  denotes the transpose and

$$\Phi = \begin{pmatrix} u - u \\ v - v \\ w - w \end{pmatrix}, A = \begin{pmatrix} d_1 & 0 & d_1^0 u \\ 0 & d_2 & d_2^0 v \\ 0 & 0 & 1 \end{pmatrix}, B = (B_{ij})_{3 \times 3}$$

with

$$\begin{aligned} B_{11} &= -\frac{\alpha_1 u}{u + v} < 0, \quad B_{12} = \frac{\alpha_2 \beta_1 u}{\beta_2(u + v)} > 0, \quad B_{13} = \frac{\beta_1 u v}{u + v} > 0, \\ B_{21} &= \frac{\alpha_1 \beta_2 v}{\beta_1(u + v)} > 0, \quad B_{22} = -\frac{\alpha_2 v}{u + v} < 0, \quad B_{23} = \frac{\beta_2 u v}{u + v} > 0, \\ B_{31} &= -\frac{\alpha_1 \beta_3 v}{\beta_1(u + v)} < 0, \quad B_{32} = -\frac{\alpha_2 \beta_3 u}{\beta_2(u + v)} < 0, \quad B_{33} = -\frac{\sigma w}{K} < 0. \end{aligned}$$

Then by the standard linear stability principle, the linear stability of  $(u, v, w)$  is determined by the eigenvalues of the matrix  $(-\mu_k A + B)$  where the sequence  $\{\mu_k\}_{k=0}^\infty$  with  $0 = \mu_0 < \mu_1 \leq \mu_2 \leq \mu_3 \leq \dots$  denotes the eigenvalues of  $-\Delta$  under the Neumann boundary condition. The characteristic equation of  $(-\mu_k A + B)$  is

$$\rho^3 + D_1(\mu_k)\rho^2 + D_2(\mu_k, j d_{ji}^0)\rho + D_3(\mu_k, j d_{ji}^0) = 0, \quad (5.2)$$

where

$$\begin{aligned} D_1(\mu_k) &:= \mu_k(d_1 + d_2 + 1) + \frac{\sigma w}{K} + \frac{\alpha_1 u + \alpha_2 v}{u + v} > 0, \\ D_2(\mu_k, j d_{ji}^0) &:= \mu_k^2(d_1 d_2 + d_1 + d_2) + \mu_k \left\{ (d_1 + d_2) \frac{\sigma w}{K} + (d_1 + 1) \frac{\alpha_2 v}{u + v} + (d_2 + 1) \frac{\alpha_1 u}{u + v} \right\} \\ &\quad + \mu_k \left\{ \frac{\beta_3 u v}{u + v} \left( -\frac{\alpha_1 d_1^0}{\beta_1} - \frac{\alpha_2 d_2^0}{\beta_2} \right) \right\} \\ &\quad + \frac{\beta_3 u v (\alpha_1 v + \alpha_2 u)}{(u + v)^2} + \frac{\sigma w (\alpha_1 u + \alpha_2 v)}{K(u + v)} > 0, \\ D_3(\mu_k, j d_{ji}^0) &:= \mu_k^3 d_1 d_2 + \mu_k^2 \left\{ d_1 d_2 \frac{\sigma w}{K} + d_1 \frac{\alpha_2 v}{u + v} + d_2 \frac{\alpha_1 u}{u + v} + \frac{\beta_3 u v}{u + v} \left( -\frac{\alpha_1 d_1^0 d_2}{\beta_1} - \frac{\alpha_2 d_1 d_2^0}{\beta_2} \right) \right\} \\ &\quad + \mu_k \left\{ d_1 \frac{\alpha_2 v}{u + v} \left( \frac{\sigma w}{K} + \frac{\beta_3 u^2}{u + v} \right) + d_2 \frac{\alpha_1 u}{u + v} \left( \frac{\sigma w}{K} + \frac{\beta_3 v^2}{u + v} \right) \right\} \\ &\quad + \mu_k \left\{ \frac{\alpha_1 \alpha_2 \beta_3 u v}{u + v} \left( -\frac{d_1^0}{\beta_1} - \frac{d_2^0}{\beta_2} \right) \right\} \\ &\quad + \frac{\alpha_1 \alpha_2 \beta_3 u v}{u + v} > 0. \end{aligned}$$

Hence, by direct calculations, one has

$$\begin{aligned} H(\mu_k, j d_{ji}^0) &:= D_1(\mu_k)D_2(\mu_k, j d_{ji}^0) - D_3(\mu_k, j d_{ji}^0) \\ &= \gamma_1 \mu_k^3 + \gamma_2 \mu_k^2 + \gamma_3 \mu_k + A_1 A_2 - A_3 + A_0(\mu_k, j d_{ji}^0) + A_4(\mu_k, j d_{ji}^0), \end{aligned} \quad (5.3)$$

where  $\gamma_1, \gamma_2, \gamma_3$  are positive constants and  $A_1 A_2 - A_3 > 0$  are independent of  $d_j^0$  ( $j = 1, 2$ ) and  $\mu_k$  (see details in the Appendix for the precise definitions of  $\gamma_i$  and  $A_i$ ,  $i = 1, 2, 3$ ), and

$$A_0(\mu_k, j d_{ji}^0) := \mu_k^2 \frac{\beta_3 u v}{u + v} \left\{ -\frac{(d_1 + 1)\alpha_1 d_1^0}{\beta_1} - \frac{(d_2 + 1)\alpha_2 d_2^0}{\beta_2} \right\} \geq 0,$$

as well as

$$A_4(\mu_k, jd_j^0) := \mu_k \frac{\beta_3 u v}{(u + v)} \left\{ \frac{(-d_1^0)\alpha_1}{\beta_1} \left( \frac{\sigma w}{K} + \frac{(\alpha_1 - \alpha_2)u}{u + v} \right) + \frac{(-d_2^0)\alpha_2}{\beta_2} \left( \frac{\sigma w}{K} + \frac{(\alpha_2 - \alpha_1)v}{u + v} \right) \right\}.$$

Then using the Routh-Hurwitz criterion, we have the results stated as below.

**Proposition 5.1.** *The  $(u, v, w)$  is linearly stable if one of the following conditions holds:*

- 1)  $d_1(w), d_2(w)$  are constants with  $\sigma > 0$ ;
- 2)  $d_j^0(w) < 0$  with  $\sigma > 0$  if  $\alpha_1 = \alpha_2$  or  $\sigma \geq \frac{K\beta_1\beta_2 f(\beta)j\alpha_1 - \alpha_2 j}{(\alpha_1\beta_2 + \alpha_2\beta_1)^2}$  if  $\alpha_1 \neq \alpha_2$ , where

$$f(\beta) := \begin{cases} \alpha_2\beta_1, & \text{if } \alpha_1 < \alpha_2, \\ \alpha_1\beta_2, & \text{if } \alpha_1 > \alpha_2, \end{cases} \quad (5.4)$$

and  $j = 1, 2$ .

*Proof.* In the case that  $d_j(w)$  ( $j = 1, 2$ ) are constants, one has

$$A_0(\mu_k, jd_j^0) = A_4(\mu_k, jd_j^0) = 0.$$

On the other hand, considering the case  $d_j^0(w) < 0$ . If  $\alpha_1 = \alpha_2$ , we get

$$A_0(\mu_k, jd_j^0), A_4(\mu_k, jd_j^0) > 0.$$

As for  $\alpha_1 \neq \alpha_2$  with

$$\sigma \geq \frac{K\beta_1\beta_2 f(\beta)j\alpha_1 - \alpha_2 j}{(\alpha_1\beta_2 + \alpha_2\beta_1)^2},$$

we can calculate to get

$$A_0(\mu_k, jd_j^0) > 0, A_4(\mu_k, jd_j^0) \geq 0.$$

All cases discussed above indicate  $D_1(\mu_k)D_2(\mu_k, jd_j^0) - D_3(\mu_k, jd_j^0) > 0$  for all  $k \in \mathbb{N}$ . Then by the Routh-Hurwitz criterion [34], we know  $(u, v, w)$  is linearly stable.  $\square$

Hence, we are left to study the possible patterns bifurcating from the co-existence steady state  $(u, v, w)$  in the following range of parameters:

$(H_1)$   $\alpha_1 \neq \alpha_2$ ,  $0 < \sigma < \frac{K\beta_1\beta_2 f(\beta)j\alpha_1 - \alpha_2 j}{(\alpha_1\beta_2 + \alpha_2\beta_1)^2}$  and  $d_i^0(w) < 0$ , where  $i = 1$  if  $\alpha_1 < \alpha_2$  or  $i = 2$  if  $\alpha_1 > \alpha_2$  and  $f(\beta)$  is defined in (5.4).

**Proposition 5.2.** *Let  $\sigma, K, \alpha_1, \alpha_2, \beta_1, \beta_2$  and  $\beta_3$  be fixed positive parameters, and suppose the assumption  $(H_1)$  holds. Then  $(u, v, w)$  is linearly unstable provided that  $|jd_1^0(w)|$  is large enough if  $\alpha_1 < \alpha_2$  (or  $|jd_2^0(w)|$  is large enough if  $\alpha_2 < \alpha_1$ ) and there is some  $k_0 \geq 1$  such that*

$$0 < \mu_{k_0} < \begin{cases} \frac{1}{1+d_1} \left| \frac{\sigma w}{K} + \frac{(\alpha_1 - \alpha_2)u}{u + v} \right|, & \text{if } \alpha_1 < \alpha_2, \\ \frac{1}{1+d_2} \left| \frac{\sigma w}{K} + \frac{(\alpha_2 - \alpha_1)v}{u + v} \right|, & \text{if } \alpha_1 > \alpha_2. \end{cases} \quad (5.5)$$

*Proof.* Noting that  $H(\mu_k, jd_1^0j)$  can be rewritten as

$$\begin{aligned} H(\mu_k, jd_1^0j) &= \gamma_1\mu_k^3 + \gamma_2\mu_k^2 + \gamma_3\mu_k + A_1A_2 - A_3 \\ &\quad + \frac{(-d_1^0)\alpha_1\beta_3u}{\beta_1(u+v)} \left\{ (d_1+1)\mu_k^2 + \left( \frac{\sigma w}{K} + \frac{(\alpha_1-\alpha_2)u}{u+v} \right) \mu_k \right\} \\ &\quad + \frac{(-d_2^0)\alpha_2\beta_3u}{\beta_2(u+v)} \left\{ (d_2+1)\mu_k^2 + \left( \frac{\sigma w}{K} + \frac{(\alpha_2-\alpha_1)v}{u+v} \right) \mu_k \right\}. \end{aligned}$$

Since  $(H_1)$  holds, then for some fixed  $\mu_{k_0} > 0$  satisfying Eq (5.5), we have

$$H(\mu_k, jd_1^0j) := D_1(\mu_k)D_2(\mu_k, jd_1^0j) - D_3(\mu_k, jd_1^0j) < 0$$

when  $|jd_1^0(w)|$  is large enough if  $\alpha_1 < \alpha_2$  (or  $|jd_2^0(w)|$  is large enough if  $\alpha_2 < \alpha_1$ ) and  $(u, v, w)$  is linearly unstable by the Routh-Hurwitz criterion [40].  $\square$

**Remark 5.3.** The above results imply that the density-suppressed diffusion can induce the instability of  $(u, v, w)$  and trigger the pattern formation.

## 5.2. Numerical simulations

In this subsection, we aim to give some numerical simulations to complement the previous analysis in Section 5.1. Without loss of generality, we assume  $\alpha_1 < \alpha_2$ , and define the set

$$H = \{(jd_1^0j, \eta) \in \mathbb{R}_+^2 : H(jd_1^0j, \eta) = 0\}$$

as the bifurcation curve [35], where the linearized system (5.1) admits an eigenvalue with zero real part and the curve  $H$  is the graph of the function

$$\begin{aligned} d_H(\eta) &= \frac{\beta_1(u+v)}{\alpha_1\beta_3u} \left( -(d_1+1)\eta - \frac{\sigma w}{K} - \frac{(\alpha_1-\alpha_2)u}{u+v} \right)^{-1} \cdot \left\{ \left( \eta^2\gamma_1 + \eta\gamma_2 + \gamma_3 + \frac{A_1A_2 - A_3}{\eta} \right) \right. \\ &\quad \left. + \frac{(-d_2^0)\alpha_2\beta_3u}{\beta_2(u+v)} \left( (d_2+1)\eta + \frac{\sigma w}{K} + \frac{(\alpha_2-\alpha_1)v}{u+v} \right) \right\}, \end{aligned} \quad (5.6)$$

where the constants  $\gamma_j, A_j > 0$  ( $j = 1, 2, 3$ ) are given in the Appendix. Then the characteristic Eq (5.2) has a pair of purely imaginary eigenvalues if  $|jd_1^0(w)| = d_H(\mu_{k_0})$  for some  $k_0 \in \mathbb{N}$  and  $\mu_{k_0}$  satisfying Eq (5.5) and the Hopf bifurcation may emerge.

We first fix parameter values as follows:

$$\beta_1 = 0.3, \beta_2 = 1, \beta_3 = 1.3, \alpha_1 = 0.1, \alpha_2 = 1, K = 2. \quad (5.7)$$

Then by the assumption  $(H_1)$ , one has

$$0 < \sigma < \frac{81}{80}.$$

Hence, we fix  $\sigma = \frac{1}{10}$  in what follows without loss of generality. By Eq (1.8) we have

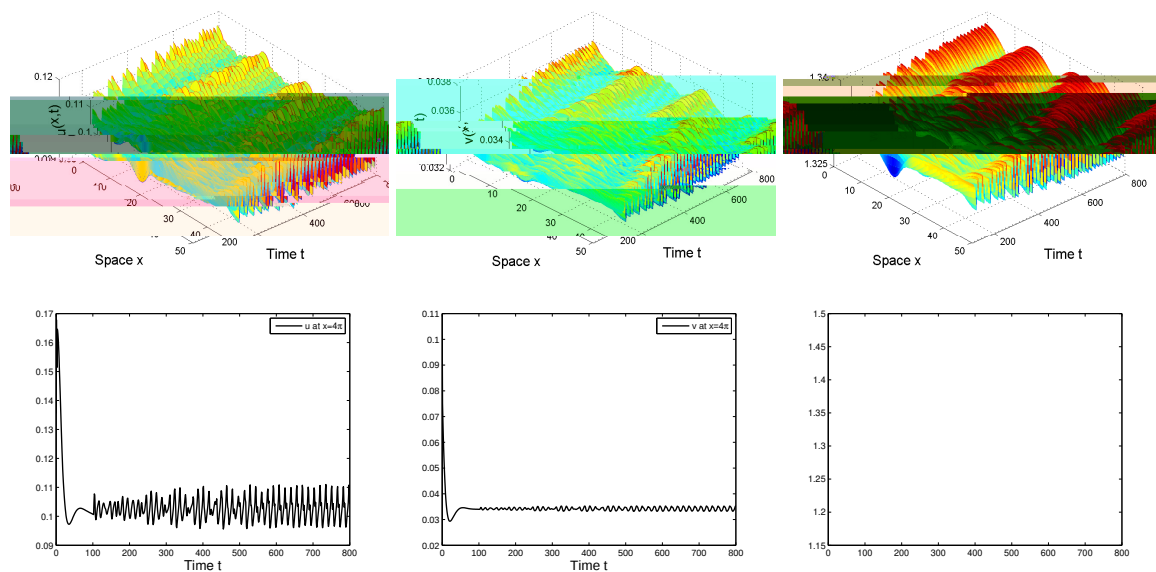
$$u = \frac{4}{39}, v = \frac{4}{117}, w = \frac{4}{3}. \quad (5.8)$$

In addition, we choose the motility function  $d_1(w)$  as

$$d_1(w) = e^{-D(w-\frac{4}{3})},$$

where the constant  $D > 0$  and  $d_1^0(w) = -De^{-D(w-\frac{4}{3})}$ . Then Eq (5.5) is equivalent to

$$0 < \mu_{k_0} < \frac{73}{240}.$$



**Figure 1.** Numerical simulation of spatio-temporal patterns (first panel) and time evolutionary profiles (second panel) generated by the model (1.5), where  $d_1(w) = e^{-D(w-\frac{4}{3})}$  with  $D = 180$  and  $d_2(w) = 1$ , other parameter values are given in Eq (5.7) with  $\sigma = 0.1$ , and the initial value  $(u_0, v_0, w_0)$  is chosen as a small random perturbation of  $(u, v, w)$  given in Eq (5.8).

Using Theorem 1.3 and Proposition 5.2, the co-existence steady state  $(\frac{4}{39}, \frac{4}{117}, \frac{4}{3})$  may lose its stability when the parameter  $D > 0$  satisfies

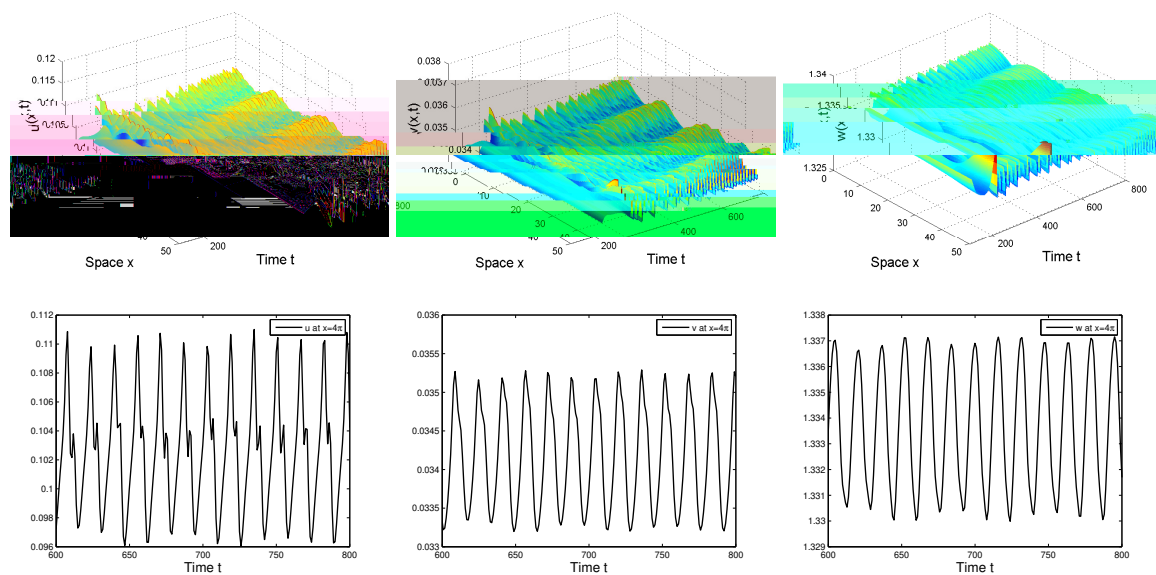
$$\frac{10D^2}{3} \max_{0 < w \leq K} e^{-D(w-\frac{4}{3})} w^2 + \max_{0 < w \leq K} \frac{|d_2^0(w)|^2 w^2}{d_2(w)} \geq 160, \quad (5.9)$$

and

$$\left| d_1^0\left(\frac{4}{3}\right) \right| = D \geq d_H(\eta) \quad (5.10)$$

with  $0 < \eta < \frac{73}{240}$ .

In our numerical simulation, we set the initial value  $(u_0, v_0, w_0)$  as a small random perturbation of  $(\frac{4}{39}, \frac{4}{117}, \frac{4}{3})$  and hence  $K = 2$ . Moreover, we perform numerical simulations in one dimension and set  $\Omega = (0, 16\pi)$ . Choosing  $D > 0$  satisfies (5.9) and (5.10) with  $\eta = (k/16)^2 < \frac{73}{240}$ , which indicates



**Figure 2.** Numerical simulation of spatio-temporal patterns (first panel) and oscillatory time-evolutionary profiles (second panel) generated by the model (1.5), where  $d_1(w) = e^{-D(w-\frac{4}{3})}$  with  $D = 268$  and  $d_2(w) = e^{-50(w-\frac{4}{3})}$ , other parameter values are given in Eq (5.7) with  $\sigma = 0.1$ , and the initial value is chosen as a small random perturbation of  $(u, v, w)$  given in Eq (5.8).

allowable unstable modes are  $k = 1, 2, 3, 4, 5, 6, 7, 8$ . Next, we will consider the following two cases for  $d_2(w)$ .

**Case 1:**  $d_2(w) = 1$ . In this case,  $d_2^0 = d_2^0(w) = 0$  and according to the definition of  $d_H(\eta)$  in Eq (5.6), we can calculate to obtain that

$$d_H(\eta) = 90 \left( \frac{73}{120} - 2\eta \right)^{-1} \left( 8\eta^2 + \frac{47}{15}\eta + \frac{2893}{7200} + \frac{733}{48000\eta} \right),$$

and Eq (5.9) can be updated as

$$e^{\frac{4}{3}D} \geq 12e^2$$

with  $D > 4$ . Then after some calculations, one gets  $d_H(\frac{1}{256}) \approx 647.963$ ,  $d_H(\frac{4}{256}) \approx 223.026$ ,  $d_H(\frac{9}{256}) \approx 159.956$ ,  $d_H(\frac{16}{256}) \approx 162.600$ ,  $d_H(\frac{25}{256}) \approx 204.933$ ,  $d_H(\frac{36}{256}) \approx 305.214$ ,  $d_H(\frac{49}{256}) \approx 548.498$  and  $d_H(\frac{64}{256}) \approx 1450.71$ . Hence,  $d_H(\frac{9}{256}) \approx 159.956$  is the critical value for possible pattern formation. In our numerical simulations shown in Figure 1, we choose  $D = 180$  and find the spatio-temporal pattern. In particular the time evolutionary profiles of solutions are oscillatory, which implies the bifurcation might be Hopf bifurcation.

**Case 2:**  $d_2(w) = e^{-50(w-\frac{4}{3})}$ . For this case,  $d_2^0 = d_2^0(w) = -50$  and hence

$$d_H(\eta) = 90 \left( \frac{73}{120} - 2\eta \right)^{-1} \left( 8\eta^2 + \frac{97}{15}\eta + \frac{2131}{2400} + \frac{733}{48000\eta} \right),$$

and Eq (5.9) can be simplified as

$$e^{\frac{200}{3}} + \frac{10}{3}e^{\frac{4}{3}D} \geq 40e^2.$$

One also can calculate to get that  $d_H(\frac{1}{256}) \approx 722.767$ ,  $d_H(\frac{4}{256}) \approx 306.961$ ,  $d_H(\frac{9}{256}) \approx 260.876$ ,  $d_H(\frac{16}{256}) \approx 291.910$ ,  $d_H(\frac{25}{256}) \approx 381.793$ ,  $d_H(\frac{36}{256}) \approx 567.953$ ,  $d_H(\frac{49}{256}) \approx 997.112$  and  $d_H(\frac{64}{256}) \approx 2546.860$ . Therefore,  $d_H(\frac{9}{256}) \approx 260.876$  is the critical value for possible pattern formation. In our numerical simulations, we choose  $D = 268$  and find the spatio-temporal pattern shown in Figure 2. Again oscillatory time evolutionary profiles indicate the solution bifurcating from the co-existence steady state may undergo a Hopf bifurcation.

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## Conflict of interest

The authors declare there is no conflict of interest.

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## Appendix

Below we show the details for the parameters  $\gamma_1, \gamma_2, \gamma_3 > 0$  and  $A_1A_2 - A_3 > 0$  present in Eq (5.3). By direct calculations, we obtain

$$\begin{aligned}\gamma_1 &= (d_1 + d_2 + 1)(d_1d_2 + d_1 + d_2) - d_1d_2 \\ &= (d_1 + d_2)(d_1d_2 + d_1 + d_2) + (d_1 + d_2) \\ &= (d_1d_2 + d_1 + d_2 + 1)(d_1 + d_2) > 0.\end{aligned}$$

As for

$$\begin{aligned}\gamma_2 &= (d_1 + d_2 + 1) \left\{ (d_1 + d_2) \frac{\sigma w}{K} + (d_1 + 1) \frac{\alpha_2 v}{u + v} + (d_2 + 1) \frac{\alpha_1 u}{u + v} \right\} \\ &\quad + (d_1d_2 + d_1 + d_2) \left( \frac{\sigma w}{K} + \frac{\alpha_1 u + \alpha_2 v}{u + v} \right) - \left( d_1d_2 \frac{\sigma w}{K} + d_1 \frac{\alpha_2 v}{u + v} + d_2 \frac{\alpha_1 u}{u + v} \right) \\ &= (d_1 + d_2 + 1) \left\{ (d_1 + d_2) \frac{\sigma w}{K} + (d_1 + 1) \frac{\alpha_2 v}{u + v} + (d_2 + 1) \frac{\alpha_1 u}{u + v} \right\} \\ &\quad + \frac{d_1d_2(\alpha_1 u + \alpha_2 v)}{u + v} + (d_1 + d_2) \frac{\sigma w}{K} + \frac{d_1\alpha_1 u + d_2\alpha_2 v}{u + v},\end{aligned}$$

which implies  $\gamma_2 > 0$  and is independent of  $d_j^0$  ( $j = 1, 2$ ) and  $\mu_k$ . Now, we calculate the other parameters:

$$\begin{aligned}\gamma_3 &= \left( \frac{\sigma w}{K} + \frac{\alpha_1 u + \alpha_2 v}{u + v} \right) \left\{ (d_1 + d_2) \frac{\sigma w}{K} + (d_1 + 1) \frac{\alpha_2 v}{u + v} + (d_2 + 1) \frac{\alpha_1 u}{u + v} \right\} \\ &\quad + (d_1 + d_2 + 1) \left\{ \frac{\beta_3 u v (\alpha_1 v + \alpha_2 u)}{(u + v)^2} + \frac{\sigma w (\alpha_1 u + \alpha_2 v)}{K(u + v)} \right\} \\ &\quad - \frac{d_1\alpha_2 v}{u + v} \left( \frac{\sigma w}{K} + \frac{\beta_3 u^2}{u + v} \right) - \frac{d_2\alpha_1 u}{u + v} \left( \frac{\sigma w}{K} + \frac{\beta_3 v^2}{u + v} \right) \\ &= \frac{d_2\alpha_2\beta_3 u^2 v}{(u + v)^2} + \frac{d_1\alpha_1\beta_3 u v^2}{(u + v)^2} + \frac{\beta_3 u v (\alpha_1 v + \alpha_2 u)}{(u + v)^2} \\ &\quad + 2(d_1 + d_2 + 1) \frac{\sigma w (\alpha_1 u + \alpha_2 v)}{K(u + v)} + (d_1 + d_2) \left( \frac{\sigma w}{K} \right)^2 \\ &\quad + ((d_1 + 1)\alpha_2 v + (d_2 + 1)\alpha_1 u) \frac{\alpha_1 u + \alpha_2 v}{(u + v)^2} \\ &> 0,\end{aligned}$$

and

$$\begin{aligned} A_1 &:= \frac{\alpha_1 u + \alpha_2 v}{u + v} + \frac{\sigma w}{K} > 0, \\ A_2 &:= \frac{\beta_3 u v (\alpha_1 v + \alpha_2 u)}{(u + v)^2} + \frac{\sigma w (\alpha_1 u + \alpha_2 v)}{K(u + v)} > 0, \\ A_3 &:= \frac{\alpha_1 \alpha_2 \beta_3 u v}{u + v} > 0. \end{aligned}$$

Hence, a direct calculation gives

$$\begin{aligned} A_1 A_2 - A_3 &> \frac{(\alpha_1 u + \alpha_2 v)(\alpha_1 v + \alpha_2 u) \beta_3 u v}{(u + v)^3} \\ &\quad - \frac{\alpha_1 \alpha_2 \beta_3 (u v^3 + u^3 v) + 2 \alpha_1 \alpha_2 \beta_3 u^2 v^2}{(u + v)^3} \\ &= \frac{(\alpha_1 - \alpha_2)^2 \beta_3 u^2 v^2}{(u + v)^3} \geq 0. \end{aligned}$$



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