Z. Angew. Math. Phys. (2021) 72:181

https://doi.org/10.1007/s00033-021-01611-w

Zeitschrift für angewandte Mathematik und Physik ZAMP

Global dynamics of a tumor invasion model with/without logistic source

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Abstract. In this paper, we shall study the following tumor invasion model

$$\begin{cases} u_t = \nabla\cdot(\gamma(w)\nabla u) - \chi\nabla\cdot(u\nabla v) + au - bu^2, & x\in\Omega,\ t>0,\\ v_t = \Delta v + wz, & x\in\Omega,\ t>0,\\ w_t = -wz, & x\in\Omega,\ t>0,\\ z_t = \Delta z - z + u, & x\in\Omega,\ t>0,\\ \frac{\partial u}{\partial\nu} = \frac{\partial v}{\partial\nu} = \frac{\partial z}{\partial\nu} = 0, & x\in\partial\Omega,\ t>0,\\ (u,v,w,z)(x,0) = (u_0,v_0,w_0,z_0)(x), & x\in\Omega, \end{cases}$$

where $\Omega\subset\mathbb{R}^n(n\geq 1)$ is a bounded domain with smooth boundary. The function $\gamma(w)$ satisfies the hypothesis: $\gamma(w)\in C^2([0,\infty))$ with $\gamma(w)>0$ for all $w\geq 0$ and $\gamma'(w)\leq 0$. Based on energy estimates, we first establish the existence of global bounded solution for $1\leq n\leq 3$ if $a=b=0$. If the logistic source is included (i.e., $a>0, b>0$), we prove that the global classical solution exists with uniform-in-time bound for all dimensions (i.e., $n\geq 1$), which solves an open problem left in Fujie (Discrete Contin Dyn Syst Ser S 13(2):203–209, 2020). Moreover, we also show that all the global bounded solution will converge to the non-trivial constant steady state exponentially.

Mathematics Subject Classification. 35A01, 35B40, 35B44, 35K57, 35Q92, 92C17.

Keywords. Boundedness, Global dynamics, Chemotaxis, Tumor invasion.

1. Introduction and main results

To describe tumor invasion phenomenon with chemotaxis effect of Chaplain–Anderson type, Fujie et al. [9] proposed the following model

$$\begin{cases} u \;= \nabla\cdot(D(u,w)\nabla u) - \chi\nabla\cdot(u\nabla v) + f(u), & x\in\Omega, t>0,\\ v \;= \Delta v + wz, & x\in\Omega, t>0,\\ w \;= -wz, & x\in\Omega, t>0,\\ z \;= \Delta z - z + u, & x\in\Omega, t>0, \end{cases} \tag{1.1}$$

where $\Omega\subset\mathbb{R}$ $(n\geq 1)$ is a bounded domain with smooth boundary. In this model, the functions u, v, w, z denote the density of tumor cells, active extracellular matrix, extracellular matrix (ECM) and matrix degrading enzymes (MDE), respectively. The coefficient $D(u,w)$ describes the random motility depending on the density of tumor cells and ECM. $\chi>0$ is the chemotactic sensitivity rate of tumor cells to active ECM, and $f(u)$ is the logistic source. The system (1.1) was proposed based on the hypothesis of certain substance called active ECM, which is produced by the biochemical reaction between ECM and MDE.

The main feature of (1.1) is that the tumor cell moves toward the chemotactic cue (active ECM), which is not released by the cells themselves (such mechanism is called indirect chemotaxis) in contrast to the classical Keller–Segel chemotaxis model

Published online: 14 September 2021

Birkhäuser

$$\begin{cases} u & = \Delta u - \chi \nabla \cdot (u \nabla z) + f(u), & x \in \Omega, t > 0, \\ z & = \Delta z - z + u, & x \in \Omega, t > 0, \end{cases} \tag{1.2}$$

whose solution behaviors including boundedness [28–30,38,41], blow-up [14,16,34,43] and large time behavior [26,44] as well as pattern formation [24,32] have been studied. (One can also see some related review papers [2,13,15] for more details.) More precisely, for the system (1.2) without logistic source (i.e., $f(u) = 0$), it has been proved that the global classical solution exists with uniform-in-time bound if $n = 1$ [30]. If $n \geq 3$, Winkler has proved that there exists some solution that blows up in finite time for any $\int_\Omega u_0 \mathrm{d}x > 0$ [40,43]. In two-dimensional spaces, a critical mass phenomenon appears, that is there exists a critical mass $m_* = \frac{4\pi}{\chi}$ such that the global boundedness of solution exists if $\int_\Omega u_0 \mathrm{d}x < m_*$ [28], while the solution will blow up if $\int_\Omega u_0 \mathrm{d}x > m_*$ [14]. The logistic source has a damping effect which can prevent the blow-up of solution. In fact, for the system (1.2) with $f(u) = \mu u(1-u)$, it has been proved that the global boundedness solution exists for any $\mu > 0$ if $n \leq 2$ [29] and for large μ if $n \geq 3$ [41]. Moreover, the global bounded solution will converge to the constant state $(1, 1)$ for large μ [44]. However, we should point out that the blow-up of solution is still possible in the presence of logistic-like growth [42,45]. In fact, if the logistic source $f(u) = au - bu^\kappa$, it has been proved in [45] that there exist some initial data such that the corresponding solutions blow up in finite time in an n-dimensional ball ($n \geq 3$) for the simplified parabolic–elliptic chemotaxis growth model provided

$$\kappa < \begin{cases} \frac{7}{6} & \text{if } \ n \in \{3, 4\}, \\ 1 + \frac{1}{2(\ \ -1)} & \text{if } \ n \geq 5. \end{cases}$$

Compared with Keller–Segel chemotaxis model (1.2), the indirect chemotaxis model has many interesting and different phenomena [11,12,17,37]. For example, the indirect chemotaxis can prevent the blow-up of solution and improve the regularization of solution. In fact, for the system (1.1) with linear diffusion (i.e., $D(u, w) = D_0 > 0$), based on the semigroup estimates, the existence of global boundedness and large time behavior of solution have been established in the case of $1 \leq n \leq$

Based on the above questions, we shall study the system (1.1) with $D(u,w)=\gamma(w)$ and $f(u)=au-bu^2$, that is

$$\begin{cases} u \ = \nabla\cdot(\gamma(w)\nabla u)-\chi\nabla\cdot(u\nabla v)+au-bu^2, \qquad x\in\Omega,\ t> \end{cases}$$

Remark 1.4. In Theorems 1.1 and 1.3, we only assume that $\gamma(w)$ satisfies the assumptions (H0) including the case $\gamma(w) = 1$. Hence, our results cover the previous results with linear diffusion [6,8]. Moreover, our results solve an open problem proposed in [6] by obtaining the boundedness of solution for any dimensions with logistic source. Furthermore, we show the solution will exponentially converge to the non-trivial constant steady state.

2. Local existence and preliminaries

We first give the local existence of solutions of (1.3), which can be proved by the fixed point theorem and regularity theorem similar as [7] (also cf. [4]). We omit the details of proof for convenience.

Lemma 2.1. (Local existence) *L* $\Omega \subset \mathbb{R}$ $(n \geq 1)$ *n m n m n . A m n* (u_0, v_0, w_0, z_0) (1.4), *n pp p (H0) i T n* $T_{\max} \in (0, \infty]$ *n n i i i n* (u, v, w, z) (1.3) *n*

$$
\begin{aligned}
&u \in C(\bar{\Omega} \times [0, T_{\max})) \cap C^{2\ 1}(\bar{\Omega} \times (0, T_{\max})),\\
&v \in C(\bar{\Omega} \times [0, T_{\max})) \cap C^{2\ 1}(\bar{\Omega} \times (0, T_{\max})) \cap L^{\infty}([0, T_{\max}); W^{1\ \infty}(\Omega)),\\
&w \in C(\bar{\Omega} \times [0, T_{\max})) \cap C^{0\ 1}(\bar{\Omega} \times (0, T_{\max})),\\
&z \in C(\bar{\Omega} \times [0, T_{\max})) \cap C^{2\ 1}(\bar{\Omega} \times (0, T_{\max})).
\end{aligned}
$$

, $T_{\max} < \infty$, *n*

$$
\lim_{\nearrow \ \max} (\|u(\cdot, t)\|_{L^{\infty}} + \|v(\cdot, t)\|_{\ 1,\infty} + \|z(\cdot, t)\|_{\ 1,\infty}) = \infty.
$$

Lemma 2.2. *L* (u, v, w, z) *i i i n* (1.3) *n n L mm* 2.1. *T n ii* $t \in (0, T_{\max})$, *n*

$$
\|u(\cdot, t)\|_{L^1} \leq M_1 := \begin{cases} \|u_0\|_{L^1}, & \text{if } \ a = b = 0, \\ \|u_0\|_{L^1} + \frac{(1+\ )^2|\Omega|}{4}, & \text{if } \ a, b > 0, \end{cases} \tag{2.1}
$$

n

$$
\|v(\cdot, t)\|_{L^1} + \|w(\cdot, t)\|_{L^1} = \|v_0\|_{L^1} + \|w_0\|_{L^1}, \tag{2.2}
$$

ii

$$
0 \leq w(x, t) \leq \|w_0\|_{L^{\infty}} := M_2. \tag{2.3}
$$

. Integrating the first equation of (1.3) over Ω and using the homogeneous Neumann boundary conditions, one has

$$
\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega} u + b \int_{\Omega} u^2 = a \int_{\Omega} u. \tag{2.4}
$$

If $a = b = 0$, then one has $\|u(\cdot, t)\|_{L^1} = \|u_0\|_{L^1}$, and hence (2.1) follows directly. On the other hand, if $a, b > 0$, then using Young's inequality, we have

$$
(1 + a) \int_{\Omega} u \leq b \int_{\Omega} u^2 + \frac{(1 + a)^2|\Omega|}{4b},
$$

which substituted into (2.4) gives

$$
\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega} u + \int_{\Omega} u \leq \frac{(1 + a)^2|\Omega|}{4b}. \tag{2.5}
$$

Then applying the Grönwall's inequality to (2.5) gives (2.1) in the case of $a, b > 0$. Adding the second and third equations of the system (1.3), and integrating the result over Ω, one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega v + \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega w = 0,$$

which gives (2.2) directly. At last, the third equation in (1.3) shows the function

$$t \mapsto \|w(\cdot,t)\|_{L^\infty}$$

is decreasing, which implies (2.3). □

Now we will show some L -L estimates of the Neumann heat group in Ω, which can be found in [40, Lemma 1.3].

Lemma 2.3. L $(e^{\ \Delta})_{\ \geq 0}$ Ω, $\lambda_1 > 0$ $-\Delta$ Ω σ $(i = 1, 2, 3)$ Ω :

(i) I $1 \leq q \leq p \leq \infty$,

$$\|e^{\ \Delta} f\|_{L^p} \leq \sigma_1 \left(1 + t^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}\right) e^{-\lambda_1\ } \|f\|_{L^q} \qquad t > 0 \tag{2.6}$$

$f \in L\ (\Omega)$ $\int_\Omega f = 0$.

(ii) I $1 \leq q \leq p \leq \infty$,

$$\|\nabla e^{\ \Delta} f\|_{L^p} \leq \sigma_2 \left(1 + t^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}\right) e^{-\lambda_1\ } \|f\|_{L^q} \qquad t > 0 \tag{2.7}$$

$f \in L\ (\Omega)$.

(iii) I $2 \leq q \leq p < \infty$,

$$\|\nabla e^{\ \Delta} f\|_{L^p} \leq \sigma_3 \left(1 + t^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}\right) e^{-\lambda_1\ } \|\nabla f\|_{L^q} \qquad t > 0 \tag{2.8}$$

$f \in W^{1\ }(\Omega)$.

To study large time behavior of solution, we shall introduce the following useful lemma, which has been proved in [40, Lemma 1.2].

Lemma 2.4. ([40]) L $\alpha_1 < 1$, δ_1 δ_2 $\delta_1 \neq \delta_2$. $\sigma_4 > 0$

$$\int_0 (1 + (t-s)^{-\alpha_1}) e^{-\delta_1(\ -s)} \cdot e^{-\delta_2 s} \mathrm{d}s \leq \sigma_4 e^{-\min\{\delta_1\ \delta_2\}}$$

$t > 0$.

Based on properties of Neumann heat semigroup shown in Lemma 2.3, one can use the similar arguments as in [8, Lemma 3.1 and Lemma 3.2] to obtain the following results.

Lemma 2.5. L (u, v, w, z) (1.3) L 2.1. :

(i) I $\frac{1}{\ } - \frac{1}{\ } < \frac{2}{\ }$ $1 \leq p, q \leq \infty$, $M_3 > 0$ t

$$\|z(\cdot,t)\|_{L^q} \leq M_3(1 + \sup_{s\in(0\ \ )} \|u(\cdot,s)\|_{L^p}) \qquad t \in (0, T_{\max}), \tag{2.9}$$

$$\|v(\cdot,t)\|_{L^q} \leq M_3(1 + \sup_{s\in(0\ \ )} \|z(\cdot,s)\|_{L^p}) \qquad t \in (0, T_{\max}). \tag{2.10}$$

(ii) I $\frac{1}{} - \frac{1}{} < \frac{1}{}$ n $1 \le p, q \le \infty$, n

$$\|\nabla z(\cdot,t)\|_{L^q} \le M_4(1 + \sup_{s\in(0\ )} \|u(\cdot,s)\|_{L^p}) \qquad t \in (0, T_{\max}), \tag{2.11}$$

n

$$\|\nabla v(\cdot,t)\|_{L^q} \le M_4(1 + \sup_{s\in(0\ )} \|z(\cdot,s)\|_{L^p}) \qquad t \in (0, T_{\max}), \tag{2.12}$$

m n n $M_4 > 0$ n p n n t.

3. Global existence: Proof of Theorem 1.1

In this section, we shall show the existence of global classical solution. To this end, we first establish the criterion of boundedness of solutions of the system (1.3) as follows.

Lemma 3.1. (Boundedness criterion) pp mp n n L mm 2.1 . L (u, v, w, z) n (1.3) fin n m m n m n $[0, T_{\max})$. L $M > 0$ n n n p n n t, n $k > \frac{}{4}$

$$\|u(\cdot,t)\|_{L^k} \le M, \quad \text{for all } t \in (0, T_{\max}), \tag{3.1}$$

n n n fin n n $M_5 > 0$ n p n n t

$$\|u(\cdot,t)\|_{L^\infty} + \|v(\cdot,t)\|_{\ ^{1,\infty}} + \|w(\cdot,t)\|_{L^\infty} + \|z(\cdot,t)\|_{\ ^{1,\infty}} \le M_5, \qquad t \in (0, T_{\max}). \tag{3.2}$$

. First, we show the following statement:

(A) If (3.1) holds, then there exists $r > \max\{n, 2\}$ such that

$$\|\nabla v(\cdot,t)\|_{L^r} \le c_1, \text{ for all } t \in (0, T_{\max}). \tag{3.3}$$

In fact, assume (3.1) holds, then by (2.9) in Lemma 2.5, we can derive that

$$\|z(\cdot,t)\|_{L^q} \le M_3(1 + M) \text{ for all } t \in (0, T_{\max}), \tag{3.4}$$

with

$$\begin{cases} 1 \le q < \frac{}{-2}, & \text{if } k \le \frac{}{2}, \\ 1 \le q \le \infty, & \text{if } k > \frac{}{2}. \end{cases}$$

If $k > \frac{}{2}$, then we have $\|z(\cdot,t)\|_{L^\infty} \le M_3(1 + M)$ and $\|\nabla v(\cdot,t)\|_{L^\infty} \le M_4(1 + M_3(1 + M))$ follows by using (2.12) in Lemma 2.5. Hence, the statement (A) holds for $k > \frac{}{2}$. On the other hand, if $\frac{}{4} < k \le \frac{}{2}$, one can derive that $\frac{}{-2} > \frac{}{2}$, which implies $\|z(\cdot,t)\|_{L^{q_1}} \le M_3(1 + M)$ for $\max\{1, \frac{}{2}\} < q_1 < \frac{}{-2}$. Then using (2.12) in Lemma 2.5 again, we can obtain (3.3) directly. Hence, the statement (A) holds for all $k > \frac{}{4}$.

Next, we shall show that (3.2) holds with the statement (A). In fact, multiplying the first equation of (1.3) by $u^{\ -1}$ with $p > 1$, and integrating the result by parts over Ω and using the Young's inequality, we end up with

$$\begin{aligned} &\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^{\ } + (p-1)\int_\Omega \gamma(w) u^{\ -2}|\nabla u|^2 + b\int_\Omega u^{\ +1} \\ &= \chi(p-1)\int_\Omega u^{\ -1}\nabla v \cdot \nabla u + a\int_\Omega u^{\ } \\ &\le \frac{p-1}{2}\int_\Omega \gamma(w) u^{\ -2}|\nabla u|^2 + \frac{(p-1)\chi^2}{2}\int_\Omega \frac{1}{\gamma(w)} u^{\ } |\nabla v|^2 + a\int_\Omega u^{\ } \end{aligned}$$

which together with the facts $0 < \gamma_1 := \gamma(\|w_0\|_{L^\infty}) \leq \gamma(w)$, Hölder's inequality and (3.3), gives

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u \ \ + \frac{p(p-1)\gamma_1}{2}\int_\Omega u^{\ \ -2}|\nabla u|^2 + bp\int_\Omega u^{\ \ +1} \\
&\leq \frac{p(p-1)\chi^2}{2\gamma_1}\int_\Omega u \ \ |\nabla v|^2 + ap\int_\Omega u \\
&\leq \frac{p(p-1)\chi^2}{2\gamma_1}\left(\int_\Omega u^{\frac{pr}{r-2}}\right)^{1-\frac{2}{r}}\|\nabla v(\cdot,t)\|^2_{L^r} + ap\int_\Omega u \\
&\leq \frac{p(p-1)\chi^2 c_1^2}{2\gamma_1}\left(\int_\Omega u^{\frac{pr}{r-2}}\right)^{1-\frac{2}{r}} + ap\int_\Omega u \ \ .
\end{aligned}
\tag{3.5}
$$

Then we can use the Gagliardo–Nirenberg inequality, Young's inequality and the fact $\|u^{\frac{p}{2}}\|_{L^{\frac{2}{p}}} = \|u\|^{\frac{p}{2}}_{L^1} \leq M_1^{\frac{p}{2}}$ to find two constants $\varepsilon_1 = \frac{\frac{p}{2}-\frac{r-2}{2r}}{\frac{p}{2}+\frac{1}{n}-\frac{1}{2}} \in (0,1)$ and $\varepsilon_2 = \frac{\frac{p}{2}-\frac{1}{2}}{\frac{p}{2}+\frac{1}{n}-\frac{1}{2}} \in (0,1)$ such that

$$
\begin{aligned}
\frac{p(p-1)\chi^2 c_1^2}{2\gamma_1}\left(\int_\Omega u^{\frac{pr}{r-2}}\right)^{1-\frac{2}{r}} &= \frac{p(p-1)\chi^2 c_1^2}{2\gamma_1}\|u^{\frac{p}{2}}\|^2_{L^{\frac{2r}{r-2}}} \\
&\leq c_2\left(\|\nabla u^{\frac{p}{2}}\|^{2\varepsilon_1}_{L^2}\|u^{\frac{p}{2}}\|^{2(1-\varepsilon_1)}_{L^{\frac{2}{p}}} + \|u^{\frac{p}{2}}\|^2_{L^{\frac{2}{p}}}\right) \\
&\leq c_2\left(\|\nabla u^{\frac{p}{2}}\|^{2\varepsilon_1}_{L^2} M_1^{\ \ (1-\varepsilon_1)} + M_1^{\ \ }\right) \\
&\leq \frac{p(p-1)\gamma_1}{4}\int_\Omega u^{\ \ -2}|\nabla u|^2 + c_3
\end{aligned}
\tag{3.6}
$$

with $c_3 := c_2^{\frac{1}{1-\varepsilon_1}} M_1^{\ \ }(1-\varepsilon_1)\left[\frac{\varepsilon_1}{(\ \ -1)\gamma_1}\right]^{\frac{\varepsilon_1}{1-\varepsilon_1}} + c_2 M_1^{\ \ }$ and

$$
\begin{aligned}
(ap+1)\int_\Omega u \ \ &= (ap+1)\|u^{\frac{p}{2}}\|^2_{L^2} \\
&\leq c_4\left(\|\nabla u^{\frac{p}{2}}\|^{2\varepsilon_2}_{L^2}\|u^{\frac{p}{2}}\|^{2(1-\varepsilon_2)}_{L^{\frac{2}{p}}} + \|u^{\frac{p}{2}}\|^2_{L^{\frac{2}{p}}}\right) \\
&\leq \frac{p(p-1)\gamma_1}{4}\int_\Omega u^{\ \ -2}|\nabla u|^2 + c_5,
\end{aligned}
\tag{3.7}
$$

where $c_5 := c_4^{\frac{1}{1-\varepsilon_2}} M_1^{\ \ }(1-\varepsilon_2)\left[\frac{\varepsilon_2}{(\ \ -1)\gamma_1}\right]^{\frac{\varepsilon_2}{1-\varepsilon_1}} + c_4 M_1^{\ \ }$.

Substituting (3.6) and (3.7) into (3.5), we get

$$
\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u \ \ + \int_\Omega u \ \ \leq c_3 + c_5,
$$

which together with Grönwall's inequality gives $\|u(\cdot,t)\|_{L^p} \leq c_6$ for all $t \in (0, T_{\max})$. Using the Moser iteration procedure (cf. [1]), one has

$$
\|u(\cdot,t)\|_{L^\infty} \leq c_7, \quad \text{for all } t \in (0, T_{\max}). \tag{3.8}
$$

Then applying the parabolic regularity to the second and fourth equations of the system (1.3), we can obtain

$$\|z(\cdot,t)\|_{\quad 1,\infty} + \|v(\cdot,t)\|_{\quad 1,\infty} \le c_8, \quad \text{for all } t \in (0, T_{\max}). \tag{3.9}$$

The combination of (2.3), (3.8) and (3.9) gives (3.2). Hence, the proof of Lemma 3.1 is completed. □

For the system (1.3), it holds $\|u(\cdot,t)\|_{L^1} \le M_1$ (see (2.1)). Hence, as a direct consequence of Lemma 3.1, we can obtain the boundedness of solution for $1 \le n \le 3$ as follows.

Lemma 3.2. L $\Omega \subset \mathbb{R}$ $(1 \le n \le 3)$, ,t,t(=F15 1 Tf .3885 01.443(u)Tj /1Φ

Next, we state the boundedness of solution for the system (1.3) with logistic source in all dimensions.

Lemma 3.5. *L* $\Omega \subset \mathbb{R}$ $(n \geq 1)$ *n m n m n . pp mp n (H0) n n* (u_0, v_0, w_0, z_0) (1.4). *A m* $a, b > 0$, *n m* (1.3) *n n n m- n- m n n m n n .*

. When $1 \leq n \leq 3$, the global boundedness of solutions has been established as in Lemma 3.2. Hence, in the following , we shall show the global boundedness of solution in the case of $n \geq 4$. First, multiplying the first equation of (1.3) by $u^{\ -1}$ $(p > \max\{\frac{}{4}, \frac{+2}{-2}\})$, and integrating it by parts, we end up with

$$\begin{aligned}\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^{\ } + (p-1)\int_\Omega \gamma(w)u^{\ -2}|\nabla u|^2 + b\int_\Omega u^{\ +1} &= \chi(p-1)\int_\Omega u^{\ -1}\nabla v\cdot\nabla u + a\int_\Omega u^{\ } \\ &= -\frac{\chi(p-1)}{p}\int_\Omega u^{\ }\Delta v + a\int_\Omega u^{\ }.\end{aligned} \tag{3.15}$$

Then using Hölder inequality and Young's inequality, one has

$$-\frac{\chi(p-1)}{p}\int_\Omega u^{\ }\Delta v \leq \frac{b}{2}\int_\Omega u^{\ +1} + c_1\int_\Omega |\Delta v|^{\ +1}, \tag{3.16}$$

with $c_1 := \frac{1}{}\left(\frac{2}{}\right)^{\ }\left(\frac{\chi(\ -1)}{\ +1}\right)^{\ +1}$ and

$$\left(ap + \frac{p+1}{2}\right)\int_\Omega u^{\ } \leq \frac{bp}{2}\int_\Omega u^{\ +1} + c_2 \tag{3.17}$$

with $c_2 := \left(\frac{2}{}\right)^{\ }\left(\frac{2\ \ +\ +1}{2(\ +1)}\right)^{\ +1}|\Omega|$.

Substituting (3.16) into (3.15) and using (3.17), we can derive that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^{\ } + \frac{p+1}{2}\int_\Omega u^{\ } \leq c_1 p\int_\Omega |\Delta v|^{\ +1} + c_2. \tag{3.18}$$

Multiplying (3.18) by $e^{\frac{(p+1)t}{2}}$ and then integrating the result with respect to time variable directly, one has

$$\begin{aligned}\int_\Omega u^{\ }(\cdot,t) &\leq c_1 p e^{-\frac{(p+1)t}{2}}\int_0^{\ } e^{\frac{(p+1)\tau}{2}}\int_\Omega |\Delta v(\cdot,\tau)|^{\ +1} \\ &\quad + c_2 e^{-\frac{(p+1)t}{2}}\int_0^{\ } e^{\frac{(p+1)\tau}{2}} + e^{-\frac{p+1}{2}(\ -\ _0)}\int_\Omega u^{\ }(\cdot,t_0) \\ &\leq c_1 p e^{-\frac{(p+1)t}{2}}\int_0^{\ } e^{\frac{(p+1)\tau}{2}}\int_\Omega |\Delta v(\cdot,\tau)|^{\ +1} + \frac{2c_2}{p+1} + \|u(\cdot,t_0)\|_{L^p}^{\ } \\ &\leq c_1 p e^{-\frac{(p+1)t}{2}}\int_0^{\ } e^{\frac{(p+1)\tau}{2}}\int_\Omega |\Delta v(\cdot,\tau)|^{\ +1} + c_3,\end{aligned} \tag{3.19}$$

where $t \in (t_0, T_{\max})$ with $t_0 := \min\left\{1, \frac{1}{2}T_{\max}\right\}$. On the other hand, applying the maximal Sobolev regularity (see [3,39]) to the following system

$$\begin{cases} v \ = \Delta v - v + wz + v, & x \in \Omega, t > 0, \\ \frac{\partial}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ v(x,0) = v_0(x), & x \in \Omega, \end{cases}$$

we have

$$\begin{aligned} &\int_0 e^{\frac{(p+1)\tau}{2}} \int_\Omega |\Delta v| \ ^{+1} \\ &\quad \leq c_4 \left(\int_0 e^{\frac{(p+1)\tau}{2}} \int_\Omega |v + wz| \ ^{+1} + e^{\frac{(p+1)}{2} \ 0} (\|v(\cdot, t_0)\|_{L^{p+1}}^{+1} + \|\Delta v(\cdot, t_0)\|_{L^{p+1}}^{+1}) \right), \end{aligned}$$

which substituted into (3.19) gives

$$\begin{aligned} \int_\Omega u \ (\cdot, t) &\leq c_1 c_4 p e^{-\frac{p+1}{2}} \ \left(\int_0 e^{\frac{p+1}{2}\tau} \int_\Omega |v(\cdot, \tau) + (wz)(\cdot, \tau)| \ ^{+1} \right) \\ &\quad + c_1 c_4 p e^{-\frac{(p+1)(t-t_0)}{2}} (\|v(\cdot, t_0)\|_{L^{p+1}}^{+1} + \|\Delta v(\cdot, t_0)\|_{L^{p+1}}^{+1}) + c_3 \\ &\leq c_1 c_4 p e^{-\frac{p+1}{2}} \ \int_0 e^{\frac{p+1}{2}\tau} \int_\Omega |v(\cdot, \tau) + (wz)(\cdot, \tau)| \ ^{+1} + c_5, \end{aligned} \tag{3.20}$$

for all $t \in (t_0, T_{\max})$, where we have used the fact $\|v(\cdot, t_0)\|_{L^{p+1}}^{+1} + \|\Delta v(\cdot, t_0)\|_{L^{p+1}}^{+1} \leq c_6$. Next, we shall show the term $\int_\Omega |v(\cdot, \tau) + (wz)(\cdot, \tau)| \ ^{+1}$ can be controlled by $\|u(\cdot, t)\|_{L^p}^{(\ +1)\theta}$ for some $\theta \in (0, 1)$. In fact, using (2.3) and (2.10), for all $\tau \in (t_0, t)$ and $t \in (0, T_{\max})$, one has

$$\begin{aligned} \int_\Omega |v(\cdot, \tau) + (wz)(\cdot, \tau)| \ ^{+1} &\leq 2 \ \int_\Omega v(\cdot, \tau) \ ^{+1} + 2 \ M_2 \ ^{+1} \int_\Omega z(\cdot, \tau) \ ^{+1} \\ &\leq 2 \ (2 \ M_3 \ ^{+1} + M_2 \ ^{+1}) \sup_{s \in (0 \ )} \|z(\cdot, s)\|_{L^{p+1}}^{+1} + 4 \ M_3 \ ^{+1}, \end{aligned} \tag{3.21}$$

where we have used the following basic inequality

$$(X + Y) \ ^{+1} \leq 2 \ (X \ ^{+1} + Y \ ^{+1}) \text{ for all } X, Y > 0 \text{ and } p \geq 0. \tag{3.22}$$

Before estimating the terms on the right-hand side of (3.21), we give some relations on the parameters. Since $p > \max\{\frac{}{4}, \frac{+2}{-2}\}$, we know that for any $2 \leq q_* < \frac{2}{-2}$, one has $p + 1 > q_*$ and $2pq_* > n$. Then after some calculations, one has

$$\frac{1}{p} - \frac{1}{n} < \frac{1}{n} + \frac{1}{p} - \frac{1}{pq_*} < \frac{1}{n} + \frac{1}{p+1}, \tag{3.23}$$

which entails us to find a constant $r \geq 1$ satisfying

$$\frac{1}{p} - \frac{1}{n} < \frac{1}{r} < \frac{1}{n} + \frac{1}{p} - \frac{1}{pq_*} \tag{3.24}$$

such that

$$\theta = \frac{\frac{1}{*} - \frac{1}{+1}}{\frac{1}{*} + \frac{1}{} - \frac{1}{r}} \in (0, 1) \tag{3.25}$$

and

$$1 < (p+1)\theta < p. \tag{3.26}$$

Since $b > 0$ and $n \geq 4$, from Lemma 3.4, one has

$$\|z(\cdot,t)\|_{L^{q_*}} \leq M_6, \text{ for all } t \in (0, T_{\max}) \text{ and } 2 \leq q_* < \frac{2n}{n-2}. \tag{3.27}$$

Then we can use the Gagliardo–Nirenberg inequality and (3.27) to obtain

$$\begin{aligned} \|z(\cdot,s)\|_{L^{p+1}}^{\ +1} &\leq c_7 \left(\|\nabla z(\cdot,s)\|_{L^r}^{(\ +1)\theta} \|z(\cdot,s)\|_{L^{q_*}}^{(\ +1)(1-\theta)} + \|z(\cdot,s)\|_{L^{q_*}}^{\ +1} \right) \\ &\leq c_7 \left(M_6^{(\ +1)(1-\theta)} \|\nabla z(\cdot,s)\|_{L^r}^{(\ +1)\theta} + M_6^{\ +1} \right), \end{aligned} \tag{3.28}$$

for all $s \in (0,t)$ with $t \in (t_0, T_{\max})$. Moreover, using (2.11), (3.22) and noting the fact $\frac{1}{\ } - \frac{1}{r} < \frac{1}{\ }$, we derive

$$\begin{aligned} \|\nabla z(\cdot,s)\|_{L^r}^{(\ +1)\theta} &\leq \frac{(2M_4)^{(\ +1)\theta}}{2} \left(1 + \sup_{\eta \in (0\ s)} \|u(\cdot,\eta)\|_{L^p}^{(\ +1)\theta} \right) \\ &\leq \frac{(2M_4)^{(\ +1)\theta}}{2} \left(1 + \sup_{\eta \in (0\ \ )} \|u(\cdot,\eta)\|_{L^p}^{(\ +1)\theta} \right). \end{aligned} \tag{3.29}$$

Combining (3.21), (3.28) and (3.29), one has

$$\int_\Omega |v(\cdot,\tau) + wz(\cdot,\tau)|^{\ +1} \leq c_8 \sup_{\eta \in (0\ \ )} \|u(\cdot,\eta)\|_{L^p}^{(\ +1)\theta} + c_9, \tag{3.30}$$

where

$$c_8 := c_7 2^{\ -1} (2\ M_3^{\ +1} + M_2^{\ +1}) M_6^{(\ +1)(1-\theta)} (2M_4)^{(\ +1)\theta}$$

and

$$c_9 := 4\ M_3^{\ +1} + c_7 2\ (2\ M_3^{\ +1} + M_2^{\ +1}) \left(M_6^{\ +1} + \frac{(2M_4)^{(\ +1)\theta}}{2} M_6^{(\ +1)(1-\theta)} \right).$$

We substitute (3.30) into (3.20) to obtain

$$\begin{aligned} \int_\Omega u(\cdot,t) &\leq c_1 c_4 p \left(c_8 \sup_{\eta \in (0\ \ )} \|u(\cdot,\eta)\|_{L^p}^{(\ +1)\theta} + c_9 \right) e^{-\frac{(p+1)t}{2}} \int_0 e^{\frac{(p+1)\tau}{2}} + c_5 \\ &\leq \frac{2pc_1c_4c_8}{p+1} \sup_{\eta \in (0\ \ )} \|u(\cdot,\eta)\|_{L^p}^{(\ +1)\theta} + \frac{2pc_1c_4c_9}{p+1} + c_5 \text{ for all } t \in (t_0, T_{\max}). \end{aligned}$$

Hence,

$$\begin{aligned} \sup_{\ \in (0\ \ )} \int_\Omega u(\cdot,t) &\leq c_{10} \left(\sup_\eta \in (0,T) \|u(\cdot,\eta)\|_{L^p}^{(\ +1)\theta} + 1 \right) \\ &= c_{10} \left(\sup_{\ \in (0\ \ )} \|u(\cdot,t)\|_{L^p}^{(\ +1)\theta} + 1 \right) \text{ for all } T \in (t_0, T_{\max}), \end{aligned}$$

by letting $c_{10} := \frac{2\ \ _1\ _4(\ _8 + \ _9) + \ _5(\ +1)}{\ +1}$. Using the fact $(p+1)\theta < p$ and Young's inequality, we can derive that

$$\|u(\cdot,t)\|_{L^p} \leq 2c_{10}^{\frac{p}{p-(p+1)\theta}} \cdot \frac{p-(p+1)\theta}{p} \cdot \left[\frac{2(p+1)\theta}{p} \right]^{\frac{(p+1)\theta}{p-(p+1)\theta}}, \text{ for all } t \in (0, T_{\max}).$$

Since $p > \max\{\frac{\ }{4}, \frac{\ +2}{\ -2}\}$, then using the boundedness criterion in Lemma 3.1, we obtain the desired results. ∎

m 1.1. Theorem 1.1 is a consequence by combining Lemmas 3.2 and 3.5.

Remark 3.6. With some appropriate modifications of the arguments in Sect. 3, we can establish the existence of global classical solution in any dimensions for the following Fujie–Senba-type indirect chemotaxis system [11,12] with $\kappa = 1$:

$$\begin{cases} u \ = \Delta u - \nabla \cdot (u\nabla v) + u(1-u^{\kappa}), & x \in \Omega, t > 0, \\ v \ = \Delta v - v + w, & x \in \Omega, t > 0, \\ w \ = \Delta w - w + u, & x \in \Omega, t > 0. \end{cases} \tag{3.31}$$

In fact, if $\kappa > \frac{}{4} - \frac{1}{2}$ and $n \geq 2$, by using the semigroup estimates method, the boundedness of solution for the system (3.31) has been established in [46]. Compared with the results obtained in [46], we need a smaller value of κ if $n \geq 6$ but a larger value of κ if $4 \leq n \leq 5$, to obtain the boundedness of solution. Hence, it is an interesting question to consider whether or not there exists an optimal value of κ_* to ensure the global existence of classical solution for the system (3.31) for all dimensions.

4. Global stabilization: Proof of Theorem 1.3

In this section, we shall show that the solution (u, v, w, z) obtained in Theorem 1.1 will exponentially converge to the unique constant state as $t \to \infty$. First, we improve the regular properties of solution as follows.

Lemma 4.1. L (u, v, w, z) i n m (1.3) n n m 1.1. n $\sigma \in (0, 1)$ n $M_7 > 0$

$$\|u(\cdot,t)\|_{\ \sigma,\frac{\sigma}{2}(\bar{\Omega}\times[\ \ +1])} + \|v(\cdot,t)\|_{\ 2+\sigma,1+\frac{\sigma}{2}(\bar{\Omega}\times[\ \ +1])} + \|z(\cdot,t)\|_{\ 2+\sigma,1+\frac{\sigma}{2}(\bar{\Omega}\times[\ \ +1])} \leq M_7, \tag{4.1}$$

ii $t \geq 1$.

. Theorem 1.1 entails us that there exists a constant $C_0 > 0$ independent of t such that

$$0 < u(x,t) \leq C_0, \ 0 < v(x,t) \leq C_0 \text{ and } 0 < |\nabla v(x,t)| \leq C_0 \text{ for all } (x,t) \in \Omega \times (0,\infty).$$

We rewrite the first equation of (1.3) as

$$u \ = \nabla \Phi(x,t,u,\nabla u) + \Psi(x,t,u) \text{ for all } x \in \Omega \text{ and } t > 0,$$

where

$$\Phi(x,t,u,\nabla u) := \gamma(w)\nabla u - \chi u \nabla v$$

and

$$\Psi(x,t,u) = au - bu^2.$$

Using the Young's inequality and the fact $0 < \gamma_1 := \gamma(\|w_0\|_{L^\infty}) \leq \gamma(w) \leq \gamma(0)$, one has

$$\begin{aligned} \Phi(x,t,u,\nabla u) \cdot \nabla u &= \gamma(w)|\nabla u|^2 - \chi u \nabla v \cdot \nabla u \\ &\geq \gamma(w)|\nabla u|^2 - \chi u |\nabla u||\nabla v| \\ &\geq \gamma_1 |\nabla u|^2 - \chi u |\nabla u||\nabla v| \\ &\geq \frac{\gamma_1}{2}|\nabla u|^2 - \frac{\chi^2}{2\gamma_1} u^2 |\nabla v|^2 \\ &\geq \frac{\gamma_1}{2}|\nabla u|^2 - \frac{\chi^2 C_0^4}{2\gamma_1}, \end{aligned} \tag{4.2}$$

and

$$|\Phi(x,t,u,\nabla u)| \leq \gamma(0)|\nabla u| + \chi C_0^2, \tag{4.3}$$

as well as

$$|\Psi(x,t,u)| \le aC_0 + bC_0^2. \tag{4.4}$$

Thus, according to (4.2)–(4.4), we can use the Hölder regularity of quasilinear parabolic equations [33, Theorem 1.3] to derive that $\|u(\cdot,t)\|_{\ \ \sigma,\frac{\sigma}{2}(\bar{\Omega}\times[\ \ \ +1])} \le c_1$ for all $t > 1$. And applying the standard parabolic regularity theory [25] to the fourth equation of (1.3), one has $\|z(\cdot,t)\|_{\ \ 2+\sigma,1+\frac{\sigma}{2}(\bar{\Omega}\times[\ \ \ +1])} \le c_2$ for all $t > 1$. Similarly, we can obtain $\|v(\cdot,t)\|_{\ \ 2+\sigma,1+\frac{\sigma}{2}(\bar{\Omega}\times[\ \ \ +1])} \le c_3$ for all $t > 1$. Hence, we complete the proof of Lemma 4.1. □

To show the convergence rate of solution u in L^∞-norm, we shall use the following lemma.

Lemma 4.2. L (u, v, w, z) i n n n m 1.1. A m m $t_0 \ge 1$

$$\|\nabla w(\cdot,t)\|_{L^\infty} \le M_8 \text{ for all } \ t \ge t_0, \tag{4.5}$$

n i

$$\|\nabla u(\cdot,t)\|_{L^{2n}} \le M_9 \text{ for all } \ t \ge t_0, \tag{4.6}$$

$M\ > 0$ $(i = 8, 9)$ n n n p n n $t.$

. Differentiating the first equation of (1.3) once and multiplying the result with $|\nabla u|^{2\ \ -2}\nabla u$, we have

$$\begin{aligned}\frac{1}{2n}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla u|^2 \ &= \int_\Omega |\nabla u|^{2\ \ -2}\nabla u\cdot\nabla(\nabla\cdot(\gamma(w)\nabla u)) \\ &\quad - \chi\int_\Omega |\nabla u|^{2\ \ -2}\nabla u\cdot\nabla(\nabla\cdot(u\nabla v)) \\ &\quad + \int_\Omega |\nabla u|^{2\ \ -2}\nabla u\cdot\nabla(au - bu^2) \\ &=: J_1 + J_2 + J_3.\end{aligned} \tag{4.7}$$

First, we can use the identity $\nabla\Delta u\cdot\nabla u = \frac{1}{2}\Delta|\nabla u|^2 - |D^2u|^2$ to estimate the term J_1 as follows

$$\begin{aligned} J_1 &= -\int_\Omega |\nabla u|^{2\ \ -2}\Delta u\nabla\cdot(\gamma(w)\nabla u) - \int_\Omega \nabla|\nabla u|^{2\ \ -2}\cdot\nabla u\nabla\cdot(\gamma(w)\nabla u) \\ &= \int_\Omega \gamma(w)|\nabla u|^{2\ \ -2}\nabla\Delta u\cdot\nabla u - \int_\Omega \gamma'(w)\nabla|\nabla u|^{2\ \ -2}\cdot\nabla u\nabla u\cdot\nabla w \\ &= \frac{1}{2}\int_\Omega \gamma(w)|\nabla u|^{2\ \ -2}\Delta|\nabla u|^2 - \int_\Omega \gamma(w)|\nabla u|^{2\ \ -2}|D^2u|^2 \\ &\quad - (n-1)\int_\Omega \gamma'(w)|\nabla u|^{2\ \ -4}\nabla|\nabla u|^2\cdot\nabla u\nabla u\cdot\nabla w \\ &= \frac{1}{2}\int_{\partial\Omega} \gamma(w)|\nabla u|^{2\ \ -2}\frac{\partial|\nabla u|^2}{\partial\nu}\mathrm{d}S - \frac{n-1}{2}\int_\Omega \gamma(w)|\nabla u|^{2\ \ -4}|\nabla|\nabla u|^2|^2 \\ &\quad + \left(\frac{1}{2} - n\right)\int_\Omega \gamma'(w)|\nabla u|^{2\ \ -2}\nabla|\nabla u|^2\cdot\nabla w - \int_\Omega \gamma(w)|\nabla u|^{2\ \ -2}|D^2u|^2. \end{aligned} \tag{4.8}$$

On the other hand, using the facts $\|u\|_{L^\infty} + \|v\|_{\ \ 1,\infty} \le C_0$ (see Theorem 1.1) and $|\Delta u| \le \sqrt{n}|D^2 u|$, we can estimate J_2 as follows

$$
\begin{aligned}
J_2 &= \chi \int_\Omega \nabla|\nabla u|^{2\ \ -2} \cdot \nabla u \nabla\cdot(u\nabla v) + \chi\int_\Omega |\nabla u|^{2\ \ -2}\Delta u \nabla\cdot(u\nabla v) \\
&\le \chi(n-1)C_0 \int_\Omega |\nabla u|^{2\ \ -3}|\nabla|\nabla u|^2|(1+|\nabla u|+|\Delta v|) \\
&\quad + \chi\sqrt{n}C_0\int_\Omega |\nabla u|^{2\ \ -2}|D^2u|(1+|\nabla u|+|\Delta v|).
\end{aligned}
\tag{4.9}
$$

At last, using the nonnegative of u, we have

$$
J_3 = \int_\Omega |\nabla u|^{2\ \ -2}\nabla u\cdot\nabla(au-bu^2) \le a\int_\Omega |\nabla u|^{2\ \ }. \tag{4.10}
$$

Moreover, using the boundedness of w and the hypothesis $\gamma(w)\in C^2([0,\infty))$, one can find some positive constants $\gamma_{\ } > 0$ $(i=1,2,3)$ such that

$$
0<\gamma_1\le\gamma(w)\le\gamma_2 \qquad \text{and} \qquad |\gamma'(w)|\le\gamma_3. \tag{4.11}
$$

Then substituting (4.8)–(4.10) into (4.7), and using (4.11) and the Young's inequality, we end up with

$$
\begin{aligned}
&\frac{1}{2n}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla u|^{2\ \ } + \frac{(n-1)\gamma_1}{2}\int_\Omega |\nabla u|^{2(\ \ -2)}|\nabla|\nabla u|^2|^2 + \gamma_1\int_\Omega |\nabla u|^{2(\ \ -1)}|D^2u|^2 \\
&\le \frac12\int_{\partial\Omega}\gamma(w)|\nabla u|^{2(\ \ -1)}\frac{\partial|\nabla u|^2}{\partial\nu}\mathrm{d}S - (n-\frac12)\int_\Omega \gamma'(w)|\nabla u|^{2(\ \ -1)}\nabla w\cdot\nabla|\nabla u|^2 \\
&\quad + \chi(n-1)C_0\int_\Omega |\nabla u|^{2\ \ -3}|\nabla|\nabla u|^2|(1+|\nabla u|+|\Delta v|) \\
&\quad + \chi\sqrt{n}C_0\int_\Omega |\nabla u|^{2\ \ -2}|D^2u|(1+|\nabla u|+|\Delta v|) + a\int_\Omega |\nabla u|^{2\ \ } \\
&\le \gamma_2\zeta\int_{\partial\Omega}|\nabla u|^{2\ \ }\ \mathrm{d}S + (n-\frac12)\gamma_3\int_\Omega |\nabla u|^{\ \ -2}|\nabla|\nabla u|^2|\cdot|\nabla u|^{\ \ }|\nabla w| \\
&\quad + \chi(n-1)C_0\int_\Omega |\nabla u|^{\ \ -2}|\nabla|\nabla u|^2|\cdot|\nabla u|^{\ \ -1}(1+|\nabla u|+|\Delta v|) \\
&\quad + \chi\sqrt{n}C_0\int_\Omega |\nabla u|^{\ \ -1}|D^2u|\cdot|\nabla u|^{\ \ -1}(1+|\nabla u|+|\Delta v|) + a\int_\Omega |\nabla u|^{2\ \ } \\
&\le \gamma_2\zeta\int_{\partial\Omega}|\nabla u|^{2\ \ }\ \mathrm{d}S + \frac{(n-1)\gamma_1}{4}\int_\Omega |\nabla u|^{2(\ \ -2)}|\nabla|\nabla u|^2|^2 + \frac{\gamma_1}{2}\int_\Omega |\nabla u|^{2(\ \ -1)}|D^2u|^2 \\
&\quad + \frac{2\chi^2C_0^2(5n-4)}{\gamma_1}\int_\Omega |\nabla u|^{2(\ \ -1)}(1+|\Delta v|^2) + \frac{2(n-\frac12)^2\gamma_3^2}{(n-1)\gamma_1}\int_\Omega |\nabla u|^{2\ \ }|\nabla w|^2 \\
&\quad + \left(a + \frac{4\chi^2(n-1)C_0^2}{\gamma_1} + \frac{\chi^2 nC_0^2}{\gamma_1}\right)\int_\Omega |\nabla u|^{2\ \ }
\end{aligned}
$$

where we have used the inequality $\frac{\partial|\nabla\ \ |^2}{\partial\nu} \leq 2\zeta|\nabla u|^2$ with some constant $\zeta > 0$ on $\partial\Omega$ ([27, Lemma 4.2]), and the above inequality can be simplified as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla u|^2 \ \ + \ ^{n(n-1)\gamma}$$

where $c_5 := c_3 + c_4 + 2 + c_2 M_8^2$ and $c_6 := \left(\frac{-1}{}\right)^{-1}\frac{n}{1} + |\Omega|$. Using the fact $\|u(\cdot,t)\|_{L^\infty} \le C_0$ and the Young's inequality again, one can find a positive constant $c_7 := \frac{(\ _5+1)^2\ \ {}_0^2(5\ -4)}{2\ \gamma_1}$ such that

$$
\begin{aligned}
(c_5+1)\int_\Omega |\nabla u|^2 &= (c_5+1)\int_\Omega |\nabla u|^{2\ -2}\nabla u\cdot\nabla u \\
&= (c_5+1)\left\{-(n-1)\int_\Omega u|\nabla u|^{2\ -4}\nabla|\nabla u|^2\cdot\nabla u - \int_\Omega u|\nabla u|^{2\ -2}\Delta u\right\} \\
&\le (c_5+1)(n-1)C_0\int_\Omega |\nabla u|^{2\ -}\ \ u| \qquad \Omega
\end{aligned}
$$

$$\begin{cases} u &= \nabla\cdot(\gamma(w)\nabla u)-\chi\nabla\cdot(u\nabla v), & x\in\Omega,\ t>0,\\ v &= \Delta v+wz, & x\in\Omega,\ t>0,\\ w &= -wz, & x\in\Omega,\ t>0,\\ z &= \Delta z-z+u, & x\in\Omega,\ t>0,\\ \frac{\partial}{\partial\nu}=\frac{\partial}{\partial\nu}=\frac{\partial}{\partial\nu}=\frac{\partial}{\partial\nu}=0, & & x\in\partial\Omega,\ t>0,\\ (u,v,w,z)(x,0)=(u_0,v_0,w_0,z_0)(x), & & x\in\Omega. \end{cases} \tag{4.18}$$

Then, we have the following results on the large time behavior of solution of the system (4.18).

Theorem 4.3. L (u,v,w,z) n (4.18). n p n n K n κ n p n n t, $t>0$

$$\|(u,v,w,z)(\cdot,t)-(\bar u_0,\bar v_0+\bar w_0,0,\bar u_0)\|_{L^\infty}\le Ke^{-\kappa},$$

$\bar u_0=\frac{1}{|\Omega|}\int_\Omega u_0$, $\bar v_0=\frac{1}{|\Omega|}\int_\Omega v_0$ n $\bar w_0=\frac{1}{|\Omega|}\int_\Omega w_0$.

Remark 4.4. For the system (4.18) with $\gamma(w)=1$, the convergence and decay rate of solution have been established in [9] and [20] based on the semigroup estimates method. However, when $\gamma(w)$ is not a constant, the semigroup estimates method cannot be used directly to obtain the converge properties of u. Instead, we shall use the energy estimate to obtain the convergence properties of solution in L^2-norm. Then using the regularity of ∇u and Gagliardo–Nirenberg inequality, we can obtain the decay information in L^∞-norm.

First, we show the decay properties of w.

Lemma 4.5. pp n n n m 4.3 n $t>0$,

$$\|w(\cdot,t)\|_{L^\infty}\le K_1e^{-\kappa_1},\tag{4.19}$$

n

$$\|\nabla w(\cdot,t)\|_{L^\infty}\le K_2e^{-\frac{\kappa_1}{2}},\tag{4.20}$$

K_1,K_2 n κ_1 p n n n p n n t.

. Integrating the first equation of (4.18) and using the homogeneous Neumann boundary conditions, one has

$$\int_\Omega u=\int_\Omega u_0.$$

On the other hand, applying the variation-of-constants formula to the fourth equation of (4.18), it follows that

$$z(t)=e^{-}e^{\Delta}z_0+\int_0 e^{-(\,-s)}e^{(\,-s)\Delta}u(s)\mathrm{d}s.$$

Then using the similar arguments as in [5, Lemma 2.2], we can find a constant $\kappa_1>0$ independent of t such that

$$\inf_{\in\Omega} z(x,t)\ge\kappa_1>0 \text{ for all } t\ge 0.\tag{4.21}$$

From the third equation of (4.18), we have

$$w(x,t)=w_0(x)e^{-\int_0^t \mathrm{d}s},\tag{4.22}$$

which combined with (4.21) gives (4.19). Using (4.22) and the facts $w_0 \in C^1(\bar{\Omega})$, (4.21) and $\|\nabla z\|_{L^\infty} \le C_0$, one has

$$\|\nabla w(\cdot,t)\|_{L^\infty} = \|\nabla w_0 e^{-\int_0^t (\quad) ds} - w_0 e^{-\int_0^t (\quad) ds} \cdot \int_0 \nabla z(\cdot,s) ds\|_{L^\infty}$$
$$\le \|\nabla w_0\|_{L^\infty} e^{-\kappa_1} + C_0 \|w_0\|_{L^\infty} e^{-\kappa_1} \cdot t,$$

which implies (4.20). Then the proof of this lemma is completed. □

Lemma 4.6. 4.3 . L $\lambda_1 > 0$ $-\Delta$ Ω $0 < \kappa_2 < \min\{\lambda_1, \kappa_1\}$. K_3 t

$$\|v(\cdot,t) - \bar{v}_0 - \bar{w}_0\|_{L^\infty} \le K_3 e^{-\kappa_2}, \ t > 0. \tag{4.23}$$

. Noting the fact that $\bar{v} = \overline{wz}$ and using the variation-of-constants, we can from second equation of (4.18) obtain

$$v(\cdot,t) - \bar{v}(t) = e^{\Delta}(v_0 - \bar{v}_0) + \int_0 e^{\Delta(- s)}\Big((wz)(\cdot,s) - \overline{wz}(s)\Big) ds. \tag{4.24}$$

Then, using the facts $\int_\Omega (v_0 - \bar{v}_0) = \int_\Omega (wz - \overline{wz}) = 0$, $\|z(\cdot,t)\|_{L^\infty} \le C_0, v_0 \in W^{1,\infty}(\Omega)$, (4.19) and (2.6), one can derive that

$$\begin{aligned}
\|v(\cdot,t) - \bar{v}(t)\|_{L^\infty} &\le \|e^{\Delta}(v_0 - \bar{v}_0)\|_{L^\infty} + \int_0 \Big\|e^{\Delta(- s)}\Big((wz)(\cdot,s) - \overline{wz}(s)\Big)\Big\|_{L^\infty} ds \\
&\le 4\sigma_1 \|v_0\|_{L^\infty} e^{-\lambda_1} + 2\sigma_1 \int_0 e^{-\lambda_1(- s)} \|(wz)(\cdot,s) - \overline{wz}(s)\|_{L^\infty} ds \\
&\le 4\sigma_1 \|v_0\|_{L^\infty} e^{-\lambda_1} + 4\sigma_1 \int_0 e^{-\lambda_1(- s)} \|w(\cdot,s)\|_{L^\infty} \|z(\cdot,s)\|_{L^\infty} ds \\
&\le 4\sigma_1 \|v_0\|_{L^\infty} e^{-\lambda_1} + 4\sigma_1 C_0 K_1 \int_0 e^{-\lambda_1(- s)} e^{-\kappa_1 s} ds \\
&\le 4\sigma_1 \|v_0\|_{L^\infty} e^{-\lambda_1} + 4\sigma_1 C_0 K_1 \int_0 e^{-\kappa_2(- s)} e^{-\kappa_1 s} ds \\
&= 4\sigma_1 \|v_0\|_{L^\infty} e^{-\lambda_1} + 4\sigma_1 C_0 K_1 e^{-\kappa_2} \int_0 e^{-(\kappa_1 - \kappa_2) s} ds \\
&\le 4\sigma_1 \|v_0\|_{L^\infty} e^{-\lambda_1} + \frac{4\sigma_1 C_0 K_1}{\kappa_1 - \kappa_2} e^{-\kappa_2}.
\end{aligned} \tag{4.25}$$

On the other hand, from (2.2), one has

$$\bar{v}(t) = \bar{v}_0 + \bar{w}_0 - \bar{w}(t),$$

which together with (4.25) and the fact (4.19) gives

$$\begin{aligned}\|v(\cdot,t)-(\bar v_0+\bar w_0)\|_{L^\infty} &\le \|v(\cdot,t)-\bar v\|_{L^\infty}+\|\bar v-(\bar v_0+\bar w_0)\|_{L^\infty}\\ &\le 4\sigma_1\|v_0\|_{L^\infty}e^{-\lambda_1} \ +\frac{4\sigma_1C_0K_1}{\kappa_1-\kappa_2}e^{-\kappa_2} \ +\|w(t)\|_{L^\infty}\\ &\le 4\sigma_1\|v_0\|_{L^\infty}e^{-\lambda_1} \ +\frac{4\sigma_1C_0K_1}{\kappa_1-\kappa_2}e^{-\kappa_2} \ +K_1e^{-\kappa_1}\\ &\le \left(4\sigma_1\|v_0\|_{L^\infty}+\frac{4\sigma_1C_0K_1}{\kappa_1-\kappa_2}+K_1\right)e^{-\kappa_2} \ ,\end{aligned}$$

and hence (4.23) follows. Then we complete the proof of Lemma 4.6. □

Next, we shall show the convergence properties of u. More precisely, one has the following results.

Lemma 4.7. 4.3

$$\|u(\cdot,t)-\bar u_0\|_{L^\infty}\le K_4e^{-\kappa_3} \ , \qquad t>0, \tag{4.26}$$

K_4 κ_3 t.

. From the first equation of (4.18), one has

$$(u-\bar u_0) \ =\nabla\cdot(\gamma(w)\nabla u)-\chi\nabla\cdot(u\nabla v). \tag{4.27}$$

Using the fact $\|w(\cdot,t)\|_{L^\infty}\le\|w_0\|_{L^\infty}$ and the properties of $\gamma(w)$, one has $0<\gamma_1\le\gamma(w)$. Then multiplying (4.27) by $u-\bar u_0$, integrating the resulting equation by parts and using the fact $\|u(\cdot,t)\|_{L^\infty}\le C_0$ (see Theorem 1.1) and the Young's inequality, we end up with

$$\begin{aligned}\frac12\frac{\mathrm d}{\mathrm dt}\int_\Omega(u-\bar u_0)^2 &= -\int_\Omega\gamma(w)|\nabla u|^2+\chi\int_\Omega u\nabla v\cdot\nabla u\\ &\le -\gamma_1\int_\Omega|\nabla u|^2+C_0\chi\int_\Omega|\nabla u||\nabla v|\\ &\le -\frac{\gamma_1}{2}\int_\Omega|\nabla u|^2+\frac{C_0^2\chi^2}{2\gamma_1}\int_\Omega|\nabla v|^2,\end{aligned}$$

which yields

$$\frac{\mathrm d}{\mathrm dt}\int_\Omega(u-\bar u_0)^2+\gamma_1\int_\Omega|\nabla u|^2\le\frac{C_0^2\chi^2}{\gamma_1}\int_\Omega|\nabla v|^2. \tag{4.28}$$

On the other hand, noting the fact $\int_\Omega(u-\bar u_0)=0$, and using the Poincaré inequality, we can derive that

$$\int_\Omega|u-\bar u_0|^2\le c_1\int_\Omega|\nabla(u-\bar u_0)|^2. \tag{4.29}$$

Substituting (4.29) into (4.28), one has

$$\frac{\mathrm d}{\mathrm dt}\int_\Omega|u-\bar u_0|^2+\frac{\gamma_1}{c_1}\int_\Omega|u-\bar u_0|^2\le\frac{C_0^2\chi^2}{\gamma_1}\int_\Omega|\nabla v|^2. \tag{4.30}$$

Moreover, using the variation-of-constants method and (2.7), (2.8), the fact $\|z(\cdot,t)\|_{L^\infty} \le C_0$ as well as (4.19), one derives that

$$
\begin{aligned}
\|\nabla v(\cdot,t)\|_{L^2} &\le \|\nabla e^{t\Delta} v_0\|_{L^2} + \int_0^t \|\nabla e^{(t-s)\Delta} w(\cdot,s)z(\cdot,s)\|_{L^2}\mathrm{d}s\\
&\le 2\sigma_3 e^{-\lambda_1 t}\|\nabla v_0\|_{L^2} + \sigma_2\int_0^t \left(1+(t-s)^{-\frac{1}{2}}\right)e^{-\lambda_1(t-s)}\|w(\cdot,s)z(\cdot,s)\|_{L^2}\mathrm{d}s\\
&\le 2\sigma_3 e^{-\lambda_1 t}\|\nabla v_0\|_{L^2} + C_0\sigma_2 K_1|\Omega|^{\frac{1}{2}}\int_0^t \left(1+(t-s)^{-\frac{1}{2}}\right)e^{-\lambda_1(t-s)}e^{-\kappa_1 s}\mathrm{d}s\\
&\le 2\sigma_3 e^{-\lambda_1 t}\|\nabla v_0\|_{L^2} + C_0\sigma_2 K_1|\Omega|^{\frac{1}{2}}\sigma_4 e^{-\min\{\kappa_1,\kappa_2\}t}\\
&\le (2\sigma_3\|\nabla v_0\|_{L^2} + C_0\sigma_2 K_1|\Omega|^{\frac{1}{2}}\sigma_4)e^{-\kappa_2 t}\\
&\le c_2 e^{-\kappa_2 t},
\end{aligned}
\tag{4.31}
$$

where we have used Lemma 2.4 and the fact $\kappa_2 < \min\{\lambda_1,\kappa_1\}$.

Then combining (4.30) and (4.31), and letting $y(t) := \int_\Omega |u-\bar{u}_0|^2$, one has

$$
y'(t) + \frac{\gamma_1}{c_1}y(t) \le \frac{C_0^2\chi^2c_2^2}{\gamma_1}e^{-2\kappa_2 t},
$$

which make us to find two constants $\mu_0 := \frac{\min\{\frac{\gamma_1}{c_1},2\kappa_2\}}{2}$ and $c_3 > 0$ such that

$$
\begin{aligned}
y(t) &\le e^{-\frac{\gamma_1}{c_1}t}y(0) + \frac{C_0^2\chi^2c_2^2}{\gamma_1}\int_0^t e^{-\frac{\gamma_1}{c_1}(t-s)}e^{-2\kappa_2 s}\mathrm{d}s\\
&\le \|u_0\|_{L^2}^2 e^{-\frac{\gamma_1}{c_1}t} + \frac{C_0^2\chi^2c_2^2}{\gamma_1}\int_0^t e^{-\mu_0(t-s)}e^{-2\kappa_2 s}\mathrm{d}s\\
&\le \|u_0\|_{L^2}^2 e^{-\frac{\gamma_1}{c_1}t} + \frac{C_0^2\chi^2c_2^2\sigma_4}{2\gamma_1}e^{-\mu_0 t}\\
&\le \left(\|u_0\|_{L^2}^2 + \frac{C_0^2\chi^2c_2^2\sigma_4}{2\gamma_1}\right)e^{-\mu_0 t}\\
&\le c_3 e^{-\mu_0 t},
\end{aligned}
\tag{4.32}
$$

where we have used Lemma 2.4 to derive that $\int_0^t e^{-\mu_0(t-s)}e^{-2\kappa_2 s}\mathrm{d}s \le \frac{\sigma_4}{2}$. Then using the definition of $y(t)$, from (4.32) one obtains

$$
\|u(\cdot,t)-\bar{u}_0\|_{L^2} \le c_3^{\frac{1}{2}}e^{-\frac{\mu_0}{2}t}, \tag{4.33}
$$

On the other hand, using (4.20), from Lemma 4.2, we can derive that

$$
\|\nabla u(\cdot,t)\|_{L^{2n}} \le M_9, \quad \text{for all } t \ge 1. \tag{4.34}
$$

Then noting (4.33) and (4.34), and using the Gagliardo–Nirenberg inequality, one has

$$
\begin{aligned}
\|u(\cdot,t)-\bar{u}_0\|_{L^\infty} &\le c_4\|\nabla(u-\bar{u}_0)\|_{L^{2n}}^{\frac{n}{n+1}}\|u-\bar{u}_0\|_{L^2}^{\frac{1}{n+1}} + c_4\|u-\bar{u}_0\|_{L^2}\\
&= c_4\|\nabla u\|_{L^{2n}}^{\frac{n}{n+1}}\|u-\bar{u}_0\|_{L^2}^{\frac{1}{n+1}} + c_4\|u-\bar{u}_0\|_{L^2}
\end{aligned}
$$

$$\leq c_4 M_9^{\frac{n}{n+1}} \|u-\bar{u}_0\|_{L^2}^{\frac{1}{n+1}} + c_4\|u-\bar{u}_0\|_{L^2}$$
$$\leq (c_4 M_9^{\frac{n}{n+1}} c_3^{\frac{1}{2(n+1)}} + c_3^{\frac{1}{2}} c_4) e^{-\frac{\mu_0}{2(n+1)}} \quad \text{for all } t \geq 1,$$

and thus (4.26) follows by letting $\kappa_3 = \frac{0}{2(\ +1)}$ and noting the boundedness of $\|u(\cdot,t)\|_{L^\infty}$. □

At last, we shall use the semigroup method to give the large time behavior of z. More precisely, we have the following result.

Lemma 4.8. 4.3 $t > 0$,

$$\|z(\cdot,t)-\bar{u}_0\|_{L^\infty} \leq K_5 e^{-\kappa_4}\ , \tag{4.35}$$

K_5 κ_4 t.

. Applying the variation-of-constants methods to the fourth equation of (4.18), one can derive that

$$z(\cdot,t)-\bar{u}_0 = e^{(\Delta-1)}(z_0-\bar{u}_0) + \int_0 e^{(\Delta-1)(\ -s)}(u(\cdot,s)-\bar{u}_0)\mathrm{d}s.$$

Then we can use the semigroup estimate, (4.26) and Lemma 2.4 to find a $\kappa_4 := \frac{\min\{\lambda_1+1\ \kappa_3\ 1\}}{2}$ such that

$$\begin{aligned}
\|z(\cdot,t)-\bar{u}_0\|_{L^\infty} &\leq \|e^{(\Delta-1)}(z_0-\bar{u}_0)\|_{L^\infty} + \int_0 \|e^{(\Delta-1)(\ -s)}(u(\cdot,s)-\bar{u}_0)\|_{L^\infty}\mathrm{d}s \\
&\leq e^{-}\|e^{\Delta}(z_0-\bar{u}_0)\|_{L^\infty} + 2\sigma_1\int_0 e^{-(\lambda_1+1)(\ -s)}\|u(\cdot,s)-\bar{u}_0\|_{L^\infty}\mathrm{d}s \\
&\leq e^{-}\|z_0-\bar{u}_0\|_{L^\infty} + 2\sigma_1 K_4\int_0 e^{-\kappa_4(\ -s)}e^{-\kappa_3 s}\mathrm{d}s \\
&\leq (\|z_0\|_{L^\infty}+\|u_0\|_{L^\infty})e^{-} + \sigma_1 K_4\sigma_4 e^{-\kappa_4} \\
&\leq (\|z_0\|_{L^\infty}+\|u_0\|_{L^\infty}+\sigma_1 K_4\sigma_4)e^{-\kappa_4}\ ,
\end{aligned}$$

which gives (4.35). Hence, the proof of Lemma 4.8 is completed. □

4.3. Theorem 4.3 is a consequence of the combination of Lemmas 4.5–4.8.

4.2. Large time behavior: $a, b > 0$

In this section, we shall study the large time behavior in the case of $a > 0$ and $b > 0$. More precisely, we have the following results.

Theorem 4.9. (Global stabilization: $a, b > 0$) 1.1 (u, v, w, z) (1.3) $(-, \bar{v}_0+\bar{w}_0, 0, -)$.

Next, we shall prove Theorem 4.9 based on the following Lemmas.

Lemma 4.10. *L* 4.9 $v_* \geq 0$

$$\|v(\cdot,t)-v_*\|_{\ 2,\infty} \to 0 \text{ as } t\to\infty. \tag{4.36}$$

. Integrating the third equation of (1.3), one can derive that

$$\int_0^\infty \int_\Omega wz < \infty.$$

Then we can use the similar arguments as in [8, lemma 4.3] to obtain the results. We omit the details for convenience.

Lemma 4.11. *L* n n n m 4.9 l n l n (u, v, w, z)

$$\|u(\cdot,t) - \frac{a}{b}\|_{L^\infty} \to 0 \text{ as } \ t \to \infty, \tag{4.37}$$

n

$$\|z(\cdot,t) - \frac{a}{b}\|_{L^\infty} \to 0 \text{ as } \ t \to \infty. \tag{4.38}$$

. The first equation of (1.3) can be rewritten as

$$u \ = \nabla\cdot(\gamma(w)\nabla u) - \chi\nabla v\cdot\nabla u - \chi u\Delta v + au - bu^2. \tag{4.39}$$

From (4.36), we have $\|\Delta v\|_{L^\infty} \to 0$ as $t \to \infty$, which implies for any $\varepsilon > 0$, there exists $t_1 > 0$ such that

$$\|\Delta v\|_{L^\infty} \le \varepsilon, \text{ for all } t \ge t_1. \tag{4.40}$$

Then, based on some ideas in [6], we can show (4.37) hold. In fact, for $x \in \Omega$ and $t \ge t_1$, from (4.39) one can derive that

$$u \ \le \nabla\cdot(\gamma(w)\nabla u) - \chi\nabla v\cdot\nabla u + (a+\chi\varepsilon)u - bu^2. \tag{4.41}$$

and

$$\nabla\cdot(\gamma(w)\nabla u) - \chi\nabla v\cdot\nabla u + (a-\chi\varepsilon)u - bu^2 \le u \tag{4.42}$$

Let $\phi(t)$ be a solution of the following system

$$\begin{cases} \phi'(t) = (a+\chi\varepsilon)\phi - b\phi^2, & t > t_1 \\ \phi(t_1) = \|u(\cdot,t_1)\|_{L^\infty}, \end{cases} \tag{4.43}$$

then noting (4.41) and using the comparison principle, one has

$$u(x,t) \le \phi(t) \quad \text{for all } x \in \Omega, t \ge t_1, \tag{4.44}$$

which gives

$$\limsup_{\to\infty} \sup_{\in\Omega} u(x,t) \le \limsup_{\to\infty} \phi(t) = \lim_{\to\infty} \phi(t) = \frac{a+\chi\varepsilon}{b},$$

and hence

$$\limsup_{\to\infty} \sup_{\in\Omega} u(x,t) \le \frac{a}{b} \tag{4.45}$$

due to the arbitrary of ε. Similarly, using (4.42) one can derive that

$$\limsup_{\to\infty} \inf_{\in\Omega} u(x,t) \ge \frac{a}{b}. \tag{4.46}$$

Then the combination of (4.45) and (4.46) gives (4.37).

Next, we shall show (4.38) holds. To this end, letting $\psi(x,t) = z(x,t) - -$, then from the fourth equation of (1.3), one has

$$\begin{cases} \psi \ - \Delta\psi + \psi = u - -, & x \in \Omega, t > 0, \\ \frac{\partial\psi}{\partial\nu} = 0, & x \in \partial\Omega, t > 0, \\ \psi(x,0) = \psi_0(x) = z_0(x) - -, & x \in \Omega. \end{cases} \tag{4.47}$$

Let $\psi^*(t)$ be the solution of the following ODE problem

$$\begin{cases} \psi^*(t) + \psi^*(t) = \|u - -\|_{L^\infty}, t > 0, \\ \psi^*(0) = \|\psi_0\|_{L^\infty}. \end{cases} \tag{4.48}$$

Then applying the comparison principle, one can show that $\psi^*(t)$ is a super-solution of problem (4.47) and satisfies

$$\psi(x,t) \le \psi^*(t) \quad \text{for all} \quad x \in \Omega, t > 0.$$

Similarly, one can show that $\psi(x,t) \ge -\psi^*(t)$ for all $x \in \Omega, t > 0$. Hence, one has

$$|\psi(x,t)| \le \psi^*(t) \quad \text{for all} \quad x \in \Omega, t > 0. \tag{4.49}$$

Since $\|u(\cdot,t) - -\|_{L^\infty} \to 0$ as $t \to \infty$, then from (4.48) we have

$$\psi^*(t) \to 0 \quad \text{as} \quad t \to \infty,$$

which combined with (4.49) gives

$$\|z(\cdot,t) - \frac{a}{b}\|_{L^\infty} = \|\psi(\cdot,t)\|_{L^\infty} \le \psi^*(t) \to 0 \quad \text{as} \quad t \to \infty,$$

and hence (4.38) follows.

Lemma 4.12. *L* n n n m 4.9 l n ll $t > 0$, n

$$\|w(\cdot,t)\|_{L^\infty} \le \mathcal{C}_1 e^{-\ell_1} \tag{4.50}$$

n

$$\|v(\cdot,t) - \bar{v}_0 - \bar{w}_0\|_{L^\infty} \le \mathcal{C}_2 e^{-\ell_2}, \tag{4.51}$$

n n $\mathcal{C}$, ℓ $> 0 (i = 1,2)$ n n n t.

. From (4.38), we know that there exists $T_1 > 0$ such that for $t \ge T_1$, it holds that

$$\frac{a}{2b} \le z(\cdot,t) \le \frac{3a}{2b}. \tag{4.52}$$

Then from the third equation of the system (1.3), one has

$$w(x,t) = w(x,T_1) e^{-\int_{T_1}^{t} (\;) \mathrm{d}s},$$

which, combined with the boundedness of w and (4.52), gives

$$\|w(\cdot,t)\|_{L^\infty} \le \|w(\cdot,T_1)\|_{L^\infty} e^{-\frac{a}{2b}(\; - \; _1)} \text{ for all } t \ge T_1,$$

and then (4.50) follows by noting the boundedness of $\|w(\cdot,t)\|_{L^\infty}$. Moreover, using the similar arguments in Lemma 4.6, one can obtain (4.51) directly. □

Next, we shall show the convergence rate of u and z.

Lemma 4.13. *L* n n n m 4.9 l n ll $t > 0$, l

$$\|u(\cdot,t) - \frac{a}{b}\|_{L^\infty} \le \mathcal{C}_3 e^{-\ell_3}, \tag{4.53}$$

n

$$\|z(\cdot,t) - \frac{a}{b}\|_{L^\infty} \le \mathcal{C}_4 e^{-\ell_4}, \tag{4.54}$$

n n $\mathcal{C}$, ℓ > 0 $(i = 3,4)$ n n n t.

. We first show (4.53) holds. To this end, we multiply the first equation of (1.3) by $\frac{-\frac{a}{b}}{}$, integrate it by parts, and apply the Young's inequality to obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(u-\frac{a}{b}\ln u\right)+b\int_\Omega\left(u-\frac{a}{b}\right)^2&=-\frac{a}{b}\int_\Omega\gamma(w)\frac{|\nabla u|^2}{u^2}+\frac{a\chi}{b}\int_\Omega\frac{\nabla u}{u}\cdot\nabla v\\&\le\frac{a\chi^2}{4b}\int_\Omega\frac{|\nabla v|^2}{\gamma(w)},\end{aligned}\tag{4.55}$$

which combined with the fact $\frac{\mathrm{d}}{\mathrm{d}}\int_\Omega\left[-\ln - - -\right]=0$, and the estimate $0<\gamma_1\le\gamma(w)$, one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left(u-\frac{a}{b}-\frac{a}{b}\ln\frac{bu}{a}\right)+b\int_\Omega\left(u-\frac{a}{b}\right)^2\le\frac{a\chi^2}{4b\gamma_1}\|\nabla v\|^2_{L^2}.\tag{4.56}$$

On the other hand, defining $\phi(\zeta)=\zeta - - \ln\zeta$, then one has $\phi'(-)=0,\ \phi''(\zeta)=\frac{}{2}$ and

$$u-\frac{a}{b}-\frac{a}{b}\ln\frac{bu}{a}=\phi(u)-\phi\left(\frac{a}{b}\right)=\frac{\phi''(\zeta_*)}{2}\left(u-\frac{a}{b}\right)^2=\frac{a}{2b\zeta_*^2}\left(u-\frac{a}{b}\right)^2,\tag{4.57}$$

where $\zeta_*>0$ between u and $-$. Since $u(\cdot,t)\to -$ as $t\to\infty$, there exists some $t_2>1$ such that for all $t\ge t_2$ it holds that

$$\frac{1}{\sqrt{2}}\frac{a}{b}\le\zeta_*\le\frac{\sqrt{3}}{\sqrt{2}}\frac{a}{b},$$

which, combined with (4.57), implies that

$$\frac{b}{3a}\left(u-\frac{a}{b}\right)^2\le u-\frac{a}{b}-\frac{a}{b}\ln\frac{bu}{a}\le\frac{b}{a}\left(u-\frac{a}{b}\right)^2,\ \text{for all}\ \ t\ge t_2.\tag{4.58}$$

Then the combination of (4.56) and (4.58) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left(u-\frac{a}{b}-\frac{a}{b}\ln\frac{bu}{a}\right)+a\int_\Omega\left(u-\frac{a}{b}-\frac{a}{b}\ln\frac{bu}{a}\right)\le\frac{a\chi^2}{4b\gamma_1}\|\nabla v\|^2_{L^2},\tag{4.59}$$

for all $t\ge t_2$. Moreover, noting (4.50) and $\|z(\cdot,t)\|_{L^\infty}\le C_0$ (see Theorem 1.1) , we can use (2.7) and (2.8) to find a constant $\mu_1:=\frac{\min\{\lambda_1\ \ell\}}{2}$ such that

$$\begin{aligned}\|\nabla v(\cdot,t)\|_{L^2}&\le\|\nabla e^{\ \Delta}v_0\|_{L^2}+\int_0\|\nabla e^{(\ -\ )\Delta}w(\cdot,s)z(\cdot,s)\|_{L^2}\mathrm{d}s\\&\le 2\sigma_3\|\nabla v_0\|_{L^2}e^{-\lambda_1}\ +\sigma_2\int_0\left(1+(t-s)^{-\frac{1}{2}}\right)e^{\lambda_1(\ -\ )}\|w(\cdot,s)z(\cdot,s)\|_{L^2}\mathrm{d}s\\&\le 2\sigma_3\|\nabla v_0\|_{L^2}e^{-\lambda_1}\ +\sigma_2C_0C_1|\Omega|^{\frac{1}{2}}\int_0\left(1+(t-s)\right.\end{aligned}$$

Letting $Z(t) := \int\limits_{\Omega} \left(u - - - - \ln -\right)$, then combining (4.59) and (4.60), one has for all $t \geq t_2$ that

$$Z'(t) + aZ(t) \leq \frac{a\chi^2 c_1^2}{4b\gamma_1} e^{-2 \quad {}_1} ,$$

which, together with the Grönwall's inequality and the fact (4.58), gives a constant $\mu_2 := \frac{\min\{\ \ 2\ \ {}_1\}}{2}$ such that (see the similar arguments as in (4.32))

$$\frac{b}{3a} \int\limits_{\Omega} \left(u - \frac{a}{b}\right)^2 \leq Z(t) \leq \left(Z(0) + \frac{a\chi^2 c_1^2 \sigma_4}{8b\gamma_1}\right) e^{- \quad 2} \leq c_2 e^{- \quad 2} \text{, for all } \ t \geq t_2,$$

and hence

$$\|u(\cdot,t) - \frac{a}{b}\|_{L^2} \leq \left(\frac{3ac_2}{b}\right)^{\frac{1}{2}} e^{-\frac{\mu_2}{2}} \text{ , for all } \ t \geq t_2. \tag{4.61}$$

Next, we shall show the convergence rate in L^∞-norm. Using the nonnegative of z and the fact $\|\nabla z\|_{L^\infty} \leq C_0$ (see Theorem 1.1), from the third equation of (1.3), one has

$$\begin{aligned} \|\nabla w(\cdot,t)\|_{L^\infty} &= \|\nabla w_0 e^{-\int_0^t (\cdot\ \ \ )\mathrm{d}\ } - w_0 e^{-\int_0^t (\cdot\ \ \ )\mathrm{d}\ } \cdot \int_0 \nabla z(\cdot,s)\mathrm{d}s\|_{L^\infty} \\ &\leq \|\nabla w_0\|_{L^\infty} + C_0\|w_0\|_{L^\infty} t \\ &\leq c_3(1+t), \end{aligned} \tag{4.62}$$

for all $t > 0$. On the other hand, using (4.38), we can find a fixed $t_3 > 0$ such that

$$\frac{a}{2b} \leq z(x,t) \leq \frac{3a}{2b}, \quad \text{for all } t \geq t_3 > 1. \tag{4.63}$$

Then using the third equation of (1.3) again and noting the facts (4.63) and $\|\nabla z\|_{L^\infty} \leq C_0$, one can derive that

$$\begin{aligned} \|\nabla w(\cdot,t)\|_{L^\infty} &= \|\nabla w(\cdot,t_3) e^{-\int_{t_3}^t (\cdot\ \ \ )\mathrm{d}\ } - w(\cdot,t_3) e^{-\int_{t_3}^t (\ \ \ )\mathrm{d}\ } \cdot \int_3 \nabla z(\cdot,s)\mathrm{d}s\|_{L^\infty} \\ &\leq \|\nabla w(\cdot,t_3)\|_{L^\infty} e^{-\frac{a}{2b}(\ -\ 3)} + C_0\|w_0\|_{L^\infty}(t-t_3)e^{-\frac{a}{2b}(\ -\ 3)} \\ &\leq c_3(1+t_3)e^{-\frac{a}{2b}(\ -\ 3)} + C_0\|w_0\|_{L^\infty}(t-t_3)e^{-\frac{a}{2b}(\ -\ 3)} \\ &\leq c_4, \end{aligned} \tag{4.64}$$

for all $t \geq t_3 > 1$. Then applying Lemma 4.2 and (4.64), one has $\|\nabla u\|_{L^{2n}} \leq M_9$ for some $t \geq t_4 \geq \max\{t_3, t_2\} > 1$. Then using the Gagliardo–Nirenberg inequality and (4.61), one has

$$\begin{aligned} \|u - \frac{a}{b}\|_{L^\infty} &\leq c_5 \|\nabla(u - \frac{a}{b})\|_{L^{2n}}^{\frac{n}{n+1}} \|u - \frac{a}{b}\|_{L^2}^{\frac{1}{n+1}} + c_5\|u - \frac{a}{b}\|_{L^2} \\ &\leq c_5 M_9^{\frac{n}{n+1}} \left(\frac{3ac_2}{b}\right)^{\frac{1}{2(n+1)}} e^{-\frac{\mu_2}{2(n+1)}} + c_5 \left(\frac{3ac_2}{b}\right)^{\frac{1}{2}} e^{-\frac{\mu_2}{2}} \\ &\leq c_5 \left[M_9^{\frac{n}{n+1}} \left(\frac{3ac_2}{b}\right)^{\frac{1}{2(n+1)}} + \left(\frac{3ac_2}{b}\right)^{\frac{1}{2}} \right] e^{-\frac{\mu_2}{2(n+1)}} \end{aligned}$$

for all $t \geq t_4$, which gives (4.53) by noting the boundedness of $\|u - -\|_{L^\infty}$. At last, applying the comparison principle as in Lemma 4.11, one has (4.54). Then the proof of this lemma is completed.

T *m 4.9.* The combination of Lemmas 4.12 and 4.13 gives the results in Theorem 4.9. □

4.3. Proof of Theorem 1.3

Theorem 1.3 is a consequence of Theorems 4.3 and 3.2.

Acknowledgements

The authors are grateful to the two referees for their valuable comments, which greatly improved the exposition of our paper. The research of H.Y. Jin was supported by Guangdong Basic and Applied Basic Research Foundation (No. 2020A1515010140) and (No. 2020B1515310015), Guangzhou Science and Technology Program (No. 202002030363), the NSF of China (No. 11871226), and the Fundamental Research Funds for the Central Universities. The research of L. Xiong was supported by the NSF of China (No. 11871203).

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(Received: January 13, 2021; revised: August 19, 2021; accepted: August 26, 2021)