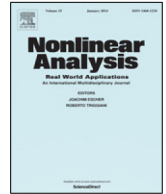




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Global dynamics of a quasilinear chemotaxis model arising from tumor invasion



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where $\Omega \subset \mathbb{R}^n (n \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$, and $\frac{\partial}{\partial\nu}$ denotes the outward normal derivative on $\partial\Omega$. The system (1.1) was recently proposed by Fujie et al. [1] to describe a tumor invasion phenomenon with chemotaxis effect of Chaplain and Anderson type [2]. The unknown functions u, v, w and z denote the concentration of tumor cells, active extracellular matrix (ECM*), extracellular matrix (ECM) and matrix degrading enzymes (MDE), respectively. The main feature of system (1.1) is that the chemotactic cue is not released by the cells themselves, which has been called indirect chemotaxis model [3,4].

It has been proved in [5,6] that the indirect chemotaxis mechanism in system (1.1) has a role in enhancing the regularity and boundedness properties of solutions. Precisely, when $D(u) = 1$ and $S(u) = u$, using the semigroup estimate method, it has been shown in [5,6] that when $n \leq 3$ and the initial data satisfies

$$(u_0, v_0, w_0, z_0) \in C^0(\bar{\Omega}) \times W^{1,\infty}(\Omega) \times C^1(\bar{\Omega}) \times C^0(\bar{\Omega}) \text{ with } u_0, v_0, w_0, z_0 \geq 0, \quad (1.2)$$

there exists a unique nonnegative global classical solution (u, v, w, z) exponentially converging to the equilibrium $(\bar{u}_0, \bar{v}_0 + \bar{w}_0, 0, \bar{u}_0)$ as $t \rightarrow +\infty$, where $\bar{u}_0 = \frac{1}{|\Omega|} \int_{\Omega} u_0(x) dx$, $\bar{v}_0 = \frac{1}{|\Omega|} \int_{\Omega} v_0(x) dx$ and $\bar{w}_0 = \frac{1}{|\Omega|} \int_{\Omega} w_0(x) dx$. It is different from the direct chemotaxis model [7]

$$\begin{cases} u_t = \nabla \cdot (D(u) \nabla u) - \nabla \cdot (S(u) \nabla z), & x \in \Omega, t > 0, \\ z_t = \Delta z - z + u, & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial z}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ (u, z)(x, 0) = (u_0, z_0)(x), & x \in \Omega, \end{cases} \quad (1.3)$$

whose solution will globally exist or blow up in a finite/infinite time depending strongly on the space dimension when $D(u) = 1$ and $S(u) = u$ (no blow-up in 1-D [8,9], critical mass blow-up in 2-D [10–15] and, generically blow-up in ≥ 3 -D [16,17]).

As pointed out in [18], the chemotaxis model with nonlinear diffusion and cross-diffusion may play a predominant role in the solution behavior. For the direct chemotaxis model (1.3), the early literature has already contained some pieces of evidence confirming the intuitive idea that the tendency toward blow-up can be weakened at large cell densities if either cross-diffusion is inhibited, or diffusion is enhanced. More precisely, the boundedness or blow-up of solutions strongly depends on the space dimensions and the power α of the ratio $\frac{S(u)}{D(u)} \approx u^\alpha$ for large values of u . If $\alpha > \frac{2}{n}$, then there exist some solutions that will blow up in finite time or infinite time [19–23], whereas if $0 < \alpha < \frac{2}{n}$ and

$$\frac{S(u)}{D(u)} \leq C(u+1)^\alpha \text{ for all } u \geq 0, \quad (1.4)$$

then the solution will globally exist with D behaves algebraically [24,25] or exponentially [26,27]. The results in [19–22,24–26] indicate that the power-type asymptotic behavior $\frac{S(u)}{D(u)} \approx u^{\frac{2}{n}}$ for the direct Keller–Segel chemotaxis model (1.3) is critical.

In contrast to the well-understood direct chemotaxis model (1.3), the theoretical understanding is much less developed in situations when a chemotactic cue is not released by the cells themselves as system (1.1). Specially, a complete understanding of the competitive interplay among diffusion, cross-diffusion and indirect chemotaxis is yet lacking. It is the purpose of the present work to achieve some insight into possible features of chemotaxis models accounting for indirect signal production mechanisms. More precisely, we will show how the indirect chemotaxis mechanisms affect the value of α in (1.4) to global classical solution. As in [24,25], we assume that the nonlinear diffusion $D(u)$ and chemosensitivity $S(u)$ satisfy

$$D, S \in C^2([0, \infty)), D(u) > 0 \text{ and } S(u) \geq 0 \text{ for all } u \geq 0, \quad (1.5)$$

and

$$K_0(u+1)^{m-1} \leq D(u) \leq K_1(u+1)^{M-1} \text{ for all } u \geq 0 \quad (1.6)$$

for some constants $m \in \mathbb{R}$, $M \in \mathbb{R}$, $K_0 > 0$ and $K_1 > 0$. Then we have the following main results.

Theorem 1.1. Let Ω be a bounded domain in $\mathbb{R}^n$ ($n \geq 1$) with smooth boundary. Assume (1.2) holds. Suppose $D(u)$ and $S(u)$ satisfy (1.4)–(1.6) and

$$0 < \alpha < \begin{cases} 1 + \frac{1}{n}, & \text{if } 1 \leq n \leq 3, \\ \frac{4}{n}, & \text{if } n \geq 4. \end{cases}$$

Then there exists a uniquely determined quadruple (u, v, w, z) of nonnegative functions defined on $\bar{\Omega} \times [0, \infty)$ which solves (1.1) classically and is bounded in the sense that there exists a constant $C > 0$ such that for all $t > 0$

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{W^{1,\infty}(\Omega)} + \|w(\cdot, t)\|_{L^\infty(\Omega)} + \|z(\cdot, t)\|_{L^\infty(\Omega)} \leq C. \quad (1.7)$$

Moreover, the global solution (u, v, w, z) will exponentially converge to $(\bar{u}_0, \bar{v}_0 + \bar{w}_0, 0, \bar{u}_0)$ as $t \rightarrow \infty$ with $\bar{u}_0 = \frac{1}{|\Omega|} \int_{\Omega} u_0(x) dx$, $\bar{v}_0 = \frac{1}{|\Omega|} \int_{\Omega} v_0(x) dx$ and $\bar{w}_0 = \frac{1}{|\Omega|} \int_{\Omega} w_0(x) dx$.

Remark 1.1. In this paper, we focus on studying the global existence of classical solution for indirect chemotaxis system (1.1) with $D(u)$ behaves algebraically. It is interesting to study the same questions in the case that $D(u)$ behaves exponentially as in [26].

Remark 1.2. We do not know whether or not $\alpha = \frac{4}{n}$ is critical when $n \geq 4$. It would be interesting to investigate whether or not the indirect chemotactic cross-diffusion could drive blow-up phenomenon if $\alpha > \frac{4}{n}$ in the case of $n \geq 4$. While, we will leave this for future explorations.

When $D(u) = 1$ and $S(u) = u$, one has $\frac{S(u)}{D(u)} = u$, which implies $\alpha = 1 < 1 + \frac{1}{n}$ in (1.6). Hence a direct application of Theorem 1.1 recovers the unconditional boundedness and convergence properties obtained in [5,6].

Corollary 1.2. Let $1 \leq n \leq 3$, and let $D(u) = 1$ and $S(u) = u$.

Lemma 2.1. Let $n \geq 1$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary and let the initial data (u_0, v_0, w_0, z_0) satisfy (1.2). If $D(u)$ and $S(u)$ satisfy (1.4)–(1.6), then there exist $T_{max} \in (0, \infty]$ and a unique, nonnegative, classical solution (u, v, w, z) of (1.1) on $\Omega \times [0, T_{max})$ such that

$$\begin{aligned} u &\in C(\bar{\Omega} \times [0, T_{max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{max})), \\ v &\in C(\bar{\Omega} \times [0, T_{max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{max})) \cap L_{loc}^\infty([0, T_{max}); W^{1,\infty}(\Omega)), \\ w &\in C(\bar{\Omega} \times [0, T_{max})) \cap C^{0,1}(\bar{\Omega} \times (0, T_{max})), \\ z &\in C(\bar{\Omega} \times [0, T_{max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{max})). \end{aligned}$$

Furthermore, if $T_{max} < \infty$, then

$$\|u(\cdot, t)\|_{L^\infty} + \|v(\cdot, t)\|_{W^{1,\infty}} + \|w(\cdot, t)\|_{L^\infty} + \|z(\cdot, t)\|_{W^{1,\infty}} \rightarrow \infty \quad \text{as } t \rightarrow T_{max}^-.$$

Lemma 2.2. Let (u, v, w, z) be the solution to (1.1) obtained in Lemma 2.1. Then for all $t \in (0, T_{max})$, one has

$$\|u(\cdot, t)\|_{L^1} = \|u_0\|_{L^1} \quad (2.1)$$

and

$$\|w(\cdot, t)\|_{L^\infty} \leq \|w_0\|_{L^\infty}. \quad (2.2)$$

Proof. Integrating the first equation of (1.1) with respect to x over Ω , and using the homogeneous Neumann boundary conditions, we obtain (2.1) directly. Moreover, noting w and z are nonnegative for $t \in (0, T_{max})$, one has $w_t(x, t) \leq 0$, which gives (2.2). $\square$

Next, we show some basic estimates which will be later used to prove the boundedness of solutions in Theorem 1.1.

Lemma 2.3 ([5]). Let (u, v, w, z) be a solution to (1.1) defined on its maximal existence interval $[0, T_{max})$. Suppose $p \geq 1$ and for $i = 1, 2$

$$\begin{aligned} q_i &\in [1, \frac{np}{n-ip}), & \text{if } p \leq \frac{n}{i}, \\ q_i &\in [1, \infty], & \text{if } p > \frac{n}{i}. \end{aligned}$$

If there exists a constant $M_0 > 0$ such that for some $T \in (0, T_{max})$, it holds that

$$\|u(\cdot, t)\|_{L^p} \leq M_0, \quad t \in (0, T),$$

then for all $t \in (0, T)$, one has

$$\|\nabla z(\cdot, t)\|_{L^{q_1}} \leq C_z(p, q_1, M_0), \quad \|z(\cdot, t)\|_{L^{q_2}} \leq C_z(p, q_2, M_0).$$

Lemma 2.4 ([5]). Let (u, v, w, z) be a solution to (1.1) defined on its maximal existence interval $[0, T_{max})$. Assume that $q_2 \geq 1$ and suppose

$$\begin{aligned} q_3 &\in [1, \frac{nq_2}{n-q_2}), & \text{if } q_2 \leq n, \\ q_3 &\in [1, \infty], & \text{if } q_2 > n. \end{aligned}$$

If there exist $M_1 > 0$ and $T \in (0, T_{max})$, such that

$$\|z(\cdot, t)\|_{L^{q_2}} \leq M_1, \quad t \in (0, T),$$

then there is a $C_v(q_2, q_3, M_1) > 0$ such that

$$\|\nabla v(\cdot, t)\|_{L^{q_3}} \leq C_v(q_2, q_3, M_1), \quad t \in (0, T).$$

Next, we state some well-known smoothing $L^p - L^q$ type estimates for the Neumann heat semigroup in Ω . We list them here for convenience and readers can find them in [16, Lemma 1.3].

Lemma 2.5. *Let $(e^{t\Delta})_{t>0}$ be the Neumann heat semigroup in Ω and let $\lambda_1 > 0$ be the first nonzero eigenvalue of $-\Delta$ with homogeneous Neumann boundary condition. Then there exist some constants c_i ($i = 1, 2, 3, 4$) depending only on Ω such that*

(i) *If $1 \leq q \leq p \leq \infty$, then*

$$\|e^{t\Delta} f\|_{L^p} \leq c_1 \left(1 + t^{-\frac{n}{2}(\frac{1}{q} - \frac{1}{p})}\right) e^{-\lambda_1 t} \|f\|_{L^q} \text{ for all } t > 0 \quad (2.3)$$

holds for all $f \in L^q(\Omega)$ satisfying $\int_{\Omega} f = 0$.

(ii) *If $1 \leq q \leq p \leq \infty$, then*

$$\|\nabla e^{t\Delta} f\|_{L^p} \leq c_2 \left(1 + t^{-\frac{1}{2} - \frac{n}{2}(\frac{1}{q} - \frac{1}{p})}\right) e^{-\lambda_1 t} \|f\|_{L^q} \text{ for all } t > 0 \quad (2.4)$$

is valid for all $f \in L^q(\Omega)$.

(iii) *If $2 \leq q \leq p < \infty$, then*

$$\|\nabla e^{t\Delta} f\|_{L^p} \leq c_3 \left(1 + t^{-\frac{n}{2}(\frac{1}{q} - \frac{1}{p})}\right) e^{-\lambda_1 t} \|\nabla f\|_{L^q} \text{ for all } t > 0 \quad (2.5)$$

is true for all $f \in W^{1,p}(\Omega)$.

(iv) *If $1 < q \leq p \leq \infty$, then*

$$\|e^{t\Delta} \nabla \cdot f\|_{L^p} \leq c_4 \left(1 + t^{-\frac{1}{2} - \frac{n}{2}(\frac{1}{q} - \frac{1}{p})}\right) e^{-\lambda_1 t} \|f\|_{L^q} \text{ for all } t > 0$$

is valid for all $f \in (W^{1,p}(\Omega))^n$.

3. Proof of Theorem 1.1

3.1. Boundedness of solution

In this section, we will study the boundedness of the solution to system (1.1) as stated in Theorem 1.1. Precisely, we have following results.

Lemma 3.1. *Suppose the conditions in Theorem 1.1 hold. Then system (1.1) has a unique non-negative global classical solution (u, v, w, z) , which satisfies*

$$\|u(\cdot, t)\|_{L^\infty} + \|v(\cdot, t)\|_{W^{1,\infty}} + \|w(\cdot, t)\|_{L^\infty} + \|z(\cdot, t)\|_{W^{1,\infty}} \leq C,$$

where $C > 0$ is a constant independent of t .

Before proving our main results on the boundedness of solutions, we first define a test function as follows:

$$\varphi(u) = \int_0^u \int_0^\rho \frac{(\sigma + 1)^{k-2}}{D(\sigma)} d\sigma d\rho, \quad u \geq 0. \quad (3.1)$$

Then using (1.5), one derives

$$C_1(u + 1)^{k+M-1} \leq \varphi(u) \leq C_2(u + 1)^{k+m-1} \quad \text{for all } u \geq 0, \quad (3.2)$$

where C_1 and C_2 are positive constants.

Lemma 3.2. Suppose the assumptions in Lemma 3.1 hold. Then the solution of system (1.1) satisfies

$$\frac{d}{dt} \int_{\Omega} \varphi(u) + \frac{1}{2} \int_{\Omega} (u+1)^{k-2} |\nabla u|^2 \leq C \int_{\Omega} (u+1)^{\ell} |\nabla v|^2 \quad (3.3)$$

for all $t \in (0, T_{\max})$, where φ is given by (3.1), $\ell = k + 2\alpha - 2$ and C is a positive constant independent of t .

Proof. Multiplying the first equation of (1.1) by $\varphi'(u)$, and integrating it over Ω by parts, we end up with

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} \varphi(u) &= \int_{\Omega} \nabla \cdot (D(u) \nabla u) \varphi'(u) - \int_{\Omega} \nabla \cdot (S(u) \nabla v) \varphi'(u) \\ &= - \int_{\Omega} D(u) \varphi''(u) |\nabla u|^2 + \int_{\Omega} S(u) \varphi''(u) \nabla u \cdot \nabla v \\ &= - \int_{\Omega} (u+1)^{k-2} |\nabla u|^2 + \int_{\Omega} \frac{S(u)}{D(u)} (u+1)^{k-2} \nabla u \cdot \nabla v. \end{aligned} \quad (3.4)$$

Using (1.4) and the Cauchy–Schwarz inequality, one derives

$$\begin{aligned} \int_{\Omega} \frac{S(u)}{D(u)} (u+1)^{k-2} \nabla u \cdot \nabla v &\leq c_1 \int_{\Omega} (u+1)^{k+\alpha-2} |\nabla u| |\nabla v| \\ &\leq -\frac{1}{2} \int_{\Omega} (u+1)^{k-2} |\nabla u|^2 + c_2 \int_{\Omega} (u+1)^{k+2\alpha-2} |\nabla v|^2. \end{aligned} \quad (3.5)$$

Then substituting (3.5) into (3.4), we obtain (3.3). $\square$

3.1.1. A priori estimates: $1 \leq n \leq 3$

When $1 \leq n \leq 3$, we can use Lemmas 2.3 and 2.4 to obtain enough regularity of v such that the term on the right hand side of (3.3) can be estimated directly. Hence we split our proof into the cases $1 \leq n \leq 3$ and $n \geq 4$. In fact, when $1 \leq n \leq 3$, we have the following results.

Lemma 3.3. Let $1 \leq n \leq 3$ and suppose the conditions in Lemma 3.1 hold. Assume that

$$\alpha < 1 + \frac{1}{n}.$$

Then for all $k > 1$, there exists a constant C independent of t such that

$$\|u(\cdot, t)\|_{L^k} \leq C. \quad (3.6)$$

Proof. When $1 \leq n \leq 3$ and $\alpha < 1 + \frac{1}{n}$, we combine (2.1), Lemmas 2.3 and 2.4 to find a constant γ with $\gamma > \frac{n}{n+1-n\alpha} > 0$, such that

$$\|\nabla v\|_{L^\gamma} \leq c_1. \quad (3.7)$$

Then using (3.7) and Hölder's inequality, from (3.3) one has

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} \varphi(u) + \frac{1}{2} \int_{\Omega} (u+1)^{k-2} |\nabla u|^2 &\leq c_2 \int_{\Omega} (u+1)^{\ell} |\nabla v|^2 \\ &\leq c_2 \int_{\Omega} (u+1)^{\frac{\ell\gamma}{\gamma-2}} \int_{\Omega} |\nabla v|^{\frac{\gamma-2}{\gamma}} \\ &\leq c_3 \int_{\Omega} (u+1)^{\frac{\ell\gamma}{\gamma-2}}. \end{aligned} \quad (3.8)$$

Using the Gagliardo–Nirenberg inequality and noting the boundedness of $\|(u+1)^{\frac{k}{2}}\|_{L^{\frac{2}{k}}}^2 = \|u+1\|_{L^1}^{\frac{k}{2}}$, we obtain

$$\begin{aligned} \|(u+1)^\ell\|_{L^{\frac{\gamma}{\gamma-2}}} &= \|(u+1)^{\frac{k}{2}}\|_{L^{\frac{2}{k}}}^{\frac{2\ell}{k}} \cdot \frac{\gamma}{\gamma-2} \\ &\leq c_4 \|\nabla(u+1)^{\frac{k}{2}}\|_{L^2}^{2\frac{\ell}{k}} \|(u+1)^{\frac{k}{2}}\|_{L^{\frac{2}{k}}}^{\frac{2\ell}{k}(1-\bar{\theta})} + c_4 \|(u+1)^{\frac{k}{2}}\|_{L^{\frac{2}{k}}}^{2\frac{\ell}{k}} \\ &\leq c_5 \|\nabla(u+1)^{\frac{k}{2}}\|_{L^2}^{2\frac{\ell}{k}\bar{\theta}} + c_6, \end{aligned} \quad (3.9)$$

where

$$\frac{\bar{\theta}}{k} = \frac{\frac{\ell}{2} - \frac{\gamma-2}{2\gamma}}{\frac{k}{2} + \frac{1}{n} - \frac{1}{2}} = \frac{\frac{k}{2} + \alpha - 1 + \frac{1}{\gamma} - \frac{1}{2}}{\frac{k}{2} + \frac{1}{n} - \frac{1}{2}} \in (0, 1),$$

due to $\gamma > \frac{n}{n+1-n\alpha}$. Then substituting (3.9) into (3.8) and using Young's inequality, we end up with

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} \varphi(u) + \frac{2}{k^2} \int_{\Omega} |\nabla(u+1)^{\frac{k}{2}}|^2 &\leq c_3 c_5 \|\nabla(u+1)^{\frac{k}{2}}\|_{L^2}^{2\frac{\ell}{k}\bar{\theta}} + c_3 c_6 \\ &\leq \frac{1}{k^2} \int_{\Omega} |\nabla(u+1)^{\frac{k}{2}}|^2 + c_7, \end{aligned}$$

which gives

$$\frac{d}{dt} \int_{\Omega} \varphi(u) + \frac{1}{k^2} \int_{\Omega} |\nabla(u+1)^{\frac{k}{2}}|^2 \leq c_7. \quad (3.10)$$

Moreover, using (3.2), the Gagliardo–Nirenberg inequality and Young's inequality, we find a constant

$\hat{\theta} = \frac{\frac{k}{2} - \frac{k}{2(k+m-1)}}{\frac{k}{2} + \frac{1}{n} - \frac{1}{2}} \in (0, 1)$ such that

$$\begin{aligned} \int_{\Omega} \varphi(u)^{\frac{k}{k+m-1}} &\leq c_8 \int_{\Omega} u^{k+m-1} \\ &= c_8 \|(u+1)^{\frac{k}{2}}\|_{L^2}^{2(k+m-1)} \end{aligned}$$

3.1.2. A priori estimate: $n \geq 4$

Next, we will show that (3.6) still holds for $n \geq 4$ if $\alpha < \frac{4}{n}$. In this case, we cannot directly obtain enough regularity of $\mathbf{v}$ as used in Lemma 3.3. Instead, we will use the coupled energy estimate motivated by [24] to control the term on the right hand side of (3.3). First, we derive the regularities of the solution on $\mathbf{z}$ and $\mathbf{v}$ as follows.

Lemma 3.4. Assume that $n \geq 4$ and the conditions in Lemma 3.1 hold. Let $(\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z})$ be a solution of system (1.1). Then there exists a constant $C > 0$ independent of t such that

$$\|\mathbf{z}\|_{L^q} + \|\nabla \mathbf{z}\|_{L^r} + \|\nabla \mathbf{v}\|_{L^{\tilde{r}}} \leq C \quad (3.13)$$

for all $q \in [1, \frac{n}{n-2})$, $r \in [1, \frac{n}{n-1})$ and $\tilde{r} \in [1, \frac{n}{n-3})$.

Proof. Noting the boundedness of $\|\mathbf{u} + 1\|_{L^1}$, we obtain (3.13) directly by combining Lemmas 2.3 and 2.4. $\square$

For the readers' convenience, we will use a similar argument as in [24] to show that when $n \geq 4$ and $\alpha < \frac{4}{n}$, there exist certain parameters that can be used in Lemma 3.8, appropriately.

Lemma 3.5 (Parameters Conditions). Let $n \geq 4$ and $\alpha < \frac{4}{n}$. Suppose that $r \in [1, \frac{n}{n-1})$, $q \in [1, \frac{n}{n-2})$, $\tilde{r} \in [1, \frac{n}{n-3})$, $\kappa_1 > n$, $\kappa_2 > n$ and $k > \max\{n, 2 - m, 3 - 2\alpha, \frac{2}{r}\kappa_1, \frac{2}{\tilde{r}}\kappa_2\}$. Then we have

$$\frac{k + 2\alpha - 2 - \frac{1}{p_1}}{k + \frac{2}{n} - 1} + \frac{\frac{2}{\tilde{r}} - 1 + \frac{1}{p_1}}{\frac{2}{\tilde{r}}\kappa_2 + \frac{2}{n} - 1} < 1, \quad (3.14)$$

$$\frac{2 - \frac{1}{p_2}}{k + \frac{2}{n} - 1} + \frac{\frac{2(\kappa_1-1)}{r} - 1 + \frac{1}{p_2}}{\frac{2}{r}\kappa_1 + \frac{2}{n} - 1} < 1, \quad (3.15)$$

and

$$\frac{\frac{2n}{(n+q)r} - \frac{1}{p_3}}{\frac{2}{r}\kappa_1 + \frac{2}{n} - 1} + \frac{\frac{2(\kappa_2-1)}{\tilde{r}} - 1 + \frac{1}{p_3}}{\frac{2}{\tilde{r}}\kappa_2 + \frac{2}{n} - 1} < 1 \quad (3.16)$$

where $p_1 = \frac{n}{n-2}$, $p_2 = \frac{n^2}{2(n-2)}$ and $p_3 = n$.

Proof. Define

$$g_1(r, q, \tilde{r}, \kappa_1, \kappa_2) := \frac{k + 2\alpha - 2 - \frac{1}{p_1}}{k + \frac{2}{n} - 1} + \frac{\frac{2}{\tilde{r}} - 1 + \frac{1}{p_1}}{\frac{2}{\tilde{r}}\kappa_2 + \frac{2}{n} - 1},$$

$$g_2(r, q, \tilde{r}, \kappa_1, \kappa_2) := \frac{2 - \frac{1}{p_2}}{k + \frac{2}{n} - 1} + \frac{\frac{2(\kappa_1-1)}{r} - 1 + \frac{1}{p_2}}{\frac{2}{r}\kappa_1 + \frac{2}{n} - 1},$$

$$g_3(r, q, \tilde{r}, \kappa_1, \kappa_2) := \frac{\frac{2n}{(n+q)r} - \frac{1}{p_3}}{\frac{2}{r}\kappa_1 + \frac{2}{n} - 1} + \frac{\frac{2(\kappa_2-1)}{\tilde{r}} - 1 + \frac{1}{p_3}}{\frac{2}{\tilde{r}}\kappa_2 + \frac{2}{n} - 1}$$

for $k > n$, $\kappa_1 > n$, $\kappa_2 > n$, $r \in [1, \frac{n}{n-1})$, $q \in [1, \frac{n}{n-2})$ and $\tilde{r} \in [1, \frac{n}{n-3})$. Letting

$$\kappa_1^* = \frac{r}{2}k \text{ and } \kappa_2^* = \frac{\tilde{r}}{2}k,$$

we have

$$g_2 \left(\frac{n}{n-1}, \frac{n}{n-2}, \frac{n}{n-3}, \kappa_1^*, \kappa_2^* \right) = \frac{2 - \frac{1}{p_2} + k - \frac{2(n-1)}{n} - 1 + \frac{1}{p_2}}{k + \frac{2}{n} - 1} = 1 \quad (3.17)$$

and

$$\begin{aligned} g_3 \left(\frac{n}{n-1}, \frac{n}{n-2}, \frac{n}{n-3}, \kappa_1^*, \kappa_2^* \right) &= \frac{\frac{2n}{(n + \frac{n}{n-2})^{\frac{n}{n-1}}} - \frac{1}{p_3} + k - \frac{2(n-3)}{n} - 1 + \frac{1}{p_3}}{k + \frac{2}{n} - 1} \\ &= \frac{\frac{2(n-2)}{n} + k - \frac{2(n-3)}{n} - 1}{k + \frac{2}{n} - 1} \\ &= 1. \end{aligned} \quad (3.18)$$

Furthermore, due to $n \geq 4$, after some calculations, we derive

$$\begin{aligned} &\frac{\partial g_2}{\partial \kappa_1} \left(\frac{n}{n-1}, \frac{n}{n-2}, \frac{n}{n-3}, \kappa_1, \kappa_2 \right) \\ &= \frac{\frac{2(n-1)}{n} \left(\frac{2(n-1)}{n} \kappa_1 + \frac{2}{n} - 1 \right) - \frac{2(n-1)}{n} \frac{2(n-1)}{n} (\kappa_1 - 1) - 1 + \frac{2(n-2)}{n^2}}{\left(\frac{2(n-1)}{n} \kappa_1 + \frac{2}{n} - 1 \right)^2} \\ &= - \frac{2(n-1)}{n \left(\frac{2(n-1)}{n} \kappa_1 + \frac{2}{n} - 1 \right)^2} \left(\frac{2(n-1)}{n} \kappa_1 + \frac{2}{n} - 1 - \frac{2(n-1)}{n} (\kappa_1 - 1) + 1 - \frac{2(n-2)}{n^2} \right) \\ &= - \frac{4(n-1)}{n \left(\frac{2(n-1)}{n} \kappa_1 + \frac{2}{n} - 1 \right)^2} \cdot \frac{n^2 - n + 2}{n^2} > 0, \end{aligned}$$

and

$$\begin{aligned} &\frac{\partial g_3}{\partial \kappa_2} \left(\frac{n}{n-1}, \frac{n}{n-2}, \frac{n}{n-3}, \kappa_1, \kappa_2 \right) \\ &= \frac{\frac{2(n-3)}{n} \left(\frac{2(n-3)}{n} \kappa_2 + \frac{2}{n} - 1 \right) - \frac{2(n-3)}{n} \frac{2(n-3)}{n} (\kappa_2 - 1) - 1 + \frac{1}{n}}{\left(\frac{2(n-3)}{n} \kappa_2 + \frac{2}{n} - 1 \right)^2} \\ &= - \frac{2(n-3)}{n \left(\frac{2(n-3)}{n} \kappa_2 + \frac{2}{n} - 1 \right)^2} \left(\frac{2(n-3)}{n} \kappa_2 + \frac{2}{n} - 1 - \frac{2(n-3)}{n} (\kappa_2 - 1) + 1 - \frac{1}{n} \right) \\ &= - \frac{2(n-3)}{n \left(\frac{2(n-3)}{n} \kappa_2 + \frac{2}{n} - 1 \right)^2} \left(2 - \frac{3}{n} \right) > 0, \end{aligned}$$

which together with (3.17) and (3.18) imply that there exist $\tilde{\kappa}_1 \in [2, \kappa_1^*)$ and $\tilde{\kappa}_2 \in [2, \kappa_2^*)$, such that taking $\kappa_1 \in [\tilde{\kappa}_1, \kappa_1^*)$ and $\kappa_2 \in [\tilde{\kappa}_2, \kappa_2^*)$, we have

$$g_2 \left(\frac{n}{n-1}, \frac{n}{n-2}, \frac{n}{n-3}, \kappa_1, \kappa_2 \right) < 1 \text{ and } g_3 \left(\frac{n}{n-1}, \frac{n}{n-2}, \frac{n}{n-3}, \kappa_1, \kappa_2 \right) < 1.$$

Besides, one can check that

$$\begin{aligned} g_1 \left(\frac{n}{n-1}, \frac{n}{n-2}, \frac{n}{n-3}, \kappa_1^*, \kappa_2^* \right) &= \frac{k + 2(\alpha + \beta) - 2 - \frac{1}{p_1} + \frac{2(n-3)}{n} - 1 + \frac{1}{p_1}}{k + \frac{2}{n} - 1} \\ &< \frac{k + \frac{8}{n} - \frac{6}{n} - 1}{k + \frac{2}{n} - 1} \\ &= 1. \end{aligned}$$

Then applying the continuity argument, we can find some $\kappa_1 \in [\bar{\kappa}_1, \kappa_1^*)$ and $\kappa_2 \in [\bar{\kappa}_2, \kappa_2^*)$, such that

$$g_1 \left(\frac{n}{n-1}, \frac{n}{n-2}, \frac{n}{n-3}, \kappa_1, \kappa_2 \right) < 1.$$

Again invoking the continuity argument, and choosing r, q and $\tilde{r}$ close enough to $\frac{n}{n-1}, \frac{n}{n-2}, \frac{n}{n-3}$ respectively, we complete the proof. $\square$

Next, we will show that if $\alpha < \frac{4}{n}$, then (3.6) still holds for $n \geq 4$ by studying the coupled energy estimate $\int_{\Omega} \varphi(u) + |\nabla z|^{2\kappa_1} + |\nabla v|^{2\kappa_2}$. To this end, we first establish the following two basic inequalities (3.19) and (3.26) by using similar arguments as in [24,25].

Lemma 3.6. Suppose the assumptions in Lemma 3.4 hold. Then the solution of (1.1) satisfies

$$\frac{d}{dt} \int_{\Omega} |\nabla z|^{2\kappa_1} + \int_{\Omega} |\nabla z|^{2\kappa_1} + \frac{\kappa_1 - 1}{\kappa_1} \|\nabla |\nabla z|^{\kappa_1}\|_{L^2}^2 \leq C \int_{\Omega} (u+1)^2 |\nabla z|^{2(\kappa_1-1)} + C \quad (3.19)$$

for all $t \in (0, T_{\max})$, where $\kappa_1 > n$ and $C > 0$ is a constant independent of t .

Proof. Differentiating the fourth equation of (1.1) once and then multiplying it by ∇z , applying the identity $\frac{1}{2} \Delta |\nabla z|^2 = \nabla \Delta z \cdot \nabla z + |D^2 z|^2$ and $\frac{1}{n} |\Delta z|^2 \leq |D^2 z|^2$, one has

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} |\nabla z|^2 &= \nabla \Delta z \cdot \nabla z - |\nabla z|^2 + \nabla u \cdot \nabla z \\ &\leq \frac{1}{2} \Delta |\nabla z|^2 - \frac{1}{n} |\Delta z|^2 - |\nabla z|^2 + \nabla u \cdot \nabla z. \end{aligned} \quad (3.20)$$

We multiply (3.20) by $2\kappa_1 |\nabla z|^{2(\kappa_1-1)}$ and integrate it by parts to get

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} |\nabla z|^{2\kappa_1} &+ \frac{2\kappa_1}{n} \int_{\Omega} |\Delta z|^2 |\nabla z|^{2(\kappa_1-1)} + 2\kappa_1 \int_{\Omega} |\nabla z|^{2\kappa_1} \\ &\leq \kappa_1 \int_{\Omega} |\nabla z|^{2(\kappa_1-1)} \Delta |\nabla z|^2 + 2\kappa_1 \int_{\Omega} |\nabla z|^{2(\kappa_1-1)} \nabla u \cdot \nabla z \\ &= \kappa_1 \int_{\partial\Omega} |\nabla z|^{2(\kappa_1-1)} \frac{\partial |\nabla z|^2}{\partial \nu} - \frac{4(\kappa_1-1)}{\kappa_1} \|\nabla |\nabla z|^{\kappa_1}\|_{L^2}^2 + 2\kappa_1 \int_{\Omega} |\nabla z|^{2(\kappa_1-1)} \nabla u \cdot \nabla z, \end{aligned}$$

which yields

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} |\nabla z|^{2\kappa_1} &+ \frac{2\kappa_1}{n} \int_{\Omega} |\Delta z|^2 |\nabla z|^{2(\kappa_1-1)} + 2\kappa_1 \int_{\Omega} |\nabla z|^{2\kappa_1} + \frac{4(\kappa_1-1)}{\kappa_1} \int_{\Omega} |\nabla |\nabla z|^{\kappa_1}|^2 \\ &\leq \kappa_1 \int_{\partial\Omega} |\nabla z|^{2(\kappa_1-1)} \frac{\partial |\nabla z|^2}{\partial \nu} + 2\kappa_1 \int_{\Omega} |\nabla z|^{2(\kappa_1-1)} \nabla u \cdot \nabla z. \end{aligned} \quad (3.21)$$

Next, we estimate the first term on the right hand side of (3.21). In fact, using the following trace inequality [29, Remark 52.9]

$$\|\square\|_{L^2(\partial\Omega)} \leq \varepsilon \|\nabla \square\|_{L^2(\Omega)} + C_\varepsilon \|\square\|_{L^2(\Omega)} \quad \text{for any } \varepsilon > 0$$

and the fact $\frac{\partial|\nabla z|^2}{\partial\nu} \leq 2\delta|\nabla z|^2$ on $\partial\Omega$ for some constants $\delta > 0$ (cf. [30, Lemma 4.2] and [25]), we have

$$\begin{aligned} \int_{\partial\Omega} |\nabla z|^{2(\kappa_1-1)} \frac{\partial|\nabla z|^2}{\partial\nu} &\leq 2\delta \int_{\partial\Omega} |\nabla z|^{2\kappa_1} \\ &= 2\delta \|\nabla z\|_{L^2(\partial\Omega)}^{2\kappa_1} \\ &\leq \frac{\kappa_1-1}{\kappa_1^2} \|\nabla|\nabla z|^{\kappa_1}\|_{L^2}^2 + c_1 \|\nabla z\|_{L^2}^{2\kappa_1}. \end{aligned} \quad (3.22)$$

Moreover, noting $\kappa_1 > n > \frac{n}{2}$ and using (3.13), then we can find $\tilde{\theta} = \frac{\frac{\kappa_1}{2}-\frac{1}{2}}{\frac{\kappa_1}{2}+\frac{1}{n}-\frac{1}{2}} \in (0, 1)$ such that the Gagliardo–Nirenberg inequality and Young’s inequality can be used to derive

$$\begin{aligned} c_1 \|\nabla z\|_{L^2}^{2\kappa_1} &\leq c_2 \|\nabla|\nabla z|^{\kappa_1}\|_{L^2}^{2\tilde{\theta}} \|\nabla z\|_{L^{\frac{f}{\kappa_1}}}^{2(1-\tilde{\theta})} + c_3 \|\nabla z\|_{L^{\frac{f}{\kappa_1}}}^{2\frac{f}{\kappa_1}} \\ &\leq c_4 \|\nabla|\nabla z|^{\kappa_1}\|_{L^2}^{2\tilde{\theta}} + c_5 \\ &\leq \frac{\kappa_1-1}{\kappa_1^2} \|\nabla|\nabla z|^{\kappa_1}\|_{L^2}^2 + c_6. \end{aligned} \quad (3.23)$$

Then the collection of (3.21)–(3.23) entails

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} |\nabla z|^{2\kappa_1} + \frac{2\kappa_1}{n} \int_{\Omega} |\Delta z|^2 |\nabla z|^{2(\kappa_1-1)} &+ \int_{\Omega} |\nabla z|^{2\kappa_1} + \frac{2(\kappa_1-1)}{\kappa_1} \|\nabla|\nabla z|^{\kappa_1}\|_{L^2}^2 \\ &\leq 2\kappa_1 \int_{\Omega} |\nabla z|^{2(\kappa_1-1)} \nabla u \cdot \nabla z + c_7. \end{aligned} \quad (3.24)$$

Moreover, we can use Hölder’s inequality to obtain

$$\begin{aligned} 2\kappa_1 \int_{\Omega} |\nabla z|^{2(\kappa_1-1)} \nabla u \cdot \nabla z &= -2\kappa_1 \int_{\Omega} (u+1) |\nabla z|^{2(\kappa_1-1)} \Delta z - 2(\kappa_1-1)\kappa_1 \int_{\Omega} (u+1) |\nabla z|^{2(\kappa_1-2)} \nabla z \cdot \nabla |\nabla z|^2 \\ &\leq \frac{2\kappa_1}{n} \int_{\Omega} |\Delta z|^2 |\nabla z|^{2(\kappa_1-1)} + \frac{\kappa_1-1}{\kappa_1} \|\nabla|\nabla z|^{\kappa_1}\|_{L^2}^2 + c_8 \int_{\Omega} (u+1)^2 |\nabla z|^{2(\kappa_1-1)}. \end{aligned} \quad (3.25)$$

Substituting (3.25) into (3.24), we obtain (3.19) directly. $\square$

Lemma 3.7. *Let the assumptions in Lemma 3.4 hold true. Let $\kappa_2 > n$. Then there exists a constant $C > 0$ independent of t such that the solution of (1.1) satisfies*

$$\frac{d}{dt} \int_{\Omega} |\nabla v|^{2\kappa_2} + \int_{\Omega} |\nabla v|^{2\kappa_2} + \frac{\kappa_2-1}{\kappa_2} \|\nabla|\nabla v|^{\kappa_2}\|_{L^2}^2 \leq C \int_{\Omega} z^2 |\nabla v|^{2(\kappa_2-1)} + C \quad (3.26)$$

for all $t \in (0, T_{\max})$.

Proof. Differentiating the second equation of (1.1) once and then multiplying the result by ∇v , we deduce

$$\frac{1}{2} \frac{d}{dt} |\nabla v|^2 \leq \frac{1}{2} \Delta |\nabla v|^2 - \frac{1}{n} |\Delta v|^2 + \nabla(wz) \cdot \nabla v. \quad (3.27)$$

Letting $\kappa_2 > n > \tilde{r}$ and using a similar argument as in Lemma 3.6, we multiply (3.27) by $2\kappa_2|\nabla v|^{2(\kappa_2-1)}$ and integrate it to show

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} |\nabla v|^{2\kappa_2} + \frac{2\kappa_2}{n} \int_{\Omega} |\Delta v|^2 |\nabla v|^{2(\kappa_2-1)} + \int_{\Omega} |\nabla v|^{2\kappa_2} + \frac{2(\kappa_2-1)}{\kappa_2} \|\nabla |\nabla v|^{\kappa_2}\|_{L^2}^2 \\ & \leq 2\kappa_2 \int_{\Omega} |\nabla v|^{2(\kappa_2-1)} \nabla(wz) \cdot \nabla v + c_1. \end{aligned} \quad (3.28)$$

Applying Hölder's inequality and noting the fact $\|w\|_{L^\infty} \leq c_2$, we derive that

$$\begin{aligned} & 2\kappa_2 \int_{\Omega} |\nabla v|^{2(\kappa_2-1)} \nabla(wz) \cdot \nabla v \\ & = -2\kappa_2 \int_{\Omega} wz |\nabla v|^{2(\kappa_2-1)} \Delta v - 2(\kappa_2-1)\kappa_2 \int_{\Omega} wz |\nabla v|^{2(\kappa_2-2)} \nabla v \cdot \nabla |\nabla v|^2 \\ & \leq \frac{2\kappa_2}{n} \int_{\Omega} |\Delta v|^2 |\nabla v|^{2(\kappa_2-1)} + \frac{\kappa_2(\kappa_2-1)}{4} \int_{\Omega} |\nabla |\nabla v|^{\kappa_2}|^2 + c_3 \int_{\Omega} w^2 z^2 |\nabla v|^{2(\kappa_2-1)} \\ & \leq \frac{2\kappa_2}{n} \int_{\Omega} |\Delta v|^2 |\nabla v|^{2(\kappa_2-1)} + \frac{\kappa_2-1}{\kappa_2} \|\nabla |\nabla v|^{\kappa_2}\|_{L^2}^2 + c_4 \int_{\Omega} z^2 |\nabla v|^{2(\kappa_2-1)}, \end{aligned}$$

which together with (3.28) gives (3.26). $\square$

Next, we will combine Lemmas 3.2, 3.6 and 3.7 to obtain the following results.

Lemma 3.8. Assume that the conditions in Lemma 3.4 hold. If $\alpha < \frac{4}{n}$, then it holds that

$$\|u(\cdot, t)\|_{L^k} \leq C$$

for all $k > 1$, where C is a positive constant.

Proof. Combining (3.3), (3.19) and (3.26), and using Hölder's inequality, one has

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} \varphi(u) + |\nabla z|^{2\kappa_1} + |\nabla v|^{2\kappa_2} + \frac{2}{k^2} \|\nabla(u+1)^{\frac{k}{2}}\|_{L^2}^2 + \frac{\kappa_1-1}{\kappa_1} \|\nabla |\nabla z|^{\kappa_1}\|_{L^2}^2 \\ & + \frac{\kappa_2-1}{\kappa_2} \|\nabla |\nabla v|^{\kappa_2}\|_{L^2}^2 + \int_{\Omega} |\nabla z|^{2\kappa_1} + \int_{\Omega} |\nabla v|^{2\kappa_2} \\ & \leq c_1 \int_{\Omega} (u+1)^\ell |\nabla v|^2 + c_2 \int_{\Omega} (u+1)^2 |\nabla z|^{2(\kappa_1-1)} + c_3 \int_{\Omega} z^2 |\nabla v|^{2(\kappa_2-1)} + c_4 \\ & \leq c_1 \|(u+1)^\ell\|_{L^{p_1}} \|\nabla v\|_{L^{p'_1}}^2 + c_2 \|(u+1)^2\|_{L^{p_2}} \|\nabla z\|_{L^{p'_2}}^{2(\kappa_1-1)} \\ & + c_3 \|z^2\|_{L^{p_3}} \|\nabla v\|_{L^{p'_3}}^{2(\kappa_2-1)} + c_4, \end{aligned} \quad (3.29)$$

where $\ell = k - 2 + 2\alpha$ and

$$\begin{aligned} p_1 &= \frac{n}{n-2}, & p'_1 &= \frac{n}{2}, \\ p_2 &= \frac{n^2}{2(n-2)}, & p'_2 &= \frac{n^2}{n^2-2n+4}, \\ p_3 &= n, & p'_3 &= \frac{n}{n-1}. \end{aligned}$$

Next, we will estimate the terms on the right hand side of (3.29). Due to the facts that $\alpha < \frac{4}{n} \leq 1$ and $p_1 = \frac{n}{n-2} > 1$, we have $\frac{2}{k} < \frac{2\ell p_1}{k} < \frac{2n}{n-2}$. Then using the Gagliardo–Nirenberg inequality and noting the

boundedness of $\|(u+1)^{\frac{k}{2}}\|_{L^{\frac{2}{k}}}$, one has

$$\begin{aligned} \|(u+1)^{\ell}\|_{L^{p_1}} &= \|(u+1)^{\frac{k}{2}}\|_{L^{\frac{2}{k}}}^{\frac{2\ell}{k}} \\ &\leq c_5 \|\nabla(u+1)^{\frac{k}{2}}\|_{L^2}^{2\frac{\theta_1}{k}} \|(u+1)^{\frac{k}{2}}\|_{L^{\frac{2}{k}}}^{2(1-\theta_1)} + c_6 \|(u+1)^{\frac{k}{2}}\|_{L^{\frac{2}{k}}}^{2\frac{\ell}{k}} \\ &\leq c_7 \|\nabla(u+1)^{\frac{k}{2}}\|_{L^2}^{2a_1} + c_8, \end{aligned} \quad (3.30)$$

where $\theta_1 = \frac{k - \frac{2}{p_1}}{k + \frac{2}{n} - 1} \in (0, 1)$ and

$$a_1 = \frac{\theta_1}{k} = \frac{1 - \frac{1}{p_1}}{k + \frac{2}{n} - 1}.$$

Similarly, because of $\kappa_2 > n > \tilde{r}$ and $p'_1 = \frac{n}{2}$, we can derive $\frac{\tilde{r}}{\kappa_2} < \frac{2p'_1}{\kappa_2} < \frac{2n}{n-2}$. Then we can apply the Gagliardo–Nirenberg inequality and use the fact $\|\nabla v\|_{L^{\frac{\tilde{r}}{\kappa_2}}}^{\frac{\tilde{r}}{\kappa_2}} = \|\nabla v\|_{L^{\tilde{r}}}^{\frac{\tilde{r}}{\kappa_2}} \leq c_9$ to derive

$$\begin{aligned} \|\nabla v\|_{L^{p'_1}}^2 &= \|\nabla v\|_{L^{\frac{2}{\kappa_2}}}^{\frac{2}{\kappa_2}} \\ &\leq c_{10} \|\nabla|\nabla v|^{\kappa_2}\|_{L^2}^{2\frac{\theta_2}{\kappa_2}} \|\nabla v\|_{L^{\frac{2}{\kappa_2}}}^{\frac{2}{\kappa_2}(1-\theta_2)} + c_{10} \|\nabla v\|_{L^{\frac{\tilde{r}}{\kappa_2}}}^{\frac{2}{\kappa_2}} \\ &\leq c_{11} \|\nabla|\nabla v|^{\kappa_2}\|_{L^2}^{2b_1} + c_{12} \end{aligned} \quad (3.31)$$

with $\theta_2 = \frac{\frac{2\kappa_2}{\tilde{r}} - \frac{\kappa_2}{p'_1}}{\frac{2\kappa_2}{\tilde{r}} + \frac{2}{n} - 1} \in (0, 1)$ and

$$b_1 = \frac{\frac{2}{\tilde{r}} - 1 + \frac{1}{p_1}}{\frac{2\kappa_2}{\tilde{r}} + \frac{2}{n} - 1}.$$

Thanks to $p_2 = \frac{n^2}{2(n-2)} > 1$ and $k > n$, we have $\frac{2}{k} < \frac{4p_2}{k} < \frac{2n}{n-2}$. Using the Gagliardo–Nirenberg inequality again, one has

$$\begin{aligned} \|(u+1)^2\|_{L^{p_2}} &= \|(u+1)^{\frac{k}{2}}\|_{L^{\frac{4}{k}}}^{\frac{4}{k}} \\ &\leq c_{13} \|\nabla(u+1)^{\frac{k}{2}}\|_{L^2}^{\frac{4}{k}\theta_3} \|(u+1)^{\frac{k}{2}}\|_{L^{\frac{2}{k}}}^{\frac{4}{k}(1-\theta_3)} + c_{14} \|(u+1)^{\frac{k}{2}}\|_{L^{\frac{2}{k}}}^{\frac{4}{k}} \\ &\leq c_{15} \|\nabla(u+1)^{\frac{k}{2}}\|_{L^2}^{2a_2} + c_{16}, \end{aligned} \quad (3.32)$$

where $\theta_3 = \frac{k - \frac{2}{p_2}}{k + \frac{2}{n} - 1} \in (0, 1)$ and

$$a_2 = \frac{2 - \frac{1}{p_2}}{k + \frac{2}{n} - 1}.$$

Noting that $\frac{r}{\kappa_1} < \frac{2(\kappa_1-1)p'_2}{\kappa_1} < \frac{2n}{n-2}$ due to $p'_2 = \frac{n^2}{n^2-2n+4} > 1$, $\kappa_1 > n$ and $r < \frac{n}{n-1} < 2$, and using the Gagliardo–Nirenberg inequality, we derive

$$\begin{aligned} \|\nabla z\|_{L^{p'_2}}^{2(\kappa_1-1)} &= \|\nabla z\|_{L^{\frac{2(\kappa_1-1)p'_2}{\kappa_1}}}^{2\frac{\kappa_1-1}{\kappa_1}} \\ &\leq c_{17} \|\nabla|\nabla z|^{\kappa_1}\|_{L^2}^{2\frac{\kappa_1-1}{\kappa_1}\theta_4} \|\nabla z\|_{L^{\frac{r}{\kappa_1}}}^{2\frac{\kappa_1-1}{\kappa_1}(1-\theta_4)} + c_{18} \|\nabla z\|_{L^{\frac{r}{\kappa_1}}}^{2\frac{\kappa_1-1}{\kappa_1}} \\ &\leq c_{19} \|\nabla|\nabla z|^{\kappa_1}\|_{L^2}^{2b_2} + c_{20}, \end{aligned} \quad (3.33)$$

where $\theta_4 = \frac{\frac{2\kappa_1}{r} - \frac{\kappa_1}{(\kappa_1-1)p'_2}}{\frac{2\kappa_1}{r} + \frac{2}{n} - 1} \in (0, 1)$ and

$$b_2 = \frac{\frac{2(\kappa_1-1)}{r} - 1 + \frac{1}{p_2}}{\frac{2\kappa_1}{r} + \frac{2}{n} - 1}.$$

The fact $p_3 = n \geq \frac{n}{n-2} > q$ implies $2p_3(n - \frac{2np_3}{n+q}) < \frac{2n^2p_3}{n+q}$, which along with the Gagliardo–Nirenberg inequality gives

$$\begin{aligned} \|z^2\|_{L^{p_3}} &= \|z\|_{L^{2p_3}}^2 \leq c_{21} \|\nabla z\|_{L^{\frac{2n}{n+q}}}^{\frac{2n}{n+q}} \|z\|_{L^q}^{\frac{2q}{n+q}} + c_{22} \|z\|_{L^q}^{\frac{2q}{n+q}} \\ &\leq c_{23} \|\nabla z\|_{L^{\frac{2n}{n+q}}}^{\frac{2n}{n+q}} + c_{24}. \end{aligned}$$

Furthermore, one has $\frac{r}{\kappa_1} < \frac{2np_3}{(n+q)\kappa_1} < \frac{2n}{n-2}$ by noting $\kappa_1 > n$, $r < n$ and $p_3 = n > q$. Then using the Gagliardo–Nirenberg inequality, we obtain

$$\begin{aligned} \|\nabla z\|_{L^{\frac{2np_3}{n+q}}}^{\frac{2n}{n+q}} &= \|\nabla z\|_{L^{\frac{2np_3}{(n+q)\kappa_1}}}^{\frac{2n}{n+q}} \\ &\leq c_{25} \|\nabla|\nabla z|^{\kappa_1}\|_{L^2}^{\frac{2n}{(n+q)\kappa_1}\theta_5} \|\nabla z\|_{L^{\frac{r}{\kappa_1}}}^{\frac{2n}{(n+q)\kappa_1}(1-\theta_5)} + c_{26} \|\nabla z\|_{L^{\frac{r}{\kappa_1}}}^{\frac{2n}{(n+q)\kappa_1}}, \end{aligned}$$

where $\theta_5 = \frac{\frac{2\kappa_1}{r} - \frac{(n+q)\kappa_1}{np_3}}{\frac{2\kappa_1}{r} + \frac{2}{n} - 1} \in (0, 1)$. Then combining the above two inequalities, we end up with

$$\|z^2\|_{L^{p_3}} \leq c_{27} \|\nabla|\nabla z|^{\kappa_1}\|_{L^2}^{2a_3} + c_{28}, \quad (3.34)$$

where

$$a_3 = \frac{\frac{2n}{(n+q)r} - \frac{1}{p_3}}{\frac{2\kappa_1}{r} + \frac{2}{n} - 1}.$$

In view of $\kappa_2 > n$, $p'_3 = \frac{n}{n-1}$ and $\tilde{r} < \frac{n}{n-3}$, we see that $\frac{\tilde{r}}{\kappa_2} < \frac{2(\kappa_2-1)p'_3}{\kappa_2} < \frac{2n}{n-2}$. The application of the Gagliardo–Nirenberg inequality gives

$$\begin{aligned} \|\nabla v\|_{L^{p'_3}}^{2(\kappa_2-1)} &= \|\nabla v\|_{L^{\frac{2(\kappa_2-1)p'_3}{\kappa_2}}}^{\frac{2(\kappa_2-1)}{\kappa_2}} \\ &\leq c_{29} \|\nabla|\nabla v|^{\kappa_2}\|_{L^2}^{2\frac{\kappa_2-1}{\kappa_2}\theta_6} \|\nabla v\|_{L^{\frac{\tilde{r}}{\kappa_2}}}^{2\frac{\kappa_2-1}{\kappa_2}(1-\theta_6)} + c_{30} \|\nabla v\|_{L^{\frac{\tilde{r}}{\kappa_2}}}^{2\frac{\kappa_2-1}{\kappa_2}}, \end{aligned}$$

with $\theta_6 = \frac{\frac{2\kappa_2}{\tilde{r}} - \frac{\kappa_2}{(\kappa_2-1)p'_3}}{\frac{2\kappa_2}{\tilde{r}} + \frac{2}{n} - 1} \in (0, 1)$. Consequently, one has

$$\|\nabla v\|_{L^{p'_3}}^{2(\kappa_2-1)} \leq c_{31} \|\nabla|\nabla v|^{\kappa_2}\|_{L^2}^{2b_3} + c_{32}, \quad (3.35)$$

where

$$b_3 = \frac{\frac{2(\kappa_2-1)}{\tilde{r}} - 1 + \frac{1}{p_3}}{\frac{2\kappa_2}{\tilde{r}} + \frac{2}{n} - 1}.$$

Moreover, using a similar argument as in (3.11) and noting $k > 2 - m$, one derives

$$\int_{\Omega} \varphi(u) \leq \frac{1}{k^2} \int_{\Omega} |\nabla(u+1)|^{\frac{k}{2}} + c_{33}. \quad (3.36)$$

From (3.14)–(3.16) we know $a_i + b_i < 1$, $i = 1, 2, 3$. Then substituting (3.30)–(3.36) into (3.29) and then invoking Young's inequality, we obtain

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} \varphi(u) + |\nabla z|^{2\kappa_1} + |\nabla v|^{2\kappa_2} + \frac{1}{k^2} \|\nabla(u+1)\|_{L^2}^{\frac{k}{2}} + \frac{\kappa_1 - 1}{\kappa_1} \|\nabla|\nabla z|^{\kappa_1}\|_{L^2}^2 \\ & + \frac{\kappa_2 - 1}{\kappa_2} \|\nabla|\nabla v|^{\kappa_2}\|_{L^2}^2 + \int_{\Omega} \varphi(u) + \int_{\Omega} |\nabla z|^{2\kappa_1} + \int_{\Omega} |\nabla v|^{2\kappa_2} \\ & \leq c_{34} \|\nabla(u+1)\|_{L^2}^{\frac{k}{2}} \|\nabla|\nabla v|^{\kappa_2}\|_{L^2}^{2b_1} + c_{35} \|\nabla(u+1)\|_{L^2}^{\frac{k}{2}} \|\nabla|\nabla z|^{\kappa_1}\|_{L^2}^{2b_2} \\ & + c_{36} \|\nabla|\nabla z|^{\kappa_1}\|_{L^2}^{2a_3} \|\nabla|\nabla v|^{\kappa_2}\|_{L^2}^{2b_3} + c_{37} \\ & \leq \frac{1}{k^2} \|\nabla(u+1)\|_{L^2}^{\frac{k}{2}} + \frac{\kappa_1 - 1}{\kappa_1} \|\nabla|\nabla z|^{\kappa_1}\|_{L^2}^2 + \frac{\kappa_2 - 1}{\kappa_2} \|\nabla|\nabla v|^{\kappa_2}\|_{L^2}^2 + c_{38}, \end{aligned}$$

which implies

$$y'(t) + c_{39}y^{\rho}(t) \leq c_{40},$$

where $\rho = \frac{k}{k+m-1}$ and

$$y(t) = \int_{\Omega} \varphi(u) + |\nabla z|^{2\kappa_1} + |\nabla v|^{2\kappa_2}.$$

Furthermore, the ODE argument indicates

$$y(t) \leq c_{41} := \max \left\{ y_0, \frac{c_{40}}{c_{39}} \right\}^{\frac{1}{\rho}}.$$

Using (3.2), we obtain

$$\|u\|_{L^{k+m-1}} \leq \|u+1\|_{L^{k+m-1}} \leq c_{43} \quad \text{for all } t \in (0, T_{\max}).$$

The proof of Lemma 3.8 is finished. $\square$

Next, we will show the boundedness of $\|u(\cdot, t)\|_{L^\infty}$ based on the following lemma, which was proved based on the iteration method (see [24, Lemma A.1] for details).

Lemma 3.9. *Let the components of the vector field $\Phi : \Omega \times (0, \infty) \rightarrow \mathbb{R}^n$ satisfy $\Phi \in L^\infty((0, T); L^{q_*})$ with $q_* > n+2$ and $\Phi \cdot v = 0$. Let $u_0 \in W^{1,\infty}(\Omega)$ with $u_0 \geq 0$. Suppose that $u \in C^0(\bar{\Omega} \times (0, T)) \cap C^{2,1}(\bar{\Omega} \times (0, T))$ is a solution of the following initial-boundary value problem:*

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u) + \nabla \cdot \Phi, & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \nu} = \Phi \cdot v = 0, & x \in \partial\Omega, t > 0, \\ u(x, 0) = u_0(x), & x \in \Omega, \end{cases}$$

where $D(u)$ satisfies (1.4)–(1.5). If we further have

$$u \in L^\infty((0, T); L^{p_0}(\Omega))$$

for some $p_0 \geq 1$ fulfilling

$$p_0 > 1 - m \cdot \frac{(n+1)q_* - (n+2)}{q_* - (n+2)}$$

and

$$p_0 > \frac{n(1-m)}{2},$$

then there exists a constant $C > 0$, only depending on $m, c_D, \Omega, \|\Phi\|_{L^\infty((0,T);L^{q_*})}, \|u\|_{L^\infty((0,T);L^{p_0}(\Omega))}$ and $\|u_0\|_{L^\infty}$ such that

$$\|u(\cdot, t)\|_{L^\infty} \leq C \quad \text{for all } t \in (0, T).$$

Then we can use Lemma 3.9 to obtain the boundedness of $\|u(\cdot, t)\|_{L^\infty}$ as follows.

Proof of Lemma 3.1. From Lemmas 3.3 and 3.8, we can find a constant $c_1 > 0$ such that for all $t \in (0, T)$

$$\|u(\cdot, t)\|_{L^p} \leq c_1 \quad \text{for all } p > 1 \quad (3.37)$$

under the condition

$$0 < \alpha < \begin{cases} 1 + \frac{1}{n}, & \text{if } 1 \leq n \leq 3, \\ \frac{4}{n}, & \text{if } n \geq 4. \end{cases}$$

Using (3.37) with $p > n$, and then from Lemmas 2.3 and 2.4, one can derive

$$\|z(\cdot, t)\|_{W^{1,\infty}} + \|v(\cdot, t)\|_{W^{1,\infty}} \leq c_2 \quad \text{for all } t \in (0, T), \quad (3.38)$$

which entails

$$\|\Phi\|_{L^{q_*}} = \|S(u)\nabla v\|_{L^{q_*}} \leq \|\nabla v\|_{L^\infty} \|S(u)\|_{L^{q_*}} \leq c_3 \quad (3.39)$$

due to $S \in C^2([0, \infty))$ and (3.37). Then using (3.37) and (3.39), from Lemma 3.9 we obtain

$$\|u(\cdot, t)\|_{L^\infty} \leq c_4 \quad \text{for all } t \in (0, T). \quad (3.40)$$

Then combining (3.38), (3.40) and the fact $\|w(\cdot, t)\|_{L^\infty} \leq c_5$, and using Lemma 2.1, we prove the results in Lemma 3.1. $\square$

3.2. Large time behavior

Next, we will study the large time behavior of the solution. Different from the case of linear diffusion [5,6], we cannot use the straightforward estimate for $u(x, t) - \bar{u}_0$ via the Duhamel formula due to the nonlinear diffusion.

Lemma 3.10. Let (u, v, w, z) be the solution of (1.1) obtained in Lemma 3.1. Then there exist two positive constants λ and C such that for all $t \geq 0$

$$\|(u, v, w, z)(\cdot, t) - (\bar{u}_0, \bar{v}_0 + \bar{w}_0, 0, \bar{u}_0)\|_{L^\infty} \leq Ce^{-\lambda t},$$

where $\bar{u}_0 = \frac{1}{|\Omega|} \int_{\Omega} u_0(x) dx$, $\bar{v}_0 = \frac{1}{|\Omega|} \int_{\Omega} v_0(x) dx$ and $\bar{w}_0 = \frac{1}{|\Omega|} \int_{\Omega} w_0(x) dx$.

First, using a similar argument as in [6, Theorem 1.1], we directly obtain the convergence rate of $\mathbf{w}$ as follows.

Lemma 3.11. *Let the assumptions in Lemma 3.10 holds. Then there exist two positive constants C and $\zeta_1 (\neq \lambda_1)$ such that*

$$\|\mathbf{w}(\cdot, t)\|_{L^\infty} \leq C e^{-\zeta_1 t}, \quad t \geq 0. \quad (3.41)$$

Next, we shall show the convergence of $\mathbf{v}$ based on the convergence of $\mathbf{w}$ in Lemma 3.11.

Lemma 3.12. *Let the assumptions in Lemma 3.10 holds. Then the component $\mathbf{v}$ satisfies*

$$\|\mathbf{v}(\cdot, t) - (\bar{\mathbf{v}}_0 + \bar{\mathbf{w}}_0)\|_{L^\infty} \leq C e^{-\zeta_2 t}, \quad t \geq 0, \quad (3.42)$$

where C and ζ_2 are positive constants independent of t .

Proof. Letting $\bar{\mathbf{f}} = \frac{1}{|\Omega|} \int_{\Omega} \mathbf{f}$ for all $\mathbf{f} \in L^1(\Omega)$, then from the third equation of (1.1) we have

$$(\mathbf{v} - \bar{\mathbf{v}})_t = \Delta \mathbf{v} + \mathbf{w}\mathbf{z} - \bar{\mathbf{w}}\bar{\mathbf{z}}. \quad (3.43)$$

Applying the variation-of-constants method, from (3.43) one has

$$\mathbf{v}(\cdot, t) - \bar{\mathbf{v}}(t) = e^{\Delta t}(\mathbf{v}_0 - \bar{\mathbf{v}}_0) + \int_0^t e^{\Delta(t-\tau)} (\mathbf{w}\mathbf{z})(\cdot, \tau) - \bar{\mathbf{w}}\bar{\mathbf{z}}(\tau) d\tau. \quad (3.44)$$

Noting the fact $\int_{\Omega} (\mathbf{v}_0 - \bar{\mathbf{v}}_0) = 0$ and $\int_{\Omega} (\mathbf{w}\mathbf{z} - \bar{\mathbf{w}}\bar{\mathbf{z}}) = 0$, and using (2.3), (3.41) and the boundedness of $\|\mathbf{z}(\cdot, t)\|_{L^\infty}$, we obtain

$$\begin{aligned} \|\mathbf{v}(\cdot, t) - \bar{\mathbf{v}}(t)\|_{L^\infty} &\leq \|e^{\Delta t}(\mathbf{v}_0 - \bar{\mathbf{v}}_0)\|_{L^\infty} + \int_0^t \|e^{\Delta(t-\tau)} (\mathbf{w}\mathbf{z})(\cdot, \tau) - \bar{\mathbf{w}}\bar{\mathbf{z}}(\tau)\|_{L^\infty} d\tau \\ &\leq c_1 e^{-\lambda_1 t} \|\mathbf{v}_0 - \bar{\mathbf{v}}_0\|_{L^\infty} + c_2 \int_0^t e^{-\lambda_1(t-\tau)} \|(\mathbf{w}\mathbf{z})(\cdot, \tau) - \bar{\mathbf{w}}\bar{\mathbf{z}}(\tau)\|_{L^\infty} d\tau \\ &\leq c_3 e^{-\lambda_1 t} + 2c_2 \int_0^t e^{-\lambda_1(t-\tau)} \|\mathbf{w}(\cdot, \tau)\|_{L^\infty} \|\mathbf{z}(\cdot, \tau)\|_{L^\infty} d\tau \\ &\leq c_3 e^{-\lambda_1 t} + c_4 \int_0^t e^{-\lambda_1(t-\tau)} e^{-\zeta_1 \tau} d\tau \\ &\leq c_5 e^{-\lambda_1 t} + c_6 e^{-\min\{\lambda_1, \zeta_1\}t}, \end{aligned}$$

which yields

$$\|\mathbf{v}(\cdot, t) - \bar{\mathbf{v}}(t)\|_{L^\infty} \leq c_7 e^{-\min\{\lambda_1, \zeta_1\}t}. \quad (3.45)$$

On the other hand, adding the second and the third equations of system (1.1), and integrating the result over Ω with zero Neumann boundary conditions, then we can derive that

$$\bar{\mathbf{v}}(t) = \bar{\mathbf{v}}_0 + \bar{\mathbf{w}}_0 - \bar{\mathbf{w}}(t). \quad (3.46)$$

Substituting (3.46) into (3.45), one has

$$\|\mathbf{v}(\cdot, t) - \bar{\mathbf{v}}_0 - \bar{\mathbf{w}}_0 + \bar{\mathbf{w}}(t)\|_{L^\infty} \leq c_7 e^{-\min\{\lambda_1, \zeta_1\}t},$$

which together with (3.41) gives

$$\begin{aligned}\|v(\cdot, t) - (\bar{v}_0 + \bar{w}_0)\|_{L^\infty} &\leq c_7 e^{-\min\{\lambda_1, \zeta_1\}t} + \|\bar{w}(t)\|_{L^\infty} \\ &\leq c_7 e^{-\min\{\lambda_1, \zeta_1\}t} + c_8 e^{-\zeta_1 t} \\ &\leq c_9 e^{-\min\{\lambda_1, \zeta_1\}t}.\end{aligned}$$

Therefore, we obtain (3.42) by choosing $\zeta_2 = \min\{\lambda_1, \zeta_1\}$. $\square$

Next, we will show that u stabilizes exponentially to its spatial mean. To this end, we first obtain the decay rate of $\|\nabla v\|_{L^2}$.

Lemma 3.13. *Suppose the assumptions in Lemma 3.10 hold true. Then it follows that*

$$\|\nabla v(\cdot, t)\|_{L^2} \leq C e^{-\zeta_2 t} \quad \text{for all } t \geq 0, \quad (3.47)$$

where C is a positive constant independent of t .

Proof. Applying the variation of constants representation to the second equation of system (1.1), then we have

$$v(\cdot, t) = e^{t\Delta} v_0 + \int_0^t e^{(t-\tau)\Delta} w(\cdot, \tau) z(\cdot, \tau) d\tau,$$

which implies

$$\|\nabla v(\cdot, t)\|_{L^2} \leq \|\nabla e^{t\Delta} v_0\|_{L^2} + \int_0^t \|\nabla e^{(t-\tau)\Delta} w(\cdot, \tau) z(\cdot, \tau)\|_{L^2} d\tau. \quad (3.48)$$

First, we can use the heat semigroup property (2.5) to derive

$$\|\nabla e^{t\Delta} v_0\|_{L^2} \leq c_1 e^{-\lambda_1 t} \|\nabla v_0\|_{L^2}. \quad (3.49)$$

Moreover, using (2.4) and Hölder's inequality, one infers from (3.41) and the boundedness of $\|z(\cdot, t)\|_{L^\infty}$ that

$$\begin{aligned}& \int_0^t \|\nabla e^{(t-\tau)\Delta} w(\cdot, \tau) z(\cdot, \tau)\|_{L^2} d\tau \\ & \leq c_2 \int_0^t \int_0^t 1 + (t-\tau)^{-\frac{1}{2}} e^{-\lambda_1(t-\tau)} \|w(\cdot, \tau) z(\cdot, \tau)\|_{L^2} d\tau \\ & \leq c_2 |\Omega|^{\frac{1}{2}} \int_0^t \int_0^t 1 + (t-\tau)^{-\frac{1}{2}} e^{-\lambda_1(t-\tau)} \|w(\cdot, \tau)\|_{L^\infty} \|z(\cdot, \tau)\|_{L^\infty} d\tau \\ & \leq c_3 \int_0^t \int_0^t 1 + (t-\tau)^{-\frac{1}{2}} e^{-\lambda_1(t-\tau)} e^{-\zeta_1 \tau} d\tau \\ & \leq c_4 e^{-\min\{\lambda_1, \zeta_1\}t}.\end{aligned} \quad (3.50)$$

Then substituting (3.49) and (3.50) into (3.48), we can derive

$$\|\nabla v(\cdot, t)\|_{L^2} \leq c_5 e^{-\min\{\lambda_1, \zeta_1\}t},$$

which gives (3.47) by noting $\zeta_2 = \min\{\lambda_1, \zeta_1\}$ as in Lemma 3.12. $\square$

Lemma 3.14. *Let the assumptions in Lemma 3.10 holds. Then there exist two positive constants C and ζ_3 such that for all $t \geq 0$*

$$\|u(\cdot, t) - \bar{u}_0\|_{L^\infty} \leq Ce^{-\zeta_3 t}. \quad (3.51)$$

Proof. Let $\bar{u} = u - \bar{u}_0$, we can obtain from the first equation of system (1.1) that

$$\bar{u}_t = \nabla \cdot (D(u) \nabla \bar{u}) - \nabla \cdot (S(u) \nabla v). \quad (3.52)$$

Then multiplying (3.52) by $\bar{u}$, and using Young's inequality, we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\Omega} \bar{u}^2 &= - \int_{\Omega} D(u) |\nabla \bar{u}|^2 + \int_{\Omega} S(u) \nabla v \cdot \nabla \bar{u} \\ &\leq -D_0 \int_{\Omega} |\nabla \bar{u}|^2 + S_0 \int_{\Omega} |\nabla \bar{u}| |\nabla v| \\ &\leq -\frac{D_0}{2} \int_{\Omega} |\nabla \bar{u}|^2 + \frac{S_0^2}{2D_0} \int_{\Omega} |\nabla v|^2, \end{aligned} \quad (3.53)$$

where we have used the boundedness of $\|u(\cdot, t)\|_{L^\infty}$ to show $D(u) \geq D_0 > 0$ and $S(u) \leq S_0$. Noting that $\int_{\Omega} \bar{u} = 0$, hence we can use the Poincaré inequality to find a constant $c_1 > 0$ such that

$$c_1 \int_{\Omega} |\bar{u}|^2 \leq D_0 \int_{\Omega} |\nabla \bar{u}|^2. \quad (3.54)$$

Substituting (3.54) into (3.53), and using (3.47), we derive

$$\frac{d}{dt} \int_{\Omega} |\bar{u}|^2 + c_1 \int_{\Omega} |\bar{u}|^2 \leq c_2 e^{-2\zeta_2 t},$$

which leads to

$$\|\bar{u}\|_{L^2}^2 \leq c_3 e^{-\min\{c_1, 2\zeta_2\}t}.$$

If there exists a positive constant c_4 such that for all $t \geq 1$

$$\|\nabla u\|_{L^{2n}} \leq c_4, \quad (3.55)$$

then using the Gagliardo–Nirenberg inequality, we obtain

$$\begin{aligned} \|\bar{u}\|_{L^\infty} &\leq c_5 \|\nabla u\|_{L^{2n}}^{\frac{n}{n+1}} \|\bar{u}\|_{L^2}^{\frac{1}{n+1}} + c_6 \|\bar{u}\|_{L^2} \\ &\leq c_7 \|\bar{u}\|_{L^2}^{\frac{1}{n+1}} + c_6 \|\bar{u}\|_{L^2} \\ &\leq c_8 e^{-\min\{c_1, 2\zeta_2\}t} \quad ; \quad D_{007} \text{ Td } [(k)3u40tl] \text{ T J T d } [7.4849.9626 \text{ T f } 5.704J / F59 \text{ 5.9776 T f9}[(\text{J})] \text{ T J / F} \end{aligned}$$

Denote $y(t) = \int_{\Omega} |\nabla u|^{2n}$. Then from (3.60), one has

$$y'(t) + 2ny(t) \leq 2nc_{16},$$

which gives (3.55) by using Grönwall's inequality. Hence the proof of this lemma is completed. $\square$

Lemma 3.15. *Suppose the assumptions in Lemma 3.10 hold true. Then there exists a positive constant C such that for all $t \geq 0$*

$$\|z(\cdot, t) - \bar{u}_0\|_{L^\infty} \leq Ce^{-\zeta_4 t}. \quad (3.61)$$

Proof. Letting $\tilde{z} = z - \bar{u}_0$ and $\tilde{u} = u - \bar{u}_0$, then from the fourth equation of system (1.1), we have

$$\tilde{z}_t = \Delta \tilde{z} - \tilde{z} + \tilde{u}.$$

Then applying the variation-of-constants representation of $\tilde{z}$, one has

$$\tilde{z} = e^{(\Delta-1)t} \tilde{z}_0 + \int_0^t e^{(\Delta-1)(t-\tau)} \tilde{u}(\cdot, \tau) d\tau.$$

Using Lemma 2.5 and (3.51), we obtain

$$\begin{aligned} \|\tilde{z}\|_{L^\infty} &\leq e^{-t} \|e^{t\Delta} \tilde{z}_0\|_{L^\infty} + \int_0^t e^{-(t-\tau)} \|e^{(t-\tau)\Delta} \tilde{u}(\cdot, \tau)\|_{L^\infty} d\tau \\ &\leq c_1 e^{-t} \|\tilde{z}_0\|_{L^\infty} + c_2 \int_0^t e^{-\lambda_1 \tau} e^{-(t-\tau)} \|\tilde{u}(\cdot, \tau)\|_{L^\infty} d\tau \\ &\leq c_3 e^{-t} + c_4 \int_0^t e^{-\lambda_1 \tau} e^{-(t-\tau)} e^{-\zeta_3 \tau} d\tau \\ &\leq c_5 e^{-t} + c_6 e^{-\min\{\zeta_3, 1, \lambda_1\}t} \\ &\leq c_7 e^{-\min\{\zeta_3, 1, \lambda_1\}t}, \end{aligned}$$

which gives (3.61). Then the proof of this lemma is completed. $\square$

Proof of Lemma 3.10. The combination of Lemmas 3.11–3.15 completes the proof of Lemma 3.10. $\square$

3.3. Proof of Theorem 1.1

Proof of Theorem 1.1. Theorem 1.1 is a consequence result by combining Lemmas 3.1 and 3.10. $\square$

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