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$$\begin{cases} u_t - (uv)_x = Du_{xx}, \\ v_t - u_x = \ , \end{cases} \qquad (\ .\ )$$

dresses: . . . (.- .), . . (.).

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/© E .

(. ., $\phi_\pm = \psi_\pm =$). ,
(.)-(.) (. ., $\phi_\pm \neq$ $\psi_\pm \neq$). ,
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× , -
. ,
, - ,
.
, (u_+, v_+) (u_-, v_-)
- -

$$\begin{cases} -s(u_+ - u_-) - (u_+v_+ - u_-v_-) = , \\ -s(v_+ - v_-) - (u_+ - u_-) = , \end{cases} \tag{ . }$$

,

$$< u_+ < u_-, \quad v_- < v_+, \tag{ . }$$

s

$$s = -\frac{v_+}{} + \frac{\sqrt{v_+ + \ u_-}}{} > .$$

, $\phi_\pm = \psi_\pm =$ $(u_\pm, v_\pm)$
(.) (.), (.)-(.)
$(u, v)(x, t) = (U, V)(\xi)$ $\xi = x - st$,
$(U, V)(\xi) \in C^\infty(\mathbb{R})$

$$\begin{cases} -sU' - (UV)' = DU'', \\ -sV' - U' = , \\ (U, V)(\xi) \to (u_\pm, v_\pm) \qquad \xi \to \pm\infty, \end{cases} \tag{ . }$$

$' = \frac{d}{d\xi}$. T (.) ξ,

$$\begin{cases} sU + UV + DU' = su_- + u_-v_- = su_+ + u_+v_+, \\ sV + U = sv_- + u_- = sv_+ + u_+. \end{cases} \tag{ . }$$

T .

Lemma 1.1 *([29]). The system* (.) *with condition* (.) *and* (.) *has a unique smooth solution* $(U, V)(x - st)$ *up to a shift with* $U' <$, $V' >$. *Moreover* U, V *and all their derivatives are bounded.*

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(u_+, v_+) (u_-, v_-) $s >$,
, .
(.) (.) (.)
. T .
T , $(\mathcal{U}, \mathcal{V})$
$\mathcal{X}(t)$ $\mathcal{Y}(t)$
. T (u, v)
$(u - \mathcal{U}, v - \mathcal{V}) \in H$ $(\mathbb{R})$, -
. , . ,
-
(.). ,
- .
,
. , $(\mathcal{U}, \mathcal{V})$ x
t, -
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T . , -
$(\mathcal{U}, \mathcal{V})$, (T .)
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2. Preliminaries and main results

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2.1. Notations

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- $\int_{\mathbb{R}} f dx$ $\int f dx$.
- $k \geq$, H^k $H^k(\mathbb{R})$

$$\|f\|_{H^k(\mathbb{R})} := \left(\sum_{j=}^{k} \int_{\mathbb{R}} |\partial_x^j f| \, dx \right)^{-} .$$

, $\|\cdot\|_{H^k(\mathbb{R})}$ $\|\cdot\|_{H^k}$.

- ,

$$\|(f,g)\|_{H^k} = \|f\|_{H^k} + \|g\|_{H^k}.$$

- , $f^{\pm} = g_{\pm} + h_{\pm}$ $f^{-} = g_{-} + h_{-}$ $f^{+} = g_{+} + h_{+}$.
- $\mathbb{T}$ $(\ ,\pi)$. , ,
  $\mathbb{T}^{-} = (\ ,\pi_{-})$ $\mathbb{T}^{+} = (\ ,\pi_{+})$.

2.2. Periodic solution and ansatz

T . , . - . . - , (.) .

Lemma 2.1. *Assume that* $(u\ ,v\ ) \in H\ (\ ,\pi)$ *is periodic with period* $\pi >$ *and average* $(\bar{u},\bar{v})$ *in the sense that* $\bar{u} = \frac{}{\pi}\int_{\mathbb{T}} u\ dx$ *and* $\bar{v} = \frac{}{\pi}\int_{\mathbb{T}} v\ dx$. *Suppose that there exists* $\varepsilon >$ *such that*

$$\varepsilon := \|(u\ - \bar{u}, v\ - \bar{v})\|_{H\ (\ ,\pi)} \le \varepsilon\ , \qquad (\ .\ )$$

then the problem (.) *with initial data* $(u\ ,v\ )$ *has a unique periodic solution*

$$(u,v)(x,t) \in C([\ ,+\infty); H\ (\ ,\pi)),$$

which has the same period and average as $(u\ ,v\ )$. *Moreover, we have*

$$\|(u-\bar{u}, v-\bar{v})\|_{H\ (\ ,\pi)} \le C\varepsilon e^{-\alpha t},\quad t >\ ,$$

where the constants $C >$ *and* $\alpha >$ *are independent of* ε *and* t.

Remark 2.2. . . . - , .

Remark 2.3. $(U^{\pm}, V^{\pm})(x,t)$ . . . (.)

$$(U^{\pm}, V^{\pm})(x,\ ) = (u_{\pm}, v_{\pm}) + (\phi\ _{\pm}, \psi\ _{\pm})(x) \qquad (\ .\ )$$

$(\phi\ _{\pm}, \psi\ _{\pm})(x)$ (.) (.) (.). T . ,

$$(U^{\pm}, V^{\pm})(x,t) = (u_{\pm}, v_{\pm}) + (\phi_{\pm}, \psi_{\pm})(x,t), \qquad (\ .\ )$$

$(\phi_{\pm}, \psi_{\pm})(x,t)$ $\int_{\mathbb{T}^{\pm}} \phi_{\pm}(x,t)dx = \int_{\mathbb{T}^{\pm}} \psi_{\pm}(x,t)dx =$

$$\|(\phi_{\pm}, \psi_{\pm})(x,t)\|_{H\ (\mathbb{T}^{\pm})} \le C\varepsilon e^{-\alpha t}. \qquad (\ .\ )$$

Constructing Ansatz: , ,

$$\begin{cases} \mathcal{U}(x,t) = U^-(x,t)\big(\quad - g_{st+\mathcal{X}(t)}(x)\big) + U^+(x,t) g_{st+\mathcal{X}(t)}(x), \\ \mathcal{V}(x,t) = V^-(x,t)\big(\quad - g_{st+\mathcal{Y}(t)}(x)\big) + V^+(x,t) g_{st+\mathcal{Y}(t)}(x), \end{cases} \qquad (\ .\ )$$

$(U^{\pm}, V^{\pm})(x,t)$ (.), $\mathcal{X}(t)$ $\mathcal{Y}(t)$ $C[\ ,+\infty)$. ,
. $g_{\sigma(t)}(x) := g(x-\sigma(t))$

$$g(x) := \frac{U(x)-u_-}{u_+-u_-} = V(x) - v$$

F F (.).

, . $\mathcal{X}(t)$ $\mathcal{Y}(t)$. (.) . ,

(.) (.) . (.),

$$\begin{cases} \frac{d}{dt}\int_{\mathbb{R}}(u-\mathcal{U})(x,t)dx = \int_{\mathbb{R}} f\ (x,t)dx, \\ \frac{d}{dt}\int_{\mathbb{R}}(v-\mathcal{V})(x,t)dx = \int_{\mathbb{R}} f\ (x,t)dx. \end{cases} \quad (\ .\ )$$

, (.) (.)

$$\int_{\mathbb{R}} f\ (x,t)dx = \int_{\mathbb{R}} f\ (x,t)dx = \ , \quad (\ .\ )$$

$$\int_{\mathbb{R}} (u\ (x)-\mathcal{U}(x,\ )\ dx = \int_{\mathbb{R}} v\ (x)-\mathcal{V}(x,\ )\ dx = \ . \quad (\ .\ )$$

. , (.)

$$\mathcal{X}'(t) = -s - \frac{\int_{\mathbb{R}}[D(U_x^+ - U_x^-) + (U^+V^+ - U^-V^-)]g'_{st+\mathcal{X}(t)}dx}{\int_{\mathbb{R}}(U^+-U^-)g'_{st+\mathcal{X}(t)}dx}, \quad (\ .\ )$$

$$\mathcal{Y}'(t) = -s - \frac{\int_{\mathbb{R}}(U^+-U^-)g'_{st+\mathcal{Y}(t)}dx}{\int_{\mathbb{R}}(V^+-V^-)g'_{st+\mathcal{Y}(t)}dx}. \quad (\ .\ )$$

, $\mathcal{X}(\ ) =: \mathcal{X}$ $\mathcal{Y}(\ ) =: \mathcal{Y}$ (.).

, . $\mathcal{U}(x,t)$, (.) $\int_{\mathbb{R}}[u\ (x)-\mathcal{U}(x,\ )]dx =$ (.),

$$\begin{aligned}
&= \int_{\mathbb{R}}[u\ (x)-\mathcal{U}(x,\ )]dx \\
&= \int_{\mathbb{R}} u\ (x)dx - \int_{\mathbb{R}}\Big(U^-(\ -g_{\mathcal{X}}\ ) + U^+g_{\mathcal{X}}\Big)(x)dx \\
&= \int_{\mathbb{R}} u\ (x)dx - \int_{\mathbb{R}}\Big(u_-(\ -g_{\mathcal{X}}\ ) + \phi\ _-(\ -g_{\mathcal{X}}\ ) + u_+g_{\mathcal{X}}\ + \phi\ _+g_{\mathcal{X}}\Big)(x)dx \\
&= \int_{\mathbb{R}}(U-U_{\mathcal{X}}\ )(x)dx + \int_{\mathbb{R}}(u\ -U)(x)dx - \int_{\mathbb{R}}\Big(\phi\ _-(\ -g_{\mathcal{X}}\ ) + \phi\ _+g_{\mathcal{X}}\Big)(x)dx \\
&= \mathcal{X}\ (u_+-u_-) + \int_{\mathbb{R}}(u\ -U)(x)dx - \int_{\mathbb{R}}\Big(\phi\ _-(\ -g_{\mathcal{X}}\ ) + \phi\ _+g_{\mathcal{X}}\Big)(x)dx,
\end{aligned}$$

$$\mathcal{X} + \frac{}{u_+ - u_-}\int_{\mathbb{R}}(u \ - U)dx - \frac{}{u_+ - u_-}\int_{\mathbb{R}}\Big(\phi_-(\ - g_{\mathcal{X}}) + \phi_+ g_{\mathcal{X}}\Big)dx = \ . \qquad (\ .\ )$$

, $\|\phi_-\|_{L^\infty(\mathbb{R})} + \|\phi_+\|_{L^\infty(\mathbb{R})}$

$$\|\phi_-\|_{L^\infty(\mathbb{R})} + \|\phi_+\|_{L^\infty(\mathbb{R})} \le -|u_+ - u_-|, \qquad (\ .\ )$$

$\mathcal{X}$ (.) . T ,

$$\mathcal{G}(z) = z + \frac{}{u_+ - u_-}\int_{\mathbb{R}}(u \ - U)dx - \frac{}{u_+ - u_-}\int_{\mathbb{R}}\Big(\phi_-(\ - g_z) + \phi_+ g_z\Big)dx.$$

(.), $\mathcal{G}(\mathcal{X}) = \ .$, $g' > \ ,$ (.), (.) (.),

$$\begin{aligned}|\mathcal{G}'(\mathcal{X}) - \ | &\le \frac{\|\phi_-\|_{L^\infty(\mathbb{R})} + \|\phi_+\|_{L^\infty(\mathbb{R})}}{|u_+ - u_-|}\|g'\|_{L\ (\mathbb{R})}\\ &= \frac{\|\phi_-\|_{L^\infty(\mathbb{R})} + \|\phi_+\|_{L^\infty(\mathbb{R})}}{|u_+ - u_-|}[g(+\infty) - g(-\infty)]\\ &= \frac{\|\phi_-\|_{L^\infty(\mathbb{R})} + \|\phi_+\|_{L^\infty(\mathbb{R})}}{|u_+ - u_-|} < -,\end{aligned}$$

$\mathcal{G}'(\mathcal{X}) \ne \ .$ T $\mathcal{X}$ (.).

, $\int_{\mathbb{R}}[v\ (x) - \mathcal{V}(x, \ )]dx = \ ,$ $\mathcal{E}(\mathcal{Y}) =$ $\mathcal{E}'(\mathcal{Y}) \ne$ $\|\psi_-\|_{L^\infty(\mathbb{R})} + \|\psi_+\|_{L^\infty(\mathbb{R})}$,

$$\mathcal{E}(\mathcal{Y}) = \mathcal{Y} + \frac{}{v_+ - v_-}\int_{\mathbb{R}}(v \ - V)dx - \frac{}{v_+ - v_-}\int_{\mathbb{R}}\Big(\psi_-(\ - g_{\mathcal{Y}}) + \psi_+ g_{\mathcal{Y}}\Big)dx = \ . \ (\ .\ )$$

T $\mathcal{Y}$ (.). ,
E (.) (.) $(\mathcal{X}, \mathcal{Y})$. ,
$\mathcal{X}(t)$ $\mathcal{Y}(t)$.

Lemma 2.4. *If the periodic perturbations $\phi_\pm, \psi_\pm$ satisfy (.), and there exists $\varepsilon > \ $ such that*

$$\varepsilon := \|(\phi_-, \psi_-)\|_{H\ (\ ,\pi_-)} + \|(\phi_+, \psi_+)\|_{H\ (\ ,\pi_+)} \le \varepsilon$$

there exists a unique solution $(\mathcal{X}, \mathcal{Y})(t) \in C\ [\ , +\infty)$ for the system (.) and (.) with initial data $(\mathcal{X}\ , \mathcal{Y}\ )$, satisfying

$$|\mathcal{X}'(t)| + |\mathcal{X}(t) - \mathcal{X}(\infty)| \le C\varepsilon e^{-\alpha t}, t \ge ,$$
$$|\mathcal{Y}'(t)| + |\mathcal{Y}(t) - \mathcal{Y}(\infty)| \le C\varepsilon e^{-\alpha t}, t \ge ,$$

where C, α are positive constants independent of ε and t. Moreover, the shift $\mathcal{X}(\infty) := \mathcal{X}_\infty$ and $\mathcal{Y}(\infty) := \mathcal{Y}_\infty$ are given as follows

$$\begin{aligned}\mathcal{X}_\infty = \mathcal{X} \ &+ \frac{}{u_+ - u_-}\Big\{\frac{}{\pi_-}\int^{\pi_-}\int^{x} \phi_-(y)dydx - \frac{}{\pi_+}\int^{\pi_+}\int^{x} \phi_+(y)dydx \\ &+ \int_{-\infty} (\phi_- - \phi_+)g_{\mathcal{X}}(x)dx - \int^{+\infty} (\phi_- - \phi_+)(- g_{\mathcal{X}})(x)dx \\ &+ \frac{}{\pi_-}\int^{+\infty}\int^{\pi_-} (U^-V^-(x,t) - u_-v_-)dxdt \\ &- \frac{}{\pi_+}\int^{+\infty}\int^{\pi_+} (U^+V^+(x,t) - u_+v_+)dxdt\Big\},\end{aligned} \tag{.}$$

and

$$\begin{aligned}\mathcal{Y}_\infty = \mathcal{Y} \ &+ \frac{}{v_+ - v_-}\Big\{\frac{}{\pi_-}\int^{\pi_-}\int^{x} \psi_-(y)dydx - \frac{}{\pi_+}\int^{\pi_+}\int^{x} \psi_+(y)dydx \\ &+ \int_{-\infty} (\psi_- - \psi_+)g_{\mathcal{Y}}(x)dx - \int^{+\infty} (\psi_- - \psi_+)(- g_{\mathcal{Y}})(x)dx\Big\}.\end{aligned} \tag{.}$$

Proof. ε , . , - (.) (.) . T - (.) (.) $t >$. T $(\mathcal{X}(t), \mathcal{Y}(t))$ (.) (.) $(\mathcal{X}, \mathcal{Y})$ - . , (.), (.), (.) . ,

$$|\mathcal{X}'(t)| + |\mathcal{Y}'(t)| \le C\varepsilon e^{-\alpha t}, \qquad t > . \tag{.}$$

, (.), $s = -\frac{u_+v_+ - u_-v_-}{u_+ - u_-}$, (.)

$$\begin{aligned}\mathcal{X}'(t) &= -s - \frac{\int_{\mathbb{R}}[D(U_x^+ - U_x^-) + (U^+V^+ - U^-V^-)]g'_{st+\mathcal{X}(t)}dx}{\int_{\mathbb{R}}(U^+ - U^-)g'_{st+\mathcal{X}(t)}dx} \\ &= \frac{u_+v_+ - u_-v_-}{u_+ - u_-} - \frac{\int_{\mathbb{R}}[D(U_x^+ - U_x^-) + (U^+V^+ - U^-V^-)]g'_{st+\mathcal{X}(t)}dx}{\int_{\mathbb{R}}(U^+ - U^-)g'_{st+\mathcal{X}(t)}dx}\end{aligned}$$

$$
\begin{aligned}
&\|(u_+v_+ - u_-v_-)(U^+ - U^-) - (u_+ - u_-)(U^+V^+ - U^-V^-)\|_{L^\infty} \\
&\le |u_+v_+ - u_-v_-|(\|U^+ - u_+\|_{L^\infty} + \|U^- - u_-\|_{L^\infty}) \\
&\quad + (u_- - u_+)(\|U^+V^+ - u_+v_+\|_{L^\infty} + \|U^-V^- - u_-v_-\|_{L^\infty}) \\
&\le C\varepsilon e^{-\alpha t}.
\end{aligned}
$$

(.) (.) (.), $|\mathcal{X}'(t)| \le |\mathcal{E}\ | + |\mathcal{E}\ | \le C\varepsilon e^{-\alpha t}$
$|\mathcal{Y}'(t)| \le C\varepsilon e^{-\alpha t}$ (.).
, , .

$$
\lim_{t\to+\infty} \mathcal{X}(t) = \mathcal{X}(\infty) = \mathcal{X}_\infty, \quad \lim_{t\to+\infty} \mathcal{Y}(t) = \mathcal{Y}(\infty) = \mathcal{Y}_\infty
$$

$$
|\mathcal{X}(t) - \mathcal{X}_\infty| + |\mathcal{Y}(t) - \mathcal{Y}_\infty| \le C\varepsilon e^{-\alpha t}.
$$

T . . □

Remark 2.5. . , (.) (.) ,
$\mathcal{X}_\infty$ $\mathcal{Y}_\infty$.

Remark 2.6. $(U(x - st - \mathcal{X}_\infty), V(x - st - \mathcal{Y}_\infty))$ -
(.)

$$
\mathcal{X}_\infty = \mathcal{Y}_\infty, \qquad (\ .\ )
$$

$$
\begin{aligned}
&s\int_{\mathbb{R}} (v\ - V)dx + \int_{\mathbb{R}} (u\ - U)dx \\
&= \int_{-\infty}^{} (\phi\ _- + s\psi\ _-)dx + \int^{+\infty} (\phi\ _+ + s\psi\ _+)dx \\
&\quad + \frac{}{\pi_-}\int^{\pi_-}\int^{x} (\phi\ _- + s\psi\ _-)(y)dydx - \frac{}{\pi_+}\int^{\pi_+}\int^{x} (\phi\ _+ + s\psi\ _+)(y)dydx \\
&\quad + \frac{}{\pi_-}\int^{+\infty}\int^{\pi_-} (U^-V^- - u_-v_-)dxdt - \frac{}{\pi_+}\int^{\infty}\int^{\pi_+} (U^+V^+ - u_+v_+)dxdt. \qquad (\ .\ )
\end{aligned}
$$

(.) - . , $\phi_\pm = \psi_\pm =$, $U^\pm V^\pm = u_\pm v_\pm$, (.)

$$s\int_{\mathbb{R}}(v\ -V)\,dx+\int_{\mathbb{R}}(u\ -U)dx=\ .$$

$$-sV'=U',\qquad\qquad(\ .\ )$$

$x\ \in\mathbb{R}$.

$$\int_{\mathbb{R}}v\ (x)-V(x-x\ )\ dx=\int_{\mathbb{R}}u\ (x)-U(x-x\ )\ dx=\ ,$$

(. , ,) -

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2.3. Main results

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$$\Phi\ (x)=\int_{-\infty}^{x}(u\ (y)-\mathcal{U}(y,\ ))dy\qquad\Psi\ (x)=\int_{-\infty}^{x}(v\ (y)-\mathcal{V}(y,\ ))dy,\qquad(\ .\ )$$

(.)

$$\Phi\ (\pm\infty)=\Psi\ (\pm\infty)=\ .$$

T .

Theorem 2.7. *Let* $u_+>$ *and the periodic perturbations in* (.) *satisfy* (.) *and* (.). *Then there exists a constant* $\delta\ >$ *such that if*

$$\|(\phi\ -,\psi\ -)\|_{H\ (\ ,\pi_-)}+\|(\phi\ +,\psi\ +)\|_{H\ (\ ,\pi_+)}+\|(\Phi\ ,\Psi\ )\|_{H\ (\mathbb{R})}\le\delta\ ,$$

where Φ *and* Ψ *are defined in* (.), *then the system* (.)-(.) *has a unique global solution* (u,v) *satisfying*

$$u-\mathcal{U}\in C\left([\ ,+\infty);H\ (\mathbb{R})\right)\cap L\ \left((\ ,+\infty);H\ (\mathbb{R})\right),$$
$$v-\mathcal{V}\in C\left([\ ,+\infty);H\ (\mathbb{R})\right)\cap L\ \left((\ ,+\infty);H\ (\mathbb{R})\right),$$

and

$$\|(u,v)(\cdot,t)-(U,V)(\cdot-st-\mathcal{X}_\infty)\|_{L^\infty(\mathbb{R})}\to\quad as\ \ t\to+\infty,\qquad(\ .\ )$$

with the constant shift $\mathcal{X}_\infty(=\mathcal{Y}_\infty)$ *given by* (.), *where* $(U,V)(x-st)$ *is the traveling wave solution obtained in Lemma 1.1.*

Remark 2.8. T .

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3. Proof of Theorem 2.7

, . T . .

3.1. Reformulation of the problem

, $(x,t) \to (\xi = x - st, t)$
(.)

$$\begin{cases} u_t = Du_{\xi\xi} + su_\xi + (uv)_\xi, \\ v_t = u_\xi + sv_\xi. \end{cases} \tag{ . }$$

(ξ, t), (.),

$$\begin{cases} \mathcal{U}(\xi,t) = U^-(\xi,t)\big(\quad - g_{\mathcal{X}(t)}(\xi)\big) + U^+(\xi,t) g_{\mathcal{X}(t)}(\xi), \\ \mathcal{V}(\xi,t) = V^-(\xi,t)\big(\quad - g_{\mathcal{Y}(t)}(\xi)\big) + V^+(\xi,t) g_{\mathcal{Y}(t)}(\xi), \end{cases} \tag{ . }$$

, (.),

$$\begin{cases} \mathcal{U}_t = D\mathcal{U}_{\xi\xi} + s\mathcal{U}_\xi + (\mathcal{U}\mathcal{V})_\xi - \mathcal{P}_\xi, \\ \mathcal{V}_t = \mathcal{U}_\xi + s\mathcal{V}_\xi - \mathcal{P}_\xi, \end{cases} \tag{ . }$$

$$\begin{aligned} \mathcal{P}(\xi,t) &:= F(\xi,t) + F(\xi,t) = F(\xi,t) + \int_{-\infty}^{\xi} f(y,t)dy, \\ \mathcal{P}(\xi,t) &:= F(\xi,t) + F(\xi,t) = F(\xi,t) + \int_{-\infty}^{\xi} f(y,t)dy. \end{aligned} \tag{ . }$$

-

$$\Phi(\xi,t) = \int_{-\infty}^{\xi} (u - \mathcal{U})(y,t)dy, \quad \Psi(\xi,t) = \int_{-\infty}^{\xi} (v - \mathcal{V})(y,t)dy, \tag{ . }$$

$$\Phi(\pm\infty, t) = \Psi(\pm\infty, t) = \quad . \tag{ . }$$

T (.) (.), . (.) (.),

$$\begin{cases} \Phi_t = D\Phi_{\xi\xi} + (s + \mathcal{V})\Phi_\xi + \Phi_\xi\Psi_\xi + \mathcal{U}\Psi_\xi + \mathcal{P}, \\ \Psi_t = \Phi_\xi + s\Psi_\xi + \mathcal{P}, \end{cases} \tag{ . }$$

$$(\Phi, \Psi)(\xi, \ ) = (\Phi \ , \Psi \ )(\xi). \qquad (\ . \ )$$

$T > \ ,$

$$X(\ \ , T) = \{(\Phi, \Psi) \in C\left((\ \ , T); H \ (\mathbb{R}) \times H \ (\mathbb{R})\right),$$

$$\Phi_\xi \in L$$

$$
\begin{aligned}
&\|\mathcal{U}(x,t) - U(x - st - \mathcal{X}_\infty)\|_{L^\infty} \\
&\le \|\mathcal{U}(x,t) - U(x - st - \mathcal{X}(t))\|_{L^\infty} + \|U(x - st - \mathcal{X}(t)) - U(x - st - \mathcal{X}_\infty)\|_{L^\infty} \\
&\le \|(U^- - u_-)(\ \ - g_{\mathcal{X}(t)}) + (U^+ - u_+)g_{\mathcal{X}(t)}\|_{L^\infty} + C\|\mathcal{X}(t) - \mathcal{X}_\infty\|_{L^\infty} \\
&\le \|U^- - u_-\|_{L^\infty} + \|U^+ - u_+\|_{L^\infty} + C\|\mathcal{X}(t) - \mathcal{X}_\infty\|_{L^\infty} \\
&\le C\varepsilon e^{-\alpha t},
\end{aligned}
$$

and

$$\|\mathcal{U}_\xi - U'_{\mathcal{X}_\infty}\|_{L^\infty} + \|\mathcal{V}_\xi - V'_{\mathcal{X}_\infty}\|_{L^\infty} \le C\varepsilon e^{-\alpha t}, \qquad (\ .\ )$$

as well as

$$\|\mathcal{U}_t\|_{L^\infty} \le C\varepsilon e^{-\alpha t}, \qquad (\ .\ )$$

where $U'_{\mathcal{X}_\infty} := U_\xi(\xi - \mathcal{X}_\infty)$ *and* $V'_{\mathcal{X}_\infty} := V_\xi(\xi - \mathcal{X}_\infty)$, $C >$ *is a constant independent of* t *and* ε.

Lemma 3.6. *Let the conditions in Theorem 2.7 hold, then we have*

$$\|\mathcal{P}\ (\cdot,t)\|_H\ + \|\mathcal{P}\ (\cdot,t)\|_H\ \le C\varepsilon e^{-\alpha t},\ \ t \ge\ ,$$

where $C >$ *is independent of* ε *ant* t, *and* $\alpha >$ *is the constant given in Lemma 2.4.*

3.2. Energy estimates

, . .

Lemma 3.7 *(*L *-estimates). Let the assumptions in Proposition 3.4 hold. Then there exists* $C >$ *independent of* t *and* ε *such that*

$$\begin{aligned}\|(\Phi,\Psi)(\cdot,t)\|_L\ &+ \int^t |U'_{\mathcal{X}_\infty}|\|\ (\Phi,\Psi)\ (\cdot,\tau)\|_L\ d\tau + D\int^t \|\Phi_\xi(\cdot,\tau)\|_L\ d\tau \\ &\le C\left(\|(\Phi\ ,\Psi\ )\|_L\ + \varepsilon + \int^t\int |\Phi\Phi_\xi\Psi_\xi| d\xi d\tau\right).\end{aligned} \qquad (\ .\ )$$

Proof. . . (.) $\frac{\Phi}{\mathcal{U}}$. (.) Ψ, . . ξ ,

$$\begin{aligned}&-\frac{d}{dt}\int\Big(\frac{\Phi}{\mathcal{U}} + \Psi\ \Big)d\xi + D\int\overline{\mathcal{U}}\Phi_\xi d\xi \\ &= -\int\Big[\Big(\overline{\mathcal{U}}\Big)_t + D\Big(\overline{\mathcal{U}}\Big)_{\xi\xi} - \Big(\frac{s+\mathcal{V}}{\mathcal{U}}\Big)_\xi\Big]\Phi\ d\xi \\ &\quad + \int\Phi\Psi_\xi d\xi + \int\Phi_\xi\Psi d\xi + \int\overline{\mathcal{U}}\Phi\Phi_\xi\Psi_\xi d\xi + \int\overline{\mathcal{U}}\mathcal{P}\ \Phi d\xi + \int\mathcal{P}\ \Psi d\xi \\ &= -\int\mathcal{M}\Phi\ d\xi + \int\overline{\mathcal{U}}\Phi\Phi_\xi\Psi_\xi d\xi + \int\overline{\mathcal{U}}\mathcal{P}\ \Phi d\xi + \int\mathcal{P}\ \Psi d\xi,\end{aligned} \qquad (\ .\ )$$

$$\int\Phi\Psi_\xi d\xi + \int\Phi_\xi\Psi d\xi =$$

$$\mathcal{M} := \Big(\frac{}{\mathcal{U}}\Big)_t + D\Big(\frac{}{\mathcal{U}}\Big)_{\xi\xi} - \Big(\frac{s+\mathcal{V}}{\mathcal{U}}\Big)_\xi.$$

(.), ,

$$\mathcal{M} = -\frac{\mathcal{U}_t + \mathcal{P}_\xi}{\mathcal{U}} + \frac{}{\mathcal{U}}\mathcal{U}_\xi(s\mathcal{U} + \mathcal{U}\mathcal{V} + D\mathcal{U}_\xi). \qquad (\ .\ )$$

T (.) (.), $-\int \frac{}{\mathcal{U}} U'_{\mathcal{X}_\infty}(sU_{\mathcal{X}_\infty} + U_{\mathcal{X}_\infty}V_{\mathcal{X}_\infty} + DU'_{\mathcal{X}_\infty})\Phi\ d\xi$ (.),

$$\begin{aligned}
&-\frac{d}{dt}\int\Big(\frac{\Phi}{\mathcal{U}} + \Psi\ \Big)d\xi - \int \frac{U'_{\mathcal{X}_\infty}}{\mathcal{U}}(sU_{\mathcal{X}_\infty} + U_{\mathcal{X}_\infty}V_{\mathcal{X}_\infty} + DU'_{\mathcal{X}_\infty})\Phi\ d\xi + D\int \frac{}{\mathcal{U}}\Phi_\xi d\xi\\
&= -\int\Big[-\frac{\mathcal{U}_t + \mathcal{P}_\xi}{\mathcal{U}} + \frac{}{\mathcal{U}}\mathcal{U}_\xi(s\mathcal{U} + \mathcal{U}\mathcal{V} + D\mathcal{U}_\xi)\\
&\quad - \frac{}{\mathcal{U}}U'_{\mathcal{X}_\infty}(sU_{\mathcal{X}_\infty} + U_{\mathcal{X}_\infty}V_{\mathcal{X}_\infty} + DU'_{\mathcal{X}_\infty})\Big]\Phi\ d\xi\\
&\quad + \int \frac{}{\mathcal{U}}\Phi\Phi_\xi\Psi_\xi d\xi + \int \frac{}{\mathcal{U}}\mathcal{P}\ \Phi d\xi + \int \mathcal{P}\ \Psi d\xi\\
&= --\int \frac{\mathcal{U}_t + \mathcal{P}_\xi}{\mathcal{U}}\Phi\ d\xi + \int \mathcal{N}\Phi\ d\xi + \int \frac{}{\mathcal{U}}\Phi\Phi_\xi\Psi_\xi d\xi + \int \frac{}{\mathcal{U}}\mathcal{P}\ \Phi d\xi + \int \mathcal{P}\ \Psi d\xi
\end{aligned} \qquad (\ .\ )$$

$$\mathcal{N} := \frac{}{\mathcal{U}}\Big[\mathcal{U}_\xi(s\mathcal{U} + \mathcal{U}\mathcal{V} + D\mathcal{U}_\xi) - U'_{\mathcal{X}_\infty}(sU_{\mathcal{X}_\infty} + U_{\mathcal{X}_\infty}V_{\mathcal{X}_\infty} + DU'_{\mathcal{X}_\infty})\Big]. \qquad (\ .\ )$$

(.), $U'_{\mathcal{X}_\infty} <$ (.) (.),

$$-\frac{U'_{\mathcal{X}_\infty}}{\mathcal{U}}(sU_{\mathcal{X}_\infty} + U_{\mathcal{X}_\infty}V_{\mathcal{X}_\infty} + DU'_{\mathcal{X}_\infty}) = -\frac{U'_{\mathcal{X}_\infty}}{\mathcal{U}}u_+(s + v_+) \geq -\kappa\ U'_{\mathcal{X}_\infty} > \qquad (\ .\ )$$

$\kappa = \frac{u_+(s+v_+)}{u_-} > .$ (.) $\mathcal{U}, \mathcal{U}_\xi, U_{\mathcal{X}_\infty}, V_{\mathcal{X}_\infty}$ $U'_{\mathcal{X}_\infty}$,

$$\begin{aligned}
\|\mathcal{N}\|_{L^\infty} &\leq \Big(\frac{}{u_+}\Big)\ \|\mathcal{U}_\xi(s\mathcal{U} + \mathcal{U}\mathcal{V} + D\mathcal{U}_\xi) - U'_{\mathcal{X}_\infty}(sU_{\mathcal{X}_\infty} + U_{\mathcal{X}_\infty}V_{\mathcal{X}_\infty} + DU'_{\mathcal{X}_\infty})\|_{L^\infty}\\
&\leq \Big(\frac{}{u_+}\Big)\ \|\mathcal{U}_\xi(s\mathcal{U} + \mathcal{U}\mathcal{V} + D\mathcal{U}_\xi) - \mathcal{U}_\xi(sU_{\mathcal{X}_\infty} + U_{\mathcal{X}_\infty}V_{\mathcal{X}_\infty} + DU'_{\mathcal{X}_\infty})\|_{L^\infty}\\
&\quad + \Big(\frac{}{u_+}\Big)\ \|\mathcal{U}_\xi(sU_{\mathcal{X}_\infty} + U_{\mathcal{X}_\infty}V_{\mathcal{X}_\infty} + DU'_{\mathcal{X}_\infty}) - U'_{\mathcal{X}_\infty}(sU_{\mathcal{X}_\infty}\\
&\quad + U_{\mathcal{X}_\infty}V_{\mathcal{X}_\infty} + DU'_{\mathcal{X}_\infty})\|_{L^\infty}\\
&\leq \Big(\frac{}{u_+}\Big)\ \|\mathcal{U}_\xi[s(\mathcal{U} - U_{\mathcal{X}_\infty}) + \mathcal{U}\mathcal{V} - U_{\mathcal{X}_\infty}V_{\mathcal{X}_\infty} + D(\mathcal{U}_\xi - U'_{\mathcal{X}_\infty})]\|_{L^\infty}
\end{aligned}$$

$$+\left(\frac{}{u_+}\right)\ \|(\mathcal{U}_\xi - U'_{\mathcal{X}_\infty})(sU_{\mathcal{X}_\infty} + U_{\mathcal{X}_\infty}V_{\mathcal{X}_\infty} + DU'_{\mathcal{X}_\infty})\|_{L^\infty}$$
$$\le C\varepsilon e^{-\alpha t}, \qquad (\ .\ )$$

$$\|\mathcal{U}\mathcal{V} - U_{\mathcal{X}_\infty}V_{\mathcal{X}_\infty}\|_{L^\infty} = \|\mathcal{U}\mathcal{V} - U_{\mathcal{X}_\infty}V_{\mathcal{Y}_\infty}\|_{L^\infty} \le \|\mathcal{U}\mathcal{V} - \mathcal{U}V_{\mathcal{Y}_\infty}\|_{L^\infty} + \|\mathcal{U}V_{\mathcal{Y}_\infty} - U_{\mathcal{X}_\infty}V_{\mathcal{Y}_\infty}\|_{L^\infty}$$
$$\le C\varepsilon e^{-\alpha t}.$$

, . $\mathcal{U}$ (.) . ,

$$\frac{\mathcal{U}_t + \mathcal{P}_\xi}{\mathcal{U}} \le C(\|\mathcal{U}_t\|_{L^\infty} + \|\mathcal{P}_\xi\|_{L^\infty}) \le C\varepsilon e^{-\alpha t}. \qquad (\ .\ )$$

T . . (.),(.) (.) (.), t,

$$\int(\Phi\ +\Psi\ )d\xi + \kappa\int_0^t\int|U'_{\mathcal{X}_\infty}|\Phi\ d\xi d\tau + \frac{D}{u_-}\int_0^t\int\Phi_\xi d\xi d\tau$$
$$\le \int(\Phi\ +\Psi\ )d\xi + C\varepsilon\int_0^t\int e^{-\alpha\tau}\Phi\ d\xi d\tau \qquad (\ .\ )$$
$$+\frac{}{u_+}\int_0^t\int|\Phi\Phi_\xi\Psi_\xi|d\xi d\tau + \int_0^t\int\left(\frac{}{u_+}|\mathcal{P}\ \Phi| + |\mathcal{P}\ \Psi|\right)d\xi d\tau.$$

T (.)

$$C\varepsilon\int_0^t\int e^{-\alpha\tau}\Phi\ d\xi d\tau \le C\varepsilon\max_{\tau\in[\ ,t]}\|\Phi(\cdot,t)\|_{L^\ }. \qquad (\ .\ )$$

, . (.) . ,

$$\int_0^t\int\left(\frac{}{u_+}|\mathcal{P}\ \Phi| + |\mathcal{P}\ \Psi|\right)d\xi d\tau \le \int_0^t\left(\frac{}{u_+}\|\mathcal{P}\ \|_{L^\ }\|\Phi\|_{L^\ } + \|\mathcal{P}\ \|_{L^\ }\|\Psi\|_{L^\ }\right)d\tau$$
$$\le \int_0^t\left(\frac{}{u_+}\|\mathcal{P}\ \|_{L^\ }(\|\Phi\|_{L^\ } + \ ) + -\|\mathcal{P}\ \|_{L^\ }(\|\Psi\|_{L^\ } + \ )\right)d\tau$$
$$\le C\varepsilon\int_0^t e^{-\alpha\tau}(\|\Phi\|_{L^\ } + \|\Psi\|_{L^\ })d\tau + C\varepsilon\int_0^t e^{-\alpha\tau}d\tau$$

$$\leq C\varepsilon \sup_{\tau\in[\ ,t]}\Big(\|\Phi\|_{L^{\ }}+\|\Psi\|_{L^{\ }}\Big)+C\varepsilon. \qquad (\ .\ )$$

(.) (.) (.), ε ,

$$\int(\Phi\ +\Psi\ )d\xi+\int^{t}\int|U'_{\mathcal{X}_\infty}|\Phi\ d\xi d\tau+D\int^{t}\int\Phi_\xi d\xi d\tau$$
$$\leq C\Big(\|\Phi\ \|_{L^{\ }}+\|\Psi\ \|_{L^{\ }}+\varepsilon+\int^{t}\int|\Phi\Phi_\xi\Psi_\xi|d\xi d\tau\Big). \qquad (\ .\ )$$

, $\int^t\int|U'_{\mathcal{X}_\infty}|\Psi\ d\xi d\tau$ . .

T , . . (.) Φ . (.)

$\mathcal{U}\Psi$, . . ξ

$$-\frac{d}{dt}\int(\Phi\ +\mathcal{U}\Psi\ )d\xi+-\int\mathcal{V}_\xi\Phi\ d\xi+D\int\Phi_\xi d\xi$$
$$=-\int(\mathcal{U}_t-s\mathcal{U}_\xi)\Psi\ d\xi-\int\mathcal{U}_\xi\Phi\Psi d\xi+\int\Phi\Phi_\xi\Psi_\xi d\xi+\int\mathcal{P}\ \Phi d\xi+\int\mathcal{U}\mathcal{P}\ \Psi d\xi. \qquad (\ .\ )$$

T $--s\int U'_{\mathcal{X}_\infty}\Psi\ d\xi$ (.),

$$--s\int U'_{\mathcal{X}_\infty}\Psi\ d\xi=-\frac{d}{dt}\int(\Phi\ +\mathcal{U}\Psi\ )d\xi+-\int V'_{\mathcal{Y}_\infty}\Phi\ d\xi+D\int\Phi_\xi d\xi+\int U'_{\mathcal{X}_\infty}\Phi\Psi d\xi$$
$$--\int\mathcal{U}_t\Psi\ d\xi+-s\int(\mathcal{U}_\xi-U'_{\mathcal{X}_\infty})\Psi\ d\xi+-\int(\mathcal{V}_\xi-V'_{\mathcal{Y}_\infty})\Phi\ d\xi$$
$$+\int(\mathcal{U}_\xi-U'_{\mathcal{X}_\infty})\Phi\Psi d\xi-\int\Phi\Phi_\xi\Psi_\xi d\xi-\int\mathcal{P}\ \Phi d\xi-\int\mathcal{U}\mathcal{P}\ \Psi d\xi. \qquad (\ .\ )$$

. , $\|\mathcal{U}_t\|_{L^\infty}\leq C\varepsilon e^{-\alpha t}$, $\|\mathcal{U}_\xi-U'_{\mathcal{X}_\infty}\|_{L^\infty}\leq C\varepsilon e^{-\alpha t}$ $\|\mathcal{V}_\xi-V'_{\mathcal{Y}_\infty}\|_{L^\infty}\leq C\varepsilon e^{-\alpha t}$,

$$--\int\mathcal{U}_t\Psi\ d\xi+-s\int(\mathcal{U}_\xi-U'_{\mathcal{X}_\infty})\Psi\ d\xi+-\int(\mathcal{V}_\xi-V'_{\mathcal{Y}_\infty})\Phi\ d\xi+\int(\mathcal{U}_\xi-U'_{\mathcal{X}_\infty})\Phi\Psi d\xi$$
$$\leq C\varepsilon e^{-\alpha t}\int(\Phi\ +\Psi\ )d\xi. \qquad (\ .\ )$$

T . . (.) (.), . . (, t) $V'_{\mathcal{Y}_\infty}=-sU'_{\mathcal{X}_\infty}$,

$$s\int^{t}\int|U'_{\mathcal{X}_\infty}|\Psi\ d\xi d\tau\leq\int(\Phi\ +\mathcal{U}\Psi\ )d\xi-\int(\Phi\ +\mathcal{U}\ \Psi\ )d\xi+s\int^{t}\int|U'_{\mathcal{X}_\infty}|\Phi\ d\xi d\tau$$

$$+ \quad D\int^t\int \Phi_\xi d\xi d\tau + \quad \int^t\int |U'_{\mathcal{X}_\infty}|\Phi\Psi d\xi d\tau$$

$$+ \quad \int^t\int |\Phi\Phi_\xi\Psi_\xi| d\xi d\tau + \quad \int^t\int |P\ \Phi| d\xi d\tau$$

$$+ \quad \int^t\int |\mathcal{U}P\ \Psi| d\xi d\tau + C\varepsilon\int^t\int e^{-\alpha\tau}(\Phi\ \ +\Psi\ ) d\xi d\tau. \qquad (\ .\ \ )$$

, ,

$$\int^t\int |U'_{\mathcal{X}_\infty}|\Phi\Psi d\xi d\tau \le \frac{}{s}\int^t\int |U'_{\mathcal{X}_\infty}|\Phi\ \ d\xi d\tau + \frac{s}{}\int^t\int |U'_{\mathcal{X}_\infty}|\Psi\ \ d\xi d\tau. \qquad (\ .\ \ )$$

, .

$$\begin{aligned}
&\int^t\int |\mathcal{P}\ \Phi| d\xi d\tau + \quad \int^t\int |\mathcal{U}\mathcal{P}\ \Psi| d\xi d\tau \\
&\le \quad \int^t \|\mathcal{P}\ \|_L\ \|\Phi\|_L\ d\tau + \ u_-\int^t \|\mathcal{P}\ \|_L\ \|\Psi\|_L\ d\tau \qquad (\ .\ \ ) \\
&\le \int^t \|\mathcal{P}\ \|_L\ \|\Phi\|_L\ d\tau + \frac{u_-}{}\int^t \|\mathcal{P}\ \|_L\ \|\Psi\|_L\ d\tau + \int^t \left(\|\mathcal{P}\ \|_L\ + \frac{u_-}{}\|\mathcal{P}\ \|_L\ \right) d\tau \\
&\le C\varepsilon \underset{\tau\in[\ ,t]}{\ } \left(\|\Phi\|_L\ + \|\Psi\|_L\ \right) + C\varepsilon.
\end{aligned}$$

T . . (.) (.) (.), (.),

$$\begin{aligned}
&\frac{s}{}\int^t\int |U'_{\mathcal{X}_\infty}|\Psi\ \ d\xi d\tau \\
&\quad \le \int (\Phi\ \ +\mathcal{U}\Psi\ )d\xi + \frac{s\ +}{s}\int^t\int |U'_{\mathcal{X}_\infty}|\Phi\ \ d\xi d\tau \\
&\qquad + \quad D\int^t\int \Phi_\xi d\xi d\tau + \quad \int^t\int |\Phi\Phi_\xi\Psi_\xi| d\xi d\tau + C\varepsilon \underset{\tau\in[\ ,t]}{\ } \left(\|\Phi\|_L\ + \|\Psi\|_L\ \right) + C\varepsilon
\end{aligned}$$

$$\leq C\left(\|\Phi\|_{L^{}} + \|\Psi\|_{L^{}} + \varepsilon + \int^{t}\int |\Phi\Phi_{\xi}\Psi_{\xi}|d\xi d\tau + \varepsilon \sup_{\tau\in[\ ,t]} (\|\Phi\|_{L^{}} + \|\Psi\|_{L^{}})\right).$$

(.)

T (.) (.), (.) . T . -
. □

L - , H - .

Lemma 3.8 *(H -estimates). Let the assumptions in Proposition 3.4 hold. Then it holds that*

$$\|(\Phi,\Psi)(\cdot,t)\|_{H^{}} + \int^{t} |U'_{\mathcal{X}_{\infty}}| \|(\Phi,\Psi)(\cdot,\tau)\|_{L^{}}\, d\tau + D\int^{t} \|\Phi_{\xi}(\cdot,\tau)\|_{H^{}}\, d\tau + \int^{t} \|\Psi_{\xi}(\cdot,\tau)\|_{L^{}}\, d\tau$$

$$\leq C\Big(\|(\Phi\ ,\Psi\ )\|_{H^{}} + \varepsilon\Big) + C\int^{t}\int \Big(|\Phi| + |\Phi_{\xi}| + |\Psi_{\xi}| + |\Phi_{\xi\xi}|\Big)|\Phi_{\xi}\Psi_{\xi}|d\xi d\tau,$$

(.)

where $C >$ is a constant independent of t and ε.

Proof. . . (.) $-\frac{\Phi_{\xi\xi}}{u}$. (.) $-\Psi_{\xi\xi}$,
. . ξ, ,

$$\begin{aligned}
&-\frac{d}{dt}\int \Big(\overline{u}\Phi_{\xi} + \Psi_{\xi}\Big)d\xi + D\int \overline{u}\Phi_{\xi\xi}d\xi \\
&= -\int \Big(\overline{u}\Big)_{\xi}\Phi_{t}\Phi_{\xi}d\xi + -\int \Big(\overline{u}\Big)_{t}\Phi_{\xi}d\xi + -\int \Big(\frac{s+\mathcal{V}}{u}\Big)_{\xi}\Phi_{\xi}d\xi \\
&\quad -\int \overline{u}\Phi_{\xi}\Psi_{\xi}\Phi_{\xi\xi}d\xi - \int \Psi_{\xi}\Phi_{\xi\xi}d\xi - \int \Phi_{\xi}\Psi_{\xi\xi}d\xi - \int \overline{u}\mathcal{P}\ \Phi_{\xi\xi}d\xi - \int \mathcal{P}\ \Psi_{\xi\xi}d\xi.
\end{aligned}$$

(.)

T . . (.), (.)

$$\begin{aligned}
-\int \Big(\overline{u}\Big)_{\xi}\Phi_{t}\Phi_{\xi}d\xi =& \frac{D}{}\int \Big(\overline{u}\Big)_{\xi\xi}\Phi_{\xi}d\xi - \int (s+\mathcal{V})\Big(\overline{u}\Big)_{\xi}\Phi_{\xi}d\xi \\
&-\int \Big(\overline{u}\Big)_{\xi}\Phi_{\xi}\Psi_{\xi}d\xi - \int \mathcal{U}\Big(\overline{u}\Big)_{\xi}\Phi_{\xi}\Psi_{\xi}d\xi - \int \Big(\overline{u}\Big)_{\xi}\mathcal{P}\ \Phi_{\xi}d\xi.
\end{aligned}$$

(.)

. . (.) (.),

$$
\begin{aligned}
&-\frac{d}{dt}\int\Big(\overline{\mathcal{U}}\Phi_\xi+\Psi_\xi\Big)d\xi+D\int\overline{\mathcal{U}}\Phi_{\xi\xi}d\xi\\
&=-\int\mathcal{N}\,\Phi_\xi d\xi-\int\mathcal{U}\Big(\overline{\mathcal{U}}\Big)_\xi\Phi_\xi\Psi_\xi d\xi-\int\Big(\overline{\mathcal{U}}\Big)_\xi\Phi_\xi\Psi_\xi d\xi-\int\overline{\mathcal{U}}\Phi_\xi\Psi_\xi\Phi_{\xi\xi}d\xi\\
&\quad-\int\Big(\overline{\mathcal{U}}\Big)_\xi\mathcal{P}\,\Phi_\xi d\xi-\int\overline{\mathcal{U}}\mathcal{P}\,\Phi_{\xi\xi}d\xi-\int\mathcal{P}\,\Psi_{\xi\xi}d\xi \qquad (\ .\ )\\
&=-\int\mathcal{N}\,\Phi_\xi d\xi-\int\mathcal{U}\Big(\overline{\mathcal{U}}\Big)_\xi\Phi_\xi\Psi_\xi d\xi-\int\Big(\overline{\mathcal{U}}\Big)_\xi\Phi_\xi\Psi_\xi d\xi-\int\overline{\mathcal{U}}\Phi_\xi\Psi_\xi\Phi_{\xi\xi}d\xi\\
&\quad+\int\frac{\mathcal{P}_\xi}{\mathcal{U}}\Phi_\xi d\xi+\int\mathcal{P}_\xi\Psi_\xi d\xi,
\end{aligned}
$$

$$
\mathcal{N}:=\Big(\overline{\mathcal{U}}\Big)_t+D\Big(\overline{\mathcal{U}}\Big)_{\xi\xi}-\ (s+\mathcal{V})\Big(\overline{\mathcal{U}}\Big)_\xi+\Big(\frac{s+\mathcal{V}}{\mathcal{U}}\Big)_\xi.
$$

T . . (.),

$$
\mathcal{N}\ =-\frac{\mathcal{U}_t+\mathcal{P}_\xi}{\mathcal{U}}+\ \frac{\mathcal{U}_\xi}{\mathcal{U}}(s\mathcal{U}+\mathcal{U}\mathcal{V}+D\mathcal{U}_\xi)+\ \frac{\mathcal{V}_\xi}{\mathcal{U}},
$$

. . (.)

$$
\begin{aligned}
&-\frac{d}{dt}\int\Big(\overline{\mathcal{U}}\Phi_\xi+\Psi_\xi\Big)d\xi+D\int\overline{\mathcal{U}}\Phi_{\xi\xi}d\xi\\
&=--\int\frac{\mathcal{U}_t+\mathcal{P}_\xi}{\mathcal{U}}\Phi_\xi d\xi+\int\left(\frac{\mathcal{U}_\xi}{\mathcal{U}}(s\mathcal{U}+\mathcal{U}\mathcal{V}+D\mathcal{U}_\xi)+\frac{\mathcal{V}_\xi}{\mathcal{U}}\right)\Phi_\xi d\xi\\
&\quad-\int\mathcal{U}\Big(\overline{\mathcal{U}}\Big)_\xi\Phi_\xi\Psi_\xi d\xi-\int\Big(\overline{\mathcal{U}}\Big)_\xi\Phi_\xi\Psi_\xi d\xi-\int\overline{\mathcal{U}}\Phi_\xi\Psi_\xi\Phi_{\xi\xi}d\xi\\
&\quad+\int\frac{\mathcal{P}_\xi}{\mathcal{U}}\Phi_\xi d\xi+\int\mathcal{P}_\xi\Psi_\xi d\xi.
\end{aligned}
\qquad (\ .\ )
$$

, (.), .
(.) . , $<\underline{u_+}\le\mathcal{U}\le\underline{u_-}$,
(.)

$$
--\int\frac{\mathcal{U}_t+\mathcal{P}_\xi}{\mathcal{U}}\Phi_\xi d\xi\le\frac{}{u_+}\int(|\ \mathcal{U}_t|+|\mathcal{P}_\xi|)\Phi_\xi d\xi\le C\int\Phi_\xi d\xi. \qquad (\ .\ )
$$

$\mathcal{U},\overline{\mathcal{U}},\mathcal{V},\mathcal{U}_\xi$ $\mathcal{V}_\xi$,
(.)

$$
\int\left(\frac{\mathcal{U}_\xi}{\mathcal{U}}(s\mathcal{U}+\mathcal{U}\mathcal{V}+D\mathcal{U}_\xi)+\frac{\mathcal{V}_\xi}{\mathcal{U}}\right)\Phi_\xi d\xi\le C\int\Phi_\xi d\xi. \qquad (\ .\ )
$$

, (.)

$$
\begin{aligned}
-\int \mathcal{U}\Big(\frac{\mathcal{U}}{\mathcal{U}}\Big)_\xi \Phi_\xi \Psi_\xi d\xi &= -\int \frac{\mathcal{U}_\xi}{\mathcal{U}} \Psi \Phi_{\xi\xi} d\xi - \int \Big(\frac{\mathcal{U}_\xi}{\mathcal{U}}\Big)_\xi \Psi \Phi_\xi d\xi \\
&= -\int \frac{\mathcal{U}_\xi}{\mathcal{U}} \Psi \Phi_{\xi\xi} d\xi + \int \frac{\mathcal{U}_\xi - \mathcal{U}\mathcal{U}_{\xi\xi}}{\mathcal{U}} \Psi \Phi_\xi d\xi \qquad (\ .\ ) \\
&= I\ + I\ .
\end{aligned}
$$

$\mathcal{U}, \mathcal{U}_\xi, \mathcal{V}_\xi, V'_{\mathcal{Y}_\infty}, U'_{\mathcal{X}_\infty}$ $\mathcal{X}_\infty = \mathcal{Y}_\infty,$

I I

$$
\begin{aligned}
I\ &= -\int \frac{\mathcal{U}_\xi}{\mathcal{U}} \Psi \Phi_{\xi\xi} d\xi \\
&= -\int \frac{\mathcal{U}_\xi - U'_{\mathcal{X}_\infty}}{\mathcal{U}} \Psi \Phi_{\xi\xi} d\xi - \int \frac{U'_{\mathcal{X}_\infty}}{\mathcal{U}} \Psi \Phi_{\xi\xi} d\xi \qquad (\ .\ ) \\
&\le \frac{D}{\underline{\mathcal{U}}} \int \Phi_{\xi\xi} d\xi + \frac{1}{D} \int \frac{|\mathcal{U}_\xi - U'_{\mathcal{X}_\infty}|}{\mathcal{U}} \Psi\ d\xi + \frac{1}{D} \int \frac{|U'_{\mathcal{X}_\infty}|}{\mathcal{U}} \Psi\ d\xi \\
&\le \frac{D}{\underline{\mathcal{U}}} \int \Phi_{\xi\xi} d\xi + C \int |\mathcal{U}_\xi - U'_{\mathcal{X}_\infty}| \Psi\ d\xi + C \int |U'_{\mathcal{X}_\infty}| \Psi\ d\xi
\end{aligned}
$$

$$
\begin{aligned}
I\ &= \int \frac{\mathcal{U}_\xi - \mathcal{U}\mathcal{U}_{\xi\xi}}{\mathcal{U}} \Psi \Phi_\xi d\xi \\
&= \int \frac{\mathcal{U}_\xi - |U'_{\mathcal{X}_\infty}|}{\mathcal{U}} \Psi \Phi_\xi d\xi + \int \frac{|U'_{\mathcal{X}_\infty}|}{\mathcal{U}} \Psi \Phi_\xi d\xi - \int \frac{\mathcal{U}_{\xi\xi}}{\mathcal{U}} \Psi \Phi_\xi d\xi \\
&= \int \frac{\mathcal{U}_\xi - |U'_{\mathcal{X}_\infty}|}{\mathcal{U}} \Psi \Phi_\xi d\xi + \int \frac{|U'_{\mathcal{X}_\infty}|}{\mathcal{U}} \Psi \Phi_\xi d\xi - \frac{1}{D} \int \frac{\mathcal{U}_t + \mathcal{P}_\xi}{\mathcal{U}} \Psi \Phi_\xi d\xi \\
&\quad + \frac{1}{D} \int \frac{s + \mathcal{V}}{\mathcal{U}} (\mathcal{U}_\xi - U'_{\mathcal{X}_\infty}) \Psi \Phi_\xi d\xi + \frac{1}{D} \int (\mathcal{V}_\xi - V'_{\mathcal{Y}_\infty}) \Psi \Phi_\xi d\xi \qquad (\ .\ ) \\
&\quad + \frac{1}{D} \int \left[\frac{(s + \mathcal{V}) U'_{\mathcal{X}_\infty}}{\mathcal{U}} + V'_{\mathcal{Y}_\infty} \right] \Psi \Phi_\xi d\xi \\
&\le C \int (|\mathcal{U}_\xi - U'_{\mathcal{X}_\infty}| + |\mathcal{V}_\xi - V'_{\mathcal{Y}_\infty}| + |\mathcal{U}_t| + |\mathcal{P}_\xi|) \Psi\ d\xi \\
&\quad + C \int |U'_{\mathcal{X}_\infty}| \Psi\ d\xi + C \int \Phi_\xi d\xi,
\end{aligned}
$$

$V'_{\mathcal{Y}_\infty} = V'_{\mathcal{X}_\infty} = -\frac{}{s} U'_{\mathcal{X}_\infty}$ (.) . T
(.) (.) (.), . .

$$
-\int \mathcal{U}\Big(\frac{\mathcal{U}}{\mathcal{U}}\Big)_\xi \Phi_\xi \Psi_\xi d\xi \le \frac{D}{\underline{\mathcal{U}}} \int \Phi_{\xi\xi} d\xi + C\varepsilon e^{-\alpha\tau} \int \Psi\ d\xi + C \int |U'_{\mathcal{X}_\infty}| \Psi\ d\xi + C \int \Phi_\xi d\xi. \qquad (\ .\ )
$$

(.), (.) (.) (.)

$$
\begin{aligned}
&\frac{d}{dt}\int \left(\overline{\mathcal{U}}\Phi_\xi + \Psi_\xi\right)d\xi + D\int \overline{\mathcal{U}}\Phi_{\xi\xi}d\xi \\
&\le C\varepsilon e^{-\alpha\tau}\int \Psi\ d\xi + C\int |U'_{\mathcal{X}_\infty}|\Psi\ d\xi + C\int \Phi_\xi d\xi \\
&\quad - \int \left(\overline{\mathcal{U}}\right)_\xi \Phi_\xi\Psi_\xi d\xi - \int \overline{\mathcal{U}}\Phi_\xi\Psi_\xi\Phi_{\xi\xi}d\xi + \int \frac{\mathcal{P}_\xi}{\mathcal{U}}\Phi_\xi d\xi + \int \mathcal{P}_\xi\Psi_\xi d\xi \\
&\le C\varepsilon e^{-\alpha\tau}\int \Psi\ d\xi + C\int \Phi_\xi d\xi + C\int |\Phi_\xi\Psi_\xi|(|\Phi_\xi| + |\Phi_{\xi\xi}|)d\xi + C\int |\mathcal{P}_\xi||\Phi_\xi|d\xi \\
&\quad + C\int |\mathcal{P}_\xi||\Psi_\xi|d\xi. \qquad (\ .\ )
\end{aligned}
$$

(.) (, t), . (.) < $\frac{u_+}{}$ ≤ $\mathcal{U}$ ≤ $\frac{u_-}{}$,

$$
\begin{aligned}
&\int \left(\Phi_\xi + \Psi_\xi\right)d\xi + D\int^t\int \Phi_{\xi\xi}d\xi d\tau \\
&\le C\int \left(\Phi_\xi + \Psi_\xi\right)d\xi + C\varepsilon\int^t\int e^{-\alpha\tau}\Psi\ d\xi d\tau + C\left(\int^t\int |U'_{\mathcal{X}_\infty}|\Psi\ d\xi d\tau + \int^t\int \Phi_\xi d\xi d\tau\right) \\
&\quad + C\int^t\int |\Phi_\xi\Psi_\xi|(|\Phi_\xi| + |\Phi_{\xi\xi}|)d\xi d\tau + C\int^t\int (|\mathcal{P}_\xi||\Phi_\xi| + |\mathcal{P}_\xi||\Psi_\xi|)d\xi d\tau \\
&\le C\left(\|(\Phi\ ,\Psi\ )\|_H + \varepsilon + \int^t\int |\Phi_\xi\Psi_\xi|(|\Phi| + |\Phi_\xi| + |\Phi_{\xi\xi}|)d\xi d\tau\right) \\
&\quad + C\varepsilon \sup_{\tau\in[\ ,t]} \|\Psi\|_L + C\int^t\int (|\mathcal{P}_\xi||\Phi_\xi| + |\mathcal{P}_\xi||\Psi_\xi|)d\xi d\tau. \qquad (\ .\ )
\end{aligned}
$$

, . . ,

$$
\begin{aligned}
&\int^t\int (|\mathcal{P}_\xi||\Phi_\xi| + |\mathcal{P}_\xi||\Psi_\xi|)d\xi d\tau \\
&\le \int^t \|\mathcal{P}_\xi\|_L\ \|\Phi_\xi\|_L\ d\tau + \int^t \|\mathcal{P}_\xi\|_L\ \|\Psi_\xi\|_L\ d\tau \\
&\le -\int^t \left(\|\mathcal{P}_\xi\|_L\ \|\Phi_\xi\|_L + \|\mathcal{P}_\xi\|_L\ \|\Psi_\xi\|_L\right)d\tau + -\int^t (\|\mathcal{P}_\xi\|_L + \|\mathcal{P}_\xi\|_L)d\tau
\end{aligned}
$$

$$\leq C\varepsilon \int^t e^{-\alpha\tau}\left(\|\Phi_\xi\|_L + \|\Psi_\xi\|_L\right)d\tau + C\varepsilon\int^t e^{-\alpha\tau}d\tau$$

$$\leq C\varepsilon \sup_{\tau\in[\ ,t]}(\|\Phi_\xi\|_L + \|\Psi_\xi\|_L) + C\varepsilon. \qquad (\ .\ )$$

T . . (.) (.), ε,

$$\|\Phi_\xi\|_L + \|\Psi_\xi\|_L + D\int^t \|\Phi_{\xi\xi}\|_L\, d\tau$$

$$\leq C\left(\|(\Phi\ ,\Psi\ )\|_H + \varepsilon\right) + C\int^t\int |\Phi_\xi\Psi_\xi|(|\Phi| + |\Phi_\xi| + |\Phi_{\xi\xi}|)d\xi d\tau + C\varepsilon \sup_{\tau\in[\ ,t]}\|\Psi\|_L,$$

, (.),

$$\|\Phi\|_H + \|\Psi\|_H + \int^t |U'_{\chi_\infty}|\left(\|\Phi\|_L + \|\Psi\|_L\right)d\tau + D\int^t \|\Phi_\xi\|_H\, d\tau$$

$$\leq C\Big(\|\Phi\ \|_H + \|\Psi\ \|_H + \varepsilon + \int^t\int (|\Phi| + |\Phi_\xi| + |\Phi_{\xi\xi}|)|\Phi_\xi\Psi_\xi|d\xi d\tau\Big). \qquad (\ .\ )$$

$\int^t\int \Psi_\xi\, d\xi d\tau$. T , . .
(.) Ψ_ξ

$$\Phi_t\Psi_\xi = D\Psi_\xi\Phi_{\xi\xi} + (s+\mathcal{V})\Phi_\xi\Psi_\xi + \Phi_\xi\Psi_\xi + \mathcal{U}\Psi_\xi + \mathcal{P}\ \Psi_\xi. \qquad (\ .\ )$$

. (.) ξ, . . . Φ

$$\Phi_t\Psi_\xi = (\Phi\Psi_\xi)_t - \Phi\Phi_{\xi\xi} - s\Phi\Psi_{\xi\xi} - \Phi\mathcal{P}_{\ \xi}. \qquad (\ .\ )$$

T (.) (.), ,

$$\int \mathcal{U}\Psi_\xi\, d\xi = \frac{d}{dt}\int \Phi\Psi_\xi d\xi + \int \Phi_\xi d\xi - D\int \Psi_\xi\Phi_{\xi\xi}d\xi - \int \mathcal{V}\Phi_\xi\Psi_\xi d\xi$$
$$- \int \Phi_\xi\Psi_\xi\, d\xi - \int \mathcal{P}\ \Psi_\xi d\xi - \int \mathcal{P}_{\ \xi}\Phi d\xi. \qquad (\ .\ )$$

$< \underline{u_+} \leq \mathcal{U}$ . . $\mathcal{V}$, (.)

$$\underline{u_+}\int \Psi_\xi\, d\xi \le \frac{d}{dt}\int \Phi\Psi_\xi d\xi + \int \Phi_\xi d\xi + D\int |\Psi_\xi \Phi_{\xi\xi}|d\xi + C\int |\Phi_\xi\Psi_\xi|d\xi + \int |\Phi_\xi|\Psi_\xi\, d\xi + \int |\mathcal{P}\ \Psi_\xi|d\xi + \int |\mathcal{P}_\xi \Phi|d\xi. \tag{.}$$

,

$$D\int |\Psi_\xi\Phi_{\xi\xi}|d\xi + C\int |\Phi_\xi\Psi_\xi|d\xi \le \underline{u_+}\int \Psi_\xi\, d\xi + \frac{D}{u_+}\int \Phi_{\xi\xi}d\xi + C\int \Phi_\xi d\xi,$$

, (.),

$$\underline{u_+}\int \Psi_\xi\, d\xi \le \frac{d}{dt}\int \Phi\Psi_\xi d\xi + C\left(\int \Phi_\xi d\xi + \int \Phi_{\xi\xi}d\xi\right) + \int |\Phi_\xi|\Psi_\xi\, d\xi + \int |\mathcal{P}\ \Psi_\xi|d\xi + \int |\mathcal{P}_\xi \Phi|d\xi. \tag{.}$$

(.) , (.) .,

$$\underline{u_+}\int^t\int \Psi_\xi\, d\xi d\tau \le \int \Phi\Psi_\xi d\xi - \int \Phi\ \Psi_\xi d\xi + C\int^t\int (\Phi_\xi + \Phi_{\xi\xi})d\xi d\tau + \int^t\int |\Phi_\xi|\Psi_\xi\, d\xi d\tau + \int^t \|\mathcal{P}\ \|_L\ \|\Psi_\xi\|_L\ d\tau + \int^t \|\mathcal{P}_\xi\|_L\ \|\Phi\|_L\ d\tau$$

$$\le \|\Phi\|_L\ + \|\Psi_\xi\|_L\ + \|\Phi\ \|_L\ + \|\Psi_\xi\|_L\ + C\int^t \|\Phi_\xi\|_H\ d\tau + \int^t\int |\Phi_\xi|\Psi_\xi\, d\xi d\tau$$

$$+ - \int^t (\|\mathcal{P}\ \|_L\ \|\Psi_\xi\|_L\ + \|\mathcal{P}_\xi\|_L\ \|\Phi\|_L\ )d\tau + - \int^t (\|\mathcal{P}\ \|_L\ + \|\mathcal{P}_\xi\|_L\ )d\tau$$

$$\le C\Big(\|\Phi\ \|_H\ + \|\Psi\ \|_H\ + \varepsilon_{\tau\in[\ ,t]}(\|\Phi\|_L\ + \|\Psi_\xi\|_L\ ) + \varepsilon\Big) + C\int^t\int (|\Phi| + |\Phi_\xi| + |\Psi_\xi| + |\Phi_{\xi\xi}|)|\Phi_\xi\Psi_\xi|d\xi d\tau,$$

(.), (.) ε. T
. . □

, H - .

Lemma 3.9 *(H -estimates). Let the assumptions in Proposition 3.4 hold. Then it holds that*

$$\|(\Phi,\Psi)(\cdot,t)\|_H + D\int \|\Phi_\xi(\cdot,t)\|_H\, d\tau + \int \|\Psi_\xi(\cdot,t)\|_H\, d\tau$$

$$\le C\Big(\|(\Phi\ ,\Psi\ )\|_H + \varepsilon\Big) + C\int\int (|\Phi| + |\Phi_\xi| + |\Psi_\xi| + |\Phi_{\xi\xi}|)|\Phi_\xi\Psi_\xi| d\xi d\tau \qquad (\ .\ )$$

$$+ C\int\int (|\Phi_{\xi\xi}\Psi_\xi| + |\Phi_\xi\Psi_{\xi\xi}|)(|\Phi_{\xi\xi}| + |\Psi_{\xi\xi}| + |\Phi_{\xi\xi\xi}|) d\xi d\tau,$$

where C > is a constant independent of t and ε.

Proof. (.) $\overline{u}$,
(.) ξ ,

$$\begin{cases} \left(\overline{u}\Phi_t\right)_{\xi\xi} = \left(\frac{D}{\overline{u}}\Phi_{\xi\xi}\right)_{\xi\xi} + \left(\frac{s+\mathcal{V}}{\overline{u}}\Phi_\xi\right)_{\xi\xi} + \left(\overline{u}\Phi_\xi\Psi_\xi\right)_{\xi\xi} + \Psi_{\xi\xi\xi} + \left(\overline{u}\mathcal{P}\ \right)_{\xi\xi}, \\ \Psi_{t\xi\xi} = s\Psi_{\xi\xi\xi} + \Phi_{\xi\xi\xi} + \mathcal{P}\ _{\xi\xi}. \end{cases} \qquad (\ .\ )$$

, . (.) $\Phi_{\xi\xi}$. (.) $\Psi_{\xi\xi}$,

$$\begin{aligned} &-\frac{d}{dt}\int \left(\overline{u}\Phi_{\xi\xi} + \Psi_{\xi\xi}\right) d\xi - -\int \left(\overline{u}\right)_t \Phi_{\xi\xi} d\xi + \int \Big[\left(\overline{u}\right)_{\xi\xi}\Phi_t + \left(\overline{u}\right)_\xi \Phi_{t\xi}\Big]\Phi_{\xi\xi} d\xi \\ &= D\int \left(\overline{u}\Phi_{\xi\xi}\right)_{\xi\xi}\Phi_{\xi\xi} d\xi + \int \left(\frac{s+\mathcal{V}}{u}\Phi_\xi\right)_{\xi\xi}\Phi_{\xi\xi} d\xi + \int \left(\overline{u}\Phi_\xi\Psi_\xi\right)_{\xi\xi}\Phi_{\xi\xi} d\xi \qquad (\ .\ ) \\ &\quad + \int \left(\overline{u}P\ \right)_{\xi\xi}\Phi_{\xi\xi} d\xi + \int P\ _{\xi\xi}\Psi_{\xi\xi} d\xi. \end{aligned}$$

, (.)

$$\begin{aligned} D\int \left(\overline{u}\Phi_{\xi\xi}\right)_{\xi\xi}\Phi_{\xi\xi} d\xi &= D\int \Big[\left(\overline{u}\right)_{\xi\xi}\Phi_{\xi\xi} + \left(\overline{u}\right)_\xi \Phi_{\xi\xi}\Phi_{\xi\xi\xi} + \overline{u}\Phi_{\xi\xi}\Phi_{\xi\xi\xi\xi}\Big] d\xi \\ &= D\int \Big[\left(\overline{u}\right)_{\xi\xi}\Phi_{\xi\xi} - \left(\overline{u}\right)_{\xi\xi}\Phi_{\xi\xi} - \left(\overline{u}\right)_\xi \Phi_{\xi\xi}\Phi_{\xi\xi\xi} - \overline{u}\Phi_{\xi\xi\xi}\Big] d\xi \\ &= \frac{D}{-}\int \left(\overline{u}\right)_{\xi\xi}\Phi_{\xi\xi} d\xi - D\int \overline{u}\Phi_{\xi\xi\xi} d\xi. \end{aligned}$$

(.)

,

$$\begin{aligned}\int \Big(\frac{s+\mathcal{V}}{u}\Phi_\xi\Big)_{\xi\xi}\Phi_{\xi\xi}d\xi &= \int\Big[\Big(\frac{s+\mathcal{V}}{u}\Big)_{\xi\xi}\Phi_\xi\Phi_{\xi\xi} + \Big(\frac{s+\mathcal{V}}{u}\Big)_{\xi}\Phi_{\xi\xi} + \frac{s+\mathcal{V}}{u}\Phi_{\xi\xi}\Phi_{\xi\xi\xi}\Big]d\xi\\ &= \int\Big[--\Big(\frac{s+\mathcal{V}}{u}\Big)_{\xi\xi\xi}\Phi_\xi + \Big(\frac{s+\mathcal{V}}{u}\Big)_{\xi}\Phi_{\xi\xi} - -\Big(\frac{s+\mathcal{V}}{u}\Big)_{\xi}\Phi_{\xi\xi}\Big]d\xi\\ &= --\int\Big(\frac{s+\mathcal{V}}{u}\Big)_{\xi\xi\xi}\Phi_\xi d\xi + -\int\Big(\frac{s+\mathcal{V}}{u}\Big)_{\xi}\Phi_{\xi\xi}d\xi.\end{aligned} \tag{.}$$

T . . (.) (.) (.)

$$\begin{aligned}&-\frac{d}{dt}\int\Big(\overline{u}\Phi_{\xi\xi}+\Psi_{\xi\xi}\Big)d\xi + D\int\overline{u}\Phi_{\xi\xi\xi}d\xi\\ &= -\int\Big[\Big(\overline{u}\Big)_{\xi\xi}\Phi_t + \Big(\overline{u}\Big)_{\xi}\Phi_{t\xi}\Big]\Phi_{\xi\xi}d\xi - -\int\Big(\frac{s+\mathcal{V}}{u}\Big)_{\xi\xi\xi}\Phi_\xi d\xi\\ &\quad+\int\Big[\overline{D}\Big(\overline{u}\Big)_{\xi\xi} + -\Big(\frac{s+\mathcal{V}}{u}\Big)_{\xi} + -\Big(\overline{u}\Big)_{t}\Big]\Phi_{\xi\xi}d\xi\\ &\quad+\int\Big(\overline{u}\mathcal{P}\Big)_{\xi\xi}\Phi_{\xi\xi}d\xi + \int\mathcal{P}_{\xi\xi}\Psi_{\xi\xi}d\xi + \int\Big(\overline{u}\Phi_\xi\Psi_\xi\Big)_{\xi\xi}\Phi_{\xi\xi}d\xi\\ &= J + \int\mathcal{N}\Phi_\xi d\xi + \int\mathcal{N}\Phi_{\xi\xi}d\xi + \int\Big[\Big(\overline{u}\Big)_{\xi\xi}\mathcal{P} + \Big(\overline{u}\Big)_{\xi}\mathcal{P}_\xi + \frac{\mathcal{P}_{\xi\xi}}{u}\Big]\Phi_{\xi\xi}d\xi\\ &\quad+\int\mathcal{P}_{\xi\xi}\Psi_{\xi\xi}d\xi + \int\Big(\overline{u}\Phi_\xi\Psi_\xi\Big)_{\xi\xi}\Phi_{\xi\xi}d\xi,\end{aligned} \tag{.}$$

$$\mathcal{N} = --\Big(\frac{s+\mathcal{V}}{u}\Big)_{\xi\xi\xi} \qquad \mathcal{N} = \overline{D}\Big(\overline{u}\Big)_{\xi\xi} + -\Big(\frac{s+\mathcal{V}}{u}\Big)_{\xi} + -\Big(\overline{u}\Big)_{t}.$$

. (.), J

$$\begin{aligned}J &= -\int\Big[\Big(\overline{u}\Big)_{\xi\xi}\Phi_t + \Big(\overline{u}\Big)_{\xi}\Phi_{t\xi}\Big]\Phi_{\xi\xi}d\xi\\ &= -\int(s+\mathcal{V})\Big(\overline{u}\Big)_{\xi}\Phi_{\xi\xi}d\xi - \int\Big[\big(s+\mathcal{V}\big)\Big(\overline{u}\Big)_{\xi\xi} + \Big(\overline{u}\Big)_{\xi}\mathcal{V}_\xi\Big]\Phi_\xi\Phi_{\xi\xi}d\xi\\ &\quad-\int\overline{u}\Big(\overline{u}\Big)_{\xi\xi}\Psi_\xi\Phi_{\xi\xi}d\xi + \int\overline{u}\Big(\overline{u}\Big)_{\xi}\Psi_\xi\Phi_{\xi\xi\xi}d\xi - \int\Big(\overline{u}\Big)_{\xi\xi}\Phi_\xi\Psi_\xi\Phi_{\xi\xi}d\xi\\ &\quad-\int\Big(\overline{u}\Big)_{\xi}(\Phi_\xi\Psi_\xi)_\xi\Phi_{\xi\xi}d\xi - \int\Big(\overline{u}\Big)_{\xi\xi}\mathcal{P}\Phi_{\xi\xi}d\xi - \int\Big(\overline{u}\Big)_{\xi}\mathcal{P}_\xi\Phi_{\xi\xi}d\xi,\end{aligned}$$

. . (.)

$$
\begin{aligned}
&-\frac{d}{dt}\int\Big(\frac{\Phi_{\xi\xi}}{\mathcal{U}}+\Psi_{\xi\xi}\Big)d\xi+D\int\frac{\Phi_{\xi\xi\xi}}{\mathcal{U}}d\xi\\
&=\int\mathcal{N}\,\Phi_{\xi}d\xi+\int\mathcal{N}\,\Phi_{\xi\xi}d\xi+\int\mathcal{N}\,\Phi_{\xi}\Phi_{\xi\xi}d\xi+\int\mathcal{N}\,\Psi_{\xi}\Phi_{\xi\xi}d\xi-\int\frac{\mathcal{U}_{\xi}}{\mathcal{U}}\Psi_{\xi}\Phi_{\xi\xi\xi}d\xi\\
&\quad+\int\frac{P_{\xi\xi}}{U}\Phi_{\xi\xi}d\xi+\int\mathcal{P}_{\xi\xi}\Psi_{\xi\xi}d\xi\\
&\quad+\int\Big(\overline{\mathcal{U}}\Phi_{\xi}\Psi_{\xi}\Big)_{\xi\xi}\Phi_{\xi\xi}d\xi-\int\Big(\overline{\mathcal{U}}\Big)_{\xi\xi}\Phi_{\xi}\Psi_{\xi}\Phi_{\xi\xi}d\xi-\int\Big(\overline{\mathcal{U}}\Big)_{\xi}(\Phi_{\xi}\Psi_{\xi})_{\xi}\Phi_{\xi\xi}d\xi,
\end{aligned}
\tag{.\ }
$$

$$
\mathcal{N}=\mathcal{N}-(s+\mathcal{V})\Big(\overline{\mathcal{U}}\Big)_{\xi},\quad \mathcal{N}=-\Big(s+\mathcal{V}\Big)\Big(\overline{\mathcal{U}}\Big)_{\xi\xi}+\Big(\overline{\mathcal{U}}\Big)_{\xi}\mathcal{V}_{\xi}\qquad \mathcal{N}=-\overline{\mathcal{U}}\Big(\overline{\mathcal{U}}\Big)_{\xi\xi}.
$$

$\mathcal{U},\mathcal{V},\overline{\mathcal{U}}$ $\mathcal{U}_{\xi},\mathcal{U}_{t},\mathcal{V}_{\xi},\mathcal{U}_{\xi\xi},\mathcal{U}_{\xi\xi\xi}$ $\mathcal{V}_{\xi\xi\xi}$, $\mathcal{N}_i(i=\ ,\ \cdots,\ )$, $C>$

$$
\int\mathcal{N}\,\Phi_{\xi}d\xi+\int\mathcal{N}\,\Phi_{\xi\xi}d\xi+\int\mathcal{N}\,\Phi_{\xi}\Phi_{\xi\xi}d\xi+\int\mathcal{N}\,\Psi_{\xi}\Phi_{\xi\xi}d\xi-\int\frac{\mathcal{U}_{\xi}}{\mathcal{U}}
$$

$$
\begin{aligned}
&\int \left(\frac{}{\mathcal{U}}\Phi_\xi\Psi_\xi\right)_{\xi\xi}\Phi_{\xi\xi}d\xi - \int \left(\frac{}{\mathcal{U}}\right)_{\xi\xi}\Phi_\xi\Psi_\xi\Phi_{\xi\xi}d\xi - \int \left(\frac{}{\mathcal{U}}\right)_{\xi}(\Phi_\xi\Psi_\xi)_\xi\Phi_{\xi\xi}d\xi\\
&= -\int \left(\frac{}{\mathcal{U}}\Phi_\xi\Psi_\xi\right)_{\xi}\Phi_{\xi\xi\xi}d\xi - \int \left(\frac{}{\mathcal{U}}\right)_{\xi\xi}\Phi_\xi\Psi_\xi\Phi_{\xi\xi}d\xi - \int \left(\frac{}{\mathcal{U}}\right)_{\xi}(\Phi_\xi\Psi_\xi)_\xi\Phi_{\xi\xi}d\xi\\
&= -\int \left(\frac{}{\mathcal{U}}\right)_{\xi}(\Phi_\xi\Psi_\xi)_\xi\Phi_{\xi\xi}d\xi - \int \frac{}{\mathcal{U}}(\Phi_\xi\Psi_\xi)_\xi\Phi_{\xi\xi\xi}d\xi \qquad (\ .\ )\\
&= \int \frac{\mathcal{U}_\xi}{\mathcal{U}}\left(\Phi_{\xi\xi}\Psi_\xi + \Phi_\xi\Psi_{\xi\xi}\right)\Phi_{\xi\xi}d\xi - \int \frac{}{\mathcal{U}}\left(\Phi_{\xi\xi}\Psi_\xi + \Phi_\xi\Psi_{\xi\xi}\right)\Phi_{\xi\xi\xi}d\xi\\
&\le C\int \left(|\Phi_{\xi\xi}\Psi_\xi| + |\Phi_\xi\Psi_{\xi\xi}|\right)|\Phi_{\xi\xi}|d\xi + C\int \left(|\Phi_{\xi\xi}\Psi_\xi| + |\Phi_\xi\Psi_{\xi\xi}|\right)|\Phi_{\xi\xi\xi}|d\xi.
\end{aligned}
$$

(.)-(.) (.), . t . <
$\frac{}{u_+} \le \mathcal{U} \le \frac{}{u_-}$ (.),

$$
\begin{aligned}
&\|\Phi_{\xi\xi}\|_L + \|\Psi_{\xi\xi}\|_L + D\int^t \|\Phi_{\xi\xi\xi}\|_L\, d\tau\\
&\le C(\|\Phi_{\xi\xi}\|_L + \|\Psi_{\xi\xi}\|_L) + C\int^t (\|\Phi_\xi\|_H + \|\Psi_\xi\|_L)d\tau\\
&\quad + C\int^t \varepsilon e^{-\alpha\tau}(\|\Phi_{\xi\xi}\|_L + \|\Psi_{\xi\xi}\|_L + \ )d\tau\\
&\quad + C\int^t\int (|\Phi_{\xi\xi}\Psi_\xi| + |\Phi_\xi\Psi_{\xi\xi}|)(|\Phi_{\xi\xi}| + |\Phi_{\xi\xi\xi}|)d\xi d\tau \qquad (\ .\ )\\
&\le C\left(\|(\Phi\ ,\Psi\ )\|_H + \varepsilon\right) + \varepsilon \sup_{\tau\in[\ ,t]}(\|\Phi_{\xi\xi}\|_L + \|\Psi_{\xi\xi}\|_L)\\
&\quad + C\int^t\int \left(|\Phi| + |\Phi_\xi| + |\Psi_\xi| + |\Phi_{\xi\xi}|\right)|\Phi_\xi\Psi_\xi|d\xi d\tau\\
&\quad + C\int^t\int (|\Phi_{\xi\xi}\Psi_\xi| + |\Phi_\xi\Psi_{\xi\xi}|)(|\Phi_{\xi\xi}| + |\Phi_{\xi\xi\xi}|)d\xi d\tau.
\end{aligned}
$$

ε (.) (.),

$$
\|(\Phi,\Psi)\|_H + D\int^t \|\Phi_\xi\|_H\, d\tau + \int^t \|\Psi_\xi\|_L\, d\tau
$$

$$\leq C\left(\|(\Phi\ ,\Psi\ )\|_{H}\ +\varepsilon\right)+C\int^{t}\int(|\Phi|+|\Phi_{\xi}|+|\Psi_{\xi}|+|\Phi_{\xi\xi}|)|\Phi_{\xi}\Psi_{\xi}|d\xi d\tau$$

$$+C\int^{t}\int(|\Phi_{\xi\xi}\Psi_{\xi}|+|\Phi_{\xi}\Psi_{\xi\xi}|)(|\Phi_{\xi\xi}|+|\Phi_{\xi\xi\xi}|)d\xi d\tau. \qquad (\ .\ )$$

, $\int^{t}\int\Psi_{\xi\xi}d\xi d\tau$. T
, . (.) $\Psi_{\xi\xi\xi}$.

$$\begin{aligned}\int\Phi_{t}\Psi_{\xi\xi\xi}d\xi =&D\int\Phi_{\xi\xi}\Psi_{\xi\xi\xi}d\xi+\int(s+\mathcal{V})\Phi_{\xi}\Psi_{\xi\xi\xi}d\xi+\int\Phi_{\xi}\Psi_{\xi}\Psi_{\xi\xi\xi}d\xi\\&+\int\mathcal{U}\Psi_{\xi}\Psi_{\xi\xi\xi}d\xi+\int\mathcal{P}\ \Psi_{\xi\xi\xi}d\xi\\=&-\int\mathcal{U}\Psi_{\xi\xi}d\xi-\int\left(D\Phi_{\xi\xi\xi}+\mathcal{V}\Phi_{\xi\xi}+\mathcal{V}_{\xi}\Phi_{\xi}+\mathcal{U}_{\xi}\Psi_{\xi}\right)\Psi_{\xi\xi}d\xi\\&-\int\Psi_{\xi}\Phi_{\xi\xi}\Psi_{\xi\xi}d\xi-\int\Phi_{\xi}\Psi_{\xi\xi}d\xi-\int\mathcal{P}_{\ \xi}\Psi_{\xi\xi}d\xi.\end{aligned} \qquad (\ .\ )$$

, . (.),

$$\begin{aligned}\int\Phi_{t}\Psi_{\xi\xi\xi}d\xi&=\int(\Phi\Psi_{\xi\xi\xi})_{t}d\xi-\int\Phi\Psi_{t\xi\xi\xi}d\xi\\&=\int(\Phi\Psi_{\xi\xi\xi})_{t}d\xi-s\int\Phi\Psi_{\xi\xi\xi\xi}d\xi-\int\Phi\Phi_{\xi\xi\xi\xi}d\xi-\int\Phi\mathcal{P}_{\ \xi\xi\xi}d\xi \qquad (\ .\ )\\&=\frac{d}{dt}\int\Phi\Psi_{\xi\xi\xi}d\xi-s\int\Phi_{\xi\xi}\Psi_{\xi\xi}d\xi-\int\Phi_{\xi\xi}d\xi-\int\mathcal{P}_{\ \xi}\Phi_{\xi\xi}d\xi.\end{aligned}$$

T (.) (.)

$$\begin{aligned}\int\mathcal{U}\Psi_{\xi\xi}d\xi=&-\frac{d}{dt}\int\Phi\Psi_{\xi\xi\xi}d\xi+s\int\Phi_{\xi\xi}\Psi_{\xi\xi}d\xi+\int\Phi_{\xi\xi}d\xi\\&-\int\left(D\Phi_{\xi\xi\xi}+\mathcal{V}\Phi_{\xi\xi}+\mathcal{V}_{\xi}\Phi_{\xi}+\mathcal{U}_{\xi}\Psi_{\xi}\right)\Psi_{\xi\xi}d\xi \qquad (\ .\ )\\&+\int\mathcal{P}_{\ \xi}\Phi_{\xi\xi}d\xi-\int\mathcal{P}_{\ \xi}\Psi_{\xi\xi}d\xi-\int\Psi_{\xi}\Phi_{\xi\xi}\Psi_{\xi\xi}d\xi-\int\Phi_{\xi}\Psi_{\xi\xi}d\xi.\end{aligned}$$

T (.) (,t), $<\frac{u_{+}}{}\leq\mathcal{U},$

$$\begin{aligned}
\underline{u_+}\int^t\int \Psi_{\xi\xi}d\xi d\tau \le & \int^t\int \mathcal{U}\Psi_{\xi\xi}d\xi d\tau \\
= & -\int \Phi\Psi_{\xi\xi\xi}d\xi + \int \Phi\ \Psi_{\ \xi\xi\xi}d\xi + s\int^t\int \Phi_{\xi\xi}\Psi_{\xi\xi}d\xi d\tau + \int^t\int \Phi_{\xi\xi}d\xi d\tau \\
& -\int^t\int \left(D\Phi_{\xi\xi\xi} + \mathcal{V}\Phi_{\xi\xi} + \mathcal{V}_\xi\Phi_\xi + \mathcal{U}_\xi\Psi_\xi\right)\Psi_{\xi\xi}d\xi d\tau \\
& +\int^t\int \mathcal{P}_{\ \xi}\Phi_{\xi\xi}d\xi d\tau - \int^t\int \mathcal{P}_{\ \xi}\Psi_{\xi\xi}d\xi d\tau \\
& -\int^t\int \Psi_\xi\Phi_{\xi\xi}\Psi_{\xi\xi}d\xi d\tau - \int^t\int \Phi_\xi\Psi_{\xi\xi}d\xi d\tau \\
\le & \|\Phi_\xi\|_L\ \|\Psi_{\xi\xi}\|_L\ + \|\Phi_{\ \xi}\|_L\ \|\Psi_{\ \xi\xi}\|_L\ + \underline{u_+}\int^t \|\Psi_{\xi\xi}\|_L\ d\tau \\
& + C\int^t \|\Phi_\xi\|_H\ d\tau + C\int^t \left(\|\mathcal{P}_{\ \xi}\|_L\ + \|\mathcal{P}_{\ \xi}\|_L\ \right)d\tau \\
& +\int^t\int |\Psi_\xi\Phi_{\xi\xi}\Psi_{\xi\xi}|d\xi d\tau + \int^t\int |\Phi_\xi|\Psi_{\xi\xi}d\xi d\tau,
\end{aligned}$$

(.) . ,

$$\begin{aligned}
\int^t\int \Psi_{\xi\xi}d\xi d\tau \le & C\left(\|(\Phi\ ,\Psi\ )\|_H\ + \varepsilon\right) + C\int^t\int (|\Phi| + |\Phi_\xi| + |\Psi_\xi| + |\Phi_{\xi\xi}|)|\Phi_\xi\Psi_\xi|d\xi d\tau \\
& + C\int^t\int (|\Phi_{\xi\xi}\Psi_\xi| + |\Phi_\xi\Psi_{\xi\xi}|)(|\Phi_{\xi\xi}| + |\Psi_{\xi\xi}| + |\Phi_{\xi\xi\xi}|)d\xi d\tau.
\end{aligned} \qquad (\ .\ )$$

T (.) (.), (.) . T . . □

T . . , . .

Proof of Proposition 3.4. (.) ,

$$\int^t \int (|\Phi| + |\Phi_\xi| + |\Psi_\xi| + |\Phi_{\xi\xi}|)|\Phi_\xi \Psi_\xi| d\xi d\tau$$

$$\leq C \sup_{\tau\in[\ ,t]} (\|\Phi\|_{L^\infty} + \|\Phi_\xi\|_{L^\infty} + \|\Psi_\xi\|_{L^\infty}) \int^t (\|\Phi_\xi\|_H + \|\Psi_\xi\|_L) d\tau \qquad (.)$$

$$\leq CN(t) \int^t (\|\Phi_\xi\|_H + \|\Psi_\xi\|_L) d\tau.$$

,

$$\int^t \int (|\Phi_{\xi\xi}\Psi_\xi| + |\Phi_\xi \Psi_{\xi\xi}|)(|\Phi_{\xi\xi}| + |\Phi_{\xi\xi\xi}| + |\Psi_{\xi\xi}|) d\xi d\tau$$

$$\leq C \sup_{\tau\in[\ ,t]} (\|\Phi_\xi\|_{L^\infty} + \|\Psi_\xi\|_{L^\infty}) \int^t (\|\Phi_\xi\|_H + \|\Psi_\xi\|_H) d\tau \qquad (.)$$

$$\leq CN(t) \int^t (\|\Phi_\xi\|_H + \|\Psi_\xi\|_H) d\tau.$$

T . . (.) (.) (.), . $N(t)$,

$$N\ (t) + \int^t \|\Phi_\xi\|_H\ d\tau + \int^t \|\Psi_\xi\|_H\ d\tau \leq C\left(N\ (\ ) + \varepsilon\right) + CN(t) \int^t \left(\|\Phi_\xi\|_H + \|\Psi_\xi\|_H\right) d\tau.$$

$N(t) \leq \overline{C}$,

$$N\ (t) + \int^t \|\Phi_\xi(\cdot, t)\|_H\ d\tau + \int^t \|\Psi_\xi(\cdot, t)\|_H\ d\tau \leq C\left(N\ (\ ) + \varepsilon\right),$$

(.). T . . □

4. Appendix

, . , . . .

Proof of Lemma 2.1. T
, *a priori* .

$$\mathbb{T} = (\ ,\pi) \qquad \phi = u - \bar{u},\ \psi = v - \bar{v} \qquad \bar{f} = \tfrac{}{\pi}\int_{\mathbb{T}} f dx,$$

$$\int_{\mathbb{T}} \phi(x,t)dx = \int_{\mathbb{T}} \psi(x,t)dx = \qquad \geq \ .$$

, . . $\phi = u - \bar{u}$ $\psi = v - \bar{v}$ (.),

$$\begin{cases} \phi_t = (\phi\psi)_x + \bar{v}\phi_x + \bar{u}\psi_x + D\phi_{xx}, \\ \psi_t = \phi_x. \end{cases} \qquad (\ .\ )$$

, . *L* - . *a priori* .

$$\|(\phi,\psi)(\cdot,t)\|_{H\ (\mathbb{T})} \leq \delta, \qquad (\ .\ )$$

δ .

T . . (.) ϕ . (.) $\bar{u}\psi$, . x,

$$-\frac{d}{dt}\int_{\mathbb{T}} (\phi\ \ + \bar{u}\psi\ )dx + D\int_{\mathbb{T}} \phi_x dx = \int_{\mathbb{T}} (\phi\psi)_x \phi dx. \qquad (\ .\ )$$

, . . (.) ψ_x . .
(.)

$$\phi_t\psi_x = (\phi\psi)_x\psi_x + \bar{v}\phi_x\psi_x + \bar{u}\psi_x + D\psi_x\psi_{tx}. \qquad (\ .\ )$$

(.) x,

$$\frac{D}{}\frac{d}{dt}\int_{\mathbb{T}} \psi_x dx + \bar{u}\int_{\mathbb{T}} \psi_x dx = \int_{\mathbb{T}} \phi_t\psi_x dx - \int_{\mathbb{T}} (\phi\psi)_x\psi_x dx - \bar{v}\int_{\mathbb{T}} \phi_x\psi_x dx. \qquad (\ .\ )$$

$$\phi_t\psi_x = (\phi\psi_x)_t - \phi\psi_{xt} = (\phi\psi_x)_t - \phi\phi_{xx},$$

$$\int_{\mathbb{T}} \phi_t\psi_x dx = \frac{d}{dt}\int_{\mathbb{T}} \phi\psi_x dx - \int_{\mathbb{T}} \phi\phi_{xx} dx,$$

. . (.)

$$-\frac{d}{dt}\int_{\mathbb{T}} (D\psi_x - \ \phi\psi_x)dx + \bar{u}\int_{\mathbb{T}} \psi_x dx = \int_{\mathbb{T}} \phi_x dx - \int_{\mathbb{T}} (\phi\psi)_x\psi_x dx - \bar{v}\int_{\mathbb{T}} \phi_x\psi_x dx. \qquad (\ .\ )$$

. (.) $M >$, . . (.)

$$
\begin{aligned}
&-\frac{d}{dt}\int_{\mathbb{T}}(M\phi\ +M\bar{u}\psi\ +D\psi_x-\ \phi\psi_x)dx+(DM-\ )\int_{\mathbb{T}}\phi_x dx+\bar{u}\int_{\mathbb{T}}\psi_x dx\\
&=M\int_{\mathbb{T}}(\phi\psi)_x\phi dx-\int_{\mathbb{T}}(\phi\psi)_x\psi_x dx-\bar{v}\int_{\mathbb{T}}\phi_x\psi_x dx.
\end{aligned}
\tag{.}
$$

, , (.),

$$
\begin{aligned}
M\int_{\mathbb{T}}(\phi\psi)_x\phi dx=&M\int_{\mathbb{T}}\phi\ \psi_x dx+M\int_{\mathbb{T}}\phi\psi\phi_x dx\\
\le&M\|\phi\|_{L^\infty(\mathbb{T})}\|\phi\|_{L\ (\mathbb{T})}\|\psi_x\|_{L\ (\mathbb{T})}+M\|\phi\|_{L^\infty(\mathbb{T})}\|\psi\|_{L\ (\mathbb{T})}\|\phi_x\|_{L\ (\mathbb{T})}\\
\le&\delta Mc_p\|\psi_x\|_{L\ (\mathbb{T})}\|\phi_x\|_{L\ (\mathbb{T})}\\
\le&\frac{\bar{u}}{}\|\psi_x\|_{L\ (\mathbb{T})}+\frac{\delta\ M\ c_p}{\bar{u}}\|\phi_x\|_{L\ (\mathbb{T})}.
\end{aligned}
\tag{.}
$$

, , , , (.),

$$
\begin{aligned}
&-\int_{\mathbb{T}}(\phi\psi)_x\psi_x dx-\bar{v}\int_{\mathbb{T}}\phi_x\psi_x dx\\
&\le\|\phi\|_{L^\infty(\mathbb{T})}\|\psi_x\|_{L\ (\mathbb{T})}+\|\psi\|_{L^\infty(\mathbb{T})}\|\phi_x\|_{L\ (\mathbb{T})}\|\psi_x\|_{L\ (\mathbb{T})}+\bar{v}\|\phi_x\|_{L\ (\mathbb{T})}\|\psi_x\|_{L}\\
&\le\delta\|\psi_x\|_{L\ (\mathbb{T})}+\frac{\bar{u}}{}\|\psi_x\|_{L\ (\mathbb{T})}+\frac{(\delta\ +\bar{v}\ )}{\bar{u}}\|\phi_x\|_{L\ (\mathbb{T})}.
\end{aligned}
\tag{.}
$$

T (.) (.) (.),

$$
\begin{aligned}
&-\frac{d}{dt}\int_{\mathbb{T}}(M\phi\ +M\bar{u}\psi\ +D\psi_x-\ \phi\psi_x)dx\\
&+\left(DM-\frac{\delta\ M\ c_p}{\bar{u}}-\frac{\delta}{\bar{u}}-\frac{\bar{v}}{\bar{u}}-\ \right)\int_{\mathbb{T}}\phi_x dx+\left(\frac{\bar{u}}{}-\delta\right)\int_{\mathbb{T}}\psi_x dx\le\ .
\end{aligned}
\tag{.}
$$

, (.) $\frac{-\phi_{xx}}{\bar{u}}$, (.) $-\psi_{xx}$, ,

$$
\begin{aligned}
&-\frac{d}{dt}\int_{\mathbb{T}}\left(\frac{\phi_x}{\bar{u}}+\psi_x\right)dx+\frac{D}{\bar{u}}\int_{\mathbb{T}}\phi_{xx}dx\\
&=-\frac{}{\bar{u}}\int_{\mathbb{T}}\phi\psi_x\phi_{xx}dx-\frac{}{\bar{u}}\int_{\mathbb{T}}\psi\phi_x\phi_{xx}dx\\
&\le\frac{}{\bar{u}}\|\phi\|_{L^\infty(\mathbb{T})}\|\psi_x\|_{L\ (\mathbb{T})}\|\phi_{xx}\|_{L\ }+\frac{}{\bar{u}}\|\psi\|_{L^\infty(\mathbb{T})}\|\phi_x\|_{L\ (\mathbb{T})}\|\phi_{xx}\|_{L\ (\mathbb{T})}
\end{aligned}
$$

$$\leq \frac{\delta}{\bar{u}}(\|\psi_x\|_{L\ (\mathbb{T})} + \|\phi_x\|_{L\ (\mathbb{T})})\|\phi_{xx}\|_{L\ (\mathbb{T})}$$
$$\leq \frac{D}{\bar{u}}\|\phi_{xx}\|_{L\ (\mathbb{T})} + \frac{\delta}{D\bar{u}}(\|\psi_x\|_{L\ (\mathbb{T})} + \|\phi_x\|_{L\ (\mathbb{T})}),$$

$$-\frac{d}{dt}\int_{\mathbb{T}}\left(\frac{\phi_x}{\bar{u}} + \psi_x\right)dx \leq \frac{\delta}{D\bar{u}}(\|\psi_x\|_{L\ (\mathbb{T})} + \|\phi_x\|_{L\ (\mathbb{T})}). \qquad (\ .\ )$$

. . (.) (.),

$$-\frac{d}{dt}\int_{\mathbb{T}}\left(M\phi\ + M\bar{u}\psi\ + (D+\ )\psi_x + \frac{\phi_x}{\bar{u}} - \ \phi\psi_x\right)dx$$
$$+\left(DM - \frac{\delta\ M\ c_p}{\bar{u}} - \frac{\delta}{\bar{u}} - \frac{\bar{v}}{\bar{u}} - \frac{\delta}{D\bar{u}} - \ \right)\int_{\mathbb{T}}\phi_x dx + \left(\frac{\bar{u}}{} - \delta - \frac{\delta}{D\bar{u}}\right)\int_{\mathbb{T}}\psi_x dx \leq \ . \qquad (\ .\ )$$

T M . δ . α α .

$$DM - \frac{\delta\ M\ c_p}{\bar{u}} - \frac{\delta}{\bar{u}} - \frac{\bar{v}}{\bar{u}} - \frac{\delta}{D\bar{u}} - \ \geq \alpha\ > \qquad \frac{\bar{u}}{} - \delta - \frac{\delta}{D\bar{u}} \geq \alpha\ > \ ,$$

(.)

$$\frac{d}{dt}\mathcal{H}(t) + \alpha\ \|\phi_x\|_{L\ (\mathbb{T})} + \alpha\ \|\psi_x\|_{L\ (\mathbb{T})} \leq \ , \qquad (\ .\ )$$

$$\mathcal{H}(t) = -\int_{\mathbb{T}}\left(M\phi\ + M\bar{u}\psi\ + (D+\ )\psi_x + \frac{\phi_x}{\bar{u}} - \ \phi\psi_x\right)dx. \qquad (\ .\ )$$

, . . ,

$$\mathcal{H}(t) \leq \frac{M}{}\int_{\mathbb{T}}\phi\ dx + \frac{M\bar{u}}{}\int_{\mathbb{T}}\psi\ dx + \frac{D+}{}\int_{\mathbb{T}}\psi_x + \frac{}{\bar{u}}\int\phi_x dx + \int_{\mathbb{T}}|\phi\psi_x|dx$$
$$\leq \frac{M+}{}\|\phi\|_{L\ (\mathbb{T})} + \frac{M\bar{u}}{}\|\psi\|_{L\ (\mathbb{T})} + \frac{D+}{}\|\psi_x\|_{L\ (\mathbb{T})} + \frac{}{\bar{u}}\|\phi_x\|_{L\ (\mathbb{T})}$$
$$\leq \frac{(M+\ )\bar{u}c_p+}{\bar{u}}\|\phi_x\|_{L\ (\mathbb{T})} + \frac{M\bar{u}c_p + D+}{}\|\psi_x\|_{L\ (\mathbb{T})},$$

$\alpha\ >$.

$$\alpha\ \|\phi_x\|_{L\ (\mathbb{T})} + \alpha\ \|\psi_x\|_{L\ (\mathbb{T})} \geq \alpha\ \mathcal{H}(t). \qquad (\ .\ )$$

T . . (.) (.),

$$\frac{d}{dt}\mathcal{H}(t) + \alpha\ \mathcal{H}(t) \leq \ ,$$

$$\mathcal{H}(t) \leq \mathcal{H}(\ )e^{-\alpha\ t} \leq C(\|\phi\ \|_{H\ (\mathbb{T})} + \|\psi\ \|_{H\ (\mathbb{T})})e^{-\alpha\ t}. \qquad (\ .\ )$$

, . $\mathcal{H}(t)$ (.), $M >$.

$$\mathcal{H}(t) = \frac{(M - \ )}{\ }\int_{\mathbb{T}} \phi\ + \frac{M\bar{u}}{\ }\int_{\mathbb{T}} \psi\ + \frac{D}{\ }\int_{\mathbb{T}} \psi_x dx + \frac{\ }{\bar{u}}\int_{\mathbb{T}} \phi_x + -\int_{\mathbb{T}} (\psi_x - \phi) \qquad (\ .\ )$$
$$\geq C(\|\phi\|_{H\ (\mathbb{T})} + \|\psi\|_{H\ (\mathbb{T})}).$$

T (.) (.)

$$\|(\phi, \psi)\|_{H\ (\mathbb{T})} \leq C\|(\phi\ , \psi\ )\|_{H\ (\mathbb{T})}e^{-\alpha t}.$$

,

$$\|(\phi, \psi)\|_{H\ (\mathbb{T})} \leq C\|(\phi\ , \psi\ )\|_{H\ (\mathbb{T})}e^{-\alpha t} = C\varepsilon e^{-\alpha t}, \qquad (\ .\ )$$

$\varepsilon := \|(\phi\ , \psi\ )\|_{H\ (\mathbb{T})}$. T ε . . $C\varepsilon < \frac{\delta}{\ }$, (.)
$\|(\phi, \psi)\|_{H\ (\mathbb{T})} < \frac{\delta}{\ }$, *a priori* . (.) . T
. . □

Proof of Lemma 3.5. (.)

$$< \frac{u_+}{\ } < U^+(\xi, t) < \frac{u_+ + u_-}{\ } < U^-(\xi, t) < \frac{u_- - u_+}{\ } \qquad \xi \in \mathbb{R},\ t > \ . \qquad (\ .\ )$$

T . $\mathcal{U}(\xi, t)$ (.), $< g <$,

$$U^+(\xi, t) < \mathcal{U}(\xi, t) < U^-(\xi, t),$$

, (.), (.). , . $\mathcal{V}$.
, . $\mathcal{U}$ $\mathcal{V}$. , (.)

$$\mathcal{U}_\xi(\xi, t) = U_\xi^-(\ - g\chi_{(t)}) + U_\xi^+ g\chi_{(t)} + (U^+ - U^-)g'_{\chi_{(t)}}, \qquad (\ .\ )$$

$$g'_{\mathcal{X}(t)} = \Big(\frac{U_{\mathcal{X}(t)} - u_-}{u_+ - u_-}\Big)' = \frac{U'_{\mathcal{X}(t)}}{u_+ - u_-} = \frac{U'(\xi - \mathcal{X}(t))}{u_+ - u_-}.$$

. , $U'(\xi - \mathcal{X}(t))$ $g'_{\mathcal{X}(t)}$. -
, . , $U_\xi^\pm$,
$< g_{\mathcal{X}(t)} <$ (.) $\mathcal{U}_\xi$. ,
$\mathcal{V}_\xi, \mathcal{U}_{\xi\xi}, \mathcal{U}_{\xi\xi\xi}$ $\mathcal{V}_{\xi\xi\xi}$.
, (.) . , (.), .
.

$$\begin{aligned}
\|\mathcal{U}_\xi - U'_{\mathcal{X}_\infty}\|_{L^\infty} \le& \|\mathcal{U}_\xi - U'_{\mathcal{X}(t)}\|_{L^\infty} + \|U'_{\mathcal{X}(t)} - U'_{\mathcal{X}_\infty}\|_{L^\infty} \\
=& \|U_\xi^-(\ - g_{\mathcal{X}(t)}) + U_\xi^+ g_{\mathcal{X}(t)} + (U^+ - U^-)g'_{\mathcal{X}(t)} - (u_+ - u_-)g'_{\mathcal{X}(t)}\|_{L^\infty} \\
&+ \|(u_+ - u_-)g'_{\mathcal{X}(t)} - (u_+ - u_-)g'_{\mathcal{X}_\infty}\|_{L^\infty} \\
\le& C\Big(\|U_\xi^-\|_{L^\infty} + \|U_\xi^+\|_{L^\infty} + \|U^- - u_-\|_{L^\infty} + \|U^+ - u_+\|_{L^\infty}\Big) \\
&+ C\|\mathcal{X}(t) - \mathcal{X}_\infty\|_{L^\infty} \\
\le& C\varepsilon e^{-\alpha t}.
\end{aligned} \tag{ . }$$

,

$$\|\mathcal{V}_\xi - V'_{\mathcal{Y}_\infty}\|_{L^\infty} \le C\varepsilon e^{-\alpha t}. \tag{ . }$$

T (.) (.) (.).
, (.) . , U^- .
(.) , .

$$\begin{aligned}
\|U_t^-\|_{L^\infty} &= \|sU_\xi^- + (U^-V^-)_\xi + DU_{\xi\xi}^-\|_{L^\infty} \\
&\le (s + \|V^-\|_{L^\infty})\|U_\xi^-\|_{L^\infty} + \|U^-\|_{L^\infty}\|V_\xi^-\|_{L^\infty} + D\|U_{\xi\xi}^-\|_{L^\infty} \\
&\le C\varepsilon e^{-\alpha t}.
\end{aligned} \tag{ . }$$

,

$$\|U_t^+\|_{L^\infty} \le C\varepsilon e^{-\alpha t}. \tag{ . }$$

T . (.), (.) . ,

$$\begin{aligned}
\|\mathcal{U}_t\|_{L^\infty} &= \|U_t{}^-(\ - g_{\mathcal{X}(t)}) + U^- g'_{\mathcal{X}(t)}\mathcal{X}'(t) + U_t{}^+ g_{\mathcal{X}(t)} - U^+ g'_{\mathcal{X}(t)}\mathcal{X}'(t)\|_{L^\infty} \\
&\le C\Big(\|U_t^-\|_{L^\infty} + \|U_t^+\|_{L^\infty} + |\mathcal{X}'(t)|\Big) \le C\varepsilon e^{-\alpha t},
\end{aligned}$$

(.). □

Proof of Lemma 3.6.

. , . . . $\mathcal{P}$. T ,
$\mathcal{P}$ (.) $\xi <$

$$\begin{aligned}\mathcal{P}(\xi,t) &= F(\xi,t) + \int_{-\infty}^{\xi} f(y,t)dy \\ &= \mathcal{U}\mathcal{V} - U^-V^- + (U^-V^- - U^+V^+)g_{\mathcal{X}(t)} + D(U^+ - U^-)g'_{\mathcal{X}(t)} \\ &\quad + (s + \mathcal{X}'(t)) \int_{-\infty}^{\xi} (U^+ - U^-)g'_{\mathcal{X}(t)}dy \\ &\quad + D\int_{-\infty}^{\xi} (U_y^+ - U_y^-)g'_{\mathcal{X}(t)}dy + \int_{-\infty}^{\xi} (U^+V^+ - U^-V^-)g'_{\mathcal{X}(t)}dy.\end{aligned} \tag{.}$$

. (.), $(U^{\pm}, V^{\pm})$

(.),

$$\begin{aligned}\mathcal{P}^- =& U_{\mathcal{X}_\infty} V_{\mathcal{X}_\infty} - u_- v_- + (u_- v_- - u_+ v_+) g_{\mathcal{X}(t)} + D(u_+ - u_-) g'_{\mathcal{X}(t)} \\ &+ (u_+ - u_-)(s + \mathcal{X}'(t)) g_{\mathcal{X}(t)} + (u_+ v_+ - u_- v_-) g_{\mathcal{X}(t)} \\ =& U_{\mathcal{X}_\infty} V_{\mathcal{X}_\infty} - u_- v_- + (u_- v_- - u_+ v_+) g_{\mathcal{X}(t)} \\ &+ D(u_+ - u_-) g'_{\mathcal{X}(t)} + (u_+ - u_-) \mathcal{X}'(t) g_{\mathcal{X}(t)}.\end{aligned} \tag{ . }$$

,

, $\mathcal{P}$ $\xi >$

$$P\ (\xi,t) = F\ (\xi,t) - \int_{\xi}^{+\infty} f\ (y,t)dy = \mathcal{P}^{+}(\xi,t) + \mathcal{P}^{+}(\xi,t)$$

$$\sum_{k=}\int^{+\infty}\left|\partial_{\xi}^{k}\mathcal{P}\ \right| d\xi \le \ \sum_{k=}\int^{+\infty}\left(\left|\partial_{\xi}^{k}\mathcal{P}^{+}\right| + \left|\partial_{\xi}^{k}\mathcal{P}^{+}\right|\ \right)d\xi \le C\varepsilon\ e^{-\ \alpha t}. \qquad (\ .\ )$$

T (.) (.)

$$\|\mathcal{P}\ (\cdot,t)\|_{H\ (\mathbb{R})} \le C\varepsilon e^{-\alpha t}. \qquad (\ .\ )$$

, $\mathcal{P}\ (\xi,t)$ (.). , $\mathcal{P}\ (\xi,t)$ $\xi <$

$$\begin{aligned}\mathcal{P}\ (\xi,t) =& (U^{+}-U^{-})(g_{\mathcal{X}(t)} - g_{\mathcal{Y}(t)}) + \int_{-\infty}^{\xi}(U^{+}-U^{-})g'_{\mathcal{Y}(t)}dy\\ &+ (s+\mathcal{Y}'(t))\int_{-\infty}^{\xi}(V^{+}-V^{-})g'_{\mathcal{Y}(t)}dy\\ =& \mathcal{P}^{-}(\xi,t) + \mathcal{P}^{-}(\xi,t),\end{aligned} \qquad (\ .\ )$$

$$\begin{aligned}\mathcal{P}^{-}(\xi,t) =& (u_{+}-u_{-})(g_{\mathcal{X}(t)} - g_{\mathcal{Y}(t)}) + (u_{+}-u_{-})g_{\mathcal{Y}(t)}\\ &+ (s+\mathcal{Y}'(t))(v_{+}-v_{-})g_{\mathcal{Y}(t)}\end{aligned} \qquad (\ .\ )$$

$$\begin{aligned}\mathcal{P}^{-}(\xi,t) =& (\phi_{+}-\phi_{-})(g_{\mathcal{X}(t)} - g_{\mathcal{Y}(t)}) + \int_{-\infty}^{\xi}(\phi_{+}-\phi_{-})g'_{\mathcal{Y}(t)}dy\\ &+ (s+\mathcal{Y}'(t))\int_{-\infty}^{\xi}(\psi_{+}-\psi_{-})g'_{\mathcal{Y}(t)}dy.\end{aligned} \qquad (\ .\ )$$

(.), $u_{+}-u_{-} = -s(v_{+}-v_{-})$, (.)

$$\mathcal{P}^{-}(\xi,t) = (u_{+}-u_{-})(g_{\mathcal{X}(t)} - g_{\mathcal{Y}(t)}) + (v_{+}-v_{-})\mathcal{Y}'(t)g_{\mathcal{Y}(t)}. \qquad (\ .\ )$$

.

$$|\mathcal{Y}(t) - \mathcal{X}(t)| + |\mathcal{Y}'(t)| \leq C\varepsilon e^{-\alpha t}. \qquad (\ .\ )$$

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Acknowledgment

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References

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