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- $a_1, a_2 \in (0,1)$ μ_1, μ_2 , (u_1, u_2, v) $\left(\frac{1-a_1}{1-a_1a_2}, \frac{1-a_2}{1-a_1a_2}, \frac{b_1+b_2-a_1b_1-a_2b_2}{1-a_1a_2}\right)$ $t \to \infty$.
- $a_1 \geq 1, a_2 \in (0,1)$ μ_2 , (u_1, u_2, v) $(0, 1, b_2)$ $t \to \infty$ $a_1 = 1$, $a_1 > 1$.

35 01; 35 40; 35 44; 35 57; 35 92; 92 17

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1. Introduction

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$$\begin{cases} u_{1t} = \nabla\cdot(d_1(v)\nabla u_1) - \nabla\cdot(\chi_1(v)u_1\nabla v) + \mu_1 u_1(1-u_1-a_1u_2), & x\in\Omega, \quad t>0, \\ u_{2t} = \nabla\cdot(d_2(v)\nabla u_2) - \nabla\cdot(\chi_2(v)u_2\nabla v) + \mu_2 u_2(1-u_2-a_2u_1), & x\in\Omega, \quad t>0, \\ v_t = \Delta v + b_1u_1 + b_2u_2 - v, & x\in\Omega, \quad t>0, \\ \frac{\partial u_1}{\partial \nu} = \frac{\partial u_2}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x\in\partial\Omega, \quad t>0, \\ u_1(x,0) = u_{10}(x), u_2(x,0) = u_{20}(x), v(x,0) = v_0(x), & x\in\Omega, \end{cases} \tag{1.1}$$

$u_i(x,t)$ $i = 1,2$, , $v(x,t)$. μ_i, a_i, b_i - $i = 1,2$. $\nabla\cdot(d_i(v)\nabla u_i)$ $d_i(v)$. $-\nabla\cdot(\chi_i(v)u_i\nabla v)$ $\chi_i(v)$ $i = 1,2$, v. $d_i(v) = d_i > 0$ $\chi_i(v) = \chi_i > 0$, (1.1) -

$$\begin{cases} u_{1t} = d_1\Delta u_1 - \chi_1\nabla\cdot(u_1\nabla v) + \mu_1 u_1(1-u_1-a_1u_2), \\ u_{2t} = d_2\Delta u_2 - \chi_2\nabla\cdot(u_2\nabla v) + \mu_2 u_2(1-u_2-a_2u_1), \\ v_t = \Delta v + b_1u_1 + b_2u_2 - v, \end{cases} \tag{1.2}$$

32,34 (3). (1.2) , , . (1.2). , - (1.2) $\mu_1 = \mu_2 = 0$ 4,5,8 10,19,36,37 . - (1.2) - (. ., $\mu_1, \mu_2 > 0$),

. (1.2) -
$0 = \Delta v + b_1 u_1 + b_2 u_2 - v$, ,
μ_i, χ_i $i = 1, 2$,

6,28,33 . , (u_1, u_2, v)
$(\frac{1-a_1}{1-a_1 a_2}, \frac{1-a_2}{1-a_1 a_2}, \frac{b_1+b_2-a_1 b_1-a_2 b_2}{1-a_1 a_2})$ $a_1 < 1, a_2 < 1$ (6,33), $a_1 > 1, a_2 < 1$,
, $(u_1, u_2, v) \to (0, 1, b_2)$ $t \to \infty$ 28 .
(1.2), ,
& 2
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-
7,12,16 .
$d_i(v) = d_i > 0$, $\chi_i(v)$ - ,
21 24 . , ,
- -
(1.1). $a_1 = b_2 = 0$, u_2
(1.1) (1.1)
-

$$\begin{cases} u_{1t} = \nabla \cdot (d_1(v) \nabla u_1) - \nabla \cdot (\chi_1(v) u_1 \nabla v) + \mu_1 u_1 (1 - u_1), \\ v_t = \Delta v + b_1 u_1 - v, \end{cases} \tag{1.3}$$

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$\chi_1(v) = -d_1'(v) > 0$. (1.3) $\chi_1(v) = -d_1'(v) > 0$, $\mu_1 = 0$
$d_1(v) = \frac{c_0}{v^k}$, $c_0 > 0, k > 0$, 35 -
$c_0 > 0$. , $d_1(v)$
, 31
$(n \geq 3)$. $\mu_1 > 0$, ,
14 35
- 31 . , J 13 -
$\chi_1(v) = -d_1'(v) > 0$
(1.3).
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(1) -
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(2) , -
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. $d_i(z), \chi_i(z) > 0$ $i = 1, 2$

(1) $(d_i(z), \chi_i(z)) \in [C^2([0,\infty)]^2$ $d_i(z), \chi_i(z) > 0$ $z \geq 0$, $d_i'(z) < 0$ $\underset{z\to\infty}{} d_i(z) = 0$.

(2) $\underset{z\to\infty}{} \frac{\chi_i(z)}{d_i(z)}$ $\underset{z\to\infty}{} \frac{d_i'(z)}{d_i(z)}$.

- - 6,21 24,28, 33 , - $d_i'(v) < 0$ $i = 1, 2$, - - . - , $d_i(v)(i = 1, 2)$ 13,14 . , , L^2- u_1 u_2, - , v (1.1), . , - u_i $(i = 1, 2)$. , , . .

Theorem 1.1 *( )* $\Omega \subset \mathbb{R}^2$ a_i, b_i, μ_i $i = 1, 2$ *( ) ( )* $(u_{10}, u_{20}, v_0) \in [W^{1,p}(\Omega)]^3$ $p > 2$ $u_{10}, u_{20}, v_0 \geq 0 (\not\equiv 0)$ (1.1) (u_1, u_2, v)

( ) $a_1 > 1$ $a_2 \in (0, 1)$ $a_1^* \in (1, a_1]$ $a_1^* a_2 < 1$,

$$\mu_2 > \max_{0 \le z \le \infty} \frac{(a_1^* b_1^2 \xi_2 + a_2 b_2^2 - a_1^* a_2 b_1 b_2 (1+\xi_2))\xi_2 \chi_2^2(z)}{4 a_2 (4\xi_2 - a_1^* a_2 (1+\xi_2)^2) d_2(z)} \tag{1.5}$$

$\xi_2 > 0$ $4\xi_2 - a_1^* a_2 (1+\xi_2)^2 > 0$,

$$\|u_1(\cdot,t)\|_{L^\infty(\Omega)} + \|u_2(\cdot,t) - 1\|_{L^\infty(\Omega)} + \|v(\cdot,t) - b_2\|_{L^\infty(\Omega)} \le C_2 e^{-\lambda_2 t}$$

$t > 0$, C_2 λ_2 t
( ) $a_1 = 1$, $a_2 \in (0, 1)$ μ_2 (1.5) $a_1^* = 1$, $C_3, \lambda_3 > 0$ t $t > 0$

$$\|u_1(\cdot,t)\|_{L^\infty(\Omega)} + \|u_2(\cdot,t) - 1\|_{L^\infty(\Omega)} + \|v(\cdot,t) - b_2\|_{L^\infty(\Omega)} \le C_3 (t+1)^{-\lambda_3}.$$

Remark 1.1. $d_i(z)$ $\chi_i(z)$ $i = 1, 2$, 1.2 2 .

Notation. , $\int_\Omega f$ $\int_0^t \int_\Omega f$ $\int_\Omega f(\cdot,t)dx$ $\int_0^t \int_\Omega f(\cdot,s)dxds$, . , $\|\cdot\|_{L^p(\Omega)} = \|\cdot\|_{L^p}$, $c_i (i = 1, 2, 3, \cdots)$.

2. Boundedness of solutions: proof of Theorem 1.1

, (1.1). . , - . , $d_i(v)$ $(i = 1, 2)$ - $\|u_1(\cdot,t)\|_{L^2}$ $\|u_2(\cdot,t)\|_{L^2}$ 13,14 . L^2- u_1 u_2, $\|v(\cdot,t)\|_{L^\infty}$, . $\|u_1(\cdot,t)\|_{L^\infty}$ $\|u_2(\cdot,t)\|_{L^\infty}$ - .

, (1.1) 14, 2.1 , .

Lemma 2.1 *( )* $\Omega \subset \mathbb{R}^2$ a_i, b_i, μ_i $i = 1, 2$ $(u_{10}, u_{20}, v_0) \in [W^{1,p}(\Omega)]^3$ $p > 2$ $(u_{10}, u_{20}, v_0) \ge 0 (\not\equiv 0)$ ( ) ( ) $T_{max} \in (0, \infty]$ (1.1) (u_1, u_2, v) $u_1, u_2, v > 0$ $t > 0$

$$u_1 \in C^0(\bar\Omega \times [0, T_{max})) \cap C^{2,1}(\bar\Omega \times (0, T_{max})),$$
$$u_2 \in C^0(\bar\Omega \times [0, T_{max})) \cap C^{2,1}(\bar\Omega \times (0, T_{max})),$$
$$v \in C^0(\bar\Omega \times [0, T_{max})) \cap C^{2,1}(\bar\Omega \times (0, T_{max})) \cap L^\infty_{loc}([0, T_{max}); W^{1,p}(\Omega)).$$

, $T_{max} = \infty$

$$\|u_1(\cdot,t)\|_{L^\infty} + \|u_2(\cdot,t)\|_{L^\infty} + \|v(\cdot,t)\|_{W^{1,\infty}} \to \infty \quad t \nearrow T_{max}.$$

Lemma 2.2. $C > 0$ t (1.1)

$$\int_\Omega u_1 + \int_\Omega u_2 \le C, \qquad \forall t \in (0, T \quad) \tag{2.1}$$

$$\int_t^{t+\tau}\int_\Omega u_1^2 + \int_t^{t+\tau}\int_\Omega u_2^2 \le C, \qquad \forall t \in (0, T \quad - \tau),$$

$\tau = \quad \{1, \frac{T}{2}\}$

Proof. (1.1) x

$$\frac{d}{dt}\int_\Omega u_1 = \mu_1\int_\Omega u_1 - \mu_1\int_\Omega u_1^2 - a_1\mu_1\int_\Omega u_1u_2, \quad \forall t \in (0, T \quad).$$

$u_i \ge 0$ $i = 1, 2,$, $c_1 = \frac{(\mu_1+1)^2}{2\mu_1}|\Omega|,$

$$\frac{d}{dt}\int_\Omega u_1 + \int_\Omega u_1 + \frac{\mu_1}{2}\int_\Omega u_1^2 \le c_1, \quad \forall t \in (0, T \quad), \tag{2.2}$$

, , ,

$$\int_\Omega u_1 \le c_2, \quad \forall t \in (0, T \quad), \tag{2.3}$$

$c_2 = c_1 + \|u_{10}\|_{L^1}$. , (2.2) $(t, t+\tau)$ $t \in (0, T \quad - \tau)$ (2.3),

$$\frac{\mu_1}{2}\int_t^{t+\tau}\int_\Omega u_1^2 \le c_1\tau - \int_\Omega u_1(\cdot, t+\tau) + \int_\Omega u_1(\cdot,t) - \int_t^{t+\tau}\int_\Omega u_1 \le c_1\tau + c_2,$$

$$\int_t^{t+\tau}\int_\Omega u_1^2 \le c_3, \quad \forall t \in (0, T \quad - \tau), \tag{2.4}$$

$c_3 = \frac{2(c_1\tau + c_2)}{\mu_1}$. , (1.1),

$$\frac{d}{dt}\int_\Omega u_2 + \int_\Omega u_2 + \frac{\mu_2}{2}\int_\Omega u_2^2 \le c_4, \quad \forall t \in (0, T \quad) \tag{2.5}$$

$c_4 = \frac{(\mu_2+1)^2}{2\mu_2}|\Omega|$. (2.5), $c_5 = c_4 + \|u_{20}\|_{L^1} > 0$

$$\int_\Omega u_2 \le c_5, \quad \forall t \in (0, T \quad)$$

$$\int_t^{t+\tau}\int_\Omega u_2^2 \le \frac{2(c_4\tau + c_5)}{\mu_2}, \quad \forall t \in (0, T \quad - \tau).$$

(2.3) (2.4), . □

Lemma 2.3. - - (u_1, u_2, v)
(1.1) -

$$\int_\Omega |\nabla v|^2 \le C, \qquad \forall t \in (0, T \quad) \tag{2.6}$$

-

$$\int_t^{t+\tau}\int_\Omega |\Delta v|^2 \le C, \qquad \forall t \in (0, T \quad - \tau), \tag{2.7}$$

$\tau =$ $\{1, \frac{T}{2}\}$ - $C > 0$ - - t

Proof. (1.1) $-\Delta v$,
x , ,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla v|^2 = -\int_\Omega |\Delta v|^2 - b_1\int_\Omega u_1\Delta v - b_2\int_\Omega u_2\Delta v + \int_\Omega v\Delta v$$
$$\le -\frac{1}{2}\int_\Omega |\Delta v|^2 + b_1^2\int_\Omega u_1^2 + b_2^2\int_\Omega u_2^2 - \int_\Omega |\nabla v|^2, \qquad \forall t \in (0, T_{max}),$$

$$\frac{d}{dt}\int_\Omega |\nabla v|^2+\int_\Omega |\Delta v|^2+2\int_\Omega |\nabla v|^2\le 2b_1^2\int_\Omega u_1^2+2b_2^2\int_\Omega u_2^2,\qquad \forall t\in(0,T_{max}).\tag{2.8}$$

(1.1) x, , , ,

$$\frac{d}{dt}\int_\Omega u_1+\int_\Omega u_1+\frac{\mu_1}{2}\int_\Omega u_1^2\le c_1,\quad \forall t\in(0,T\quad)\tag{2.9}$$

$$\frac{d}{dt}\int_\Omega u_2+\int_\Omega u_2+\frac{\mu_2}{2}\int_\Omega u_2^2\le c_2,\quad \forall t\in(0,T\quad)\tag{2.10}$$

$c_1=\frac{(\mu_1+1)^2}{2\mu_1}|\Omega|$ $c_2=\frac{(\mu_2+1)^2}{2\mu_2}|\Omega|$. (2.9) $\frac{4b_1^2}{\mu_1}$ (2.10) $\frac{4b_2^2}{\mu_2}$, - , (2.8),

$$y'+y+\int_\Omega |\Delta v|^2\le\frac{4c_1b_1^2}{\mu_1}+\frac{4c_2b_2^2}{\mu_2},\quad \forall t\in(0,T\quad)\tag{2.11}$$

$y=\int_\Omega|\nabla v|^2+\frac{4b_1^2}{\mu_1}\int_\Omega u_1+\frac{4b_2^2}{\mu_2}\int_\Omega u_2$. , (2.11) (2.1), (2.6). , (2.11) $(t,t+\tau)$ $t\in(0,T\quad-\tau)$ (2.6), (2.7). 2.3 . □

u_1,u_2 L^2 v L^∞

Lemma 2.4.

$$\|u_1(\cdot,t)\|_{L^2}+\|u_2(\cdot,t)\|_{L^2}+\|v(\cdot,t)\|_{L^\infty}\le C,\qquad \forall t\in(0,T_{max}),\tag{2.12}$$

$C>0$ t

Proof. (2), K_1 K_2

$$\frac{|d_1'(z)|}{d_1(z)}\le K_1\qquad \frac{|\chi_1(z)|}{d_1(z)}\le K_2\qquad z\ge0\qquad t\in(0,T_{max}).\tag{2.13}$$

$u_1,u_2>0$, u_1 (1.1) Ω

$$\frac{1}{2}\frac{d}{dt}\int\limits_\Omega u_1^2+\int\limits_\Omega d_1(v)|\nabla u_1|^2+\mu_1\int\limits_\Omega u_1^3+a_1\mu_1\int\limits_\Omega u_1^2u_2$$
$$=\int\limits_\Omega \chi_1(v)u_1\nabla u_1\cdot\nabla v+\mu_1\int\limits_\Omega u_1^2$$
$$\le\frac{1}{2}\int\limits_\Omega d_1(v)|\nabla u_1|^2+\frac{1}{2}\int\limits_\Omega\frac{\chi_1^2(v)}{d_1(v)}u_1^2|\nabla v|^2+\frac{\mu_1}{2}\int\limits_\Omega u_1^3+\frac{16\mu_1}{27}|\Omega|$$

$t\in(0,T_{max})$,

$$\frac{d}{dt}\int\limits_\Omega u_1^2+\int\limits_\Omega d_1(v)|\nabla u_1|^2+\mu_1\int\limits_\Omega u_1^3\le\int\limits_\Omega\frac{\chi_1^2(v)}{d_1(v)}u_1^2|\nabla v|^2+c_1,\quad\forall t\in(0,T_{max}),\tag{2.14}$$

$c_1=\frac{32\mu_1}{27}|\Omega|$. 13,14 , $\int_\Omega d_1(v)|\nabla u_1|^2$ $\int_\Omega\frac{\chi_1^2(v)}{d_1(v)}u_1^2|\nabla v|^2$. ,

$$\nabla(d_1^{\frac{1}{2}}(v)u_1)=d_1^{\frac{1}{2}}(v)\nabla u_1+\frac{1}{2}\frac{d_1'(v)}{d_1^{\frac{1}{2}}(v)}u_1\nabla v,\qquad\forall t\in(0,T_{max}),$$

$$\frac{1}{2}|\nabla(d_1^{\frac{1}{2}}(v)u_1)|^2-\frac{1}{4}\frac{|d_1'(v)|^2}{d_1(v)}u_1^2|\nabla v|^2\le d_1(v)|\nabla u_1|^2,\qquad\forall t\in(0,T_{max})\tag{2.15}$$

$\frac{1}{2}A^2-B^2\le|A-B|^2$. (2.15) (2.14), (2.13),

$$\begin{aligned}&\frac{d}{dt}\int\limits_\Omega u_1^2+\frac{1}{2}\int\limits_\Omega|\nabla(d_1^{\frac{1}{2}}(v)u_1)|^2+\mu_1\int\limits_\Omega u_1^3\\
&\le\frac{1}{4}\int\limits_\Omega\frac{|d_1'(v)|^2}{d_1(v)}u_1^2|\nabla v|^2+\int\limits_\Omega\frac{\chi_1^2(v)}{d_1(v)}u_1^2|\nabla v|^2+c_1\\
&\le\left(\frac{K_1^2}{4}+K_2^2\right)\int\limits_\Omega d_1(v)u_1^2|\nabla v|^2+c_1\\
&\le\left(\frac{K_1^2}{4}+K_2^2\right)\left(\int\limits_\Omega|d_1^{\frac{1}{2}}(v)u_1|^4\right)^{\frac{1}{2}}\left(\int\limits_\Omega|\nabla v|^4\right)^{\frac{1}{2}}+c_1,\qquad\forall t\in(0,T_{max}).\end{aligned}\tag{2.16}$$

$d_1'(v)<0$ (1), $d_1(v)\le d_1(0)=:c_2$. , ,

$$\begin{aligned}\left(\int_\Omega |d_1^{\frac{1}{2}}(v)u_1|^4\right)^{\frac{1}{2}} &= \|d_1^{\frac{1}{2}}(v)u_1\|_{L^4}^2 \\ &\le c_3\big(\|\nabla(d_1^{\frac{1}{2}}(v)u_1)\|_{L^2}\|d_1^{\frac{1}{2}}(v)u_1\|_{L^2} + \|d_1^{\frac{1}{2}}(v)u_1\|_{L^2}^2\big) \\ &\le c_4\big(\|\nabla(d_1^{\frac{1}{2}}(v)u_1)\|_{L^2}\|u_1\|_{L^2} + \|u_1\|_{L^2}^2\big), \quad \forall t\in(0,T_{max}),\end{aligned} \tag{2.17}$$

$c_4 = c_3(c_2^{\frac{1}{2}} + c_2)$. , 14, 2.5

$$\|\nabla v\|_{L^4} \le c_5(\|\Delta v\|_{L^2}^{\frac{1}{2}}\|\nabla v\|_{L^2}^{\frac{1}{2}} + \|\nabla v\|_{L^2}), \qquad \forall t\in(0,T_{max}),$$

(2.6),

$$\begin{aligned}\left(\int_\Omega |\nabla v|^4\right)^{\frac{1}{2}} = \|\nabla v\|_{L^4}^2 &\le c_6\left(\|\Delta v\|_{L^2}\|\nabla v\|_{L^2} + \|\nabla v\|_{L^2}^2\right) \\ &\le c_7\left(\|\Delta v\|_{L^2} + 1\right), \qquad \forall t\in\left(0,T_{max}\right).\end{aligned} \tag{2.18}$$

(2.17) (2.18), , ,

$$\begin{aligned}&\left(\frac{K_1^2}{4}+K_2^2\right)\left(\int_\Omega |d_1^{\frac{1}{2}}(v)u_1|^4\right)^{\frac{1}{2}}\left(\int_\Omega |\nabla v|^4\right)^{\frac{1}{2}} \\ &\le \left(\frac{K_1^2}{4}+K_2^2\right)c_4c_7(\|\nabla(d_1^{\frac{1}{2}}(v)u_1)\|_{L^2}\|u_1\|_{L^2} + \|u_1\|_{L^2}^2)(\|\Delta v\|_{L^2}+1) \\ &= \left(\frac{K_1^2}{4}+K_2^2\right)c_4c_7(\|\nabla(d_1^{\frac{1}{2}}(v)u_1)\|_{L^2}\|u_1\|_{L^2}\|\Delta v\|_{L^2} \\ &\quad + \|\nabla(d_1^{\frac{1}{2}}(v)u_1)\|_{L^2}\|u_1\|_{L^2} + \|u_1\|_{L^2}^2\|\Delta v\|_{L^2} + \|u_1\|_{L^2}^2) \\ &\le \frac{1}{2}\|\nabla(d_1^{\frac{1}{2}}(v)u_1)\|_{L^2}^2 + c_8\|u_1\|_{L^2}^2\|\Delta v\|_{L^2}^2 + c_9^2\|u_1\|_{L^2}^2, \qquad \forall t\in(0,T_{max}),\end{aligned} \tag{2.19}$$

$c_8 = 2c_4^2c_7^2\left(\frac{K_1^2}{4}+K_2^2\right)^2$ $c_9 = c_4c_7\left(\frac{K_1^2}{4}+K_2^2\right)+\frac{1}{2}$. (2.19) (2.16) $c_{10} = c_1 + \frac{4c_9^6}{27\mu_1^2}|\Omega|$,

$$\frac{d}{dt}\|u_1\|_{L^2}^2 \le c_8\|\Delta v\|_{L^2}^2\|u_1\|_{L^2}^2 + c_{10}, \qquad \forall t\in(0,T_{max}), \tag{2.20}$$

, 2.2 2.3, (2.12). , $t\in(0,T_{max})$, $t\in(0,\tau)$ $t>\tau$ $\tau =$ $\{1,\frac{1}{2}T_{max}\}$, $t_0\in((t-\tau)_+,t)$ $t_0\ge 0$

$$\int\limits_{\Omega} u_1^2(x,t_0) \leq c_{11}. \tag{2.21}$$

, 2.3

$$\int\limits_{t_0}^{t_0+\tau}\int\limits_{\Omega} |\Delta v|^2 \leq c_{12}, \qquad \forall t_0 \in (0, T_{max}-\tau). \tag{2.22}$$

, (2.20) (t_0, t), (2.21) (2.22),

$$\begin{aligned}\|u_1(\cdot,t)\|_{L^2}^2 &\leq \|u_1(\cdot,t_0)\|_{L^2}^2 \cdot e^{c_8 \int_{t_0}^{t} \|\Delta v(\cdot,\rho)\|_{L^2} d\rho} + c_{10}\int\limits_{t_0}^{t} e^{c_8 \int_{\rho}^{t} \|\Delta v(\cdot,s)\|_{L^2} ds} d\rho \\ &\leq c_{11}e^{c_8 c_{12}} + c_{10}e^{c_8 c_{12}}\end{aligned}$$

$t \in (0, T_{max}-\tau)$,

$$\|u_1(\cdot,t)\|_{L^2} \leq c_{13}, \qquad \forall t \in (0, T_{max}). \tag{2.23}$$

,

$$\|u_2(\cdot,t)\|_{L^2} \leq c_{14}, \qquad \forall t \in (0, T_{max}). \tag{2.24}$$

, (2.23) (2.24)

$$\|b_1 u_1(\cdot,t) + b_2 u_2(\cdot,t)\|_{L^2} \leq b_1 c_{13} + b_2 c_{14}. \tag{2.25}$$

(1.1), (2.25)

$$\|v(\cdot,t)\|_{W^{1,4}} \leq c_{15}, \qquad \forall t \in (0, T_{max}), \tag{2.26}$$

$\|v(\cdot,t)\|_{L^\infty} \leq c_{16}$ $t \in (0, T_{max})$.
2.4. □

u_1, u_2 L^∞, v $W^{1,\infty}$

Lemma 2.5. $C > 0$
t,

$$\|u_1(\cdot,t)\|_{L^\infty} + \|u_2(\cdot,t)\|_{L^\infty} + \|v(\cdot,t)\|_{W^{1,\infty}} \leq C, \qquad \forall t \in (0, T_{max}). \tag{2.27}$$

Proof. (1.1) u_1^3 Ω,

$$\begin{aligned}
&\frac{1}{4}\frac{d}{dt}\int\limits_{\Omega} u_1^4 + 3\int\limits_{\Omega} d_1(v)u_1^2|\nabla u_1|^2 + \mu_1\int\limits_{\Omega} u_1^5 + a_1\mu_1\int\limits_{\Omega} u_1^4u_2 \\
&= 3\int\limits_{\Omega} \chi_1(v)u_1^3\nabla u_1\cdot\nabla v + \mu_1\int\limits_{\Omega} u_1^4
\end{aligned}\tag{2.28}$$

$t \in (0, T_{max})$. $\|v(\cdot,t)\|_{L^\infty} \le c_1$ 2.4
(1) (2),

$$d_1(v) > d_1(c_1) := c_2 > 0 \qquad |\chi_1(v)| \le c_3, \qquad \forall\, t \in (0, T_{max}).$$

(2.28) , $t \in (0, T_{max})$

$$\begin{aligned}
&\frac{1}{4}\frac{d}{dt}\int\limits_{\Omega} u_1^4 + 3c_2\int\limits_{\Omega} u_1^2|\nabla u_1|^2 + \frac{1}{4}\int\limits_{\Omega} u_1^4 \\
&\le 3c_3\int\limits_{\Omega} u_1^3|\nabla u_1||\nabla v| + \left(\frac{1}{4}+\mu_1\right)\int\limits_{\Omega} u_1^4 - \mu_1\int\limits_{\Omega} u_1^5 \\
&\le \frac{3c_2}{2}\int\limits_{\Omega} u_1^2|\nabla u_1|^2 + \frac{3c_3^2}{2c_2}\int\limits_{\Omega} u_1^4|\nabla v|^2 + c_4,
\end{aligned}\tag{2.29}$$

$c_4 = \left(\frac{4\mu_1+1}{5}\right)^5 \frac{|\Omega|}{4\mu_1^4}$. (2.29), -
$\|\nabla v(\cdot,$ 2
1

, , ,

$$\|u_1(\cdot,t)\|_{L^4}^4 \le \|u_1(\cdot,0)\|_{L^4}^4 + 4(c_4 + c_8), \qquad \forall t \in (0, T_{max}). \tag{2.31}$$

, (1.1),

$$\|u_2(\cdot,t)\|_{L^4}^4 \le \|u_2(\cdot,0)\|_{L^4}^4 + 4c_9, \qquad \forall t \in (0, T_{max}). \tag{2.32}$$

(2.31) (2.32) , $c_{10} > 0$ $\|\nabla v(\cdot,t)\|_{W^{1,\infty}} \le c_{10}$. - (. 29,32), (2.27). □

Proof of Theorem 1.1. 1.1 2.1 2.5. □

3. Stabilization and convergence rate

, (1.1) 2,15,30 .

$a_1 \in (0,1)$ $a_2 \in (0,1)$

, (1.1) $a_i \in (0,1)$ $i = 1,2$

$$\begin{aligned} \mathcal{E}_1(t) :=& \int_\Omega \left(u_1 - u_1^* - u_1^* \quad \frac{u_1}{u_1^*}\right) + \zeta_1 \int_\Omega \left(u_2 - u_2^* - u_2^* \quad \frac{u_2}{u_2^*}\right) \\ &+ \frac{\eta_1}{2}\int_\Omega (v - v^*)^2, \end{aligned} \tag{3.1}$$

ζ_1 η_1 . (u_1^*, u_2^*, v^*)

$$u_1^* = \frac{1-a_1}{1-a_1a_2}, \quad u_2^* = \frac{1-a_2}{1-a_1a_2}, \quad v^* = \frac{b_1 + b_2 - a_1b_1 - a_2b_2}{1-a_1a_2}.$$

Lemma

$$\mathcal{F}_1(t) := \int\limits_\Omega (u_1 - u_1^*)^2 + \int\limits_\Omega (u_2 - u_2^*)^2 + \int\limits_\Omega (v - v^*)^2. \tag{3.3}$$

Proof. $0 < a_1 < 1$ $0 < a_2 < 1$, $\xi_1 > 0$

$$4\xi_1 - a_1 a_2 (1 + \xi_1)^2 > 0, \tag{3.4}$$

$$a_1 b_1^2 \xi_1 + a_2 b_2^2 - a_1 a_2 b_1 b_2 (1 + \xi_1) > 0 \tag{3.5}$$

(3.5) $(\Delta := a_1 a_2 b_2^2 [a_1 a_2 (1 + \xi_1)^2 - 4\xi_1] < 0)$.
μ_1, μ_2 (1.4), $\eta_1 > 0$

$$\eta_1 \in \left(\frac{u_2^* a_1 \mu_1 \xi_1 \chi_2^2(v)}{4 a_2 \mu_2 d_2(v)} + \frac{u_1^* \chi_1^2(v)}{4 d_1(v)} \, , \; \frac{a_1 \mu_1 (4\xi_1 - a_1 a_2 (1 + \xi_1)^2)}{a_1 b_1^2 \xi_1 + a_2 b_2^2 - a_1 a_2 b_1 b_2 (1 + \xi_1)} \right). \tag{3.6}$$

$\zeta_1 = \frac{a_1 \mu_1 \xi_1}{a_2 \mu_2}$ η_1 (3.6), $\mathcal{E}_1(t)$

$$\mathcal{E}_1(t) = I_1(t) + \frac{a_1 \mu_1 \xi_1}{a_2 \mu_2} I_2(t) + \eta_1 I_3(t), \tag{3.7}$$

$$\begin{cases} I_1(t) & = \int_\Omega (u_1 - u_1^* - u_1^* \quad \frac{u_1}{u_1^*}), \\ I_2(t) & = \int_\Omega (u_2 - u_2^* - u_2^* \quad \frac{u_2}{u_2^*}), \\ I_3(t) & = \frac{1}{2} \int_\Omega (v - v^*)^2. \end{cases}$$

, 2, . 568 569 , I_1 I_2
$\mathcal{E}_1(t) \geq 0$.

, (3.2). ,

$$
\begin{aligned}
\frac{d}{dt}I_1(t)=&\int\limits_\Omega \left(u_{1t}-\frac{u_1^*}{u_1}u_{1t}\right)\\
=&\mu_1\int\limits_\Omega (u_1-u_1^*)(1-u_1-a_1u_2)+u_1^*\int\limits_\Omega d_1(v)\nabla u_1\cdot\nabla\left(\frac{1}{u_1}\right)\\
&-u_1^*\int\limits_\Omega \chi_1(v)u_1\nabla v\cdot\nabla\left(\frac{1}{u_1}\right)\\
=&-\mu_1\int\limits_\Omega (u_1-u_1^*)^2-a_1\mu_1\int\limits_\Omega (u_1-u_1^*)(u_2-u_2^*)\\
&-u_1^*\int\limits_\Omega d_1(v)\left|\frac{\nabla u_1}{u_1}\right|^2+u_1^*\int\limits_\Omega \chi_1(v)\frac{\nabla u_1}{u_1}\cdot\nabla v,
\end{aligned}
\tag{3.8}
$$

$u_1^*+a_1u_2^*=1.$, $a_2u_1^*+u_2^*=1,$

$$
\begin{aligned}
\frac{d}{dt}I_2(t)=&\int\limits_\Omega \left(u_{2t}-\frac{u_2^*}{u_2}u_{2t}\right)\\
=&-\mu_2\int\limits_\Omega (u_2-u_2^*)^2-a_2\mu_2\int\limits_\Omega (u_1-u_1^*)(u_2-u_2^*)\\
&-u_2^*\int\limits_\Omega d_2(v)\left|\frac{\nabla u_2}{u_2}\right|^2+u_2^*\int\limits_\Omega \chi_2(v)\frac{\nabla u_2}{u_2}\cdot\nabla v.
\end{aligned}
\tag{3.9}
$$

, $b_1u_1^*+b_2u_2^*=v^*,$

$$
\begin{aligned}
\frac{d}{dt}I_3(t)=&\int\limits_\Omega (v-v^*)v_t\\
=&-\int\limits_\Omega |\nabla v|^2+b_1\int\limits_\Omega (u_1-u_1^*)(v-v^*)\\
&+b_2\int\limits_\Omega (u_2-u_2^*)(v-v^*)-\int\limits_\Omega (v-v^*)^2.
\end{aligned}
\tag{3.10}
$$

(3.8) (3.10) (3.7),

$$
\frac{d}{dt}\mathcal{E}_1(t)=-\int\limits_\Omega X_1A_1X_1^T-\int\limits_\Omega Y_1B_1Y_1^T,
\tag{3.11}
$$

X_1 Y_1

$$X_1(x,t) := (u_1(x,t) - u_1^*, u_2(x,t) - u_2^*, v(x,t) - v^*)$$

$$Y_1(x,t) := \left(\frac{|\nabla u_1(x,t)|}{u_1(x,t)}, \frac{|\nabla u_2(x,t)|}{u_2(x,t)}, |\nabla v(x,t)| \right)$$

$\Omega \times (0, \infty)$, A_1 B_1

$$A_1 := \begin{pmatrix} \mu_1 & \frac{a_1\mu_1(1+\xi_1)}{2} & -\frac{\eta_1 b_1}{2} \\ \frac{a_1\mu_1(1+\xi_1)}{2} & \frac{a_1\mu_1\xi_1}{a_2} & -\frac{\eta_1 b_2}{2} \\ -\frac{\eta_1 b_1}{2} & -\frac{\eta_1 b_2}{2} & \eta_1 \end{pmatrix}$$

$$B_1 := \begin{pmatrix} u_1^* d_1(v) & 0 & -\frac{u_1^*\chi_1(v)}{2} \\ 0 & \frac{u_2^* a_1\mu_1\xi_1 d_2(v)}{a_2\mu_2} & -\frac{u_2^* a_1\mu_1\xi_1\chi_2(v)}{2a_2\mu_2} \\ -\frac{u_1^*\chi_1(v)}{2} & -\frac{u_2^* a_1\mu_1\xi_1\chi_2(v)}{2a_2\mu_2} & \eta_1 \end{pmatrix}.$$

, A_1 B_1 . (3.4) (3.6),

$$|\mu_1| > 0$$

$$\begin{vmatrix} \mu_1 & \frac{a_1\mu_1(1+\xi_1)}{2} \\ \frac{a_1\mu_1(1+\xi_1)}{2} & \frac{a_1\mu_1\xi_1}{a_2} \end{vmatrix} = \frac{a_1\mu_1^2}{4a_2}\left(4\xi_1 - a_1a_2(1+\xi_1)^2\right) > 0$$

$$|A_1| = \frac{\mu_1\eta_1}{4a_2}\left(a_1\mu_1(4\xi_1 - a_1a_2(1+\xi_1)^2) - \eta_1(a_1b_1^2\xi_1 + a_2b_2^2 - a_1a_2b_1b_2(1+\xi_1))\right) > 0$$

$a_i > 0$ $(i = 1, 2)$. A_1 , -
. , $\zeta_1 = \frac{a_1\mu_1\xi_1}{a_2\mu_2} > 0$,

$|u_1^* d_1(v)| > 0$ $\begin{vmatrix} u_1^* d_1(v) & 0 \\ 0 & u_2^*\zeta_1 d_2(v) \end{vmatrix} = u_1^* u_2^* \zeta_1 d_1(v) d_2(v) > 0,$

(3.6)

$$|B_1| = u_1^* d_1(v) \begin{vmatrix} \frac{u_2^* a_1 \mu_1 \xi_1 d_2(v)}{a_2 \mu_2} & -\frac{u_2^* a_1 \mu_1 \xi_1 \chi_2(v)}{2 a_2 \mu_2} \\ -\frac{u_2^* a_1 \mu_1 \xi_1 \chi_2(v)}{2 a_2 \mu_2} & \eta_1 \end{vmatrix}$$

$$- \frac{u_1^* \chi_1(v)}{2} \begin{vmatrix} 0 & \frac{u_2^* a_1 \mu_1 \xi_1 d_2(v)}{a_2 \mu_2} \\ -\frac{u_1^* \chi_1(v)}{2} & -\frac{u_2^* a_1 \mu_1 \xi_1 \chi_2(v)}{2 a_2 \mu_2} \end{vmatrix}$$

$$= \frac{u_1^* u_2^* a_1 \mu_1 \xi_1}{a_2 \mu_2} \Bigg($$

$a_1 \geq 1$ $a_2 < 1$

$a_1 \geq 1$ $a_2 < 1$, t $a\,t$

$$\begin{cases} J_1(t) &= \int_\Omega u_1, \\ J_2(t) &= \int_\Omega (u_2 - 1 - \quad u_2), \\ J_3(t) &= \frac{1}{2}\int_\Omega (v-b_2)^2. \end{cases}$$

,

$$\begin{aligned} \frac{d}{dt}J_1(t) = \frac{d}{dt}\int_\Omega u_1 =& \mu_1 \int_\Omega u_1(1-u_1-a_1u_2) \\ =& -\mu_1\int_\Omega u_1^2 - a_1^*\mu_1\int_\Omega u_1(u_2-1) - \mu_1(a_1-a_1^*)\int_\Omega u_1u_2 \\ & -\mu_1(a_1^*-1)\int_\Omega u_1, \end{aligned} \tag{3.18}$$

$$\begin{aligned} \frac{d}{dt}J_2(t) =& -\mu_2\int_\Omega (u_2-1)^2 - a_2\mu_2\int_\Omega u_1(u_2-1) - \int_\Omega d_2(v)\left|\frac{\nabla u_2}{u_2}\right|^2 \\ & + \int_\Omega \chi_2(v)\frac{\nabla u_2}{u_2}\cdot\nabla v, \end{aligned} \tag{3.19}$$

$$\begin{aligned} \frac{d}{dt}J_3(t) = \int_\Omega (v-b_2)v_t =& -\int_\Omega |\nabla v|^2 - \int_\Omega (v-b_2)^2 \\ & + b_1\int_\Omega u_1(v-b_2) + b_2\int_\Omega (u_2-1)(v-b_2). \end{aligned} \tag{3.20}$$

, (3.18) (3.20) (3.17)

$$\begin{aligned} \frac{d}{dt}\mathcal{E}_2(t) =& -X_2A_2X_2^T - Y_2B_2Y_2^T - \mu_1(a_1-a_1^*)\int_\Omega u_1u_2 - \mu_1(a_1^*-1)\int_\Omega u_1 \\ \le& -X_2A_2X_2^T - Y_2B_2Y_2^T - \mu_1(a_1^*-1)\int_\Omega u_1, \end{aligned} \tag{3.21}$$

$X_2 = (u_1, u_2 - 1, v - b_2)$ $\quad Y_2 = \left(\frac{|\nabla u_2|}{u_2}, |\nabla v|\right)$,

$$A_2 := \begin{pmatrix} \mu_1 & \frac{a_1^*\mu_1(1+\xi_2)}{2} & -\frac{\eta_2 b_1}{2} \\ \frac{a_1^*\mu_1(1+\xi_2)}{2} & \frac{a_1^*\mu_1\xi_2}{a_2} & -\frac{\eta_2 b_2}{2} \\ -\frac{\eta_2 b_1}{2} & -\frac{\eta_2 b_2}{2} & \eta_2 \end{pmatrix} \qquad B_2 := \begin{pmatrix} \frac{a_1^*\mu_1\xi_2 d_2(v)}{a_2\mu_2} & -\frac{a_1^*\mu_1\xi_2\chi_2(v)}{2a_2\mu_2} \\ -\frac{a_1^*\mu_1\xi_2\chi_2(v)}{2a_2\mu_2} & \eta_2 \end{pmatrix}.$$

, A_2 B_2. $4\xi_2 - a_1^* a_2(1+\xi_2)^2 > 0,$

$|\mu_1| > 0$

$$\begin{vmatrix} \mu_1 & \frac{a_1^*\mu_1(1+\xi_2)}{2} \\ \frac{a_1^*\mu_1(1+\xi_2)}{2} & \frac{a_1^*\mu_1\xi_2}{a_2} \end{vmatrix} = \frac{a_1^*\mu_1^2}{4a_2}\left(4\xi_2 - a_1^* a_2(1+\xi_2)^2\right) > 0.$$

, η_2 (3.16),

$$|A_2| = \frac{\mu_1\eta_2}{4a_2}\left(a_1^*\mu_1(4\xi_2 - a_1^* a_2(1+\xi_2)^2) - \eta_2(a_1^* b_1^2\xi_2 + a_2 b_2^2 - a_1^* a_2 b_1 b_2(1+\xi_2))\right) > 0.$$

,

$$\left|\frac{a_1^*\mu_1\xi_2 d_2(v)}{a_2\mu_2}\right| > 0$$

$$|B_2| = \frac{a_1^*\mu_1\xi_2 d_2(v)}{a_2\mu_2}\left(\eta_2 - \frac{a_1^*\mu_1\xi_2\chi_2^2(v)}{4a_2\mu_2 d_2(v)}\right) > 0$$

$\eta_2 > \frac{a_1^*\mu_1\xi_2\chi_2^2(v)}{4a_2\mu_2 d_2(v)}$. , A_2 B_2
. , (3.21) u_i $(i = 1, 2)$ (3.14). □

Lemma 3.4. - - -

$$\|u_1(\cdot,t)\|_{L^\infty} + \|u_2(\cdot,t) - 1\|_{L^\infty} + \|v(\cdot,t) - b_2\|_{L^\infty} \to 0 \quad t \to +\infty. \tag{3.22}$$

Proof. 3.3, $\mathcal{E}_2(t) \geq 0$, $\frac{d}{dt}\mathcal{E}_2(t) \leq 0$ $\frac{d}{dt}\mathcal{E}_2(t) = 0$ $(u_1, u_2, v) = (0, 1, b_2)$. , 3.2, (3.22) . □

-

, L^∞- . , L^p- $p \geq 1$ 3.1 3.3.

, C t $\|\nabla u_1\|_{L^4} + \|\nabla u_2\|_{L^4} + \|v\|_{W^{1,\infty}} \leq C$ $t > 1$, L^p- - L^∞- . , .

Lemma 3.5. (u_1, u_2, v) (1.1) $\sigma \in (0,1)$ $C > 0$

$$\|v\|_{C^{2+\sigma,1+\frac{\sigma}{2}}(\bar{\Omega}\times[t,t+1])} \le C, \qquad t > 1. \tag{3.23}$$

Proof. 1.1 c_1, c_2, c_3

$$0 < u_1(x,t), u_2(x,t) \le c_1, 0 < v(x,t) \le c_2 \qquad |\nabla v(x,t)| \le c_3$$

$x \in \Omega$ $t > 0$. ,
25, 1.3 1.4

$$\|u_1\|_{C^{\sigma,\frac{\sigma}{2}}(\bar{\Omega}\times[t,t+1])} \le c_4, \qquad t > 1. \tag{3.24}$$

, (1.1)

$$u_{1t} = \nabla \cdot f(x,t,\nabla u_1) + g(x,t) \qquad x \in \Omega \qquad t > 0$$

$$f(x,t,\nabla u_1) := d_1(v)\nabla u_1 - \chi_1(v) u_1 \nabla v$$

$$g(x,t) := \mu_1 u_1 (1 - u_1 - a_1 u_2).$$

(1) ' ,

$$\begin{aligned} f(x,t,\nabla u_1)\cdot\nabla u_1 &= d_1(v)|\nabla u_1|^2 - \chi_1(v) u_1 \nabla v \cdot \nabla u_1 \\ &\ge d_1(v)|\nabla u_1|^2 - |\chi_1(v)|u_1|\nabla v||\nabla u_1| \\ &\ge \frac{d_1(v)}{2}|\nabla u_1|^2 - \frac{|\chi_1(v)|^2}{2d_1(v)} u_1^2 |\nabla v|^2 \\ &\ge \frac{d_1(c_2)}{2}|\nabla u_1|^2 - c_5 \end{aligned} \tag{3.25}$$

$$|f(x,t,\nabla u_1)| \le d(0)|\nabla u_1| + c_6, \qquad |g(x,t)| \le c_7 \tag{3.26}$$

$x \in \Omega$ $t > 0$. , (3.24)
(3.25) (3.26). , (1.1),

$$\|u_2\|_{C^{\sigma,\frac{\sigma}{2}}(\bar{\Omega}\times[t,t+1])} \le c_8$$

$t > 1$. 17
(1.1), (3.23). 3.5 . □

Lemma 3.6. $\Omega \subset \mathbb{R}^2$ (u_1, u_2, v) (1.1) $\mu_i,\ a_i \geq 0$ $i = 1, 2$, $p > 1,$ $C > 0$ $t > 1$

$$\|\nabla u_1(\cdot, t)\|_{L^{2p}} + \|\nabla u_2(\cdot, t)\|_{L^{2p}} \leq C.$$

Proof. (1.1) $|\nabla u_1|^{p-2}\nabla u_1$

$$\begin{aligned}\frac{1}{2p}\frac{d}{dt}\int |\nabla u_1|^{2p} &= \int_\Omega |\nabla u_1|^{2p-2}\nabla u_1 \cdot \nabla(\nabla \cdot (d_1(v)\nabla u_1)) \\ &\quad - \int_\Omega |\nabla u_1|^{2p-2}\nabla u_1 \cdot \nabla(\nabla \cdot (\chi_1(v)u_1\nabla v)) \\ &\quad + \mu_1 \int_\Omega |\nabla u_1|^{2p-2}\nabla u_1 \cdot \nabla(u_1 - u_1^2 - a_1 u_1 u_2) \\ &=: G_1 + G_2 + G_3.\end{aligned} \tag{3.27}$$

$\nabla\Delta u_1 \cdot \nabla u_1 = \frac{1}{2}\Delta|\nabla u_1|^2 - |D^2 u_1|^2,$ G_1

$$\begin{aligned}G_1 &= -\int_\Omega |\nabla u_1|^{2p-2}\Delta u_1 \nabla \cdot (d_1(v)\nabla u_1) - \int_\Omega \nabla|\nabla u_1|^{2p-2} \cdot \nabla u_1 \nabla \cdot (d_1(v)\nabla u_1) \\ &= \int_\Omega d_1(v)|\nabla u_1|^{2p-2}\nabla\Delta u_1 \cdot \nabla u_1 - \int_\Omega d_1'(v)\nabla|\nabla u_1|^{2p-2} \cdot \nabla u_1 \nabla u_1 \cdot \nabla v \\ &= \frac{1}{2}\int_\Omega d_1(v)|\nabla u_1|^{2p-2}\Delta|\nabla u_1|^2 - \int_\Omega d_1(v)|\nabla u_1|^{2p-2}|D^2 u_1|^2 \\ &\quad - (p-1)\int_\Omega d_1'(v)|\nabla u_1|^{2p-4}\nabla|\nabla u_1|^2 \cdot \nabla u_1 \nabla u_1 \cdot \nabla v.\end{aligned} \tag{3.28}$$

, (3.28)

$$\frac{1}{2}\int_\Omega d_1(v)|\nabla u_1|^{2p-2}\Delta|\nabla u_1$$

$\|v(\cdot,t)\|_{W^{1,\infty}}$ $d_1(v)\in C^2([0,\infty))$,
c_1, c_2, c_3 c_4

$$0<c_1\le d_1(v)\le c_2,\qquad \|\nabla v(\cdot,t)\|_{L^\infty}\le c_3,\qquad |d_1'(v)|\le c_4.$$

$\frac{\partial|\nabla u_1|^2}{\partial\nu}\le 2\kappa|\nabla u_1|^2$ $\kappa>0$ $\partial\Omega$ (20, 4.2)
26, 52.9

$$\|\phi\|_{L^2(\partial\Omega)}\le\varepsilon\|\nabla\phi\|_{L^2(\Omega)}+c_\varepsilon\|\phi\|_{L^2(\Omega)}$$

$\varepsilon>0$,

$$\begin{aligned}\frac{1}{2}\int\limits_{\partial\Omega}d_1(v)|\nabla u_1|^{2p-2}\frac{\partial|\nabla u_1|^2}{\partial\nu}dS&\le c_2\kappa\int\limits_{\partial\Omega}|\nabla u_1|^{2p}dS\\&=c_2\kappa\||\nabla u_1|^p\|^2_{L^2(\partial\Omega)}\\&\le\frac{2c_1(p-1)}{3p^2}\|\nabla|\nabla u_1|^p\|^2_{L^2(\Omega)}+c_5\||\nabla u_1|^p\|^2_{L^2(\Omega)},\end{aligned}$$

(3.29)

$$\begin{aligned}&\frac{1}{2}\int\limits_\Omega d_1(v)|\nabla u_1|^{2p-2}\Delta|\nabla u_1|^2\\&\le-\frac{c_1(p-1)}{3}\int\limits_\Omega|\nabla u_1|^{2p-4}|\nabla|\nabla u_1|^2|^2+\frac{c_3c_4}{2}\int\limits_\Omega|\nabla u_1|^{2p-2}|\nabla|\nabla u_1|^2|\\&\quad+c_5\int\limits_\Omega|\nabla u_1|^{2p}.\end{aligned}\tag{3.30}$$

(3.30) (3.28) ’ ,

$$\begin{aligned}G_1\le&-\frac{c_1(p-1)}{3}\int\limits_\Omega|\nabla u_1|^{2p-4}|\nabla|\nabla u_1|^2|^2-c_1\int\limits_\Omega|\nabla u_1|^{2p-2}|D^2u_1|^2\\&+c_6\int\limits_\Omega|\nabla u_1|^{2p-2}|\nabla|\nabla u_1|^2|+c_5\int\limits_\Omega|\nabla u_1|^{2p}\\\le&-\frac{c_1(p-1)}{6}\int\limits_\Omega|\nabla u_1|^{2p-4}|\nabla|\nabla u_1|^2|^2-c_1\int\limits_\Omega|\nabla u_1|^{2p-2}|D^2u_1|^2\\&+c_7\int\limits_\Omega|\nabla u_1|^{2p}\end{aligned}\tag{3.31}$$

$c_7 = \frac{3c_6^2}{2c_1(p-1)} + c_5.$, $u_1,\ v$ ∇v
$\chi_1(\cdot) \in C^2([0,\infty)),$

$$\nabla \cdot (\chi_1(v)u_1\nabla v) \le c_8(1 + |\nabla u_1| + |\Delta v|).$$

$|\Delta u_1| \le \sqrt{n}|D^2u_1|$,
$c_9 = \frac{9c_8^2(6(p-1)+n)}{2c_1}$

$G_2 = \int$

2.166 51929.3Tm (2)Tj /T1_4 1 Tf 9.9

$$\frac{1}{2p}\frac{d}{dt}\int_{\Omega} |\nabla u_1|^{2p} + \frac{1}{p}\int_{\Omega} |\nabla u_1|$$

$$\leq -\frac{c_1(p-1)}{12}\int_{\Omega} |\nabla u_1|^{2p-4}|\nabla|\nabla u_1|^2|^2 - \frac{c_1}{2}\int_{\Omega} |\nabla u_1|^{2p-2}|D^2 u_1|^2$$

$$+c$$

$$\frac{d}{dt}\left(\int_\Omega |\nabla u_1|^{2p} + \int_\Omega |\nabla u_2|^{2p}\right) + \left(\int_\Omega |\nabla u_1|^{2p} + \int_\Omega |\nabla u_2|^{2p}\right) \le (2pc_9 + c_{15})\int_\Omega |\Delta v|^{2p} + c_{14} + c_{15}. \tag{3.37}$$

(3.37) (3.23)

$$\|\nabla u_1(\cdot,t)\|_{L^{2p}} + \|\nabla u_2(\cdot,t)\|_{L^{2p}} \le c_{16}$$

$t > 1$. , 3.6 . □

, (1.1) : $0 < a_1 < 1$ $0 < a_2 < 1$.

Lemma 3.7 *(* - - $a_1, a_2 \in (0,1)$*)* - - - λ_1 - C_1, -

$$\|u_1(\cdot,t) - u_1^*\|_{L^\infty} + \|u_2(\cdot,t) - u_2^*\|_{L^\infty} + \|v(\cdot,t) - v^*\|_{L^\infty} \le C_1 e^{-\lambda_1 t} \tag{3.38}$$

- $t > 0$

Proof. 2, 3.7 , , . , $\varphi(w) := w - u_1^*$ w $w > 0$

$$\frac{\varphi(w) - \varphi(u_1^*)}{(w - u_1^*)^2}_{w \to u_1^*} = \frac{\varphi'(w)}{2(w - u_1^*)}_{w \to u_1^*} = \frac{1}{2u_1^*}.$$

$\|u_1(\cdot,t) - u_1^*\|_{L^\infty} \to 0$ $t \to \infty$ (3.2), $t_1 > 0$ $t > t_1$

$$\int_\Omega \left(u_1 - u_1^* - u_1^* \quad \frac{u_1}{u_1^*}\right) = \int_\Omega (\varphi(u_1) - \varphi(u_1^*)) \le \frac{1}{u_1^*}\int_\Omega (u_1 - u_1^*)^2 \tag{3.39}$$

$$\int_\Omega \left(u_1 - u_1^* - u_1^* \quad \frac{u_1}{u_1^*}\right) \ge \frac{1}{4u_1^*}\int_\Omega (u_1 - u_1^*)^2. \tag{3.40}$$

, $\|u_2(\cdot,t) - u_2^*\|_{L^\infty} \to 0$ $t \to \infty$ $t_2 > 0$ $t > t_2$

$$\frac{1}{4u_2^*}\int_\Omega (u_2 - u_2^*)^2 \le \int_\Omega \left(u_2 - u_2^* - u_2^* \quad \frac{u_2}{u_2^*}\right) \le \frac{1}{u_2^*}\int_\Omega (u_2 - u_2^*)^2. \tag{3.41}$$

$t_3 =$ $\{1, t_1, t_2\}$. $\mathcal{E}_1(t)$ $\mathcal{F}_1(t)$ (3.1) (3.3), - (3.39) (3.41) $\mathcal{E}_1(t) \le \left(\frac{1}{u_1^*} + \frac{\zeta_1}{u_2^*} + \frac{\eta_1}{2}\right)\mathcal{F}_1(t)$ $t > t_3$, (3.2)

$$\frac{d}{dt}\mathcal{E}_1(t) \le -\varepsilon_1\mathcal{F}_1(t) \le -c_1\mathcal{E}_1(t), \quad \forall t > t_3, \tag{3.42}$$

$c_1 = \frac{2u_1^*u_2^*\varepsilon_1}{2\zeta_1 u_1^* + 2u_2^* + \eta_1 u_1^* u_2^*}$. , , (3.42) $\mathcal{E}_1(t_3)$,

$$\mathcal{E}_1(t) \le c_2 e^{-c_1 t}, \quad \forall t > t_3,$$

(3.40) (3.41)

$$\int\limits_\Omega (u_1 - u_1^*)^2 + \int\limits_\Omega (u_2 - u_2^*)^2 + \int\limits_\Omega (v - v^*)^2 \le c_3 e^{-c_1 t}, \quad \forall t > t_3. \tag{3.43}$$

$p = 2$ 3.6 $\|v\|_{W^{1,\infty}}$ 1.1, $\|u_1(\cdot$ [illegible]

Proof. (3.41), $t_0 > 1$

$$\frac{1}{4}\int_\Omega (u_2-1)^2 \le \int_\Omega (u_2 - 1 - \quad u_2) \le \int_\Omega (u_2-1)^2, \quad \forall t > t_0. \tag{3.47}$$

$\mathcal{F}_3(t) := \int_\Omega u_1 + \int_\Omega (u_2-1)^2 + \int_\Omega (v-b_2)^2.$ (3.13),
(3.47) $c_1 > 0$

$$\mathcal{E}_2(t) \le c_1 \mathcal{F}_3(t), \quad \forall t > t_0. \tag{3.48}$$

$a_1^* > 1$ $0 < a_2 < 1$, (3.14) (3.48),

$$\begin{aligned} \frac{d}{dt}\mathcal{E}_2(t) &\le -\varepsilon_2 \mathcal{F}_2(t) - \mu_1(a_1^* - 1)\int_\Omega u_1 \\ &\le -c_2 \mathcal{F}_3(t) \le -c_3 \mathcal{E}_2(t), \quad \forall t > t_0, \end{aligned} \tag{3.49}$$

, ,

$$\mathcal{E}_2(t) \le c_4 e^{-c_3 t}, \quad \forall t > t_0$$

$\mathcal{E}_2(t_0)$. $\mathcal{E}_2(t)$ (3.13)
(3.47)

$$\int_\Omega u_1 + \int_\Omega (u_2-1)^2 + \int_\Omega (v-b_2)^2 \le c_5 e^{-c_3 t}, \quad \forall t > t_0. \tag{3.50}$$

, 3.6 $\|\nabla u_1(\cdot,t)\|_{L^4} + \|\nabla u_2(\cdot,t)\|_{L^4} \le c_6$ $t > t_0$. -
1.1 ,

$$\|u_1\|_{L^\infty} \le c_7 \left(\|\nabla u_1\|_{L^4}^{\frac{4}{5}} \|u_1\|_{L^1}^{\frac{1}{5}} + \|u_1\|_{L^1} \right) \le c_8 \|u_1\|_{L^1}^{\frac{1}{6}}. \tag{3.51}$$

,

$$\|u_2 - 1\|_{L^\infty} \le c_9 \left(\|\nabla u_2\|_{L^4}^{\frac{2}{3}} \|u_2-1\|_{L^2}^{\frac{1}{3}} + \|u_2 - 1\|_{L^2} \right) \le c_{10} \|u_2 - 1\|_{L^2}^{\frac{1}{3}} \tag{3.52}$$

$$\|v - b_2\|_{L^\infty} \le c_{11} \left(\|\nabla v\|_{L^4}^{\frac{2}{3}} \|v-b_2\|_{L^2}^{\frac{1}{3}} + \|v - b_2\|_{L^2} \right) \le c_{12} \|v - b_2\|_{L^2}^{\frac{1}{3}}. \tag{3.53}$$

(3.51) (3.53) (3.50)

$$\|u_1\|_{L^\infty} + \|u_2 - 1\|_{L^\infty} + \|v - b_2\|_{L^\infty} \le c_{13} e^{-\frac{c_3 t}{6}}, \quad \forall t > t_0,$$

(3.46) C_2 . 3.8

. □

$a_1 = 1$, $a_1^* = 1$, $-\mu_1(a_1^* - 1)\int_\Omega u_1$ (3.14) .
(3.14) (3.49),

. , .

Lemma 3.9. $a_1 = 1$ 1 There we

$$\|u_1\|_{L^\infty} + \|u_2 - 1\|_{L^\infty} + \|v - b_2\|_{L^\infty} \le c_4 \left(\|u_1\|_{L^1}^{\frac{1}{6}} + \|u_2 - 1\|_{L^2}^{\frac{1}{3}} + \|v - b_2\|_{L^2}^{\frac{1}{3}} \right)$$

$t > t_0$, (3.58) (3.54) . □

Proof of Theorem 1.2. 3.7 3.9 1.2. □

Acknowledgments

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