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GLOBAL STABILIZATION OF THE FULL ATTRACTION-REPULSION KELLER-SEGEL SYSTEM

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To Professor Wei-Ming Ni on the occasion of his 70th birthday, with our best wishes

Abstract. We are concerned with the following full Attraction-Repulsion Keller-Segel (ARKS) system

$$\begin{cases} u_t = \ \ u - \nabla\cdot(\chi u\nabla v) + \nabla\cdot(\xi u\nabla w), & x\in \ , \ t>0,\\ v_t = D_1 \ \ v + \alpha u - \beta v, & x\in \ , \ t>0,\\ w_t = D_2 \ \ w + \gamma u - \delta w, & x\in \ , \ t>0,\\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), w(x,0) = w_0(x) & x\in \ , \end{cases} \qquad (*)$$

in a bounded domain $\ \subset \mathbb{R}^2$ with smooth boundary subject to homogeneous Neumann boundary conditions. By constructing an appropriate Lyapunov functions, we establish the boundedness and asymptotical behavior of solutions to the system $(*)$ with large initial data $(u_0, v_0, w_0) \in [W^{1,1}(\ )]^3$. Precisely, we show that if the parameters satisfy $\frac{\xi\gamma}{\chi\alpha} \geq \max\left\{\frac{D_1}{D_2}, \frac{D_2}{D_1}, \frac{\beta}{\delta}, \frac{\delta}{\beta}\right\}$ for all positive parameters $D_1, D_2, \chi, \xi, \alpha, \beta, \gamma$ and δ, the system $(*)$ has a unique global classical solution (u, v, w), which converges to the constant steady state $(u_0, \frac{\alpha}{\beta}u_0, \frac{\gamma}{\delta}u_0)$ as $t \to +\infty$, where $u_0 = \frac{1}{j\ j}\int \ u_0 dx$. Furthermore, the decay rate is exponential if $\frac{\xi\gamma}{\chi\alpha} > \max\left\{\frac{\beta}{\delta}, \frac{\delta}{\beta}\right\}$. This paper provides the rst results on the full ARKS system with unequal chemical di usion rates (i.e. $D_1 \neq D_2$) in multi-dimensions.

1. **Introduction.** To describe the aggregation of *Microglia* in the central nervous system in Alzhemer's disease due to the interaction of chemoattractant (i.e. -amyloid) and chemorepellent (i.e. TNF-), Luca et al. [21] proposed the following

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attraction-repulsion chemotaxis system

$$\begin{cases} u_t = \Delta u \quad \nabla \quad (\; u \nabla v) + \nabla \quad (\; u \nabla w), & x \in \Omega, \; t > 0, \\ {}_1 v_t = D_1 \Delta v + \quad u \quad\quad v, & x \in \Omega, \; t > 0, \\ {}_2 w_t = D_2 \Delta w + \quad u \quad\quad w, & x \in \Omega, \; t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega, \; t > 0, \\ u(x,0) = u_0(x), \quad {}_1 v(x,0) = \quad {}_1 v_0(x), \quad {}_2 w(x,0) = \quad {}_2 w_0(x), & x \in \Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and denotes the outward normal vector of $\partial\Omega$. The density of *Microglia* cells is denoted by $u(x,t)$, while $v(x,t)$ and $w(x,t)$ denote the concentration of chemoattractant and chemorepllent, respectively. The model (1) can also be regarded as a particularized model proposed in [23] to model the quorum sensing effect in chemotaxis.

When $= 0$, the variable w can be decoupled from the system (1), where the variables u and v satisfy the classical attractive Keller-Segel (KS) system

$$\begin{cases} u_t = \Delta u \quad\quad \nabla \quad (u \nabla v), & x \in \Omega, \; t > 0, \\ v_t = D_1 \Delta v + \quad u \quad\quad v, & x \in \Omega, \; t > 0. \end{cases} \tag{2}$$

The KS model (2) has been extensively studied in the past four decades in various perspectives and massive results are available (cf. survey articles [6, 1] and references therein). One of the mostly studied topics for the KS model (2) is the boundedness and blowup of solutions in two or higher dimensions [22, 30, 7] based on the following Lyapunov function:

$$E_1(u,v) = \int_\Omega u \ln u \quad\quad \int_\Omega uv + \frac{\quad}{2} \int_\Omega v^2 + \frac{D_1}{2} \int_\Omega |\nabla v|^2.$$

If $= 0$, the variable v can be decoupled and (u,w) satisfies the following repulsive Keller-Segel model

$$\begin{cases} u_t = \Delta u + \quad \nabla \quad (u \nabla w), & x \in \Omega, \; t > 0, \\ w_t = D_2 \Delta w + \quad u \quad\quad w, & x \in \Omega, \; t > 0. \end{cases} \tag{3}$$

Compared to the attractive KS model (2), the results on the repulsive KS model (3) are much less. The global existence of classical solutions in two dimensions and weak solutions in three or four dimensions were established in [4] based on the following Lyapunov function

$$E_2(u,w) = \int_\Omega u \ln u + \frac{{}^2}{2} \int_\Omega |\nabla w|^2$$

which is difference from the one for the attractive KS model. A further investigation on the repulsive KS model was made in [26].

Roughly speaking, the attraction-repulsion Keller-Segel model (1) can be regarded as a superposition of the attractive and repulsive KS models. Hence one may expect the ARKS model should behave more or less the same as the attractive or repulsive models. However it is not straightforward to justify this suspicion due to the interaction between attraction and repulsion. In particular, as we recalled above, the understanding of the attractive and repulsive KS models heavily rely on the finding of Lyapunov functions. Therefore to have a comprehensive understanding for the ARKS model, finding appropriate Lyapunov function is indispensable. This is by no means an easy work for a strongly coupled cross-diffusion system of PDEs like ARKS model. A sequence of works thus have been stimulated to reveal the mystery underlying the model gradually. The first such progress was made by Tao and Wang [27] who found that the solution behavior of the ARKS model was essentially determined by the sign of

$$_1 = \quad ;$$

which is an index measuring the competition between attraction and repulsion. More precisely, they showed that if $D_1 = D_2 = 1$ and $_1 = \ _2 = 0$, the ARKS system (1) has a unique classical solution with uniform-in-time bound if $_1 \quad 0$ (i.e. repulsion dominates or cancels attraction) in higher dimensions ($n \quad 2$). The main idea of [27] was a transformation $s = \ w \quad v$ which may significantly simplify the system and has become a major source for many of subsequent researches on the ARKS model. For the opposite case $_1 < 0$ (i.e. attraction dominates repulsion), it was shown that the solution of system (5) may blow up in finite time if initial mass is large [5, 14] and exist globally for small initial mass [5] in two dimensions. If $_1 = 1$ and $_2 = 0$, Jin and Wang [11] constructed a Lyapunov function

$$E_3(u;v;w) = \int_\Omega u \ln u \quad \int_\Omega uv + \frac{}{2}\int_\Omega v^2 + \frac{D_1}{2}\int_\Omega j\, r\, vj^2 + \frac{}{2}\int_\Omega w^2 + \frac{D_2}{2}\int_\Omega j\, r\, wj^2;$$

to establish the global existence of uniformly-in-time bounded classical solutions in two dimensions for large initial data if $_1 \quad 0$. Conversely if $_1 < 0$, they showed there exists a critical mass m_* such that the solution blows up if $\int_\Omega u_0 > m_*$ and globally exists if $\int_\Omega u_0 < m_*$.

If the three equations of the ARKS model (1) are all parabolic (i.e. $_1 = \ _2 = 1$), it is much harder to study and much less results are available. We recall the known results below. In one dimension, the global existence of classical solutions, non-trivial stationary state, asymptotic behavior and pattern formation of the system (1) have been studied in [10, 19, 20]. In two dimensions, when $D_1 = D_2$, it was shown in [27] that global classical solutions exist for large data if $\ = \ $ and for small data if $\ \neq \ $ when $_1 \quad 0$ (i.e. repulsion dominates or cancels attraction). Subsequently the global existence of large-data solutions was extended to the case $\ \neq \ $ in [8, 18]. Moreover, for $\ \neq \ $, when cell mass is small, it was shown that the global classical solution will exponentially converge to the unique constant steady state $(\bar{u}_0; \frac{\alpha}{\beta}\bar{u}_0; \frac{\gamma}{\delta}\bar{u}_0)$ with $\bar{u}_0 = \frac{1}{|\Omega|}\int_\Omega u_0$ in [15, 16], which was further elaborated by assuming

$$\bar{u}_0 < \frac{4}{(\qquad)^2} \quad \text{and} \quad > \frac{4 \quad (\quad \bar{u}_0 + 1)}{[4 \quad (\qquad)^2 \quad \bar{u}_0]\bar{u}_0} \tag{4}$$

in [17] wherein the convergence rate was, however, not given. Whether or not the same results holds for large initial data in multi-dimensions still remains unknown. Part of above-mentioned results have been extended to the multi-dimensional whole space in [9, 25]. We should underline that all existing results in two or higher dimensions recalled above for the case $_1 = \ _2 = 1$ are essentially based on the assumption $D_1 = D_2$ so that the idea of making a change of variable $s = \ w \quad v$ introduced in [27] can be employed. To the best of our knowledge, no result for the case $_1 = \ _2 = 1$ and $D_1 \neq D_2$ has been available to (1) in multi-dimensions to date. It is the purpose of this paper to exploit this challenging case and contribute some results, where the corresponding ARKS model (1) reads as

$$\begin{cases} u_t = \Delta u \quad \nabla \quad (\quad u \nabla v) + \nabla \quad (\quad u \nabla w); & x \in \Omega;\ t > 0; \\ v_t = D_1 \Delta v + \quad u \qquad v; & x \in \Omega;\ t > 0; \\ w_t = D_2 \Delta w + \quad u \qquad w; & x \in \Omega;\ t > 0; \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0; & x \in \partial\Omega;\ t > 0; \\ u(x; \end{cases}$$

Hence to show the global existence of classical solutions in two dimensions, we only need to prove (8). When $D_1 = D_2$ and $\chi_1 > 0$, using the transformation $s = \xi w - \chi v$ as in [27], one can derive the following entropy inequality (cf. [18, 8])

$$\frac{d}{dt}\left(\int_\Omega u\ln u + \frac{1}{2\chi_1}\int_\Omega |\nabla s|^2\right) + \int_\Omega \frac{|\nabla u|^2}{u} + \frac{D_1}{2\chi_1}\int_\Omega |\Delta s|^2 + \frac{\delta}{\chi_1}\int_\Omega |\nabla s|^2 \le c_2, \tag{9}$$

which can be used to derive (8) and hence the boundedness of solutions.

However, when $D_1 \neq D_2$, the transformation idea fails to work. Luckily, we are able to find a different Lyapunov function $E(u,v,w)$ (see the definition in (31)) for the system (5) under the condition (6), which satisfies

$$\frac{d}{dt}E(u,v,w) + F(u,v,w) = 0, \tag{10}$$

where $F(u,v,w)$ is defined by (32). We remark that the form of $E(u,v,w)$ is quite different from the one (9) for $D_1 = D_2$. To prove $E(u,v,w)$ is a Laypunov function, we organize the estimates into a quadratic form which is the new idea developed in the paper. Then using (10), we show that under the condition (6), there exists a constant $c_3 > 0$ such that $\|u\ln u\|_{L^1} + \|\nabla v\|_{L^2} + \|\nabla w\|_{L^2} \le c_3$ and $\int_0^t\int_\Omega \frac{|\nabla u|^2}{u} \le c_3$ (see Lemma 4.1 for details). The former estimate leads to the boundedness of solutions in two dimensions and the later estimate gives the convergence properties of u. The convergence of v and w can be derived by the parabolic comparison principle. To study the decay rate, we first show that there exists a constant $\mu > 0$ such that

$$E(u,v,w) \le \mu F(u,v,w),$$

which together with (10) gives $E(u,v,w) \le E(u_0,v_0,w_0)e^{-\frac{1}{\mu}t}$. Using the definition of $E(u,v,w)$ and noting the fact $\|u-\bar u\|_{L^1} \le 2\bar u\int_\Omega u\ln\frac{u}{\bar u}$ in Lemma 2.1, the exponential decay of $\|u-\bar u\|_{L^1}$ under the condition (6) is obtained. Then using the ideas in [27] or [15], we derive the decay rate of $\|u-\bar u\|_{L^\infty}$ and hence the exponential decay rate of $\|v-\frac{\alpha}{\beta}\bar u_0\|_{L^\infty}$ and $\|w-\frac{\gamma}{\delta}\bar u_0\|_{L^\infty}$.

In the end of this section, we remark that Theorem 1.1 only present some first-hand results on the full ARKS model for $D_1 \neq D_2$ under the parameter regime given in (6) and leave out many interesting questions due to technical difficulty. For example, whether the condition (6) is necessary for global existence of solutions and how solutions behave (in particular whether solutions blow up) if the condition (6) fails remain unsolved in our paper. We hope our studies in this paper will provide useful clues to further explore the ARKS model in future.

2. **Some basic inequalities.** In what follows, without confusion, we shall abbreviate $\int_\Omega f\,dx$ as $\int_\Omega f$ for simplicity. Moreover, we shall use $c_i\,(i = 1,2,3,\cdots)$ to denote a generic constant which may vary in the context. For reader's convenience, we present some known inequalities for later use.

Lemma 2.1. *Suppose that $f(x,t)$ is a positive function on $(x,t) \in \Omega\times(0,\infty)$. Define $\bar f = \frac{1}{|\Omega|}\int_\Omega f$, then it has that*

$$0 \le \frac{1}{2\bar f}\|f-\bar f\|_{L^1}^2 \le \int_\Omega f\ln\frac{f}{\bar f} \le \frac{1}{\bar f}\|f-\bar f\|_{L^2}^2. \tag{11}$$

Proof. Using the Csiszár-Kullback-Pinsker inequality (see [3]), one has

$$\int_\Omega f\ln\frac{f}{\bar f} \ge \frac{1}{2\bar f}\|f-\bar f\|_{L^1}^2. \tag{12}$$

On the other hand, choosing $\ \ = \frac{f}{\bar f}$ and using the fact that $\ \ \ln\ \ \ \ +1\ \ \ (\ \ \ 1)^2$ for $\ \ \ 0$, it holds that

$$\int_\Omega f \ln \frac{f}{\bar f} \leq \bar f \int_\Omega \left[\frac{f}{\bar f} - 1 + \left(\frac{f}{\bar f} - 1\right)^2\right] = \frac{1}{\bar f}\|f - \bar f\|_{L^2}^2. \tag{13}$$

Then the combination of (12) and (13) gives (11). □

Lemma 2.2. *Let Ω be a bounded domain in $\mathbb{R}^2$ with smooth boundary. Then, for any $\varphi \in W^{3,2}(\Omega)$ satisfying $\frac{\partial\varphi}{\partial\nu}|_{\partial\Omega} = 0$, there exists a positive constant C depending only on Ω such that*

$$\|\Delta\varphi\|_{L^3} \leq C(\|\nabla\Delta\varphi\|_{L^2}^{\frac{2}{3}}\|\nabla\varphi\|_{L^2}^{\frac{1}{3}} + \|\nabla\varphi\|_{L^2}). \tag{14}$$

Proof. Using Gagliardo-Nirenberg inequality, we have

$$\|\Delta\varphi\|_{L^3} = \|\nabla\cdot\nabla\varphi\|_{L^3} \leq \|D\nabla\varphi\|_{L^3} \leq c_1\|D^2\nabla\varphi\|_{L^2}^{\frac{2}{3}}\|\nabla\varphi\|_{L^2}^{\frac{1}{3}} + c_2\|\nabla\varphi\|_{L^2}, \tag{15}$$

where $|D^k\nabla\varphi| = (\sum_{|i|=k}|D^i\nabla\varphi|^2)^{\frac{1}{2}}$ and i is a multi-index of order. On the other hand, one can check that

$$\|D^2\nabla\varphi\|_{L^2} \leq c_3\|\nabla\varphi\|_{H^2}. \tag{16}$$

Moreover, under the homogeneous Neumann boundary condition (i.e., $\frac{\partial\varphi}{\partial\nu}|_{\partial\Omega} = 0$), it follows from [2, Lemma 1] that $\|\nabla\varphi\|_{H^2} \leq c_4\|\Delta\varphi\|_{H^1}$, which applied to (16) gives

$$\|D^2\nabla\varphi\|_{L^2} \leq c_3c_4\|\Delta\varphi\|_{H^1}. \tag{17}$$

Note that $|\Delta\varphi|^2 = \nabla\cdot(\nabla\varphi\Delta\varphi) - \nabla\varphi\cdot\nabla\Delta\varphi$. Then using the boundary condition $\frac{\partial\varphi}{\partial\nu}|_{\partial\Omega} = 0$ and Hölder inequality, we have

$$\|\Delta\varphi\|_{L^2}^2 = -\int_\Omega \nabla\varphi\cdot\nabla\Delta\varphi \leq \|\nabla\varphi\|_{L^2}\|\nabla\Delta\varphi\|_{L^2}. \tag{18}$$

Then substituting (17) into (15), and using (18), one derives

$$\begin{aligned}
\|\Delta\varphi\|_{L^3} \leq\ & c_5\left(\|\Delta\varphi\|_{H^1}^{\frac{2}{3}}\|\nabla\varphi\|_{L^2}^{\frac{1}{3}} + \|\nabla\varphi\|_{L^2}\right)\\
=&c_5\left(\|\nabla\Delta\varphi\|_{L^2} + \|\Delta\varphi\|_{L^2}\right)^{\frac{2}{3}}\|\nabla\varphi\|_{L^2}^{\frac{1}{3}} + c_5\|\nabla\varphi\|_{L^2}\\
\leq\ & c_6\left(\|\nabla\Delta\varphi\|_{L^2} + \|\nabla\Delta\varphi\|_{L^2}^{\frac{1}{2}}\|\nabla\varphi\|_{L^2}^{\frac{1}{2}}\right)^{\frac{2}{3}}\|\nabla\varphi\|_{L^2}^{\frac{1}{3}} + c_6\|\nabla\varphi\|_{L^2}\\
\leq\ & c_6\left(2\|\nabla\Delta\varphi\|_{L^2} + \|\nabla\varphi\|_{L^2}\right)^{\frac{2}{3}}\|\nabla\varphi\|_{L^2}^{\frac{1}{3}} + c_6\|\nabla\varphi\|_{L^2}\\
\leq\ & c_7\|\nabla\Delta\varphi\|_{L^2}^{\frac{2}{3}}\|\nabla\varphi\|_{L^2}^{\frac{1}{3}} + c_7\|\nabla\varphi\|_{L^2}
\end{aligned}$$

which yields (14), and hence completes the proof. □

3. **Boundedness criterion and Lyapunov function.**

3.1. **Local existence.** The local existence theorem of system (5) can be proved by the fixed point theorem and maximum principle along the same line as in [27]. Hence we only present the results without proof for brevity.

Lemma 3.1. *Let Ω be a bounded domain in $\mathbb{R}^n (n \geq 2)$ with smooth boundary. Suppose that $0 \leq (u_0, v_0, w_0) \in [W^{1,\infty}(\Omega)]^3$. Then there exist a $T_{max} \in (0, \infty]$ such that the system (5) has a unique solution (u, v, w) of nonnegative functions from $[C^0(\bar{\Omega} \times [0, T_{max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{max}))]^3$. Moreover $u > 0$ in $\Omega \times (0, T_{max})$ and*

$$\text{if } T_{max} < \infty, \text{ then } \|u(\cdot, t)\|_{L^\infty} \to \infty \text{ as } t \nearrow T_{max}. \tag{19}$$

Furthermore, the cell mass is conservative:

$$\|u(\cdot, t)\|_{L^1} = \|u_0\|_{L^1}. \tag{20}$$

3.2. **Boundedness criterion.** To extend the local solutions to global ones, we derive a boundedness criterion for the solution of system (5). The idea of our proof is essentially inspired by [1, lemma 3.2] and we present necessary details below for clarity.

Lemma 3.2. *Suppose the conditions in Lemma 3.1 hold. Let (u, v, w) be the solution of system (5) defined on its maximal existence time interval $[0, T_{max})$. If there exist $p > \frac{n}{2}$ and a constant M_0 such that*

$$\sup_{t \in (0, T_{max})} \|u(\cdot, t)\|_{L^p} \leq M_0,$$

then one can find a constant $C > 0$ independent of t such that

$$\|u(\cdot, t)\|_{L^\infty} + \|v(\cdot, t)\|_{W^{1,\infty}} + \|w(\cdot, t)\|_{W^{1,\infty}} \leq C \quad \text{for all} \quad t \in (0, T_{max}). \tag{21}$$

Furthermore, there exists $\sigma \in (0, 1)$ such that for all $t > 1$

$$\|u\|_{C^{\sigma, \frac{\sigma}{2}}(\bar{\Omega} \times [t, t+1])} \leq C. \tag{22}$$

Proof. Since $\|u(\cdot, t)\|_{L^p} \leq M_0$, then applying the parabolic regularity estimates in [12, Lemma 1] to the second and third equations of system (5) we have

$$\|\nabla v(\cdot, t)\|_{L^r} + \|\nabla w(\cdot, t)\|_{L^r} \leq c_1, \text{ for all } t \in (0, T_{max}) \tag{23}$$

where

$$r \in \begin{cases} [1, \frac{np}{n-p}), & \text{if} \quad p \leq n, \\ [1, \infty], & \text{if} \quad p > n. \end{cases} \tag{24}$$

Without loss of generality, we assume that $\frac{n}{2} < p \leq n$ which yields $\frac{np}{n-p} > n$. Then we can find a constant $r > 0$ with $n < r < \frac{np}{n-p}$ such that (23) holds. Now, for each $T \in (0, T_{max})$, we define

$$M(T) := \sup_{t \in (0, T)} \|u(\cdot, t)\|_{L^\infty}, \tag{25}$$

which is finite due to the local existence results in Lemma 3.1. Next, we will estimate $M(T)$. Fix $t \in (0, T)$ and let $t_0 = (t - 1)_+$. Then applying the variation-of-constants formula to the first equation of system (5), we get

$$\begin{aligned} u(\cdot, t) = & e^{(t-t_0)\Delta} u(\cdot, t_0) - \int_{t_0}^{t} e^{(t-\tau)\Delta} \nabla \cdot (u(\cdot, \tau) \nabla v(\cdot, \tau)) d\tau \\ & + \int_{t_0}^{t} e^{(t-\tau)\Delta} \nabla \cdot (u(\cdot, \tau) \nabla w(\cdot, \tau)) d\tau \end{aligned}$$

which implies

$$
\begin{aligned}
\|u(\cdot,t)\|_{L^\infty} \le\ & \|e^{(t-t_0)\Delta}u(\cdot,t_0)\|_{L^\infty} + \int_{t_0}^{t}\|e^{(t-\tau)\Delta}\nabla\cdot(\chi u(\cdot,\tau)\nabla v(\cdot,\tau))\|_{L^\infty}d\tau \\
& + \int_{t_0}^{t}\|e^{(t-\tau)\Delta}\nabla\cdot(\xi u(\cdot,\tau)\nabla w(\cdot,\tau))\|_{L^\infty}d\tau \\
=\ & I_1 + I_2 + I_3.
\end{aligned}
\tag{26}
$$

We first estimate the term I_1. If $t \le 1$, then $t_0 = 0$ and we can use the maximum principle for the heat equation to obtain

$$I_1 = \|e^{t\Delta}u_0\|_{L^\infty} \le \|u_0\|$$

At last, from (21) we know that $\nabla u\cdot\nabla v$ and $\nabla u\cdot\nabla w$ are bounded in $L^\infty(\Omega\times(0,\infty))$. Then applying the standard parabolic regularity theory (e.g. see [24, Theorem 1.3] and [28, Lemma 3.2]) and parabolic Schauder theory [13], we immediately obtain the estimate (22). Then the proof of Lemma 3.2 is completed. □

3.3. **Lyapunov function.** As mentioned in Remark 1, we consider the case $D_1\neq D_2$ or $\beta\neq\delta$ which implies that $\theta_1>0$ from (6). When $D_1=D_2$, the boundedness of solutions shown in Theorem 1.1 has been proved in [27] with $\beta=\delta$ and in [18, 8] with $\beta\neq\delta$ by constructing entropy inequality based on an idea of using the transformation $s=\xi w-\chi v$. However this transformation is no longer helpful for the case $D_1\neq D_2$. Hence, we need to find a new way. Here we achieve our results by constructing a Lyapunov function for the system (5). First, we define

$$E(u,v,w):=\frac{\theta_1}{2\xi\chi}\int_\Omega u\ln\frac{u}{\bar u}+\frac{\theta_2}{4\xi\alpha}\int_\Omega|\nabla v|^2+\frac{\theta_2}{4\gamma\chi}\int_\Omega|\nabla w|^2-\int_\Omega\nabla w\cdot\nabla v \tag{31}$$

and

$$\begin{aligned}F(u,v,w):=&\frac{\theta_1}{2\xi\chi}\int_\Omega\frac{|\nabla u|^2}{u}+\frac{\theta_2 D_1}{2\xi\alpha}\int_\Omega|\Delta v|^2+\frac{\theta_2 D_2}{2\gamma\chi}\int_\Omega|\Delta w|^2+\frac{\theta_2\beta}{2\xi\alpha}\int_\Omega|\nabla v|^2\\&+\frac{\theta_2\delta}{2\gamma\chi}\int_\Omega|\nabla w|^2-(D_1+D_2)\int_\Omega\Delta w\Delta v-(\beta+\delta)\int_\Omega\nabla w\cdot\nabla v,\end{aligned} \tag{32}$$

where $\theta_1:=\xi\gamma-\chi\alpha$ and $\theta_2:=\xi\gamma+\chi\alpha$. Then, we will show that $E(u,v,w)$ is indeed a Lyapunov function under (6). More precisely, we have the following results.

Lemma 3.3. *Let (u,v,w) be the solution of system (5). Then we have*

$$\frac{d}{dt}E(u,v,w)+F(u,v,w)=0 \tag{33}$$

where $E(u,v,w)$ and $F(u,v,w)$ are defined by (31) and (32), respectively. Moreover, if (6) holds, then

$$E(u,v,w)\geq 0\ \text{ and }\ F(u,v,w)\geq 0\ \text{ for all }\ t>0. \tag{34}$$

Proof. Multiplying the first equation of system (5) by $\ln\frac{u}{\bar u}$, we have

$$\frac{d}{dt}\int_\Omega u\ln\frac{u}{\bar u}+\int_\Omega\frac{|\nabla u|^2}{u}=\chi\int_\Omega\nabla u\cdot\nabla v-\xi\int_\Omega\nabla u\cdot\nabla w. \tag{35}$$

Similarly, we multiply the second and third equations of system (5) by $-\Delta v$ and $-\Delta w$, respectively, to obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega|\nabla v|^2+D_1\int_\Omega|\Delta v|^2+\beta\int_\Omega|\nabla v|^2=\alpha\int_\Omega\nabla u\cdot\nabla v \tag{36}$$

and

$$\frac{1}{2}\frac{d}{dt}\int_\Omega|\nabla w|^2+D_2\int_\Omega|\Delta w|^2+\delta\int_\Omega|\nabla w|^2=\gamma\int_\Omega\nabla u\cdot\nabla w. \tag{37}$$

Multiplying (35) by $\frac{\theta_1}{2\xi\chi}$, (36) by $\frac{\theta_2}{2\xi\alpha}$ and (37) by $\frac{\theta_2}{2\gamma\chi}$, and adding them, we end up with

$$\frac{d}{dt}\left(\frac{\theta_1}{2\xi\chi}\int_\Omega u\ln\frac{u}{\bar u}+\frac{\theta_2}{4\xi\alpha}\int_\Omega|\nabla v|^2+\frac{\theta_2}{4\gamma\chi}\int_\Omega|\nabla w|^2\right)+\frac{\theta_1}{2\xi\chi}\int_\Omega\frac{|\nabla u|^2}{u}$$

$$
\begin{aligned}
&+\frac{{}_2D_1}{2}\int_\Omega j\Delta vj^2+\frac{{}_2D_2}{2}\int_\Omega j\Delta wj^2+\frac{2}{2}\int_\Omega j\,r\,vj^2+\frac{2}{2}\int_\Omega j\,r\,wj^2\\
&=\quad\int_\Omega r\,u\quad r\,v+\quad\int_\Omega r\,u\quad r\,w:
\end{aligned}\tag{38}
$$

On the other hand, the second and third equations of system (5) give us that

$$
\begin{aligned}
\int_\Omega r\,u\quad r\,v&=\int_\Omega r\,(w_t+\quad w\quad D_2\Delta w)\quad r\,v\\
&=\int_\Omega r\,w_t\quad r\,v+\quad\int_\Omega r\,w\quad r\,v\quad D_2\int_\Omega r\,(\Delta w)\quad r\,v\\
&=\frac{d}{dt}\int_\Omega r\,w\quad r\,v+\int_\Omega\Delta wv_t+\quad\int_\Omega r\,w\quad r\,v+D_2\int_\Omega\Delta w\Delta v\\
&=\frac{d}{dt}\int_\Omega r\,w\quad r\,v+\int_\Omega\Delta w(D_1\Delta v+\quad u\qquad v)\\
&\quad+\quad\int_\Omega r\,w\quad r\,v+D_2\int_\Omega\Delta w\Delta v\\
&=\frac{d}{dt}\int_\Omega r\,w\quad r\,v+(D_1+D_2)\int_\Omega\Delta w\Delta v\\
&\quad+(\quad+\quad)\int_\Omega r\,w\quad r\,v\qquad\int_\Omega r\,u\quad r\,w
\end{aligned}
$$

which yields

$$
\begin{aligned}
\int_\Omega r\,u\quad r\,v+\quad\int_\Omega r\,u\quad r\,w&=\frac{d}{dt}\int_\Omega r\,w\quad r\,v+(D_1+D_2)\int_\Omega\Delta w\Delta v\\
&\quad+(\quad+\quad)\int_\Omega r\,w\quad r\,v:
\end{aligned}\tag{39}
$$

The combination of (38) and (39) gives (33).

Next, we will show the nonnegative of $E(u;v;w)$ and $F(u;v;w)$ under (6). First, we rewrite $E(u;v;w)$ in (31) as

$$E(u;v;w)=\frac{1}{2}\int_\Omega u\ln\frac{u}{\bar u}+\int_\Omega\Theta_1^TA_1\Theta_1$$

where Θ_1^T denotes the transpose of Θ_1 and

$$\Theta_1=\begin{bmatrix} r\,v\\ r\,w\end{bmatrix}\quad\text{and}\quad A_1=\begin{bmatrix}\frac{\theta_2}{4\xi\alpha} & \frac{1}{2}\\ \frac{1}{2} & \frac{\theta_2}{4\gamma\chi}\end{bmatrix}:$$

Since ${}_1>0$, one has ${}^2_2>{}^2_2\quad{}^2_1=4\qquad$. This implies the matrix A_1 is positive definite and hence there exists a constant $c_1>0$ such that

$$E(u;v;w)\qquad\frac{1}{2}\int_\Omega u\ln\frac{u}{\bar u}+c_1\int_\Omega(j\,r\,vj^2+j\,r\,wj^2)\qquad 0;\tag{40}$$

where we have used the fact $\int_\Omega u\ln\frac{u}{\bar u}\qquad 0$ from Lemma 2.1. Similarly, we rewrite $F(u;v;w)$ as

$$F(u;v;w)=\frac{1}{2}\int_\Omega\frac{j\,r\,uj^2}{u}+\int_\Omega\Theta_2^TA_2\Theta_2+\int_\Omega\Theta_1^TA_3\Theta_1;\tag{41}$$

where

$$\Theta_2=\begin{bmatrix}\Delta v\\ \Delta w\end{bmatrix};\quad A_2=\begin{bmatrix}\frac{\theta_2D_1}{2\xi\alpha} & \frac{D_1+D_2}{2}\\ \frac{D_1+D_2}{2} & \frac{\theta_2D_2}{2\gamma\chi}\end{bmatrix}\ \text{and}\ A_3=\begin{bmatrix}\frac{\theta_2\beta}{2\xi\alpha} & \frac{\beta+\delta}{2}\\ \frac{\beta+\delta}{2} & \frac{\theta_2\delta}{2\gamma\chi}\end{bmatrix}:\tag{42}$$

Clearly, the matrix A_2 is nonnegative definite if

$$\,_2^2 D_1 D_2 \quad \frac{(D_1+D_2)^2(\,_2^2 \quad \,_1^2)}{4} \quad 0.$$

Similarly, the matrix A_3 is nonnegative definite under the condition

$$\,_2^2 \quad \frac{(\;+\;)^2(\,_2^2 \quad \,_1^2)}{4} \quad 0.$$

Hence, the nonnegativity of the matrices A_2 and A_3 are satisfied simultaneously if

$$\begin{cases} 4\,\,_2^2 D_1 D_2 \quad (D_1+D_2)^2(\,_2^2 \quad \,_1^2) \quad 0, \\ 4\,\,_2^2 \quad (\;+\;)^2(\,_2^2 \quad \,_1^2) \quad 0, \end{cases}$$

which is equivalent to

$$\begin{cases} \,_2^2(D_1 \quad D_2)^2 \quad (D_1+D_2)^2\,\,_1^2, \\ \,_2^2(\quad)^2 \quad (\;+\;)^2\,\,_1^2. \end{cases} \tag{43}$$

One can check that (43) holds if $\frac{\xi\gamma}{\chi\alpha} \quad \max\{\frac{D_1}{D_2}, \frac{D_2}{D_1}, \frac{\beta}{\delta}, \frac{\delta}{\beta}\}$ or $\frac{\xi\gamma}{\chi\alpha} \quad \min\{\frac{D_1}{D_2}, \frac{D_2}{D_1}, \frac{\beta}{\delta}, \frac{\delta}{\beta}\}$. However, the latter is impossible due to $\,_1 > 0$. Hence, if (6) holds, one has $E(u, v, w) \quad 0$ and $F(u, v, w) \quad 0$. The proof of (34) is completed. □

4. **Proof of Theorem 1.1.** In this section, we are devoted to proving Theorem 1.1 based on the Lyapunov function constructed in Lemma 3.3.

4.1. **Boundedness of solutions.** In this subsection, we show the boundedness of solutions for system (5) under the condition (6). First, we give a core lemma concerning the boundedness and asymptotical behavior of solution for system (5) in two dimensions.

Lemma 4.1. *Suppose that $(u_0, v_0, w_0) \in [W^{1,\infty}(\Omega)]^3$ and (6) hold. Then the solution (u, v, w) of system (5) satisfies*

$$\|u \ln u\|_{L^1} + \|\nabla v\|_{L^2} + \|\nabla w\|_{L^2} \quad C \tag{44}$$

and

$$\int_0^t \int_\Omega \frac{|\nabla u|^2}{u} \quad C, \tag{45}$$

where $C > 0$ is a constant independent of t.

Proof. The nonnegativity of $E(u, v, w)$ and $F(u, v, w)$ has been proved in Lemma 3.3 under the condition (6). Then integrating (33) and using (40) and (41), along with the nonnegativity of A_2 and A_3, we have two positive constants c_1, c_2 such that

$$\frac{1}{2}\int_\Omega u \ln \frac{u}{\bar u} + c_1 \int_\Omega (|\nabla v|^2 + |\nabla w|^2) \quad c_2, \tag{46}$$

which, together with the fact $\int_\Omega u \ln \frac{u}{\bar u} \quad 0$ from Lemma 2.1, gives

$$\|\nabla v\|_{L^2}^2 + \|\nabla w\|_{L^2}^2 \quad \frac{c_2}{c_1} = c_3. \tag{47}$$

On the other hand, from (46), we directly obtain

$$\frac{1}{2}\int_\Omega u \ln u \quad c_2 + \frac{1}{2}|\Omega|\bar u \ln \bar u \quad c_4,$$

which, along with the fact $-u\ln u \le \frac{1}{e}$ for all $u \ge 0$, gives

$$\int_\Omega |u\ln u| \le \int_\Omega \left|u\ln u + \frac{1}{e} - \frac{1}{e}\right| \le \int_\Omega \left(u\ln u + \frac{1}{e}\right) + \int_\Omega \frac{1}{e} \le \frac{2\, c_4}{1} + \frac{2|\Omega|}{e}. \tag{48}$$

Then the combination of (47) and (48) gives (44). Hence the proof of this lemma is completed. □

Lemma 4.2. *Let the assumptions in Lemma 4.1 hold. Then the solution (u, v, w) of system (5) satisfies*

$$\|u(\cdot, t)\|_{L^2} \le C \tag{49}$$

where the constant $C > 0$ is independent of t.

Proof. Multiplying the first equation of system (5) by u and integrating it by parts, we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega u^2 + \int_\Omega |\nabla u|^2 &= \int_\Omega u\nabla u \cdot \nabla v - \int_\Omega u \nabla u \cdot \nabla w \\ &= -\frac{}{2}\int_\Omega u^2\Delta v + \frac{}{2}\int_\Omega u^2 \Delta v \\ &\le c_1\|u\|_{L^3}^2(\|\Delta v\|_{L^3} + \|\Delta w\|_{L^3}). \end{aligned} \tag{50}$$

Noting the fact $\|u\ln u\|_{L^1} \le c_2$ and $\|u\|_{L^1} \le c_3$, one can find a small $\varepsilon > 0$ such that

$$\|u\|_{L^3}^2 = \left(\|u\|_{L^3}^3\right)^{\frac{2}{3}} \le \left(\varepsilon\|\nabla u\|_{L^2}^2 + 1\right)^{\frac{2}{3}} \le \varepsilon\|\nabla u\|_{L^2}^{\frac{4}{3}} + c_4, \tag{51}$$

where we have used the following fact (see [22]): when $n = 2$, for any $\varepsilon > 0$, there exists a constant C_ε such that

$$\|u\|_{L^3} \le \varepsilon\|\nabla u\|_{L^2}^{\frac{2}{3}}\|u\ln u\|_{L^1}^{\frac{1}{3}} + C_\varepsilon(\|u\ln u\|_{L^1} + \|u\|_{L^1}^{\frac{1}{3}}).$$

On the other hand, noting the facts $\frac{\partial v}{\partial \nu}\Big|_{\partial\Omega} = \frac{\partial w}{\partial \nu}\Big|_{\partial\Omega} = 0$ on $\partial\Omega$ and using the boundedness of $\|\nabla v\|_{L^2}$ and $\|\nabla w\|_{L^2}$ (see (44)), from Lemma 2.2, one has

$$\begin{aligned} &\|\Delta v\|_{L^3} + \|\Delta w\|_{L^3} \\ &\le c_5(\|\nabla\Delta v\|_{L^2}^{\frac{2}{3}}\|\nabla v\|_{L^2}^{\frac{1}{3}} + \|\nabla v\|_{L^2}) + c_5(\|\nabla\Delta w\|_{L^2}^{\frac{2}{3}}\|\nabla w\|_{L^2}^{\frac{1}{3}} + \|\nabla w\|_{L^2}) \\ &\le c_6(\|\nabla\Delta v\|_{L^2}^{\frac{2}{3}} + \|\nabla\Delta w\|_{L^2}^{\frac{2}{3}} + 1). \end{aligned} \tag{52}$$

Then combining (51) and (52), and using Young's inequality and noting the fact $\varepsilon > 0$ is small, we find a small $\delta > 0$ such that

$$\begin{aligned} &c_1\|u\|_{L^3}^2(\|\Delta v\|_{L^3} + \|\Delta w\|_{L^3}) \\ &\le c_7\left(\varepsilon\|\nabla u\|_{L^2}^{\frac{4}{3}} + c_4\right)\left(\|\nabla\Delta v\|_{L^2}^{\frac{2}{3}} + \|\nabla\Delta w\|_{L^2}^{\frac{2}{3}} + 1\right) \\ &= c_7\varepsilon\|\nabla u\|_{L^2}^{\frac{4}{3}}\left(\|\nabla\Delta v\|_{L^2}^{\frac{2}{3}} + \|\nabla\Delta w\|_{L^2}^{\frac{2}{3}}\right) + c_7\varepsilon\|\nabla u\|_{L^2}^{\frac{4}{3}} \\ &\quad + c_1c_7\left(\|\nabla\Delta v\|_{L^2}^{\frac{2}{3}} + \|\nabla\Delta w\|_{L^2}^{\frac{2}{3}}\right) + c_1c_7 \\ &\le \frac{1}{2}\|\nabla u\|_{L^2}^2 + \delta(\|\nabla\Delta v\|_{L^2}^2 + \|\nabla\Delta w\|_{L^2}^2) + c_8. \end{aligned} \tag{53}$$

Substituting (53) into (50) gives

$$\frac{d}{dt}\int_\Omega u^2 + \int_\Omega |\nabla u|^2 \le 2\delta(\|\nabla\Delta v\|_{L^2}^2 + \|\nabla\Delta w\|_{L^2}^2) + c_9. \tag{54}$$

Differentiating the second equation of system (5) once, and multiplying the result by $-\nabla\Delta v$, and then we integrate the product in Ω to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega|\Delta v|^2 + D_1\int_\Omega|\nabla\Delta v|^2 + \beta\int_\Omega|\Delta v|^2 &= \alpha\int_\Omega \nabla\Delta v\cdot\nabla u\\ &\le \frac{D_1}{2}\|\nabla\Delta v\|_{L^2}^2 + \frac{\alpha^2}{2D_1}\|\nabla u\|_{L^2}^2,\end{aligned}$$

which yields

$$\frac{d}{dt}\int_\Omega|\Delta v|^2 + D_1\int_\Omega|\nabla\Delta v|^2 + 2\beta\int_\Omega|\Delta v|^2 \le \frac{\alpha^2}{D_1}\|\nabla u\|_{L^2}^2. \tag{55}$$

Similarly, we have the following estimates for w:

$$\frac{d}{dt}\int_\Omega|\Delta w|^2 + D_2\int_\Omega|\nabla\Delta w|^2 + 2\beta\int_\Omega|\Delta w|^2 \le \frac{\gamma^2}{D_2}\|\nabla u\|_{L^2}^2. \tag{56}$$

Letting $\lambda = \frac{\alpha^2 D_2+\gamma^2 D_1}{D_1 D_2}$, and multiplying (54) by 2λ, then adding it with (55) and (56), we end up with

$$\begin{aligned}&\frac{d}{dt}\left(2\lambda\|u\|_{L^2}^2 + \|\Delta v\|_{L^2}^2 + \|\Delta w\|_{L^2}^2\right) + \lambda\|\nabla u\|_{L^2}^2\\ &\quad + D_1\|\nabla\Delta v\|_{L^2}^2 + D_2\|\nabla\Delta w\|_{L^2}^2 + 2\beta\|\Delta v\|_{L^2}^2 + 2\beta\|\Delta w\|_{L^2}^2\\ &\le 4\varepsilon\lambda\left(\|\nabla\Delta v\|_{L^2}^2 + \|\nabla\Delta w\|_{L^2}^2\right) + c_{10}.\end{aligned} \tag{57}$$

Letting ε small such that $4\varepsilon\lambda \le \min\{D_1, D_2\}$, one has

$$\begin{aligned}&\frac{d}{dt}\left(2\lambda\|u\|_{L^2}^2 + \|\Delta v\|_{L^2}^2 + \|\Delta w\|_{L^2}^2\right) + \lambda\|\nabla u\|_{L^2}^2\\ &\quad + 2\beta\|\Delta v\|_{L^2}^2 + 2\beta\|\Delta w\|_{L^2}^2 \le c_{10}.\end{aligned} \tag{58}$$

On the other hand, using the Gagliardo-Nirenberg inequality and (20), we can show that

$$\|u\|_{L^2}^2 \le c_{11}\left(\|\nabla u\|_{L^2}\|u\|_{L^1} + \|u\|_{L^1}^2\right) \le \frac{1}{2}\|\nabla u\|_{L^2}^2 + c_{12}. \tag{59}$$

Substituting (59) into (58) and letting $y(t) := 2\lambda\|u\|_{L^2}^2 + \|\Delta v\|_{L^2}^2 + \|\Delta w\|_{L^2}^2$, we can find two positive constants c_{13} and c_{14} such that

$$y'(t) + c_{13}y(t) \le c_{14},$$

which, along with Gronwall's inequality gives (49). □

Next, we will show the existence of global classical solutions.

Lemma 4.3. *Suppose that the conditions in Lemma 4.1 hold. Then the problem (5) has a unique global classical solution $(u, v, w) \in [C^0([0,\infty)\times\bar{\Omega}) \cap C^{2,1}((0,\infty)\times\bar{\Omega})]^3$ satisfying (7).*

Proof. From Lemma 4.2, we know that there exists a constant $c_1 > 0$ such that $\|u(\cdot,t)\|_{L^2} \le c_1$. Noting $n = 2$ and using Lemma 3.2, one has

$$\|u(\cdot,t)\|_{L^\infty} \le c_2,$$

which together with the local existence results in Lemma 3.1 completes the proof of this lemma. □

4.2. **Convergence.** In this subsection, we will show the convergence of solutions.

Lemma 4.4. *Let (u,v,w) be the solution of system (5) satisfying (7) and (45). Then one has*

$$\|u(\cdot,t)-\bar{u}_0\|_{L^\infty}\to 0 \quad \text{as} \quad t\to\infty. \tag{60}$$

Proof. The combination of (7) and (45) implies that there exist a constant $c_1>0$ such that

$$\int_0^\infty \|\nabla u\|_{L^2}^2 \le c_1. \tag{61}$$

Noting the conservation of cell mass and using the Poincaré inequality, we will derive

$$\|u(\cdot,t)-\bar{u}_0\|_{L^2}^2=\|u(\cdot,t)-\bar{u}\|_{L^2}^2\le c_2\|\nabla u\|_{L^2}^2. \tag{62}$$

Combining (61) and (62), one can find a constant $c_3>0$ such that

$$\int_0^\infty \|u(\cdot,t)-\bar{u}_0\|_{L^2}^2 \le c_3. \tag{63}$$

Motivated by the ideas in [28, Lemma 3.10], we next show (63) implies (60). Indeed, if one can show that

$$\|u(\cdot,t)-\bar{u}_0\|_{C^0}\to 0, \quad \text{as} \quad t\to\infty, \tag{64}$$

then (60) follows directly. We shall show (64) by the argument of contradiction. Suppose that (64) is wrong, then for some constant $c_4>0$, there exist some sequences $(x_j)_{j\in\mathbb{N}}\subset\Omega$ and $(t_j)_{j\in\mathbb{N}}\subset(0,\infty)$ satisfying $t_j\to\infty$ as $j\to\infty$ such that

$$|u(x_j,t_j)-\bar{u}_0|\ge c_4, \quad \text{for all} \quad j\in\mathbb{N}.$$

From Lemma 3.2, we know $u-\bar{u}_0$ is uniformly continuous in $\Omega\times(1,\infty)$. Then there exist $r>0$ and $T_1>0$ such than for any $j\in\mathbb{N}$,

$$|u(x,t)-\bar{u}_0|\ge\frac{c_4}{2} \quad \text{for all} \quad x\in B_r(x_j)\cap\Omega \text{ and } t\in(t_j,t_j+T_1). \tag{65}$$

Because of the smoothness of $\partial\Omega$, we can get a constant $c_5>0$ such that

$$|B_r(x_j)\cap\Omega|\ge c_5, \quad \text{for all } x_j\in\Omega. \tag{66}$$

Using (65) and (66), for all $j\in\mathbb{N}$, we have

$$\begin{aligned}\int_{t_j}^{t_j+T_1}\int_\Omega |u(x,t)-\bar{u}_0|^2dxdt &\ge \int_{t_j}^{t_j+T_1}\int_{B_r(x_j)\cap\Omega}|u(x,t)-\bar{u}_0|^2dxdt\\ &\ge \int_{t_j}^{t_j+T_1}|B_r(x_j)\cap\Omega|\left(\frac{c_4}{2}\right)^2dt\\ &\ge \frac{c_4^2c_5T_1}{4}.\end{aligned} \tag{67}$$

However, by the fact $t_j\to\infty$ as $j\to\infty$, we have from (63) that

$$\int_{t_j}^{t_j+T_1}\int_\Omega (u(x,t)-\bar{u}_0)^2dxdt\le\int_{t_j}^{\infty}\int_\Omega (u(x,t)-\bar{u}_0)^2dxdt\to 0, \text{as} \quad j\to\infty,$$

which contradicts (67). Hence (64) holds by the argument of contradiction. Thus the proof of Lemma 4.4 is completed. □

Next, we will show the convergence of v and w by the comparison principle.

Lemma 4.5. *Let the conditions in Lemma 4.4 hold. Then it holds that*

$$\|v(\cdot;t) - \bar{u}_0\|_{L^\infty} \to 0, \quad \text{as} \quad t \to \infty;$$

and

$$\|w(\cdot;t) - \bar{u}_0\|_{L^\infty} \to 0, \quad \text{as} \quad t \to \infty.$$

Proof. Let (x,t) [illegible]

$$\begin{cases} \phi_t - D_1\Delta\phi + \phi = (u - \bar{u}_0), & x \in \Omega, t > 0, \\ \frac{\partial\phi}{\partial\nu} = 0, & x \in \partial\Omega, t > 0, \\ \phi(x,0) = \phi_0(x) = v_0(x) - \frac{\alpha}{\beta}\bar{u}_0, & x \in \Omega. \end{cases} \tag{68}$$

Let $\phi^*(t)$ be the solution of ODE problem

$$\begin{cases} \phi^*_t(t) + \phi^*(t) = \|u - \bar{u}_0\|_{L^\infty}, t > 0, \\ \phi^*(0) = \|\phi_0\|_{L^\infty}. \end{cases} \tag{69}$$

. Then

k 0kL∞ ! 3926 Tf 12.7 0 Td [(0)]TJ/F11 9.926 Tf 4.981 0 Td [(:)]TJ/F8 9.926 Tf 11.55

In fact, using the definition of $E(u;v;w)$ and $F(u;v;w)$ in (31) and (32), respectively, we derive that

$$\begin{aligned} D(u;v;w) = &\quad F(u;v;w) \quad E(u;v;w) \\ =&\frac{1}{2}\left(\quad \int_\Omega \frac{j\, r\, uj^2}{u} \quad \int_\Omega u \ln \frac{u}{\bar{u}}\right) + D_1(u;v;w) + D_2(u;v;w) \end{aligned} \tag{75}$$

where

$$D_1(u;v;w) = \quad \left(\frac{{}_2 D_1}{2} \int_\Omega j\Delta vj^2 + \frac{{}_2 D_2}{2} \int_\Omega j\Delta wj^2 \quad (D_1 + D_2) \int_\Omega \Delta w \quad \Delta v \right)$$

and

$$\begin{aligned} D_2(u;v;w) =& \frac{2}{2}(\qquad \frac{1}{2}) \int_\Omega j\, r\, vj^2 + \frac{2}{2}(\qquad \frac{1}{2}) \int_\Omega j\, r\, wj^2 \\ &+ [1 \qquad (\quad + \quad)] \int_\Omega r\, w \quad r\, v: \end{aligned}$$

To show the nonnegativity of $D(u;v;w)$, we first show the nonnegativity of first term on the right hand of (75). From Lemma 2.1, we have

$$\int_\Omega u \ln \frac{u}{\bar{u}} \qquad \frac{1}{\bar{u}} ku \quad \bar{u}k^2_{L^2}: \tag{76}$$

On the other hand, using (62) and the fact $kuk_{L^\infty} \quad c_1$, one derives

$$\frac{1}{\bar{u}} ku \quad \bar{u}k^2_{L^2} \quad c_2 k\, r\, uk^2_{L^2} \quad c_2 kuk_{L^\infty} \int_\Omega \frac{j\, r\, uj^2}{u} \quad c_1 c_2 \int_\Omega \frac{j\, r\, uj^2}{u};$$

which combined with (76) gives

$$\int_\Omega u \ln \frac{u}{\bar{u}} \quad c_1 c_2 \int_\Omega \frac{j\, r\, uj^2}{u} = \quad {}_1 \int_\Omega \frac{j\, r\, uj^2}{u};$$

where ${}_1 = c_1 c_2$. Hence, we can choose $\qquad {}_1$ such that the first term on the right hand of (75) is nonnegative.

Next, we will show the nonnegativity of $D_1(u;v;w)$. In fact, we can rewrite $D_1(u;v;w)$ as

$$D_1(u;v;w) = \quad \int_\Omega \Theta_2^T A_2 \Theta_2;$$

where A_2 and Θ_2 are defined in (42). The condition (6) gives $\frac{\xi\gamma}{\chi\alpha} \quad \max\left\{\frac{D_1}{D_2}; \frac{D_2}{D_1}\right\}$. Then hence the matrix A_2 is nonnegative definite and hence $D_1(u;v;w) \quad 0$ for any > 0.

Similarly, to show the nonnegativity of $D_2(u;v;w)$, we rewrite it as

$$D_2(u;v;w) = \int_\Omega \Theta_1^T A_4 \Theta_1;$$

where

$$\Theta_1 = \begin{bmatrix} r\, v \\ r\, w \end{bmatrix} \quad \text{and} \quad A_4 = \begin{bmatrix} \frac{\theta_2}{2\xi\alpha}(\qquad \frac{1}{2}) & \frac{1-\mu(\beta+\delta)}{2} \\ \frac{1-\mu(\beta+\delta)}{2} & \frac{\theta_2}{2\gamma\chi}(\qquad \frac{1}{2}) \end{bmatrix}:$$

Using the matrix analysis, we know that A_4 is nonnegative definite if $\quad > \quad {}_2 :=$ $\max f\frac{1}{2\beta}; \frac{1}{2\beta}g$ and

$$[4 \,{}^2_2 \qquad (\,{}^2_2 \qquad {}^2_1)(\quad + \quad)^2] \;{}^2 \quad 2(\quad + \quad) \,{}^2_1 \; + \,{}^2_1 \quad 0: \tag{77}$$

Since $\frac{\xi\gamma}{\chi\alpha} > \max\left\{\frac{\beta}{\delta}; \frac{\delta}{\beta}\right\}$, one has

$$4 \ {}_2^2 \qquad (\ {}_2^2 \qquad {}_1^2)(\ + \ )^2 = 4(\qquad\qquad)(\qquad\qquad) > 0:$$

Hence (77) holds if $\frac{\xi\gamma}{\chi\alpha} > \max\left\{\frac{\beta}{\delta}; \frac{\delta}{\beta}\right\}$ and

$$> \ {}_3 := \max\left\{\frac{1}{2(\qquad\qquad)}; \frac{1}{2(\qquad\qquad)}\right\}:$$

Then if $\ > \max f \ {}_2; \ {}_3 g$, the function $D_2(u; v; w)$ is nonnegative. Hence, choosing $> \max f \ {}_1; \ {}_2; \ {}_3 g$, the function $D(u; v; w)$ is nonnegative and (74) holds.

Substituting (74) into (33), we have

$$\frac{d}{dt} E(u; v; w) + \frac{1}{\ } E(u; v; w) \quad 0;$$

which implies

$$E(u; v; w) \quad c_3 e^{-\frac{1}{\mu} t}: \tag{78}$$

On the other hand, from (40), we have

$$\frac{1}{2} \int_\Omega u \ln \frac{u}{\bar{u}} \quad E(u; v; w);$$

which along with (78) and Lemma 2.1 gives

$$k u(\ ; t) \quad \bar{u}_0 k_{L^1}^2 = k u(\ ; t) \quad \bar{u} k_{L^1}^2 \quad \frac{4 \quad \bar{u} c_3}{1} e^{-\frac{1}{\mu} t}:$$

This yields (73) and concludes the proof. □

Next, we will derive the decay rate of solutions in L^∞-norm based on the decay rate of $k u(\ ; t) \quad \bar{u}_0 k_{L^1}$.

Lemma 4.7. *Let $(u; v; w)$ be the global classical solution of system (5). Suppose that there exist two positive constant $C;$ such that*

$$k u(\ ; t) \quad \bar{u}_0 k_{L^1} \quad C e^{-\lambda t}; \tag{79}$$

then the solution $(u; v; w)$ will exponentially decay to $(\bar{u}_0; \frac{\alpha}{\beta}\bar{u}_0; \frac{\gamma}{\delta}\bar{u}_0)$ with L^∞-norm as $t \ ! \ 1$.

Proof. With (79) in hand, we can use the Moser-Alikakos iteration procedure as in [27] or the semigroup estimate method in [15] to obtain

$$k u \quad \bar{u}_0 k_{L^\infty} \quad c_1 e^{-c_1 t}:$$

Then applying the comparison principle as in [27], one can show that there exists a constant $c_2 > 0$ such that

$$k v \quad - \bar{u}_0 k_{L^\infty} + k w \quad - \bar{u}_0 k_{L^\infty} \quad c_2 e^{-c_2 t}:$$

Then the proof of this lemma is completed. □

4.4. **Proof of Theorem 1.1.** Under the condition (6), we show the boundedness of solution for system (5) with $D_1 \neq D_2$ in Lemma 4.3, which implies there exists a constant $c_1 > 0$ such that $\|u(\cdot,t)\|_{L^\infty} \leq c_1$. Moreover, from Lemma 4.1, one has a constant $c_2 > 0$ such that

$$\int_0^t \int_\Omega \frac{|\nabla u|^2}{u} \leq c_2,$$

which together with the fact $\|u(\cdot,t)\|_{L^\infty} \leq c_1$ implies $\|u - \bar{u}_0\|_{L^\infty} \to 0$ as $t \to \infty$ as shown in Lemma 4.4. Then using the comparison principle for parabolic equations, from the second and third equations of system (5), we show that the solution (v, w) converges to $(\frac{\alpha}{\beta}\bar{u}_0, \frac{\gamma}{\beta}\bar{u}_0)$ as $t \to \infty$ in Lemma 4.5. Moreover, if $\; > \; \max\left\{\frac{\beta}{\delta}, \frac{\delta}{\beta}\right\}$, then using Lemma 4.6, we can obtain

$$\|u(\cdot,t) - \bar{u}_0\|_{L^1} \leq c_3 e^{-\lambda t},$$

which, along with Lemma 4.7, gives the exponential decay rate as shown in Theorem 1.1. Then Theorem 1.1 is proved.

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