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A dual-gradient chemotaxis system modeling the spontaneous aggregation of microglia in Alzheimer's disease

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In this paper, we consider the following dual-gradient chemotaxis model

$$\begin{cases} u_t = \Delta u - \nabla \cdot (\chi u \nabla v) + \nabla \cdot (\xi f(u) \nabla w), & x \in \Omega, t > 0, \\ \tau_1 v_t = \Delta v + \alpha u - \beta v, & x \in \Omega, t > 0, \\ \tau_2 w_t = \Delta w + \gamma u - \delta w, & x \in \Omega, t > 0, \end{cases}$$

with $f(u) = u^m$ for $m \geq 1$ and $f(u) = u(u+1)^{m-1}$ for $0 < m < 1$, where Ω is a bounded domain in $\mathbb{R}^n (n \geq 2)$ with smooth boundary, $m > 0, \chi \geq 0, \xi > 0, \alpha, \beta, \gamma, \delta > 0$ and $\tau_1, \tau_2 \in \{0, 1\}$. The model was proposed to interpret the spontaneous aggregation of microglia in Alzheimer's disease due to the interaction of attractive and repulsive chemicals released by the microglia. It has been shown in the literature that, when $m = 1$, the solution of the model with homogeneous Neumann boundary conditions either blows up or asymptotically decays to a constant in multi-dimensions depending on the sign of $\theta = \chi\alpha - \xi\gamma$, which means there is no pattern formation. In this paper, we shall show as $m > 1$, the uniformly-in-time bounded global classical solutions exist in multi-dimensions and hence pattern formation can develop. This is significantly different from the results for the case $m = 1$. We perform the numerical simulations to illustrate the various patterns generated by the model, verify our analytical results and predict some unsolved questions. Biological applications of our results are discussed and open problems are presented.

Keywords: Dual-gradient; spontaneous aggregation; chemotaxis; attraction–repulsion; boundedness; higher dimensions; pattern formation.

Mathematics Subject Classification 2010: 35A01, 35B40, 35B44, 35K57, 35Q92, 92C17

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1. Introduction and Main Results

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$$\begin{cases} u_t = \Delta u - \nabla \cdot (\chi u \nabla v) + \nabla \cdot (\xi u \nabla w), \\ \tau_1 v_t = \Delta v + \alpha u - \beta v, \\ \tau_2 w_t = \Delta w + \gamma u - \delta w, \end{cases} \tag{1.1}$$

u , v

$$\begin{cases} u_t = \Delta u - \nabla\cdot(\chi u \nabla v) + \nabla\cdot(\xi f(u)\nabla w), \\ \tau_1 v_t = \Delta v + \alpha u - \beta v, \\ \tau_2 w_t = \Delta w + \gamma u - \delta w, \end{cases} \tag{1.2}$$

$$f(u) = \begin{cases} u^m & m \geq 1, \\ u(u+1)^{m-1} & 0 < m < 1. \end{cases} \tag{1.3}$$

m 1, (1.2) (1.1)

$f(u)$ fi (1.3). (1.) m 1. n 2 n 3, 4 (1.) τ_2 1 $n \geq 3$ $m \neq 1$ 43 m $0 < m < \frac{4}{n+2}$, (1.) τ_2 1 $n \geq 2$. $m \geq \frac{4}{n+2}$, fi (1.), $m > 0$ $n \geq 2$.

Proposition 1.2. *Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary. Assume $0 \leq u_0 \in W^{1,\infty}(\Omega)$. Then for any $m > 0$ and $n \geq 2$, the system (1.) with τ_2 0 admits a unique global classical solution $(u, w) \in C^0(\bar{\Omega} \times\ 0, \infty)) \cap C^{2,1}(\bar{\Omega} \times (0, \infty))$ which is uniformly bounded in time.*

(1.2) (. . τ_1 τ_2 1) $\chi, \xi > 0$ β δ $\frac{v_0}{\alpha} \equiv \frac{w_0}{\gamma}$, (1.2) -fi 3 , 4

Theorem 1.3. *Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary. Assume β δ and $0 \leq (u_0, v_0, w_0) \in\ W^{1,\infty}(\Omega)\ ^3$ with $\frac{v_0}{\alpha}$ $\frac{w_0}{\gamma}$. Then for any $m > 1$ and $n \geq 2$, the system (1.2)–(1.4) with τ_1 τ_2 1 has a unique classical solution $(u, v, w) \in C^0(\bar{\Omega} \times\ 0, \infty)) \cap C^{2,1}(\bar{\Omega} \times (0, \infty))$. Moreover, if $0 \leq u_0 \leq (\frac{\chi\alpha}{\xi\gamma})^{\frac{1}{m-1}}$, it follows that*

$$0 < u \leq \left(\frac{\chi\alpha}{\xi\gamma}\right)^{\frac{1}{m-1}} \quad \textit{for all } t > 0.$$

2. Local Existence and Preliminaries

(1.2) (1.4) fi 4 , 4 .

Lemma 2.1. *Assume that $0 \leq (u_0, \tau_1 v_0, \tau_2 w_0) \in\ W^{1,\infty}(\Omega)\ ^3$. Then there exists $T_{\max} \in (0, \infty$ such that the model (1.2)–(1.4) has a unique nonnegative classical solution $(u, v, w) \in C(\bar{\Omega} \times\ 0, T_{\max})\ \mathbb{R}^3) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\max})\ \mathbb{R}^3)$. Moreover, $u > 0$ in $\Omega \times (0, T_{\max})$ and*

$$\textit{if } T_{\max} < \infty, \quad \textit{then } \|u(\cdot, t)\|_{L^\infty} \to \infty \quad \textit{as } t \nearrow T_{\max}. \tag{2.1}$$

(1.2) (1.4),

Lemma 2.2. *The solution (u, v, w) of (1.2)–(1.4) satisfies the following properties*

$$\|u(\cdot,t)\|_{L^1} = \|u_0\|_{L^1},$$

$$\|v(\cdot,t)\|_{L^1} \le \tau_1\|v_0\|_{L^1} + \frac{\alpha}{\beta}\|u_0\|_{L^1},$$

$$\|w(\cdot,t)\|_{L^1} \le \tau_2\|w_0\|_{L^1} + \frac{\gamma}{\delta}\|u_0\|_{L^1}.$$

[illegible]

Lemma 2.3 ([23]). *Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary. Assume there is a constant $C > 0$ such that*

$$\|u\|_{L^s} \le C \quad \text{for all } t \in (0,T).$$

If $v_0 \in W^{1,\infty}(\Omega)$, then there exists some constant C_q such that for every $t \in (0,T)$ and $1 \le s < n$, the solution of the problem

$$v_t = \Delta v + \alpha u - \beta v \quad \text{in } \Omega, \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\Omega$$

satisfies

$$\|v(t)\|_{W^{1,q}} \le C_q \tag{2.2}$$

for all $q < \frac{ns}{n-s}$. If $s = n$, then (2.2) holds for all $q < \infty$, and if $s > n$, (2.2) holds for $q = \infty$.

Lemma 2.4 (Gagliardo Nirenberg inequalit). *Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary. Let $1 \le p, q \le \infty$ satisfying $(n - kq)p < nq$ for some $k > 0$ and $r \in (0,p)$. Then, for any $h \in W^{k,q}(\Omega) \cap L^r(\Omega)$, there exist two constants c_1 and c_2 depending only on Ω, q, k, r and n such that*

$$\|h\|_{L^p} \le c_1\|D^k h\|^a_{L^q}\|h\|^{1-a}_{L^r} + c_2\|h\|_{L^r}, \tag{2.3}$$

where $a \in (0,1)$ fulfilling

$$\frac{1}{p} = a\left(\frac{1}{q} - \frac{k}{n}\right) + (1-a)\frac{1}{r}.$$

[illegible] 3). [illegible] $r \ge 1$, [illegible] $r \in (0,p)$ [illegible] 4 , [illegible] 3.2 [illegible] 4).

3. Proof of Theorem 1.1(i)

[illegible] 2.1, [illegible] $\|u\|_{L^\infty} < \infty$ [illegible] $t > 0$ [illegible] $\|u\|_{L^p}$ [illegible] $p > n$ [illegible] c_i [illegible] fi 4 ,

Lemma 3.1. *Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary. For any $\varepsilon > 0$ and $p > 1$, there exists a constant $C > 0$ such that for each $u \in L^1(\Omega)$, the solution of the problem*

$$-\Delta w + \delta w = \gamma u \quad in\ \Omega, \quad \frac{\partial w}{\partial \nu} = 0 \quad on\ \partial\Omega \tag{3.1}$$

satisfies

$$\int_\Omega w^p dx \le \varepsilon \int_\Omega u^p dx + C.$$

Proof. L^p- 1, 2 (3.1) , fi $c_1 > 0$

$$\|w(\cdot,t)\|_{W^{2,p}} \le c_1 \|u(\cdot,t)\|_{L^p}. \tag{3.2}$$

2.2 2.4 (3.2) $c_2 > 0$ $c_3 > 0$

$$\begin{aligned}\int_\Omega w^p dx = \|w\|_{L^p}^p &\le c_2 \|D^2 w\|_{L^p}^{p\theta} \|w\|_{L^1}^{p(1-\theta)} + c_2 \|w\|_{L^1}^p \\ &\le c_3 \|u\|_{L^p}^{p\theta} + c_3,\end{aligned}$$

$$\theta = \frac{1 - \frac{1}{p}}{1 + \frac{2}{n} - \frac{1}{p}} \in (0,1)$$

$p > 1$. , $\theta \in (0,1)$,

$$\int_\Omega w^p dx \le c_3 \|u\|_{L^p}^{p\theta} + c_3 \le \varepsilon \textstyle\int_\Omega u^p dx + c_4(\varepsilon), \tag{3.3}$$

. □

Lemma 3.2. *Let $m > 1$. Then for any $p > \max\{\frac{n}{2}, 1\}$, there exists a constant $C > 0$ such that the solution of (1.2)–(1.4) with $\tau_1 = \tau_2 = 0$ satisfies*

$$\int_\Omega u^p dx \le C \quad for\ all\ t \in (0, T_{\max}).$$

Proof. fi (1.2) pu^{p-1}, (1.2) $\tau_1 = \tau_2 = 0$,

$$\frac{d}{dt}\int_\Omega u^p dx + p(p-1)\int_\Omega u^{p-2}|\nabla u|^2 dx + \frac{p(p-1)\xi\gamma}{p-1+m}\int_\Omega u^{p+m}dx$$
$$\le (p-1)\chi\alpha\int_\Omega u^{p+1}dx + \frac{p(p-1)\xi\delta}{p-1+m}\int_\Omega u^{p-1+m}wdx. \qquad (3.4)$$

$m > 1$,

$$(p-1)\chi\alpha\int_\Omega u^{p+1}dx \le \frac{p(p-1)\xi\gamma}{4(p-1+m)}\int_\Omega u^{p+m}dx + c_1. \qquad (3.\ )$$

fi $c_2, c_3 > 0$

$$\begin{aligned}\frac{p(p-1)\xi\delta}{p-1+m}\int_\Omega u^{p-1+m}wdx &\le \frac{p(p-1)\xi\gamma}{4(p-1+m)}\int_\Omega u^{p+m}dx + c_2\int_\Omega w^{p+m}dx\\ &\le \frac{p(p-1)\xi\gamma}{2(p-1+m)}\int_\Omega u^{p+m}dx + c_3,\end{aligned} \qquad (3.\ )$$

3.1 (3.) (3.) (3.4),

$$\frac{d}{dt}\int_\Omega u^p dx + p(p-1)\int_\Omega u^{p-2}|\nabla u|^2 dx + \frac{p(p-1)\xi\gamma}{4(p-1+m)}\int_\Omega u^{p+m}dx \le c_1 + c_3. \quad (3.\ )$$

$$\int_\Omega u^p dx \le \frac{p(p-1)\xi\gamma}{4(p-1+m)}\int_\Omega u^{p+m}dx + c_4, \qquad (3.\ )$$

(3.) (3.)

$$\frac{d}{dt}\int_\Omega u^p dx + \int_\Omega u^p dx \le c_5.$$

$$\int_\Omega u^p dx \le \int_\Omega u_0^p dx + c_5.$$

3.2 . ◻

(1.) τ_2 0,

Lemma 3.3. *Let $m > 0$. Then for any $p >$ $\{\frac{n}{2}, 1\}$, there exists a constant $C > 0$ such that the solution of* (1.) *with* τ_2 0 *satisfies*

$$\int_\Omega u^p dx \le C \quad \textit{for all } t \in (0, T_{\max}). \qquad (3.\ 0)$$

Proof. fi $m \ge 1$ $f(u)$ u^m.
fi (1.) pu^{p-1},

t (1.) t $\tau_2 = 0$. t

$$\frac{d}{dt}\int_\Omega u^p dx + p(p-1)\int_\Omega u^{p-2}|\nabla u|^2 dx$$
$$= -p(p-1)\xi\int_\Omega u^{p-2+m}\nabla u\cdot\nabla w dx$$
$$= \frac{p(p-1)\xi\delta}{p-1+m}\int_\Omega u^{p-1+m} w dx - \frac{p(p-1)\xi\gamma}{p-1+m}\int_\Omega u^{p+m} dx. \quad (3.10)$$

t t t (3.) t (3.10), fi t $c_1 > 0$ t t

$$\frac{d}{dt}\int_\Omega u^p dx + p(p-1)\int_\Omega u^{p-2}|\nabla u|^2 dx + \frac{p(p-1)\xi\gamma}{2(p-1+m)}\int_\Omega u^{p+m} dx \le c_1,$$

t t t (3.)

$$\frac{d}{dt}\int_\Omega u^p dx + \int_\Omega u^p dx \le c_2. \quad (3.11)$$

t t (3.11), (3.) t $m \ge 1$. t t , $0 < m < 1$ $f(u) = u(u+1)^{m-1}$. t , t t fi t t (1.) $p(u+1)^{p-1}$, t t t $m \ge 1$, t

$$\frac{d}{dt}\int_\Omega (u+1)^p dx + p(p-1)\int_\Omega (u+1)^{p-2}|\nabla u|^2 dx$$
$$= -p(p-1)\xi\int_\Omega (u+1)^{p+m-2} - (u+1)^{p+m-3}\ \nabla u\cdot\nabla w dx$$
$$= \frac{p(p-1)\xi}{p-1+m}\int_\Omega (u+1)^{p-1+m}(\delta w - \gamma u) dx$$
$$- \frac{p(p-1)\xi}{p-2+m}\int_\Omega (u+1)^{p-2+m}(\delta w - \gamma u) dx$$
$$\le \frac{p(p-1)\xi\delta}{p-1+m}\int_\Omega (u+1)^{p-1+m} w dx - \frac{p(p-1)\xi\gamma}{p-1+m}\int_\Omega (u+1)^{p+m} dx$$
$$+ \frac{p(p-1)(2p+2m-3)\xi\gamma}{(p-1+m)(p-2+m)}\int_\Omega (u+1)^{p+m-1} dx$$
$$\le -\frac{p(p-1)\xi\gamma}{2(p-1+m)}\int_\Omega (u+1)^{p+m} dx + c_3$$
$$\le -\int_\Omega (u+1)^p dx + c_4,$$

t t t t t (3.). t 3.3 t . □

1.1( ) 1.2.

Proof of Theorem 1.1(i). L^p- 1, 2 -

$$\begin{cases} -\Delta v + \beta v = \alpha u, & x \in \Omega, \quad t \in (0, T_{\max}), \\ -\Delta w + \delta w = \gamma u, & x \in \Omega, \quad t \in (0, T_{\max}), \\ \dfrac{\partial v}{\partial \nu} = \dfrac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega, \quad t \in (0, T_{\max}), \end{cases}$$

$\beta, \delta > 0$. $c_1 > 0$

$$\|(v, w)(\cdot, t)\|_{W^{2,p}} \le c_1 \|u(\cdot, t)\|_{L^p}, \tag{3.12}$$

3.2

$$\|(v, w)(\cdot, t)\|_{W^{2,p}} \le c_2. \tag{3.13}$$

$p > n$ (3.13) , fi $c_3 > 0$

$$\|(\nabla v, \nabla w)(\cdot, t)\|_{L^\infty} \le c_3.$$

(. 3, 4), fi $c_4 > 0$

$$\|u(\cdot, t)\|_{L^\infty} \le c_4 \qquad t \in (0, T_{\max}). \tag{3.14}$$

1.1( ) (3.14) 2.1. □

Proof of Proposition 1.2. 3.3 (3.12)

$$\|w(\cdot, t)\|_{W^{2,p}} \le c_1. \qquad (3.1\ )$$

$p > n$ (3.1)

$$\|\nabla w(\cdot, t)\|_{L^\infty} \le c_2.$$

, (. 3, 4),

$$\|u(\cdot, t)\|_{L^\infty} \le c_3 \qquad t \in (0, T_{\max}). \qquad (3.1\ )$$

1.2 (3.1) 2.1. □

4. Proof of Theorem 1.1(ii)

4.1. *Basic energy estimates for $n \geq 2$*

$\tau_1 = 1, \tau_2 = 0$ $m > 1,$ (1.2) (1.4)

$$\begin{cases} u_t = \Delta u - \nabla\cdot(\chi u\nabla v) + \nabla\cdot(\xi u^m\nabla w), & x\in\Omega,\ t>0,\\ v_t = \Delta v + \alpha u - \beta v, & x\in\Omega,\ t>0,\\ 0 = \Delta w + \gamma u - \delta w, & x\in\Omega,\ t>0,\\ \dfrac{\partial u}{\partial\nu} = \dfrac{\partial v}{\partial\nu} = \dfrac{\partial w}{\partial\nu} = 0, & x\in\partial\Omega,\ t>0,\\ u(x,0) = u_0(x), v(x,0) = v_0(x), & x\in\Omega. \end{cases} \tag{4.1}$$

Lemma 4.1. *Assume that* $0 \leq (u_0, v_0) \in W^{1,\infty}(\Omega)^2$. *If there exists a constant* $c_1 > 0$ *such that*

$$\|u\|_{L^p} \leq c_1 \quad \textit{for all } p > n, \tag{4.2}$$

then there exists a unique triple (u, v, w) *of nonnegative functions belonging to* $C(\bar\Omega \times 0,\infty)) \cap C^{2,1}(\bar\Omega \times (0,\infty))$ *which solves* (4.1) *classically such that* $\|u(\cdot,t)\|_{L^\infty} \leq c_2$, *where* c_2 *is a constant independent of* t.

Proof. (4.2) 2.3, fi $c_1 > 0$

$$\|\nabla v\|_{L^\infty} \leq \|v\|_{W^{1,\infty}} \leq c_1. \tag{4.3}$$

(3.12) (4.2)

$$\|w\|_{W^{2,p}} \leq c_2 \qquad p > n,$$

$$\|\nabla w\|_{L^\infty} \leq c_3. \tag{4.4}$$

(4.3), (4.4) (. 3) $c_4 > 0$

$$\|u(\cdot,t)\|_{L^\infty} \leq c_4 \qquad t \in (0, T_{\max}). \tag{4. }$$

4.1 (4.) 2.1. □

(4.2) 1.1(.). 43, 44 , $\int_\Omega u^p dx + \int_\Omega |\nabla v|^{\frac{2(p+m)}{m}} dx$ $t > 0$ $\int_\Omega u^p dx$ $m > 1$ $\|\nabla v\|_{L^2}$.

Lemma 4.2. *Let $m > 1$ and $0 \leq (u_0, v_0) \in W^{1,\infty}(\Omega)\ ^2$. Then the solution of* (4.1) *satisfies*

$$\int_\Omega |\nabla v|^2 \, dx \leq C, \qquad (4.\ )$$

where $C > 0$ is a constant independent of t.

Proof. fi (4.1) u Ω

$$\frac{d}{dt}\int_\Omega u \quad u dx + \int_\Omega \frac{|\nabla u|^2}{u} dx \quad \chi \int_\Omega \nabla u \cdot \nabla v dx - \xi \int_\Omega u^{m-1} \nabla u \cdot \nabla w dx. \qquad (4.\ )$$

(4.1) $-\Delta v$ Ω

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla v|^2 \, dx + \int_\Omega |\Delta v|^2 \, dx + \beta \int_\Omega |\nabla v|^2 \, dx \quad \alpha \int_\Omega \nabla u \cdot \nabla v dx. \qquad (4.\ )$$

(4.1)

$$-\xi \int_\Omega u^{m-1} \nabla u \cdot \nabla w dx \quad \frac{\xi}{m}\int_\Omega u^m \Delta w dx \quad \frac{\xi\delta}{m}\int_\Omega u^m w dx - \frac{\xi\gamma}{m}\int_\Omega u^{m+1} dx. \qquad (4.\ )$$

(4.), (4.) (4.)

$$\begin{aligned} &\frac{d}{dt}\int_\Omega \left(u \quad u + \frac{1}{2}|\nabla v|^2\right) dx + \int_\Omega \frac{|\nabla u|^2}{u} dx \\ &\quad + \int_\Omega |\Delta v|^2 \, dx + \beta \int_\Omega |\nabla v|^2 \, dx + \frac{\xi\gamma}{m}\int_\Omega u^{m+1} dx \\ &\quad -(\chi+\alpha)\int_\Omega u \Delta v dx + \frac{\xi\delta}{m}\int_\Omega u^m w dx \\ &\leq \frac{(\chi+\alpha)^2}{2}\int_\Omega u^2 dx + \frac{1}{2}\int_\Omega |\Delta v|^2 dx + \frac{\xi\delta}{m}\int_\Omega u^m w dx, \end{aligned}$$

$2\beta \int_\Omega u \quad u dx \leq 2\beta \int_\Omega u^2 dx$,

$$\begin{aligned} &\frac{d}{dt}\int_\Omega \left(u \quad u + \frac{1}{2}|\nabla v|^2\right) dx + \int_\Omega \frac{|\nabla u|^2}{u} dx + \frac{1}{2}\int_\Omega |\Delta v|^2 dx \\ &\quad + \beta \int_\Omega |\nabla v|^2 \, dx + 2\beta \int_\Omega u \quad u dx + \frac{\xi\gamma}{m}\int_\Omega u^{m+1} dx \\ &\leq \frac{(\chi+\alpha)^2 + 4\beta}{2}\int_\Omega u^2 dx + \frac{\xi\delta}{m}\int_\Omega u^m w dx. \end{aligned} \qquad (4.10)$$

$m > 1$,

$$\frac{(\chi+\alpha)^2 + 4\beta}{2}\int_\Omega u^2 dx \leq \frac{\xi\gamma}{4m}\int_\Omega u^{m+1} dx + c_1. \qquad (4.11)$$

(4.10);

$$\begin{aligned}\frac{\xi\delta}{m}\int_\Omega u^m w dx &\le \frac{\xi\gamma}{4m}\int_\Omega u^{m+1}dx + c_2\int_\Omega w^{m+1}dx \\ &\le \frac{\xi\gamma}{2m}\int_\Omega u^{m+1}dx + c_3,\end{aligned} \tag{4.12}$$

3.1 $m+1>1$. $y(t) = \int_\Omega (u \quad u + \frac{1}{2}|\nabla v|^2)dx$. (4.11) (4.12) (4.10),

$$y'(t) + 2\beta y(t) \le c_1 + c_3,$$

$$y(t) \le y(0) + c_4 \qquad t \in (0, T_{\max}). \tag{4.13}$$

N $-u \quad u \le \frac{1}{e}$ $u > 0$ fi $y(t)$, (4.13),

$$\begin{aligned}\frac{1}{2}\int_\Omega |\nabla v|^2 dx &\le \int_\Omega u_0 \quad u_0 dx + \frac{1}{2}\int_\Omega |\nabla v_0|^2 dx - \int_\Omega u \quad u dx + c_4 \\ &\le \int_\Omega u_0 \quad u_0 dx + \frac{1}{2}\int_\Omega |\nabla v_0|^2 dx + \frac{|\Omega|}{e} + c_4,\end{aligned}$$

(4.).

$$\frac{d}{dt}\int_\Omega u^p dx + \frac{p(p-1)}{2}\int_\Omega u^{p-2}|\nabla u|^2 dx + \frac{p(p-1)\xi\gamma}{p-1+m}\int_\Omega u^{p+m}dx$$
$$\le \frac{p(p-1)\chi^2}{2}\int_\Omega u^p|\nabla v|^2 dx + \frac{p(p-1)\xi\delta}{p-1+m}\int_\Omega u^{p-1+m}w dx. \qquad (4.1\ \ )$$

3.1

(4.1)

$$\begin{aligned}\frac{p(p-1)\xi\delta}{p-1+m}\int_\Omega u^{p-1+m}w dx &\le \frac{p(p-1)\xi\gamma}{4(p-1+m)}\int_\Omega u^{p+m}dx + c_1\int_\Omega w^{p+m}dx\\ &\le \frac{p(p-1)\xi\gamma}{2(p-1+m)}\int_\Omega u^{p+m}dx + c_2.\end{aligned} \qquad (4.1\ \ )$$

(4.1) (4.1),

$$\frac{d}{dt}\int_\Omega u^p dx + \frac{2(p-1)}{p}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 dx + \frac{p(p-1)\xi\gamma}{2(p-1+m)}\int_\Omega u^{p+m}dx$$
$$\le \frac{p(p-1)\chi^2}{2}\int_\Omega u^p|\nabla v|^2 dx + c_2. \qquad (4.1\ \ )$$

ff (4.1)

$2\nabla v$

$$\begin{aligned}(|\nabla v|^2)_t &= 2\nabla v\cdot\nabla\Delta v + 2\alpha\nabla u\cdot\nabla v - 2\beta|\nabla v|^2\\ &= \Delta|\nabla v|^2 - 2|D^2v|^2 + 2\alpha\nabla u\cdot\nabla v - 2\beta|\nabla v|^2,\end{aligned} \qquad (4.1\ \ )$$

$\Delta|\nabla v|^2 = 2\nabla v\cdot\nabla\Delta v + 2|D^2v|^2$.

(4.1) $q|\nabla v|^{2q-2}(q\ge 2)$,

$$\frac{d}{dt}\int_\Omega |\nabla v|^{2q}dx + q(q-1)\int_\Omega |\nabla v|^{2q-4}|\nabla|\nabla v|^2|^2 dx$$
$$+ 2q\int_\Omega |\nabla v|^{2q-2}|D^2v|^2 dx + 2q\beta\int_\Omega |\nabla v|^{2q}dx$$
$$= q\int_{\partial\Omega}|\nabla v|^{2q-2}\frac{\partial|\nabla v|^2}{\partial\nu}dS + 2q\alpha\int_\Omega |\nabla v|^{2q-2}\nabla u\cdot\nabla v dx$$
$$\le \frac{q(q-1)}{4}\int_\Omega |\nabla v|^{2q-4}|\nabla|\nabla v|^2|^2 dx + 2q\alpha\int_\Omega |\nabla v|^{2q-2}\nabla u\cdot\nabla v dx + c_3, \qquad (4.1\ \ )$$

(1 , (3.10))

(4.)

$$q\int_{\partial\Omega}|\nabla v|^{2q-2}\frac{\partial|\nabla v|^2}{\partial\nu}dS \le \frac{q(q-1)}{4}\int_\Omega |\nabla v|^{2q-4}|\nabla|\nabla v|^2|^2 dx + c_3.$$

(4.1[illegible])

$$
\begin{aligned}
&2q\alpha\int_\Omega |\nabla v|^{2q-2}\nabla u\cdot\nabla v dx\\
&\quad = -2q(q-1)\alpha\int_\Omega u|\nabla v|^{2q-4}\nabla v\cdot\nabla|\nabla v|^2dx - 2q\alpha\int_\Omega u|\nabla v|^{2q-2}\Delta v dx\\
&\quad\le \frac{q(q-1)}{4}\int_\Omega |\nabla v|^{2q-4}|\nabla|\nabla v|^2|^2dx + 4q(q-1)\alpha^2\int_\Omega u^2|\nabla v|^{2q-2}dx\\
&\qquad + \frac{2q}{n}\int_\Omega |\nabla v|^{2q-2}|\Delta v|^2dx + \frac{nq\alpha^2}{2}\int_\Omega u^2|\nabla v|^{2q-2}dx. \qquad (4.20)
\end{aligned}
$$

(4.20) (4.1[illegible]),

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega |\nabla v|^{2q}dx + \frac{q(q-1)}{2}\int_\Omega |\nabla v|^{2q-4}|\nabla|\nabla v|^2|^2dx\\
&\quad\le \frac{2q}{n}\int_\Omega |\nabla v|^{2q-2}|\Delta v|^2dx - 2q\int_\Omega |\nabla v|^{2q-2}|D^2v|^2dx\\
&\qquad + \left(\frac{nq\alpha^2}{2} + 4q(q-1)\alpha^2\right)\int_\Omega u^2|\nabla v|^{2q-2}dx + c_3\\
&\quad\le \left(\frac{nq\alpha^2}{2} + 4q(q-1)\alpha^2\right)\int_\Omega u^2|\nabla v|^{2q-2}dx + c_3, \qquad (4.21)
\end{aligned}
$$

$$
\frac{2q}{n}\int_\Omega |\nabla v|^{2q-2}|\Delta v|^2dx \le 2q\int_\Omega |\nabla v|^{2q-2}|D^2v|^2dx,
$$

$|\Delta v|^2 \le n|D^2v|^2$. (4.1[illegible]) (4.21) (4.14). □

(4.1). $\|u\|_{L^p}$ $p > n$ 4.1. $n = 2$ $n \ge 3$

4.2. *Boundedness for $n = 2$*

L^2-

Lemma 4.4. *Let $m > 1$ and $n = 2$. Then there exists a constant $C_3 > 0$ such that the solution of* (4.1) *satisfies*

$$
\|u(\cdot,t)\|_{L^2} \le C_3 \quad \textit{for all } t\in(0,T_{\max}). \qquad (4.22)
$$

Proof. $p = 2, q = 2$ 4.3,

$$\frac{d}{dt}\left(\int_\Omega u^2dx+\int_\Omega|\nabla v|^4dx\right)+\int_\Omega|\nabla u|^2dx+\int_\Omega|\nabla|\nabla v|^2|^2dx+\frac{\xi\gamma}{1+m}\int_\Omega u^{2+m}dx$$
$$\le c_1\int_\Omega u^2|\nabla v|^2dx+c_2. \tag{4.23}$$

$m > 1$,

$$c_1\int_\Omega u^2|\nabla v|^2dx \le c_1\left(\int_\Omega u^3dx\right)^{\frac{2}{3}}\left(\int_\Omega|\nabla v|^6dx\right)^{\frac{1}{3}}$$
$$\le \mu\|\nabla v\|_{L^6}^6+c_3\|u\|_{L^3}^3 \tag{4.24}$$
$$\le \mu\|\nabla v\|_{L^6}^6+\frac{\xi\gamma}{2(1+m)}\int_\Omega u^{2+m}dx+c_4.$$

$\||\nabla v|^2\|_{L^1} = \|\nabla v\|_{L^2}^2 \le C$ (4.)

$$\|\nabla v\|_{L^6}^6 = \||\nabla v|^2\|_{L^3}^3 \le c_5\|\nabla|\nabla v|^2\|_{L^2}^2\||\nabla v|^2\|_{L^1}+c_5\||\nabla v|^2\|_{L^1}^3$$
$$\le c_6\|\nabla|\nabla v|^2\|_{L^2}^2+c_7. \tag{4.2 }$$

(4.24) (4.2) (4.23), $\mu = \frac{1}{2c_6}$,

$$\frac{d}{dt}\left(\int_\Omega u^2dx+\int_\Omega|\nabla v|^4dx\right)+\int_\Omega|\nabla u|^2dx$$
$$+\frac{1}{2}\int_\Omega|\nabla|\nabla v|^2|^2dx+\frac{\xi\gamma}{2(1+m)}\int_\Omega u^{2+m}dx \le c_8. \tag{4.2 }$$

2.2 (4.)

$$\int_\Omega u^2dx \le c_9(\|\nabla u\|_{L^2}\|u\|_{L^1}+\|u\|_{L^1}^2) \le \|\nabla u\|_{L^2}^2+c_{10} \tag{4.2 }$$

$$\int_\Omega|\nabla v|^4dx \le c_{11}(\|\nabla|\nabla v|^2\|_{L^2}\||\nabla v|^2\|_{L^1}+\||\nabla v|^2\|_{L^1}^2) \le \frac{1}{2}\|\nabla|\nabla v|^2\|_{L^2}^2+c_{12}. \tag{4.2 }$$

$z(t) = \int_\Omega u^2dx+\int_\Omega|\nabla v|^4dx.$ (4.2), (4.2) (4.2)

$$z'(t)+z(t) \le c_{13},$$

$c_{14} > 0$

$$z(t) = \int_\Omega u^2dx+\int_\Omega|\nabla v|^4dx \le c_{14}. \tag{4.2 }$$

4.4 . □

Lemma 4.5. *Let $m > 1, n$ 2. For $p > 2$, there exists a constant C independent of t such that the solution (u, v, w) of (4.1) satisfies*

$$\|u\|_{L^p} \le C. \tag{4.30}$$

Proof. fi (4.1) pu^{p-1}, x Ω, (4.1)

$$\frac{d}{dt}\int_\Omega u^p dx + \frac{2(p-1)}{p}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 dx + \frac{p(p-1)\xi\gamma}{2(p-1+m)}\int_\Omega u^{p+m} dx \le c_1 \int_\Omega u^p |\nabla v|^2 dx + c_2. \tag{4.31}$$

$$\begin{aligned} c_1 \int_\Omega u^p |\nabla v|^2 dx &\le c_1 \left(\int_\Omega u^{p+m} dx\right)^{\frac{p}{p+m}} \left(\int_\Omega |\nabla v|^{\frac{2(p+m)}{m}} dx\right)^{\frac{m}{p+m}} \\ &\le \frac{p(p-1)\xi\gamma}{4(p-1+m)}\int_\Omega u^{p+m} dx + c_3 \int_\Omega |\nabla v|^{\frac{2(p+m)}{m}} dx. \end{aligned} \tag{4.32}$$

2.3 (4.22) n 2, fi $c_4 > 0$

$$\int_\Omega |\nabla v|^{\frac{2(p+m)}{m}} dx \le c_4. \tag{4.33}$$

(4.31), (4.32) (4.33)

$$\frac{d}{dt}\int_\Omega u^p dx + \int_\Omega u^p dx \le c_5, \tag{4.34}$$

$$\int_\Omega u^p dx \le \frac{2(p-1)}{p}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 dx + c_6,$$

(4.34) (4.30) 4. □

4.3. *Boundedness for $n \ge 3$*

Lemma 4.6. *Let $m \ge 2$ and $n \ge 3$. Then for all $m \le p < \infty$, there exists a constant $C > 0$ independent of t such that the solution of (4.1) satisfies*

$$\|u(\cdot, t)\|_{L^p} \le C, \quad \textit{for all } t \in (0, T_{\max}) \qquad (4.3\ \ )$$

and

$$\|\nabla v(\cdot, t)\|_{L^{\frac{2(p+m)}{m}}} \le C, \quad \textit{for all } t \in (0, T_{\max}). \qquad (4.3\ \ )$$

Proof. $2 \le m \le p < \infty$ $q = \frac{p+m}{m} > 2$, 4.3, fi $c_1 > 0$ $c_2 > 0$

$$\frac{d}{dt}\left(\int_\Omega u^p dx + \int_\Omega |\nabla v|^{2q} dx\right) + \frac{2(p-1)}{p}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 dx$$

$$+ \frac{2(q-1)}{q}\int_\Omega |\nabla|\nabla v|^q|^2 dx + \frac{p(p-1)\xi\gamma}{2(p-1+m)}\int_\Omega u^{p+m} dx$$

$$\le c_1 \int_\Omega u^p|\nabla v|^2 dx + c_1\int_\Omega u^2|\nabla v|^{2q-2} dx + c_2. \tag{4.3 }$$

$q = \frac{m+p}{m} > 2$ $\lambda = \frac{p+m}{p+m-2} > 1$

$$\int_\Omega u^p|\nabla v|^2 dx \le \left(\int_\Omega u^{p+m} dx\right)^{\frac{p}{p+m}} \cdot \left(\int_\Omega |\nabla v|^{2q} dx\right)^{\frac{1}{q}} \tag{4.3 }$$

$$\int_\Omega u^2|\nabla v|^{2q-2} dx \le \left(\int_\Omega u^{p+m} dx\right)^{\frac{2}{p+m}} \cdot \left(\int_\Omega |\nabla v|^{2(q-1)\lambda} dx\right)^{\frac{1}{\lambda}}. \tag{4.3 }$$

$$\left(\int_\Omega |\nabla v|^{2q} dx\right)^{\frac{1}{q}} = \||\nabla v|^q\|_{L^2}^{\frac{2}{q}} \le c_3\|\nabla|\nabla v|^q\|_{L^2}^{\frac{2\theta_1}{q}} \cdot \||\nabla v|^q\|_{L^{\frac{2}{q}}}^{\frac{2(1-\theta_1)}{q}} + c_3\||\nabla v|^q\|_{L^{\frac{2}{q}}}^{\frac{2}{q}}, \tag{4.40}$$

$$\theta_1 = \frac{\frac{q}{2} - \frac{1}{2}}{\frac{q}{2} + \frac{1}{n} - \frac{1}{2}} \in (0,1).$$

(4.) (4.40)

$$\left(\int_\Omega |\nabla v|^{2q} dx\right)^{\frac{1}{q}} \le c_4\left(\int_\Omega |\nabla|\nabla v|^q|^2 dx\right)^{\frac{1}{q}\cdot\frac{\frac{q}{2}-\frac{1}{2}}{\frac{q}{2}+\frac{1}{n}-\frac{1}{2}}} + c_4. \tag{4.41}$$

fi $\theta_2 = \frac{\frac{q}{2} - \frac{p+m-2}{2p}}{\frac{q}{2} + \frac{1}{n} - \frac{1}{2}}$, (4.), fi $c_5 > 0$

$$\left(\int_\Omega |\nabla v|^{2(q-1)\lambda} dx\right)^{\frac{1}{\lambda}} = \||\nabla v|^q\|_{L^{\frac{2(q-1)\lambda}{q}}}^{\frac{2(q-1)}{q}}$$

$$= \||\nabla v|^q\|_{L^{\frac{2p}{p+m-2}}}^{\frac{2(q-1)}{q}}$$

$$\le c_5\|\nabla|\nabla v|^q\|_{L^2}^{\frac{2(q-1)\theta_2}{q}} \cdot \||\nabla v|^q\|_{L^{\frac{2}{q}}}^{\frac{2(q-1)(1-\theta_2)}{q}} + c_5\||\nabla v|^q\|_{L^{\frac{2}{q}}}^{\frac{2(q-1)}{q}}$$

$$\leq c_5 \|\nabla|\nabla v|^q\|_{L^2}^{\frac{2(q-1)\theta_2}{q}} \cdot \|\nabla v\|_{L^2}^{2(q-1)(1-\theta_2)} + c_5\|\nabla v\|_{L^2}^{2(q-1)}$$

$$\leq c_6 \left(\int_\Omega |\nabla|\nabla v|^q|^2 dx\right)^{\frac{(q-1)}{q}\cdot\frac{\frac{q}{2}-\frac{p+m-2}{2p}}{\frac{q}{2}+\frac{1}{n}-\frac{1}{2}}} + c_6. \tag{4.42}$$

(4.3), (4.3), (4.41) (4.42)

$$c_1\int_\Omega u^p|\nabla v|^2 dx + c_1\int_\Omega u^2|\nabla v|^{2q-2}dx$$

$$\leq c_1c_4\left(\int_\Omega u^{p+m}dx\right)^{\frac{p}{p+m}}\left(\int_\Omega |\nabla|\nabla v|^q|^2dx\right)^{\frac{1}{q}\cdot\frac{\frac{q}{2}-\frac{1}{2}}{\frac{q}{2}+\frac{1}{n}-\frac{1}{2}}}$$

$$+c_1c_4\left(\int_\Omega u^{p+m}dx\right)^{\frac{p}{p+m}} + c_1c_6\left(\int_\Omega u^{p+m}dx\right)^{\frac{2}{p+m}}$$

$$\times\left(\int_\Omega |\nabla|\nabla v|^q|^2dx\right)^{\frac{(q-1)}{q}\cdot\frac{\frac{q}{2}-\frac{p+m-2}{2p}}{\frac{q}{2}+\frac{1}{n}-\frac{1}{2}}} + c_1c_6\left(\int_\Omega u^{p+m}dx\right)^{\frac{2}{p+m}}. \tag{4.43}$$

q $\frac{p+m}{m}$,

$$\frac{p}{p+m}+\frac{1}{q}\cdot\frac{\frac{q}{2}-\frac{1}{2}}{\frac{q}{2}+\frac{1}{n}-\frac{1}{2}} \quad \frac{p}{p+m}+\frac{m}{p+m}\cdot\frac{\frac{q}{2}-\frac{1}{2}}{\frac{q}{2}+\frac{1}{n}-\frac{1}{2}}<1. \tag{4.44}$$

, q $\frac{p+m}{m}$ $m\geq 2$,

$$\frac{2}{p+m}+\frac{q-1}{q}\cdot\frac{\frac{q}{2}-\frac{p+m-2}{2p}}{\frac{q}{2}+\frac{1}{n}-\frac{1}{2}}<1. \qquad (4.4\ \ )$$

$\varepsilon > 0$ $X, Y \geq 0$ $C > 0$

$$X^aY^b \leq \varepsilon(X+Y)+C, \qquad (4.4\ \ )$$

$a > 0$ $b > 0$ $a + b < 1$. (4.4) (4.43) (4.44) (4.4),

$$c_1\int_\Omega u^p|\nabla v|^2dx + c_1\int_\Omega u^2|\nabla v|^{2q-2}dx \leq \varepsilon\left(\int_\Omega u^{p+m}dx + \int_\Omega|\nabla|\nabla v|^q|^2dx\right)+c_7. \qquad (4.4\ \ )$$

(4.4) (4.3) ε $\{\frac{q-1}{q}, \frac{p(p-1)\xi\gamma}{4(p-1+m)}\}$,

$$\frac{d}{dt}\left(\int_\Omega u^pdx + \int_\Omega |\nabla v|^{2q}dx\right) + \frac{2(p-1)}{p}\int_\Omega|\nabla u^{\frac{p}{2}}|^2dx + \frac{(q-1)}{q}\int_\Omega|\nabla|\nabla v|^q|^2dx$$

$$+\frac{p(p-1)\xi\gamma}{4(p-1+m)}\int_\Omega u^{p+m}dx \leq c_8. \qquad (4.4\ \ )$$

$$\int_\Omega |\nabla v|^{2q}dx = \||\nabla v|^q\|_{L^2}^2 \le c_9\|\nabla|\nabla v|^q\|_{L^2}^{2\theta_3}\cdot\||\nabla v|^q\|_{L^{\frac{2}{q}}}^{2(1-\theta_3)} + c_9\||\nabla v|^q\|_{L^{\frac{2}{q}}}^2, \tag{4.4}$$

$$\theta_3 = \frac{\frac{q}{2}-\frac{1}{2}}{\frac{q}{2}+\frac{1}{n}-\frac{1}{2}} \in (0,1).$$

$\||\nabla v|^q\|_{L^{\frac{2}{q}}} = \||\nabla v|\|_{L^2}^q \le c_{10}$ ((4.)), (4.4)

$$\int_\Omega |\nabla v|^{2q}dx \le \tfrac{(q-1)}{q}\|\nabla|\nabla v|^q\|_{L^2}^2 + c_{11}. \tag{4. 0}$$

$$\int_\Omega u^p dx \le \frac{p(p-1)\xi\gamma}{4(p-1+m)}\int_\Omega u^{p+m}dx + c_{12}. \tag{4. 1}$$

(4.4), (4. 0) (4. 1) (4.4)

$$\frac{d}{dt}\left(\int_\Omega u^p dx + \int_\Omega |\nabla v|^{2q}dx\right) + \int_\Omega u^p dx + \int_\Omega |\nabla v|^{2q}dx \le c_{13},$$

c_{14}

$$\int_\Omega u^p dx + \int_\Omega |\nabla v|^{2q}dx \le c_{14},$$

(4.3) (4.3) $q = \frac{p+m}{m}$. 4. . □

4.4. *Proof of Theorem* 1.1(ii)

1.1( ) 4.1, 4. 4. .

5. Proof of Theorem 1.3

(1.2) (1.4) $\tau_1 = \tau_2 = 1$. 1.3, (1.2) (1.4) -fi . 1.3.

Proof of Theorem 1.3. $s = \frac{v}{\alpha} - \frac{w}{\gamma}$ $\beta = \delta$, $\tau_1 = \tau_2 = 1$ $\frac{v_0}{\alpha} = \frac{w_0}{\gamma}$, (1.2) (1.4),

$$\begin{cases} s_t = \Delta s - \beta s, & x \in \Omega,\ t > 0, \\ \dfrac{\partial s}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ s(x,0) = s_0(x) = \dfrac{v_0}{\alpha} - \dfrac{w_0}{\gamma} = 0, & x \in \Omega, \end{cases}$$

$s = \frac{v}{\alpha} - \frac{w}{\gamma} = 0$ … $w = \frac{\gamma v}{\alpha}$ … fi … (1.2),

$$\begin{cases} u_t = \Delta u - \kappa \nabla \cdot (u(l - u^{m-1})\nabla v), & x \in \Omega,\ t > 0, \\ v_t = \Delta v + \alpha u - \beta v, & x \in \Omega,\ t > 0, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), v(x,0) = v_0(x), & x \in \Omega, \end{cases} \qquad (\ .1)$$

$l = \frac{\chi\alpha}{\xi\gamma}, \kappa = \frac{\chi}{l}$. … (.1) … -fi … 3 … 4 . $m = 2$, … (.1) … 0 , … 12 … 0 … 1 , 1 . … 12, 0 … $m > 1$, … *maximum principle* … (.1) … $m > 1$. … , fi … (.1) … $u(x,t) > 0$ … $t > 0$ … $u_0(x) \geq 0$. … $U = l - u^{m-1} < l$ … fi … (.1). … , fi … $U(x,t)$ … fi

$$U_t = \Delta U + \frac{m-2}{m-1}\frac{1}{l-U}|\nabla U|^2 + \kappa(m-1)U(l-U)\Delta v$$
$$+ \kappa(m-1)(l-U)\nabla U \nabla v - \kappa U \nabla U \nabla v. \qquad (\ .2)$$

… $U_0(x) = l - u_0^{m-1} \geq 0$ … $u_0 \leq (\frac{\chi\alpha}{\xi\gamma})^{\frac{1}{m-1}} = l^{\frac{1}{m-1}}$ … U … fi … , … (.2) … $U(x,t) > 0$ … $t > 0$. … $0 < u(x,t) < l^{\frac{1}{m-1}}$. … , … -fi … (.1) … fi … (. . 12, 4 … $m = 2$). … , … , fi … 1.3. ☐

Remark 5.1. … 1.3, … (1.2) … - … (.1). … , $m = 1$, … (.1)

$$\Theta = \begin{pmatrix} u-\bar{u} \\ v-\bar{v} \\ w-\bar{w} \end{pmatrix}, \quad A = \begin{pmatrix} 1 & -\chi\bar{u} & \xi\bar{u}^m \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ \alpha & -\beta & 0 \\ \gamma & 0 & -\delta \end{pmatrix}.$$

$Y_k(x)$

$$\Delta Y_k(x) + k^2 Y_k(x) = 0, \quad (\nu\cdot\nabla)Y_k(x) = 0,$$

k (.1) $\Theta(x,t)$

$$\Theta(x,t) = \sum_{k\geq 0} c_k e^{\lambda t} Y_k(x). \qquad (.2)$$

(.2) (.1), $k \geq 0$,

$$\lambda Y_k(x) = -k^2 A Y_k(x) + B Y_k(x).$$

λ

$$M_k = \begin{pmatrix} -k^2 & \chi\bar{u}k^2 & -\xi\bar{u}^m k^2 \\ \alpha & -k^2-\beta & 0 \\ \gamma & 0 & -k^2-\delta \end{pmatrix}.$$

λ M_k

$$\lambda^3 + a_2(\chi,k^2)\lambda^2 + a_1(\chi,k^2)\lambda + a_0(\chi,k^2) = 0, \qquad (.3)$$

$$a_2(\chi,k^2) = 3k^2 + \beta + \delta,$$

$$a_1(\chi,k^2) = 3k^4 + 2(\beta+\delta) + \xi\gamma\bar{u}^m - \alpha\chi\bar{u}\; k^2 + \delta\beta,$$

$$a_0(\chi,k^2) = k^6 + \beta + \delta + \xi\gamma\bar{u}^m - \alpha\chi\bar{u}\; k^4 + \beta\delta + \beta\xi\gamma\bar{u}^m - \delta\alpha\chi\bar{u}\; k^2.$$

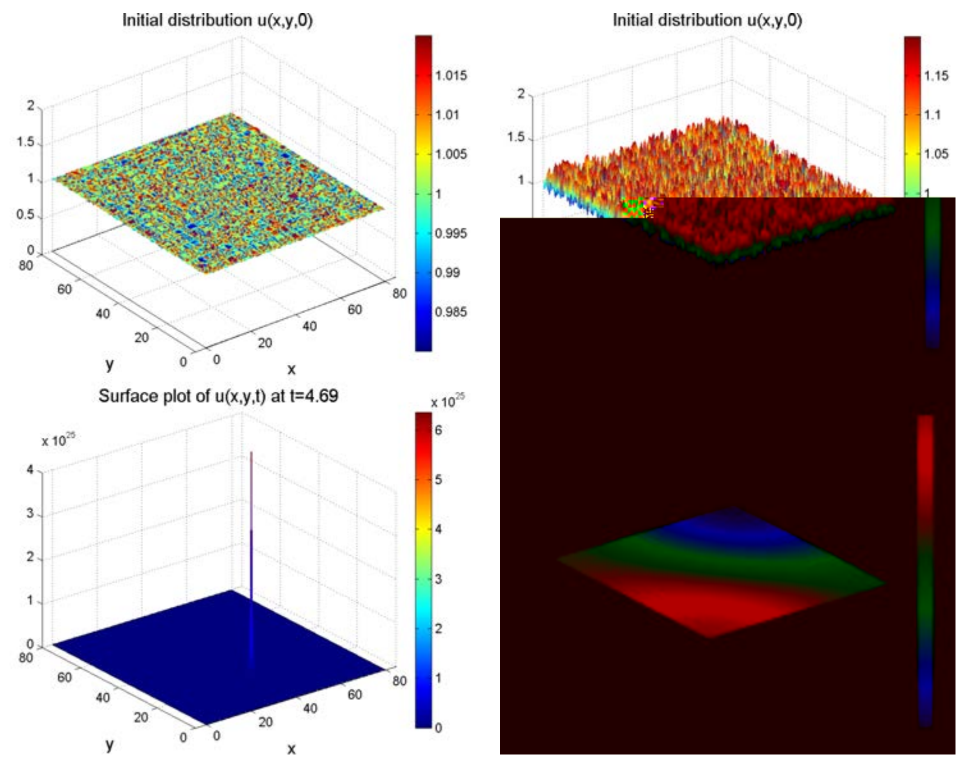
M_k $(\bar{u},\bar{v},\bar{w})$ (32), $(\bar{u},\bar{v},\bar{w})$.

Proposition 6.1. *Let $(\bar{u},\bar{v},\bar{w})$ be the homogeneous steady state of (1.2)–(1.4), where $\alpha,\beta,\gamma,\delta,\xi > 0$. Then the following results hold.*

(1) *$(\bar{u},\bar{v},\bar{w})$ is locally asymptotically stable with respect to the system (1.2)–(1.4) if $\chi \geq 0$ and satisfies*

$$\chi \leq \frac{\xi\gamma}{\alpha}\bar{u}^{m-1} \left\{\frac{\beta}{\delta}, \frac{\delta}{\beta}\right\}. \qquad (.4)$$

(2) *$(\bar{u},\bar{v},\bar{w})$ is unstable with respect to (1.2)–(1.4) if $\alpha\chi$ is large enough.*

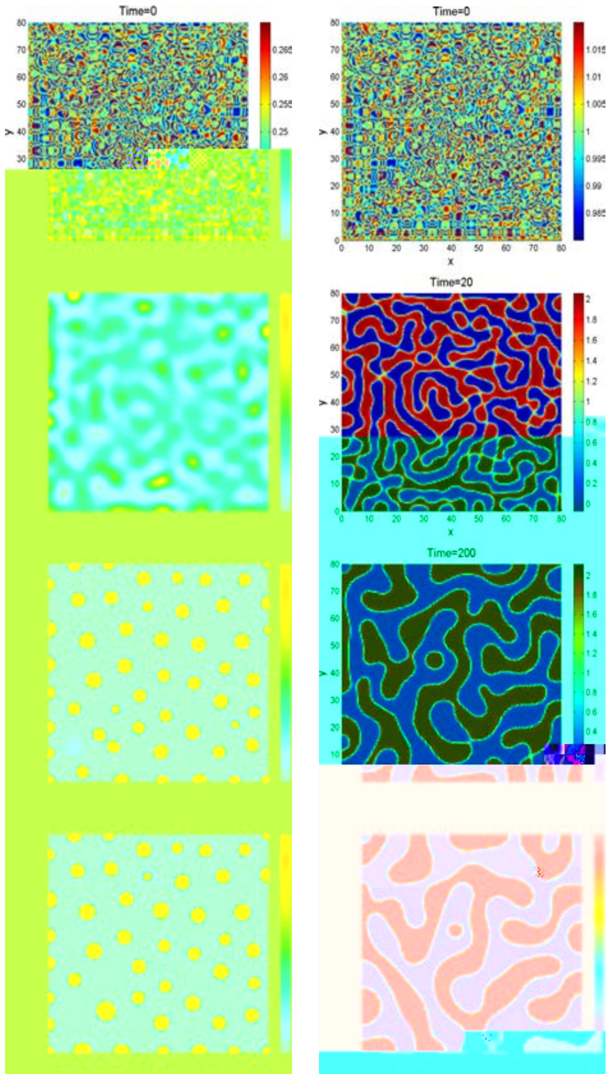


(a) $\theta > 0$ (b) $\theta < 0$

Fig. 1. Numerical illustration of solution component $u(x, y, t)$ of the system (1.2)–(1.4) with $m = 1$ and $\tau_1 = \tau_2 = 1$ in two dimensions for the case $\theta > 0$ and $\theta < 0$ where $\theta = \chi\alpha - \xi\gamma$. The initial data is set as a perturbation of the homogeneous background state $(\bar{u}, \bar{v}, \bar{w}) = (\bar{u}, \frac{\alpha}{\beta}\bar{u}, \frac{\gamma}{\delta}\bar{u})$, namely $u_0 = \bar{u} + r, v_0 = \bar{v} + r, w_0 = \bar{w} + r$, where r is a 2% small perturbation of the background state $(\bar{u}, \bar{v}, \bar{w})$ in (a) and a 20% perturbation of the background state $(\bar{u}, \bar{v}, \bar{w})$ in (b). The parameter values are: (a) $\bar{u} = 1, \chi = 8, \xi = 4, \alpha = \gamma = \beta = 1, \delta = 2$; (b) $\bar{u} = 1, \chi = 8, \xi = 10, \alpha = \gamma = \beta = 1, \delta = 2$.

β δ $\frac{v_0}{\alpha}$ $\frac{w_0}{\gamma}$ … 1.3. … 2, … $u(x, y, t)$ … fi … $0 < u \leq 2$ … 1.3. … fi … 2. … $\beta \neq \delta$, … (1.2) (1.4) … $\beta = \delta$ (…).

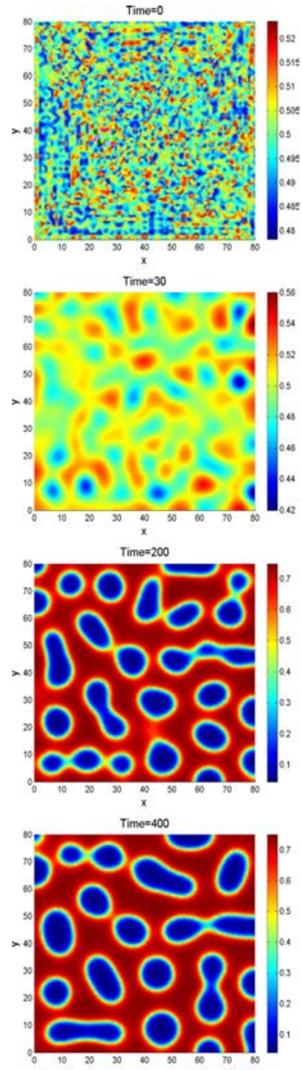
… $m > 1$ … $\theta < 0$. … $u(x, y, t)$ … 3. … ff … $m = 1$ … $\theta < 0$ … 1( ). … $\theta > 0$ …



(a) (b) (c)

Fig. 2. Numerical patterns formed by the solution component $u(x, y, t)$ to the system 1.2–(1.4) with $m > 1$ in two dimensions for the case $\theta = \chi\alpha - \xi\gamma > 0$. The initial data is set as a perturbation of the homogeneous background state $(\bar{u}, \bar{v}, \bar{w}) = (\bar{u}, \frac{\alpha}{\beta}\bar{u}, \frac{\gamma}{\delta}\bar{u})$, namely $u_0 = \bar{u} + r, v_0 = \bar{v} + r, w_0 = \bar{w} + r$, where r is a 2% small perturbation of the homogeneous background state $(\bar{u}, \bar{v}, \bar{w})$. The parameter values are $m = 2, \chi = 8, \xi = 4, \alpha = \gamma = \beta = \delta = 1$ and $\bar{u} = 0.25$ in (a), $\bar{u} = 0.5$ in (b) and $\bar{u} = 0.75$ in (c).

$(\bar{u}, \bar{v}, \bar{w})$ 1.3



(a) (b) (c)

Fig. 3. Numerical patterns of the solution component $u(x,y,t)$ to the system (1.2)–(1.4) with $m>1$ and $\tau_1=\tau_2=1$ in two dimensions for the case $\theta=\chi\alpha-\xi\gamma<0$. The initial data is set as a perturbation of the homogeneous background state $(\bar u,\bar v,\bar w)=(\bar u,\frac{\alpha}{\beta}\bar u,\frac{\gamma}{\delta}\bar u)$, namely $u_0=\bar u+r, v_0=\bar v+r, w_0=\bar w+r$, where r is a 2% small perturbation of the homogeneous background state $(\bar u,\bar v,\bar w)$. The parameter values are $m=2,\chi=8,\xi=10,\alpha=\gamma=\beta=\delta=1$ and $\bar u=0.25$ in (a), $\bar u=0.4$ in (b) and $\bar u=0.5$ in (c).

θ 0 θ 0

1.1 1.3,

(1.2) (1.4) $m > 1$

$\theta = \chi\alpha - \xi\gamma$

$m = 1$

$(\bar{u}, \bar{v}, \bar{w})$,

$(\bar{u}, \bar{v}, \bar{w})$.

$(\bar{u}, \bar{v}, \bar{w})$ $(\bar{u}, \bar{v}, \bar{w})$

(1.2) (1.4) $m > 1$

($m = 1$),

($m > 1$),

$(\bar{u}, \bar{v}, \bar{w})$

$\xi = 0$ (1.2)

11, 13, 14, 31, 33, 34, 4). $\chi = 0$

(1.2) (1.)

, 43, 4).

(1.2) $m > 1$

(1.1)

3 , 4

(1.1) 3 .

$m > 1$

(1) (1.2) (1.4) $m > 1$
τ_1 τ_2 1
(2) (1.2) (1.4)
$m > 1$
(3) (1.2) (1.4) $m > 1$

(1.2) (1.4) $m > 1$.
$m < 1$
(1.2) (1.4) $m < 1$ θ.

Acknowledgments

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