

ANALYSIS OF THE ROLE OF CONVECTION IN A SYSTEM DESCRIBING THE TUMOR-INDUCED ANGIOGENESIS

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Abstract. In this paper, we shall study the initial-boundary value problem of a mathematical model describing the branching of capillary sprouts during angiogenesis in one dimensional space. Under homogeneous Neumann boundary conditions, we show the existence of a unique global classical solution with uniform-in-time bound for all suitably regular initial data. Moreover, we show that the unique solution will exponentially converge to a non-trivial constant steady state as time tends to infinity under some appropriate conditions on the parameters.

Keywords. boundedness; chemotaxis; haptotaxis; convergence rate.

AMS subject classifications. 35A01; 35B40; 35K55; 35Q92; 92C17.

1. Introduction

Angiogenesis describes the formation of blood vessels from pre-existing vasculature. Tumor-induced angiogenesis plays an important role in the tumor invasion of the surrounding host tissue and blood system. Many mathematical models have been proposed to describe tumor angiogenesis, where chemotaxis, as the principal mechanism for cell motion, has been studied in angiogenesis [4, 28]. However, there exists some experimental evidence showing that in the early stage of angiogenesis, haptotaxis and convection play important roles [27, 30, 33]. In fact, during angiogenesis, the endothelial cells (ECs) secrete a matrix consisting of fibronectin, laminin, and collagen IV such that the movement of the ECs can be affected by the distribution of adhesive sites on this matrix [27]. Moreover, as the matrix spreads out, the convection can also play a major role in the transport of ECs [23].

To study the effect of haptotaxis and convection on the endothelial cells within the growing and developing capillary sprouts and secondary branching as they migrate towards the tumor cells, Orme & Chaplain [25] proposed the following system

$$\begin{cases} u_t = d_1 \Delta u - \nabla \cdot (u \nabla v) + \alpha_1 \nabla \cdot (u \nabla w); \\ v_t = d_2 \Delta v + \alpha_2 \nabla \cdot (v \nabla w) + \beta_1 u - \beta_2 v; \\ w_t = d_3 \Delta w + \beta_3 u - \beta_4 w; \end{cases} \quad (1.1)$$

where $u(x; t)$ describes the density of endothelial cells (ECs), $v(x; t)$ is the density of adhesive sites, and $w(x; t)$ stands for the density of the matrix. Here $d_1; d_2; d_3$ are diffusion coefficients; α_1 is the rate at which cells move up an adhesive gradient; $\beta_1; \beta_2; \beta_3; \beta_4$ are the decay of adhesive sites and matrix, respectively; $\alpha_2; \alpha_3$ describe the rate of secretion of adhesive sites and matrix per cell, respectively. In the model (1.1), the movement of the ECs is governed by a combination of random motility, haptotaxis and convection. More precisely, the cells move up an adhesive gradient (such as fibronectin secreted by the ECs during the angiogenic process), where the convection of cells depending on the spreading of the matrix with the convective flux is proportional to $-\alpha_1 \nabla w$. The

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movement of adhesive sites depends on the random diffusion and the convection flux is proportional to $-\chi_2 \nabla w$.

The key feature of (1.1) is the coupling of haptotaxis and convection, which increases the mathematical complexity of (1.1) compared with the classical Keller-Segel (KS) model

$$\begin{cases} u_t = d_1 \Delta u - \nabla \cdot (u \nabla v); \\ v_t = d_2 \Delta v + \chi_1 u - \chi_2 v; \end{cases} \quad (1.2)$$

which can be obtained by letting $\chi_1 = \chi_2 = 0$ in (1.1). The KS model (1.2) has been extensively studied in various perspectives in the past four decades, one can find the survey articles [2, 7] and references therein for details. One of the most studied topics for the KS model (1.2) is the boundedness and blowup of solutions in two or higher dimensions [8, 24, 31, 32] based on the following Lyapunov function:

$$\mathcal{E}_1(u; v) = d_1 \int_{\Omega} u \ln u - \int_{\Omega} uv + \frac{1}{2} \int_{\Omega} v^2 + \frac{d_2}{2} \int_{\Omega} |\nabla v|^2;$$

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states. Precisely, we shall study the following system

$$\left\{ \begin{array}{l} u \\ \end{array} \right.$$

that the system (1.4) has a unique classical solution $(u; v; w) \in [C^0(\bar{\Omega} \times [0; T_{max}); \mathbb{R}^3)]^3 \cap [C^{2,1}(\bar{\Omega} \times (0; T_{max}); \mathbb{R}^3)]^3$ satisfying $u; v; w > 0$. Moreover, we have

either $T_{max} = \infty$ or $\|u(\cdot; t)\|_{W^{1,2}} + \|v(\cdot; t)\|_{W^{1,2}} + \|w(\cdot; t)\|_{W^{1,2}} \rightarrow \infty$ as $t \nearrow T_{max}$.

Proof. Define $(u; v; w) \in \mathbb{R}^3$. Then the system (1.4) can be rewritten as

$$\begin{cases} t - \nabla \cdot (A(\cdot) \nabla \cdot) = B(\cdot); & x \in \Omega; t > 0; \\ \frac{\partial \Phi}{\partial \nu} = 0; & x \in \partial \Omega; t > 0; \\ (\cdot; 0) = (u_0; v_0; w_0); & x \in \partial \Omega; \end{cases} \quad (2.1)$$

where

$$A(\cdot) = \begin{pmatrix} d_1 - u & u & 1 \\ 0 & d_2 & 2v \\ 0 & 0 & d_3 \end{pmatrix}; \quad B(\cdot) = \begin{pmatrix} 0 \\ u - v \\ u - w \end{pmatrix};$$

Since all the eigenvalues of $A(\cdot)$ are positive and hence the system (2.1) is normally parabolic. Then using the results in [1, Theorem 14.6], we obtain the local existence of solutions and the extension criterion is a direct consequence of [1, Theorem 15.3]. At last, the positivity of the solutions follows from [1, Theorem 15.1]. $\square$

2.2. L^2 -estimate of v . To extend the local solution to global one, we need to establish the *a priori* estimates of solution. First, we have the following results.

LEMMA 2.2. Suppose the conditions in Lemma 2.1 hold and $\alpha_2 \in (0; 1]$. Then the solution $(u; v; w)$ of (1.4) satisfies

$$\|u(\cdot; t)\|_{L^1} + \|w(\cdot; t)\|_{H^1} \leq C(\|u_0\|_{L^1} + \|w_0\|_{H^1}) \quad (2.2)$$

and

$$\|v(\cdot; t)\|_{L^2} \leq C(\|u_0\|_{L^1} + \|v_0\|_{L^2} + \|w_0\|_{H^1}); \quad (2.3)$$

where $C > 0$ is a constant independent of α_2 and t .

Proof. Integrating the first equation of (1.4), we derive that $\frac{d}{dt} \int_{\Omega} u = 0$, which implies that

$$\|u\|_{L^1} = \|u_0\|_{L^1}. \quad (2.4)$$

Using (2.4) and applying the parabolic regularity (see [13, Lemma 1]) to the third equation of (1.4), one has

$$\|w\|_{H^1} \leq c_1(\|u_0\|_{L^1} + \|w_0\|_{H^1}). \quad (2.5)$$

To proceed, we integrate the equation of v in (1.4) over Ω to obtain

$$\frac{d}{dt} \int_{\Omega} v + \int_{\Omega} v = \int_{\Omega} u = \int_{\Omega} u_0;$$

which together with Grönwall's inequality yields that

$$\|v\|_{L^1} \leq \|v_0\|_{L^1} + \int_{\Omega} u_0. \quad (2.6)$$

Multiplying the second equation of (1.4) by v and using Cauchy-Schwarz inequality, one obtains that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\Omega} v^2 + \int_{\Omega} v^2 + d_2 \int_{\Omega} v_x^2 &= \int_{\Omega} uv - \int_{\Omega} vv_x w_x \\ &\leq \|v\|_{L^1} \|u\|_{L^1} + \frac{1}{2} \|v\|_{L^1} \|v_x\|_{L^2} \|w_x\|_{L^2} \\ &\leq c_2 (\|u_0\|_{L^1} + \|w_0\|_{H^1}) (\|v\|_{L^1} + \frac{1}{2} \|v\|_{L^1} \|v_x\|_{L^2}): \end{aligned} \quad (2.7)$$

Noting (2.6) and using Gagliardo-Nirenberg inequality, we have

$$\|v\|_{L^1} \leq c_3 (\|v_x\|_{L^2}^{\frac{2}{3}} \|v\|_{L^1}^{\frac{1}{3}} + \|v\|_{L^1}) \leq c_4 (\|v_x\|_{L^2}^{\frac{2}{3}} + 1): \quad (2.8)$$

Then substituting (2.8) into (2.7), and using (2.4){(2.5)}, one has

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\Omega} v^2 + \int_{\Omega} v^2 + d_2 \int_{\Omega} v_x^2 &\leq c_5 (\frac{1}{2} + 1) (\|v_x\|_{L^2}^{\frac{5}{3}} + \|v_x\|_{L^2} + \|v_x\|_{L^2}^{\frac{2}{3}} + 1) \\ &\leq \frac{d_2}{2} \|v_x\|_{L^2}^2 + c_6 (\frac{6}{2} + 1); \end{aligned}$$

which gives

$$\frac{d}{dt} \int_{\Omega} v^2 + 2 \int_{\Omega} v^2 \leq 2c_6 (\frac{6}{2} + 1) \leq 4c_6 \quad (2.9)$$

due to $0 < \frac{1}{2} \leq 1$. Applying Gronwall's inequality to (2.9) gives

$$\|v\|_{L^2}^2 \leq \|v_0\|_{L^2}^2 + \frac{2c_6}{\frac{1}{2}};$$

which yields (2.3). The proof of Lemma 2.2 is finished. $\square$

2.3. L^2 -estimate of u . In this subsection, we shall use the estimates of u and v to derive the L^2 -norm of u .

LEMMA 2.3. *Suppose the conditions in Lemma 2.2 hold. Then we have*

$$\begin{aligned} &\|u(\cdot; t)\|_{L^2} + \|v(\cdot; t)\|_{H^1} + \|w(\cdot; t)\|_{H^2} \\ &\leq C (\|u_0\|_{L^1} + \|u_0\|_{L^2} + \|v_0\|_{H^1} + \|w_0\|_{H^2}) \end{aligned} \quad (2.10)$$

for all $t > 0$, where C is a positive constant independent of t and $\frac{1}{2}$.

Proof. We multiply the first equation of (1.4) by u and integrate the result to obtain that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\Omega} u^2 + d_1 \int_{\Omega} u_x^2 &= \int_{\Omega} uu_x v_x - \frac{1}{2} \int_{\Omega} uu_x w_x \\ &= -\frac{1}{2} \int_{\Omega} u^2 v_{xx} + \frac{1}{2} \int_{\Omega} u^2 w_{xx}. \end{aligned} \quad (2.11)$$

Noting the fact $\|u\|_{L^1} + \|w\|_{H^1} \leq c_1$ (see (2.2) in Lemma 2.2), we apply Gagliardo-Nirenberg inequality to derive that

$$\|u\|_{L^4}^4 \leq c_2 (\|u_x\|_{L^2}^2 \|u\|_{L^1}^2 + \|u\|_{L^1}^4) \leq c_3 (\|u_x\|_{L^2}^2 + 1)$$

and

$$\|w_{xx}\|_{L^2}^2 \leq c_4 (\|w_{xxx}\|_{L^2} \|w_x\|_{L^2} + \|w_x\|_{L^2}^2) \leq \frac{d_1^2 d_3^2}{3c_3^{\frac{2}{2} - \frac{1}{1}}} \|w_{xxx}\|_{L^2}^2 + c_5.$$

Then we can use Young's inequality to get that

$$\begin{aligned} & \int_{\Omega} u^2 - \frac{1}{2} \int_{\Omega} u^2 v_{xx} + \frac{1}{2} \int_{\Omega} u^2 w_{xx} \\ & \leq \frac{d_1}{2c_3} \int_{\Omega} u^4 + \frac{3c_3}{8d_1} \int_{\Omega} v_{xx}^2 + \frac{3c_3}{8d_1} \int_{\Omega} w_{xx}^2 + \frac{3c_3}{2d_1} \int_{\Omega} u^2 \\ & \leq \frac{d_1}{2} \int_{\Omega} u_x^2 + \frac{3c_3}{8d_1} \int_{\Omega} v_{xx}^2 + \frac{d_1 d_3^2}{8} \int_{\Omega} w_{xxx}^2 + c_6. \end{aligned} \quad (2.12)$$

Adding the term $\int_{\Omega} u^2$ to both sides of (2.11) and combining it with (2.12), one gets

$$\frac{d}{dt} \int_{\Omega} u^2 + 2 \int_{\Omega} u^2 + d_1 \int_{\Omega} u_x^2 \leq \frac{3c_3}{4d_1} \int_{\Omega} v_{xx}^2 + \frac{d_1 d_3^2}{4} \int_{\Omega} w_{xxx}^2 + 2c_6. \quad (2.13)$$

Multiplying the equation of v in (1.4) by $-v_{xx}$ and integrating it by parts, we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\Omega} v_x^2 + \int_{\Omega} v_x^2 + d_2 \int_{\Omega} v_{xx}^2 \\ & = - \frac{1}{2} \int_{\Omega} v_x w_x v_{xx} - \frac{1}{2} \int_{\Omega} v v_{xx} w_{xx} - \int_{\Omega} u v_{xx} \\ & \leq \frac{d_2}{4} \int_{\Omega} v_{xx}^2 + \frac{3}{d_2} \int_{\Omega} v_x^2 w_x^2 + \frac{3}{d_2} \int_{\Omega} v^2 w_{xx}^2 + \frac{3}{d_2} \int_{\Omega} u^2. \end{aligned} \quad (2.14)$$

Applying Gagliardo-Nirenberg inequality and Young's inequality, one can derive that

$$\begin{aligned} \frac{3}{d_2} \int_{\Omega} v_x^2 w_x^2 & \leq \frac{3}{d_2} \|v_x\|_{L^2}^2 \|w_x\|_{L^4}^2 \\ & \leq c_7 \left(\|v_{xx}\|_{L^2} \|v\|_{L^2} + \|v\|_{L^2}^2 \right) \left(\|w_{xxx}\|_{L^2}^{\frac{1}{2}} \|w_x\|_{L^2}^{\frac{3}{2}} + \|w_x\|_{L^2}^2 \right) \\ & \leq c_8 \left(\frac{6}{2} + 1 \right) (\|v_{xx}\|_{L^2} + 1) (\|w_{xxx}\|_{L^2}^{\frac{1}{2}} + 1) \\ & \leq \frac{d_2}{8} \|v_{xx}\|_{L^2}^2 + \frac{d_1^2 d_2 d_3^2}{6c_3^{\frac{2}{2} - \frac{1}{2}}} \|w_{xxx}\|_{L^2}^2 + c_9 \left(\frac{20}{2} + 1 \right) \end{aligned} \quad (2.15)$$

where we have used (2.2) and (2.3). Similarly, using Gagliardo-Nirenberg inequality and Cauchy-Schwarz inequality, the third term on the right-hand side of (2.14) can be estimated as follows

$$\begin{aligned} \frac{3}{d_2} \int_{\Omega} v^2 w_{xx}^2 & \leq \frac{3}{d_2} \|v\|_{L^4}^2 \|w_{xx}\|_{L^4}^2 \\ & \leq c_{10} \left(\|v_{xx}\|_{L^2}^{\frac{1}{4}} \|v\|_{L^2}^{\frac{7}{4}} + \|v\|_{L^2}^2 \right) \left(\|w_{xxx}\|_{L^2}^{\frac{5}{4}} \|w_x\|_{L^2}^{\frac{3}{4}} + \|w_x\|_{L^2}^2 \right) \\ & \leq c_{11} \left(\left(\frac{6}{2} + 1 \right)^{\frac{7}{8}} \|v_{xx}\|_{L^2}^{\frac{1}{4}} + \frac{6}{2} + 1 \right) (\|w_{xxx}\|_{L^2}^{\frac{5}{4}} + \|w_x\|_{L^2}^2) \end{aligned}$$

Moreover, since $\|u\|_{L^1} = \|u_0\|_{L^1}$, the last term on the right-hand side of (2.14) can be estimated as

$$\frac{3}{d_2} \int_{\Omega} u^2 = \frac{3}{d_2} \|u\|_{L^2}^2 \leq c_{13} \left(\|u_x\|_{L^2}^{\frac{2}{3}} \|u\|_{L^1}^{\frac{4}{3}} + \|u\|_{L^1}^2 \right) \leq \frac{d_1^2 d_3}{6c_3^2} \|u_x\|_{L^2}^2 + c_{14}. \quad (2.17)$$

Substituting (2.15)–(2.17) into (2.14), one has

$$\begin{aligned}
&= - \frac{1}{2} \int_{\Omega} v_{xx} v_{xxx} w_x - \frac{1}{2} \int_{\Omega} v_x v_{xxx} w_{xx} - \frac{1}{2} \int_{\Omega} v v_{xxx} w_{xxx} - \int_{\Omega} u_x v_{xxx} \\
&\leq \frac{d_2}{4} \int_{\Omega} v_{xxx}^2 + \frac{4}{d_2} \int_{\Omega} v_{xx}^2 w_x^2 + \frac{16}{d_2} \int_{\Omega} v_x^2 w_{xx}^2 + \frac{4}{d_2} \int_{\Omega} v^2 w_{xxx}^2 + \frac{4}{d_2} \int_{\Omega} u_x^2. \quad (2.24)
\end{aligned}$$

Since $\|u\|_{L^2} + \|v\|_{H^1} + \|w\|_{H^2} \leq c_1$ (see (2.10) in Lemma 2.3), then we can employ the Sobolev embedding to find constants $c_2, c_3 > 0$ such that

$$\|v\|_{L^4} \leq c_2 \quad \text{and} \quad \|w\|_{L^4} + \|w_x\|_{L^4} \leq c_3. \quad (2.25)$$

Using the Gagliardo-Nirenberg inequality and Young's inequality, we derive that

$$\begin{aligned}
\frac{4}{d_2} \int_{\Omega} v_{xx}^2 w_x^2 &\leq \frac{4}{d_2} \|w_x\|_{L^4}^2 \|v_{xx}\|_{L^2}^2 \\
&\leq \frac{4}{d_2} c_3^2 c_4 \left(\|v_{xxx}\|_{L^2} \|v_x\|_{L^2} + \|v_x\|_{L^2}^2 \right) \\
&\leq \frac{d_2}{8} \|v_{xxx}\|_{L^2}^2 + c_5
\end{aligned} \quad (2.26)$$

and

$$\begin{aligned}
\frac{16}{d_2} \int_{\Omega} v_x^2 w_{xx}^2 &\leq \frac{16}{d_2} \|v_x\|_{L^4}^2 \|w_{xx}\|_{L^4}^2 \\
&\leq c_6 \left(\|v_{xxx}\|_{L^2}^{\frac{1}{4}} \|v_x\|_{L^2}^{\frac{7}{4}} + \|v_x\|_{L^2}^2 \right) \left(\|w_{xxx}\|_{L^2}^{\frac{5}{4}} \|w_x\|_{L^2}^{\frac{3}{4}} + \|w_x\|_{L^2}^2 \right) \\
&\leq c_7 \left(\|v_{xxx}\|_{L^2}^{\frac{1}{4}} \|v_x\|_{L^2}^{\frac{7}{4}} + \|v_x\|_{L^2}^2 \right) \left(\|w_{xxx}\|_{L^2}^{\frac{5}{4}} + 1 \right) \\
&\leq \frac{d_2}{8} \|v_{xxx}\|_{L^2}^2 + \|w_{xxx}\|_{L^2}^2 + c_8
\end{aligned} \quad (2.27)$$

due to $\alpha \in (0; 1]$. By (2.25), we can estimate the term $\frac{4\kappa_2^2}{d_2} \int_{\Omega} v^2 w_{xxx}^2$ as follows

$$\frac{4}{d_2} \int_{\Omega} v^2 w_{xxx}^2 \leq \frac{4}{d_2} \|v\|_{L^4}^2 \|w_{xxx}\|_{L^4}^2 \quad (2.28)$$

$$\leq \frac{d_1 d_3}{8c_{11}} \|u_{xx}\|_{L^2}^2 + c_{14} (\|u_0\|_{L^2}^2 + \|v_0\|_{H^2}^2 + \|w_0\|_{H^2}^2): \quad (2.31)$$

To control the term $\|u_{xx}\|_{L^2}$ on the right hand in (2.31), we apply $-u_{xx}$ as a test function to the first equation of (1.4) and integrate the result to obtain that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\Omega} u_x^2 + \int_{\Omega} u_x^2 + d_1 \int_{\Omega} u_{xx}^2 \\ &= \int_{\Omega} u_x u_{xx} v_x + \int_{\Omega} u u_{xx} v_{xx} - \frac{1}{2} \int_{\Omega} u_x u_{xx} w_x - \frac{1}{2} \int_{\Omega} u u_{xx} w_{xx} + \int_{\Omega} u_x^2 \\ &\leq \frac{d_1}{8} \|u_{xx}\|_{L^2}^2 + \frac{8}{d_1} \|u_x\|_{L^2}^2 \|v_x\|_{L^4}^2 + \frac{8}{d_1} \|u\|_{L^4}^2 \|v_{xx}\|_{L^4}^2 + \frac{8}{d_1} \|u_x\|_{L^2}^2 \|w_x\|_{L^4}^2 \\ &\quad + \frac{8}{d_1} \|u\|_{L^4}^2 \|w_{xx}\|_{L^2}^2 + \|u_x\|_{L^2}^2. \end{aligned} \quad (2.32)$$

By using Gagliardo-Nirenberg inequality, we have

$$\begin{aligned} & \frac{8}{d_1} \|u_x\|_{L^2}^2 \|v_x\|_{L^4}^2 \\ &\leq \frac{8c_{15}^2}{d_1} (\|u_{xx}\|_{L^2} \|u\|_{L^2} + \|u\|_{L^2}^2) \left(\|v_{xxx}\|_{L^2}^{\frac{1}{2}} \|v_x\|_{L^2}^{\frac{3}{2}} + \|v_x\|_{L^2}^2 \right) \\ &\leq \frac{d_1}{16} \|u_{xx}\|_{L^2}^2 + \frac{d_2}{8} \|v_{xxx}\|_{L^2}^2 + c_{16} \end{aligned} \quad (2.33)$$

and

$$\begin{aligned} & \frac{8}{d_1} \|u\|_{L^4}^2 \|v_{xx}\|_{L^4}^2 \\ &\leq \frac{8c_{17}^2}{d_1} \left(\|u_{xx}\|_{L^2}^{\frac{1}{4}} \|u\|_{L^2}^{\frac{7}{4}} + \|u\|_{L^2}^2 \right) \left(\|v_{xxx}\|_{L^2}^{\frac{5}{4}} \|v_x\|_{L^2}^{\frac{3}{4}} + \|v_x\|_{L^2}^2 \right) \\ &\leq \frac{d_1}{16} \|u_{xx}\|_{L^2}^2 + \frac{d_2}{8} \|v_{xxx}\|_{L^2}^2 + c_{18}. \end{aligned} \quad (2.34)$$

Besides, we can use Gagliardo-Nirenberg inequality and Young's inequality to obtain that

$$\frac{8}{d_1} \|u_x\|_{L^2}^2 \|w_x\|_{L^4}^2 \leq \frac{8c_3^2 c_{19}^2}{d_1} (\|u_{xx}\|_{L^2} \|u\|_{L^2} + \|u\|_{L^2}^2) \leq \frac{d_1}{16} \|u_{xx}\|_{L^2}^2 + c_{20} \quad (2.35)$$

and

$$\frac{8}{d_1} \|u\|_{L^4}^2 \|w_{xx}\|_{L^2}^2 \leq \frac{8c_1^2 c_{21}^2}{d_1} \left(\|u_{xx}\|_{L^2}^{\frac{1}{2}} \|u\|_{L^2}^{\frac{3}{2}} + \|u\|_{L^2}^2 \right) \leq \frac{d_1}{16} \|u_{xx}\|_{L^2}^2 + c_{22} \quad (2.36)$$

as well as

$$\|u_x\|_{L^2}^2 \leq c_{23} (\|u_{xx}\|_{L^2} \|u\|_{L^2} + \|u\|_{L^2}^2) \leq \frac{d_1}{8} \|u_{xx}\|_{L^2}^2 + c_{24}. \quad (2.37)$$

Substituting (2.33)-(2.37) into (2.32), we arrive at

$$\frac{d}{dt} \int_{\Omega} u_x^2 + 2 \int_{\Omega} u_x^2 + d_1 \int_{\Omega} u_{xx}^2 \leq \frac{d_2}{2} \int_{\Omega} v_{xxx}^2 + c_{25} (\|u_0\|_{L^2}^2 + \|v_0\|_{H^2}^2 + \|w_0\|_{H^2}^2): \quad (2.38)$$

The combination of (2.30), (2.31) and (2.38) gives that

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} \left(u_x^2 + v_{xx}^2 + \frac{2c_{11}}{d_3} w_{xx}^2 \right) + 2 \int_{\Omega} \left(u_x^2 + v_{xx}^2 + \frac{2}{d_3} c_{11} w_{xx}^2 \right) \\ & + \frac{1}{2} \int_{\Omega} (d_1 u_{xx}^2 + d_2 v_{xxx}^2 + 2c_{11} w_{xxx}^2) \\ & \leq c_{26} (\|u_0\|_{L^2}^2 + \|v_0\|_{H^2}^2 + \|w_0\|_{H^2}^2); \end{aligned}$$

Then, by Grönwall's inequality, we arrive at

$$\|u_x\|_{L^2}^2 + \|v_{xx}\|_{L^2}^2 + \frac{2c_{11}}{d_3} \|w_{xx}\|_{L^2}^2 \leq c_{27} (\|u_0\|_{H^1}^2 + \|v_0\|_{H^2}^2 + \|w_0\|_{H^2}^2);$$

which together with (2.10) yields (2.22). Then the proof of this Lemma 2.4 is completed. $\square$

Proof. (Proof of Theorem 1.1.) Based on the results in Lemma 2.4, we derive that

$$\|u(\cdot; t)\|_{W^{1,2}} + \|v(\cdot; t)\|_{W^{1,2}} + \|w(\cdot; t)\|_{W^{1,2}} \leq c_1 (\|u_0\|_{H^1} + \|v_0\|_{H^2}^2 + \|w_0\|_{H^2}^2);$$

Then using Lemma 2.1, we obtain Theorem 1.1 directly. $\square$

3. Convergence of constant steady state

In this section, motivated by some ideas in [12], we shall study the asymptotic behavior of solutions for (1.4) by constructing some energy functional. First, we define an energy functional

$$\mathcal{E}(t) := \frac{1}{2} \int_{\Omega} u \ln \frac{u}{u_0} + \frac{1}{4} \int_{\Omega} v_x^2 + \frac{1}{4} \int_{\Omega} w_x^2 - \int_{\Omega} v_x w_x; \quad (3.1)$$

where $u = \frac{1}{|\Omega|} \int_{\Omega} u_0$ and

$$v_1 := v_1 - \frac{1}{|\Omega|} \int_{\Omega} v_1; \quad v_2 := v_1 + \frac{1}{|\Omega|} \int_{\Omega} v_1; \quad (3.2)$$

Let M be the constant in Theorem 1.1, then one has $\|v\|_{L^1} + \|v_x\|_{L^1} \leq M$. We have the following results.

LEMMA 3.1. *Suppose the condition (1.5) holds. Let $(u; v; w)$ be the solution of (1.4). Then the energy functional $\mathcal{E}(t)$ defined in (3.1) is nonnegative and satisfies*

$$\frac{d}{dt} \mathcal{E}(t) + \mathcal{F}(t) \leq 0; \quad (3.3)$$

where $\mathcal{F}(t)$ is defined as

$$\begin{aligned} \mathcal{F}(t) := & \frac{d_1}{2} \int_{\Omega} \frac{u_x^2}{u} + \frac{d_2}{2} \int_{\Omega} v_{xx}^2 + \left(\frac{d_3}{2} - M \right) \int_{\Omega} w_{xx}^2 + \frac{1}{2} \int_{\Omega} v_x^2 \\ & + \frac{1}{2} \int_{\Omega} w_x^2 - \left(\frac{1}{2} + M \right) \int_{\Omega} v_x w_x - \frac{M}{2} \int_{\Omega} |v_{xx}| |w_x| \\ & - \left(\frac{M}{2} + d_2 + d_3 \right) \int_{\Omega} |v_{xx}| |w_{xx}| - M \int_{\Omega} |w_x| |w_{xx}|; \end{aligned} \quad (3.4)$$

Proof. First, we show $\mathcal{E}(t)$ and $\mathcal{F}(t)$ satisfy (3.3). To this end, we multiply the first equation of (1.4) by $\ln \frac{u}{\bar{u}}$ and integrate the result to derive that

$$\frac{d}{dt} \int_{\Omega} u \ln \frac{u}{\bar{u}} = -d_1 \int_{\Omega} \frac{u_x^2}{u} + \int_{\Omega} u_x v_x - \int_{\Omega} u_x w_x. \quad (3.5)$$

Employing $-v_{xx}$ as a test function to the second equation of (1.4) and integrating the result, we have

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} v_x^2 + \int_{\Omega} v_x^2 + d_2 \int_{\Omega} v_{xx}^2 = \int_{\Omega} u_x v_x - \int_{\Omega} (vw_x)_x v_{xx}. \quad (3.6)$$

Similarly, it follows from the third equation of (1.4) that

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} w_x^2 + \int_{\Omega} w_x^2 + d_3 \int_{\Omega} w_{xx}^2 = \int_{\Omega} u_x w_x. \quad (3.7)$$

We multiply (3.5) by $\frac{\theta_1}{2\kappa_1\chi}$, (3.6) by $\frac{\theta_2}{2\kappa_1\alpha}$ and (3.7) by $\frac{\theta_2}{2\gamma\chi}$, respectively, and add the results to derive that

$$\begin{aligned} & \frac{d}{dt} \left(\frac{1}{2} \int_{\Omega} u \ln \frac{u}{\bar{u}} + \frac{1}{4} \int_{\Omega} v_x^2 + \frac{1}{4} \int_{\Omega} w_x^2 \right) + \frac{d_1}{2} \int_{\Omega} \frac{u_x^2}{u} \\ & + \frac{d_2}{2} \int_{\Omega} v_{xx}^2 + \frac{d_3}{2} \int_{\Omega} w_{xx}^2 + \frac{1}{2} \int_{\Omega} v_x^2 + \frac{1}{2} \int_{\Omega} w_x^2 \\ & = \int_{\Omega} u_x v_x + \int_{\Omega} u_x w_x - \frac{1}{2} \int_{\Omega} (vw_x)_x v_{xx}. \end{aligned} \quad (3.8)$$

Recalling the third equation of (1.4), we have $u_x = w_{xt} - d_3 w_{xxx} + w_x$. Then the term $\int_{\Omega} u_x v_x$ on the right-hand side of (3.8) can be rewritten as

$$\begin{aligned} \int_{\Omega} u_x v_x &= \int_{\Omega} (w_{xt} - d_3 w_{xxx} + w_x) v_x \\ &= \frac{d}{dt} \int_{\Omega} v_x w_x - \int_{\Omega} v_{xt} w_x + d_3 \int_{\Omega} v_{xx} w_{xx} + \int_{\Omega} v_x w_x. \end{aligned} \quad (3.9)$$

In view of the equation of v in (1.4), we substitute $v_{xt} = d_2 v_{xxx} + (vw_x)_{xx} + u_x - v_x$ into (3.9) and integrate the result by parts to obtain

$$\begin{aligned} \int_{\Omega} u_x v_x &= \frac{d}{dt} \int_{\Omega} v_x w_x + (d_2 + d_3) \int_{\Omega} v_{xx} w_{xx} + \int_{\Omega} v_x w_x + \int_{\Omega} (vw_x)_x w_{xx} \\ &\quad - \int_{\Omega} u_x w_x; \end{aligned}$$

which together with (3.8) gives that

$$\begin{aligned} & \frac{d}{dt} \mathcal{E}(t) + \frac{1}{2} \int_{\Omega} \frac{u_x^2}{u} + \frac{1}{2} \int_{\Omega} v_x^2 + \frac{1}{2} \int_{\Omega} w_x^2 + \frac{1}{2} \int_{\Omega} v_{xx}^2 \\ & + \frac{1}{2} \int_{\Omega} w_{xx}^2 - (d_2 + d_3) \int_{\Omega} v_{xx} w_{xx} - \int_{\Omega} w_x v_x \\ & = \int_{\Omega} (vw_x)_x w_{xx} - \frac{1}{2} \int_{\Omega} (vw_x)_x v_{xx}. \end{aligned} \quad (3.10)$$

Since $\|v\|_{L^1} + \|v_x\|_{L^1} \leq M$, one has

$$\begin{aligned} & \int_{\Omega} (vw_x)_x w_{xx} - \frac{2}{2} \int_{\Omega} (vw_x)_x v_{xx} \\ &= \int_{\Omega} v_x w_x w_{xx} + \int_{\Omega} vw_{xx}^2 - \frac{2}{2} \int_{\Omega} v_x w_x v_{xx} - \frac{2}{2} \int_{\Omega} vw_{xx} v_{xx} \\ &\leq \frac{2}{2} \int_{\Omega} (|w_x| + |w_{xx}|) |v_{xx}| + 2M \int_{\Omega} |w_x| |w_{xx}| + 2M \int_{\Omega} w_{xx}^2; \end{aligned}$$

which together with (3.10) yields (3.3).

Next, we show the nonnegativity of $\mathcal{E}(t)$ to complete the proof of this lemma. Let $X = (v_x, w_x)$, then we rewrite $\mathcal{E}(t)$ as

$$\mathcal{E}(t) = \frac{1}{2} \int_{\Omega} u \ln \frac{u}{u} + \int_{\Omega} X A X^>;$$

where

$$A := \begin{pmatrix} \frac{\theta_2}{4\kappa_1\alpha} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{\theta_2}{4\gamma\chi} \end{pmatrix};$$

Using (1.5) and the definition of θ_i ($i = 1, 2$) in (3.2), we can check $\theta_1 > 0$ and $\theta_2 > 0$ directly. Then, one can derive that

$$|A| = \frac{\theta_2}{16} - \frac{1}{4} = \frac{\theta_2}{16} > 0;$$

Hence, the matrix A is positive-definite by Sylvester's criterion. Then we can find a constant $c_1 > 0$ such that

$$\int_{\Omega} X A X^> \geq c_1 \int_{\Omega} (v_x^2 + w_x^2);$$

Moreover, we have $\frac{\theta_1}{2\kappa_1\chi} \int_{\Omega} u \ln \frac{u}{u} \geq 0$ by applying the inequality (see [3, 5])

$$\int_{\Omega} f \ln \frac{f}{f} \geq \frac{1}{2f} \|f - f\|_{L^1}^2 \geq 0; \quad \forall f > 0;$$

where $f = \frac{1}{|\Omega|} \int_{\Omega} f$. In summary, we obtain

$$\mathcal{E}(t) \geq \frac{1}{2} \int_{\Omega} u \ln \frac{u}{u} + c_1 \int_{\Omega} (v_x^2 + w_x^2) \geq 0; \quad (3.11)$$

Thus the proof of Lemma 3.1 is completed. $\square$

Next, we show that the energy functional $\mathcal{E}(t)$ defined in (3.1) is monotone decreasing under certain conditions on the parameters. To this end, we must show the nonnegativity of $\mathcal{F}(t)$.

LEMMA 3.2. *Suppose the conditions in Lemma 3.1 hold. Then there exists a constant $\delta > 0$ such that if $0 < \theta_2 \leq \delta$, we have*

$$\mathcal{E}(t) \leq \mathcal{F}(t); \quad (3.12)$$

for some $\delta > 0$ independent of ϵ_2 and t .

Proof. By the definition of $\mathcal{E}(t)$ and $\mathcal{F}(t)$ (see (3.1) and (3.4)), we have

$$\mathcal{F}(t) - \mathcal{E}(t) \geq \frac{1}{2} \int_{\Omega} \left(d_1 \frac{u_x^2}{u} - u \ln \frac{u}{u} \right) + J;$$

with

$$A_2 := \begin{pmatrix} \frac{\beta\theta_2}{2\kappa_1\alpha} - \frac{\beta+\delta}{2} \\ -\frac{\beta+\delta}{2} & \frac{\delta\theta_2}{2\gamma\chi} \end{pmatrix}.$$

Condition (1.5) guarantees that

$$|A_2| = \frac{\frac{2}{4} - \left(\frac{\beta+\delta}{2}\right)^2}{4} = \frac{1}{4} \left(1 - \left(\frac{\beta+\delta}{2}\right)^2\right) > 0;$$

Applying Sylvester's criterion, we conclude that the matrix A_2 is positive-definite, which entails us to find a constant $c_4 > 0$ independent of μ_2 such that

$$\begin{aligned} J_2 &\geq c_4 \int_{\Omega} (v_x^2 + w_x^2) - \frac{2}{4} \int_{\Omega} v_x^2 - \frac{2}{4} \int_{\Omega} w_x^2 - \int_{\Omega} |v_x| |w_x| \\ &\geq \left(c_4 - \frac{2}{4} - \frac{1}{2} \right) \int_{\Omega} v_x^2 + \left(c_4 - \frac{2}{4} - \frac{1}{2} \right) \int_{\Omega} w_x^2 \\ &\geq \frac{c_4}{2} \int_{\Omega} (v_x^2 + w_x^2) \end{aligned} \quad (3.18)$$

by selecting $\mu_3 := \max\left\{\frac{2\kappa_1\alpha+\theta_2}{2c_4\kappa_1\alpha}, \frac{2\gamma\chi+\theta_2}{2c_4\gamma\chi}\right\}$. Furthermore, one uses Young's inequality to derive that

$$J_3 \leq \int_{\Omega} (v_{xx}^2 + w_x^2 + w_{xx}^2); \quad (3.19)$$

where $\mu_2 := M(2 + \frac{\theta_2}{2\kappa_1\alpha})$. Then letting $\mu_2 > \max\{\mu_1, \mu_2, \mu_3\}$, and substituting (3.17), (3.18) and (3.19) into (3.13), one has

$$\begin{aligned} J &= J_1 + J_2 - \mu_2 J_3 \geq c_3 \int_{\Omega} (v_{xx}^2 + w_{xx}^2) + \frac{c_4}{2} \int_{\Omega} (v_x^2 + w_x^2) \\ &\quad - \mu_2 \int_{\Omega} (v_{xx}^2 + w_x^2 + w_{xx}^2) \\ &\geq \left(c_3 - \mu_2\right) \int_{\Omega} (v_{xx}^2 + w_{xx}^2) + \left(\frac{c_4}{2} - \mu_2\right) \int_{\Omega} w_x^2. \end{aligned} \quad (3.20)$$

Letting $\mu_2 := \min\left\{\frac{c_3}{\xi}, \frac{c_4}{2\xi}, 1\right\}$ and noting that the constants c_3, c_4 and ξ are independent of μ_2 , then if $\mu_2 \leq \xi$, from (3.20) we can derive that $J \geq 0$, which together with (3.16) gives (3.12). $\square$

LEMMA 3.3. *Let the conditions in Lemma 3.2 hold. Then it holds that*

$$\|u(\cdot; t) - u\|_{L^1} \leq C e^{-\frac{t}{3\eta}}; \quad (3.21)$$

where C and η are positive constants independent of t .

Proof. The combination of (3.3) and (3.12) gives that

$$\frac{d}{dt} \mathcal{E}(t) + \frac{1}{\eta} \mathcal{E}(t) \leq 0;$$

which implies that

$$\mathcal{E}(t) \leq c_1 e^{-\frac{t}{\eta}}; \quad (3.22)$$

Recalling the fact (3.11) and (3.14), (3.22) yields

$$\|u(\cdot; t) - u\|_{L^1}^2 \leq \frac{4c_1 u_{-1}}{1} e^{-\frac{t}{\eta}}.$$

By the Gagliardo-Nirenberg inequality and Theorem 1.1, we obtain that

$$\|u - u\|_{L^1} \leq c_2 (\|u_x\|_{L^2}^{\frac{2}{3}} \|u - u\|_{L^1}^{\frac{1}{3}} + \|u - u\|_{L^1}) \leq c_3 \|u - u\|_{L^1}^{\frac{1}{3}} \leq c_4 e^{-\frac{t}{3\eta}};$$

which yields (3.21). $\square$

LEMMA 3.4. Suppose that the assumptions in Lemma 3.2 hold. Then there exist two positive constants $C;_{-1}$ independent of t such that

$$\|v(\cdot; t) - u\|_{L^1} + \|w(\cdot; t) - u\|_{L^1} \leq C e^{-\lambda_1 t};$$

Proof. From Theorem 1.1, we have $\|u\|_{L^1} + \|v\|_{L^1} + \|w\|_{L^1} \leq c_1$. Then we multiply the second equation of (1.4) by $v - \frac{\alpha}{\beta}u$ and integrate the result to derive that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\Omega} \left(v - u \right)^2 \\ &= -d_2 \int_{\Omega} v_x^2 - \frac{1}{2} \int_{\Omega} v v_x w_x + \int_{\Omega} (u - v) \left(v - u \right) \\ &\leq -\frac{d_2}{2} \int_{\Omega} v_x^2 + \frac{c_1^2}{2d_2} \int_{\Omega} w_x^2 + \int_{\Omega} (u - u) \left(v - u \right) - \int_{\Omega} \left(v - u \right)^2 \\ &\leq -\frac{d_2}{2} \int_{\Omega} v_x^2 + \frac{c_1^2}{2d_2} \int_{\Omega} w_x^2 - \frac{1}{2} \int_{\Omega} \left(v - u \right)^2 + \frac{1}{2} \int_{\Omega} (u - u)^2; \end{aligned}$$

which implies that

$$\frac{d}{dt} \int_{\Omega} \left(v - u \right)^2 + \int_{\Omega} \left(v - u \right)^2 + d_2 \int_{\Omega} v_x^2 \leq \frac{c_1^2}{d_2} \int_{\Omega} w_x^2 + \frac{1}{2} \int_{\Omega} (u - u)^2; \quad (3.23)$$

Similarly, it follows from the third equation of (1.4) that

$$\frac{d}{dt} \int_{\Omega} \left(w - u \right)^2 + \int_{\Omega} \left(w - u \right)^2 + 2d_3 \int_{\Omega} w_x^2 \leq \frac{1}{2} \int_{\Omega} (u - u)^2; \quad (3.24)$$

Multiplying (3.24) by $\frac{c_1^2 \kappa_2^2}{d_2 d_3}$ and adding it to (3.23), letting $y(t) := \int_{\Omega} \left(v - \frac{\alpha}{\beta} u \right)^2 + \frac{c_1^2 \kappa_2^2}{d_2 d_3} \int_{\Omega} \left(w - \frac{\gamma}{\delta} u \right)^2$ and $c_2 := \min\{ \cdot; \cdot \}$, one can derive that

$$y'(t) + c_2 y(t) \leq c_3 \|u - u\|_{L^1}^2 \leq c_4 e^{-\frac{2t}{3\eta}}; \quad (3.25)$$

where the last inequality holds due to (3.21). Then solving (3.25) gives two positive constants c_5 and λ_0 such that

$$y(t) \leq c_5 e^{-\lambda_0 t};$$

which, together with the definition of $y(t)$, implies

$$\|v - u\|_{L^2}^2 + \|w - u\|_{L^2}^2 \leq c_6 e^{-\lambda_0 t}.$$

Then using Gagliardo-Nirenberg inequality, we can find a constant $\epsilon_1 > 0$ such that

$$\|v - u\|_{L^1} + \|w - u\|_{L^1} \leq c_7 \left(\|v - u\|_{L^2}^{\frac{1}{2}} + \|w - u\|_{L^2}^{\frac{1}{2}} \right) \leq c_8 e^{-\lambda_1 t}.$$

Thus, we complete the proof of this lemma. $\square$

Proof. (Proof of Theorem 1.2.) Theorem 1.2 is a consequence of the combination of Lemma 3.3 and Lemma 3.4. $\square$

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