

Asymptotic dynamics of the one-dimensional attraction–repulsion Keller–Segel model

Hai-Yang Jin^{*†} and Zhi-An Wang

Communicated by M. Efendiev

The asymptotic behavior of the attraction–repulsion Keller–Segel model in one dimension is studied in this paper. The global existence of classical solutions and nonconstant stationary solutions of the attraction–repulsion Keller–Segel model in one dimension were previously established by Liu and Wang (2012), which, however, only provided a time-dependent bound for solutions. In this paper, we improve the results of Liu and Wang (2012) by deriving a uniform-in-time bound for solutions and furthermore prove that the model possesses a global attractor. For a special case where the attractive and repulsive chemical signals have the same degradation rate, we show that the solution converges to a stationary solution algebraically as time tends to infinity if the attraction dominates. Copyright © 2014 John Wiley & Sons, Ltd.

Keywords: chemotaxis; attraction–repulsion; classical solutions; stationary solutions; global dynamics

1. Introduction

Chemotaxis describes the cell movement in the direction of the chemical concentration gradient. The prototypical chemotaxis model was proposed by Keller–Segel [2], which reads

$$\begin{cases} u_t = D\Delta u - \nabla \cdot (\chi u \nabla v), \\ v_t = \Delta v + k_1 u - k_2 v, \end{cases} \quad (1.1)$$

where $u(x, t)$ is the cell density, $v(x, t)$ stands for the chemical concentration, $D > 0$ is the cell diffusion coefficients, and $\chi \in \mathbb{R}$ is referred to as the chemotactic coefficient. $k_1 > 0$ and $k_2 > 0$ denote production and degradation rates of the chemical v , respectively. The chemotaxis is said to be attractive (or repulsive) if $\chi > 0$ (or $\chi < 0$).

A striking feature of the classical attractive Keller–Segel system (1.1) is the finite-time blow up of solutions in two dimensions when the cell mass is larger than a threshold number, which has trigger a tremendous amount of mathematical studies in the past four decades (see a review article [3] and references therein for details). In the recent two decades, a number of mechanisms have been proposed to modify the Keller–Segel system (1.1) such that the solution of the modified system is bounded for any magnitude of cell mass (see a review article [4] and recent development in [5, 6]). When the chemotaxis is repulsive (i.e., $\chi < 0$), the classical global uniform-in-time solutions in two dimensions and weak solutions in three and four dimensions were established in [7]. In most of theoretical studies, the attraction and repulsion are treated separately and the time-monotone Lyapunov function was essentially applied.

The Keller–Segel model with both attraction and repulsion was first proposed in [8] to describe the quorum effect in chemotaxis and in [9] to describe the aggregation of microglia in Alzheimer's disease. The same model was proposed in [8, 9] and reads

$$\begin{cases} u_t = u_{xx} - (\chi uv_x)_x + (\xi uw_x)_x, & x \in \Omega, t > 0, \\ v_t = v_{xx} + \alpha u - \beta v, & x \in \Omega, t > 0, \\ w_t = w_{xx} + \gamma u - \delta w, & x \in \Omega, t > 0, \\ \frac{u}{x} = \frac{v}{x} = \frac{w}{x} = 0, & x \in \partial\Omega, t > 0, \\ u(x, 0) = u_0(x), v(x, 0) = v_0(x), w(x, 0) = w_0(x), & x \in \Omega, \end{cases} \quad (1.2)$$

Department of Applied Mathematics, Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong

* Correspondence to: Hai-Yang Jin, Department of Applied Mathematics, Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong.

† E-mail: haiyang.jin@connect.polyu.hk

where $\chi, \xi, \alpha, \gamma > 0$ and $\beta, \delta \geq 0$, ν denote the unit outward normal vector to the boundary $\partial\Omega$. Although the attraction–repulsion Keller–Segel model (1.2) is a direct generalization of the classical Keller–Segel system (1.1), it does not have an obvious Lyapunov function, and hence, mathematical analysis confronts great challenges. Until recently, it was found in [10] that model (1.2) can be reduced to the classical Keller–Segel system for a special case $\beta = \delta$, and hence, the existing methods based on the Lyapunov function can be employed to establish the global existence and large-time behavior of solutions. When $\beta \neq \delta$, the mathematical results of model (1.2) in any aspect still largely remain open except some results obtained in one dimension by Liu and Wang [1] where the global classical solutions and the stationary solutions in certain parameter regimes were established via the method of energy estimates and the phase plane analysis. However, only the time-dependent solution bound was derived therein, and whether the solution blows up at infinite time was not clear.

It is the purpose of this paper to explore the global dynamics of the attraction–repulsion Keller–Segel model for any $\beta > 0$ and $\delta > 0$ in one dimension further. Our main results consist of three main components. First, we derive a uniform-in-time bound for the solution, which substantially improve the results in [1] where the solution bound depends on time. Second, when $\beta \neq \delta$, we use the dynamical approach to prove that model (1.2) possesses a global attractor. Finally, for the special case $\beta = \delta$, we show that the solution of (1.2) converges to a stationary solutions as time tends to infinity algebraically if the attraction dominates (i.e., $\chi\alpha - \xi\gamma > 0$).

The paper is organized as follows. In Section 2, the main results of this paper will be stated. In Section 3, we prove the existence of global uniform-in-time solutions. The existence of global attractor will be shown in Section 4. Finally, the convergence of solutions to a stationary solution with algebraic decay rates will be established in Section 5.

Notations. Hereafter, $\Omega = I = (a, b)$ is a bounded open interval in $\mathbb{R}$, and ∂I denotes the boundary of the interval I . C denotes a generic constant, which may vary in the context. $L^p = L^p(I)$ ($1 \leq p \leq \infty$) denotes the usual Lebesgue space in a bounded open interval $I \subset \mathbb{R}$ with norm $\|f\|_{L^p} = \left(\int_I |f(x)|^p dx\right)^{1/p}$ for $1 \leq p < \infty$ and $\|f\|_{L^\infty} = \text{ess sup}_{x \in I} |f(x)|$. H^m denotes the m -th order Sobolev space $W^{m,2}$ with the norm $\|f\|_{H^m} = \left(\sum_{i=0}^m \|\partial_x^i f\|^2\right)^{1/2}$. Moreover, we denote $\|(f, g)\|_{L^p} = \|f\|_{L^p} + \|g\|_{L^p}$ for $1 \leq p \leq \infty$ and $\|(f, g)\|_{H^m} = \|f\|_{H^m} + \|g\|_{H^m}$ for $m = 1, 2, 3, \dots$.

2. Statement of main results

This section is devoted to stating the main results of this paper. First, we address the global existence of classical solutions with a uniform bound in time.

Theorem 2.1

Let $u_0 \in H^1(I)$, $(v_0, w_0) \in [H^2(I)]^2$. Then, system (1.2) has a unique global classical solution $(u, v, w) \in [C^0(\bar{I} \times [0, \infty); \mathbb{R}^3)]^3 \cap [C^{2,1}(\bar{I} \times (0, \infty); \mathbb{R}^3)]^3$ such that $u, v, w \geq 0$ if $u_0, v_0, w_0 \geq 0$.

Next, we consider the large time behavior of the solution of (1.2). Define

$$\mathcal{X} = \{(u, v, w) \in H^1(I) \times H^2(I) \times H^2(I) | u \geq 0, v \geq 0, w \geq 0\}.$$

From Theorem 2.1, we know that for any initial function $U_0 = (u_0, v_0, w_0) \in \mathcal{X}$, system (1.2) has a unique solution $U(t; U_0) = (u, v, w)$ for all $t > 0$. Hence, we can define a dynamical system $(\{S(t)_{t \geq 0}\}, \mathcal{X})$ by a nonlinear

Then, our second result is as follows.

Theorem 2.2

The dynamical system $(\{S(t)_{t \geq 0}\}, \mathcal{X}^*)$ possesses a global attractor.

Next, we explore the asymptotical behavior of solution for a special case $\beta = \delta$. First, noticing that the integration of the first equation of (1.2) in x entails that the cell preserves the mass:

$$\|u(t)\|_{L^1} = \|u_0\|_{L^1} =: M \quad (2.1)$$

where $M > 0$ is a prescribed constant denoting the cell mass. Therefore, the stationary solution $(U, V, W)(x)$ of (1.2) satisfies

$$\begin{cases} 0 = U_{xx} - (\chi UV_x)_x + (\xi UW_x)_x, & x \in I, \\ 0 = V_{xx} + \alpha U - \beta V, & x \in I, \\ 0 = W_{xx} + \gamma U - \delta W, & x \in I, \\ U_x = V_x = W_x = 0, & x \in \partial I, \\ \int_I U(x) dx = M, & x \in I. \end{cases} \quad (2.2)$$

When $\beta = \delta$ and $\xi\gamma - \chi\alpha \geq 0$ (i.e., repulsion dominates), the results of [10, Proposition 2.3 and Proposition 2.4] showed that (2.2) has a unique constant solution $(\bar{u}_0, -\bar{u}_0, -\bar{u}_0)$ where $\bar{u}_0 := M/|I|$, and the solution of (1.2) approaches this constant solution exponentially as time goes to infinity in two dimension. When $\beta = \delta$ and $\xi\gamma - \chi\alpha < 0$ (i.e., attraction dominates), the existence of nonconstant solution (U, V, W) has been established in [10, Proposition 2.3], whereas the asymptotical behavior of the solution to (1.2) has not been obtained for this case. In this paper, we shall explore this question and show that the solution of (1.2) converges to a solution of (2.2) algebraically as time tends to infinity in one dimension.

Theorem 2.3

Let $u_0 \in H^1(I)$, $(v_0, w_0) \in [H^2(I)]^2$. If $\beta = \delta$ and $\xi\gamma - \chi\alpha < 0$, then the global solution (u, v, w) of (1.2) converges to a stationary solution $(U(x), V(x), W(x))$ in $[H^1(I)]^3$ as time tends to infinity. Moreover, there exist a $\theta \in (0, \frac{1}{2})$ and a positive constant C such that for all $t \geq 0$, it holds that

$$\|u(x, t) - U(x)\|_{H^1} + \|v(x, t) - V(x)\|_{H^1} + \|w(x, t) - W(x)\|_{H^1} \leq C(1 + t)^{-1-2\theta}. \quad (2.3)$$

3. Uniform-in-time global solutions

Theorem 2.1 is a consequence of local existence theorem (Propositions 3.1) and the a priori estimates (Propositions 3.2) by the continuation argument. The local existence of solutions to (1.2) has been proved in [1] and is cited here for convenience.

Proposition 3.1 (Local existence)

Let I be a bounded open interval in $\mathbb{R}$. Then,

- (i) For any initial data $u_0 \in H^1(I)$, $(v_0, w_0) \in [H^2(I)]^2$, there exists a maximal existence time $T_0 \in (0, \infty]$ depending on (u_0, v_0, w_0) , such that (1.2) has a unique solution (u, v, w) satisfying

$$(u, v, w) \in [C^0(\bar{I} \times [0, T_0]; \mathbb{R}^3)]^3 \cap [C^{2,1}(\bar{I} \times (0, T_0); \mathbb{R}^3)]^3.$$

- (ii) If $\sup_{0 \leq t \leq \min\{T_0, T\}} \|(u, v, w)(t)\|_{L^\infty} < \infty$ for any $T > 0$, then $T_0 = \infty$, namely, (u, v, w) is a global classical solution of the system (1.2). Moreover $u, v, w \geq 0$ if $u_0, v_0, w_0 \geq 0$.

Proposition 3.2 (A priori estimates)

Let $u_0 \in H^1(I)$, $(v_0, w_0) \in [H^2(I)]^2$ and (u, v, w) be a solution of (1.2). Then, for any $T > 0$, (u, v, w) satisfies for all $0 < t \leq T$ that

$$\|u(t)\|_{H^1} + \|(v, w)(t)\|_{H^2} \leq C, \quad (3.1)$$

where C is a constant independent of time.

Next, we are devoted to proving the Proposition 3.2. To this end, we first give some inequalities that will be essentially used in the paper.

Lemma 3.3 ([12])

Let I be a bounded domain in $\mathbb{R}$ with ∂I in C^m , and let u be any function in $H^m(I) \cap L^q(I)$, $1 \leq q \leq \infty$. For any integer j with $0 \leq j < m$, and for any number a with $j/m \leq a \leq 1$, there exist two positive constant C_1 and C_2 depending only on I, q, m such that the following Gagliardo–Nirenberg inequality holds

$$\|D^j u\|_{L^p} \leq C_1 \|D^m u\|_{L^q}^a \|u\|_{L^q}^{1-a} + C_2 \|u\|_{L^q}, \quad a = \frac{1/p - 1/q - j}{1/2 - m - 1/q}. \quad (3.2)$$

For a special case $m = 1, j = 0$, we may employ the inequality $(c + d)^2 \leq 2(c^2 + d^2)$ for any $c, d \in \mathbb{R}$, and obtain the following inequality

$$\|u\|_{L^p}^2 \leq C \left(\|u_x\|_{L^2}^{2a} \|u\|_{L^q}^{2-2a} + \|u\|_{L^q}^2 \right), \quad a = \frac{1/q - 1/p}{1/q + 1/2}. \quad (3.3)$$

Proposition 3.2 will be verified by the following two lemmas.

Lemma 3.4

Let the assumption in Proposition 3.2 hold. If (u, v, w) is a solution of (1.2), then for any $T > 0$, there is a constant C independent of T such that the following inequality holds for each $0 < t \leq T$

$$\|u(t)\|_{L^2}^2 + \|(v, w)(t)\|_{H^1}^2 \leq C \left(\|(v_0, w_0)\|_{H^1}^2 + \|u_0\|_{L^2}^2 + \|u_0\|_{L^1}^2 \right). \quad (3.4)$$

Proof

First, the result of [13, Eq. (4.3)] gives that $\|(v, w)(t)\|_{H^1}^2 \leq C \left(\|(v_0, w_0)\|_{H^1}^2 + \|u_0\|_{L^2}^2 \right)$. Hence, it remains to derive that

$$\|u(t)\|_{L^2}^2 \leq C \left(\|u_0\|_{L^2}^2 + \|(v_0, w_0)\|_{H^1}^2 \right). \quad (3.5)$$

Multiplying the first equation of (1.2) by u and integrating the resulting equation with respect to x over I gives rise to

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_I u^2 dx + \int_I u_x^2 dx &= \chi \int_I uv_x u_x dx - \xi \int_I uw_x u_x dx \\ &\leq \int_I \chi^2 (uv_x)^2 dx + \int_I \xi^2 (uw_x)^2 dx + \frac{1}{2} \int_I u_x^2 dx, \end{aligned} \quad (3.6)$$

where we have used the Young's inequality

$$ab \leq \varepsilon a^p + (\varepsilon p)^{-q} p q^{-1} b^q, \quad \text{for any } a, b \geq 0, \varepsilon > 0, 1 < p, q < \infty, \frac{1}{p} + \frac{1}{q} = 1. \quad (3.7)$$

Applying (3.3) and (2.1) to (3.6) and using the Hölder inequality as well as the Young's inequality, we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_I u^2 dx + \frac{1}{2} \int_I u_x^2 dx &\leq \int_I \chi^2 u^2 v_x^2 dx + \int_I \xi^2 u^2 w_x^2 dx \\ &\leq \chi^2 \|u\|_{L^4}^2 \|v_x\|_{L^4}^2 + \xi^2 \|u\|_{L^4}^2 \|w_x\|_{L^4}^2 \\ &\leq C \left(\|u_x\|_{L^2} \|u\|_{L^1} + \|u\|_{L^1}^2 \right) \left(\|v_{xx}\|_{L^2}^{\frac{1}{2}} \|v_x\|_{L^2}^{\frac{3}{2}} + \|v_x\|_{L^2}^2 + \|w_{xx}\|_{L^2}^{\frac{1}{2}} \|w_x\|_{L^2}^{\frac{3}{2}} + \|w_x\|_{L^2}^2 \right) \\ &\leq \frac{1}{8} \left(\|u_x\|_{L^2}^2 + \|v_{xx}\|_{L^2}^2 + \|w_{xx}\|_{L^2}^2 \right) + C \left(\|v_x\|_{L^2}^6 + \|w_x\|_{L^2}^6 \right). \end{aligned} \quad (3.8)$$

Multiplying the second equation of (1.2) by $-v_{xx}$, the third equation by $-w_{xx}$, and adding them, then integrating the resulting equation with respect to x , we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_I (v_x^2 + w_x^2) dx + \int_I (v_{xx}^2 + w_{xx}^2) dx &+ \beta \int_I v_x^2 dx + \delta \int_I w_x^2 dx \\ &= -\alpha \int_I uv_{xx} dx - \gamma \int_I uw_{xx} dx \\ &\leq \frac{1}{2} \int_I (v_{xx}^2 + w_{xx}^2) dx + \frac{\alpha^2 + \gamma^2}{2} \int_I u^2 dx, \end{aligned} \quad (3.9)$$

where the Cauchy–Schwarz inequality has been used.

Using (3.3) and (3.7), we have

$$\int_I u^2 dx \leq C \left(\|u_x\|_{L^2}^{\frac{2}{3}} \|u\|_{L^1}^{\frac{4}{3}} + \|u\|_{L^1}^2 \right) \leq \frac{1}{4(\alpha^2 + \gamma^2 + 2)} \|u_x\|_{L^2}^2 + C \|u\|_{L^1}^2. \quad (3.10)$$

Adding (3.8), (3.9), and (3.10), and applying (3.4) to the resulting inequality, we obtain

$$\begin{aligned} & \frac{d}{dt} \int_I (u^2 + v_x^2 + w_x^2) dx + \int_I (u_x^2 + v_{xx}^2 + w_{xx}^2) dx + \int_I (u^2 + \beta v_x^2 + \delta w_x^2) dx \\ & \leq C (1 + \|v_x\|_{L^2}^6 + \|w_x\|_{L^2}^6) \leq C, \end{aligned} \quad (3.11)$$

where the constant C only depends on $\|(v_0, w_0)\|_{H^1} + \|u_0\|_{L^1}$. Solving (3.11) yields (3.5). Then, the proof of Lemma 3.4 is completed. $\square$

Lemma 3.5

Let the assumption in Proposition 3.2 hold, and (u, v, w) be a solution of (1.2). Then, for any $0 < t \leq T$, the solution satisfies that

$$\|u_x(t)\|_{L^2}^2 + \|(v_{xx}, w_{xx})(t)\|_{L^2}^2 \leq C$$

Therefore, substituting (3.17) and (3.18) back to (3.14) gives

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_I u_x^2 dx + \frac{1}{2} \int_I u_{xx}^2 dx \\ & \leq \frac{1}{4} (\|u_{xx}\|_{L^2}^2 + \|v_{xxx}\|_{L^2}^2 + \|w_{xxx}\|_{L^2}^2) + C (\|u_0\|_{L^2}^2 + \|(v_0, w_0)\|_{H^1}^2), \end{aligned} \quad (3.19)$$

where Lemma 3.4 and (3.7) have been used.

Differentiating the second and third equations of (1.2) with respect to x once, we have

$$\begin{cases} v_{tx} = v_{xxx} + \alpha u_x - \beta v_x, \\ w_{tx} = w_{xxx} + \gamma u_x - \delta w_x. \end{cases} \quad (3.20)$$

Multiplying the first equation of (3.20) by $-v_{xxx}$, the second by $-w_{xxx}$, and adding them, we end up with the following results after integrating the resulting equation with respect to x

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_I (v_{xx}^2 + w_{xx}^2) dx + \int_I (v_{xxx}^2 + w_{xxx}^2) dx + \beta \int_I v_{xx}^2 dx + \delta \int_I w_{xx}^2 dx \\ & = -\alpha \int_I u_x v_{xxx} dx - \gamma \int_I u_x w_{xxx} dx \\ & \leq \frac{1}{2} \int_I (v_{xxx}^2 + w_{xxx}^2) dx + \frac{\alpha^2 + \gamma^2}{2} \int_I u_x^2 dx, \end{aligned} \quad (3.21)$$

where we have used the Cauchy–Schwarz inequality and

$$-\int_I v_{xt} v_{xxx} dx - \int_I w_{xt} w_{xxx} dx = \frac{1}{2} \frac{d}{dt} \int_I (v_{xx}^2 + w_{xx}^2) dx$$

Noting (3.2) and (3.7) entails that

$$\|u_x\|_{L^2}^2 \leq C (\|u_{xx}\|_{L^2} \|u\|_{L^2} + \|u\|_{L^2}^2) \leq \frac{1}{4(\alpha^2 + \gamma^2 + 2)} \|u_{xx}\|_{L^2}^2 + C \|u\|_{L^2}^2. \quad (3.22)$$

Then, combining (3.5), (3.19), (3.21), and (3.22) yields that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_I (u_x^2 + v_{xx}^2 + w_{xx}^2) dx + \frac{1}{4} \int_I (u_{xx}^2 + v_{xxx}^2 + w_{xxx}^2) dx + \int_I (u_x^2 + \beta v_{xx}^2 + \delta w_{xx}^2) dx \\ & \leq \int_I u_x^2 dx + \frac{\alpha^2 + \gamma^2}{2} \int_I u_x^2 dx + C (\|u_0\|_{L^2}^2 + \|(v_0, w_0)\|_{H^1}^2) \\ & \leq \frac{1}{8} \|u_{xx}\|_{L^2}^2 + C (\|u_0\|_{L^2}^2 + \|(v_0, w_0)\|_{H^1}^2), \end{aligned}$$

which implies that

$$\begin{aligned} & \frac{d}{dt} \int_I (u_x^2 + v_{xx}^2 + w_{xx}^2) dx + \int_I (u_{xx}^2 + v_{xxx}^2 + w_{xxx}^2) dx + \int_I (u_x^2 + \beta v_{xx}^2 + \delta w_{xx}^2) dx \\ & \leq C (\|u_0\|_{L^2}^2 + \|(v_0, w_0)\|_{H^1}^2). \end{aligned}$$

Therefore, it follows that

$$\|u_x(t)\|_{L^2}^2 + \|(v_{xx}, w_{xx})(t)\|_{L^2}^2 \leq C (\|u_0\|_{H^1}^2 + \|(v_0, w_0)\|_{H^2}^2),$$

and

$$\begin{aligned} & \int_0^t (\|u_{xx}(s)\|_{L^2}^2 + \|(v_{xxx}, w_{xxx})(s)\|_{L^2}^2) ds \\ & \leq C (\|u_0\|_{H^1}^2 + \|(v_0, w_0)\|_{H^2}^2 + t (\|u_0\|_{L^2}^2 + \|(v_0, w_0)\|_{H^1}^2)) \end{aligned}$$

which completes the proof of Lemma 3.5. □

With the aforementioned results in hand, we are in a position to prove Theorem 2.1.

Proof of Theorem 2.1

On the basis of the estimates obtained in Lemmas 3.4, 3.5, and Sobolev embedding $H^1 \hookrightarrow L^\infty$, we have for any $T > 0$

$$\sup_{0 \leq t \leq \min\{T_0, T\}} \|(u, v, w)\|_{L^\infty} \leq C(\|u_0\|_{H^1} + \|(v_0, w_0)\|_{H^2}).$$

The local solution (u, v, w) can be extended to all $t > 0$ by statement (ii) of Proposition 3.1. The nonnegativity of the solution follows from statement (ii) of Proposition 3.1 directly. The regularity of the solution is obtained by the standard parabolic regularity argument (see for details in [10]). The proof of Theorem 2.1 is finished. $\square$

4. Existence of global attractor

4.1. Higher-order energy estimates

To study the large time behavior of solutions of (1.2), we need higher-order energy estimates.

Lemma 4.1

Let $u_0 \in H^2(I)$, $(v_0, w_0) \in [H^3(I)]^2$. Let (u, v, w) be a solution of (1.2). Then, for any $T > 0$, it holds that for any $0 < t \leq T$

$$\|u_{xx}(t)\|_{L^2}^2 + \|(v_{xxx}, w_{xxx})(t)\|_{L^2}^2 \leq C(\|u_0\|_{H^2}^2 + \|(v_0, w_0)\|_{H^3}^2). \quad (4.1)$$

Proof

We differentiate the first equation of (1.2) with respect to x , multiply the resulting equation by $-u$

where we have used the facts $v_{xxx} = w_{xxx} = 0$ on ∂I , which can be derive from (3.20) and the boundary conditions $u_x = v_x = w_x = 0$ on ∂I . Furthermore, (3.2) and (3.7) entail that

$$\int_I u_{xx}^2 dx \leq C (\|u_{xxx}\|_{L^2} \|u_x\|_{L^2} + \|u_x\|_{L^2}^2) \leq \frac{1}{4(\alpha^2 + \gamma^2 + 2)} \|u_{xxx}\|_{L^2}^2 + C \|u_x\|_{L^2}^2. \quad (4.7)$$

Jointing (4.4), (4.6), and (4.7) yields that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_I (u_{xx}^2 + v_{xxx}^2 + w_{xxx}^2) dx + \frac{1}{2} \int_I (u_{xxx}^2 + v_{xxxx}^2 + w_{xxxx}^2) dx \\ & + \int_I (u_{xx}^2 + \beta v_{xxx}^2 + \delta w_{xxx}^2) dx \\ & \leq \frac{1}{8} (\|u_{xxx}\|_{L^2}^2 + \|(v_{xxxx}, w_{xxxx})\|_{L^2}^2) + \left(1 + \frac{\alpha^2 + \gamma^2}{2}\right) \int_I u_{xx}^2 dx \\ & + C (\|u_x\|_{L^2}^2 + \|(v_{xx}, w_{xx})\|_{L^2}^2) \\ & \leq \frac{1}{4} \|u_{xxx}\|_{L^2}^2 + \frac{1}{8} \|(v_{xxxx}, w_{xxxx})\|_{L^2}^2 + C (\|u_x\|_{L^2}^2 + \|(v_{xx}, w_{xx})\|_{L^2}^2), \end{aligned}$$

which implies

$$\begin{aligned} & \frac{d}{dt} \int_I (u_{xx}^2 + v_{xxx}^2 + w_{xxx}^2) dx + \int_I (u_{xxx}^2 + v_{xxxx}^2 + w_{xxxx}^2) dx \\ & + \int_I (u_{xx}^2 + \beta v_{xxx}^2 + \delta w_{xxx}^2) dx \\ & \leq C (\|u_x\|_{L^2}^2 + \|(v_{xx}, w_{xx})\|_{L^2}^2). \end{aligned} \quad (4.8)$$

Then, the application of Lemma 3.5 and the Gronwall's inequality to (4.8) gives

$$\|u_{xx}(t)\|_{L^2}^2 + \|(v_{xxx}, w_{xxx})(t)\|_{L^2}^2 \leq C (\|u_0\|_{H^2}^2 + \|(v_0, w_0)\|_{H^2}^2).$$

Thus, the proof of Lemma 4.1 is completed. $\square$

Next, we derive the estimates of $\|u(t)\|_{H^2}$ and $\|(v, w)(t)\|_{H^3}$ for $(u_0, v_0, w_0) \in H^1(I) \times H^2(I) \times H^2(I)$.

Proposition 4.2

Let $u_0 \in H^1(I)$, $(v_0, w_0) \in [H^2(I)]^2$ and (u, v, w) be the global solution obtained in Theorem 2.1. Then, we have the following estimate

$$\|u(t)\|_{H^2}^2 + \|(v, w)(t)\|_{H^3}^2 \leq C \left(\frac{1}{t} + \|u_0\|_{H^1}^2 + \|(v_0, w_0)\|_{H^2}^2 \right). \quad (4.9)$$

Proof

Using (3.12) and integrating (4.8) in the interval $[s, t]$, we have

$$\begin{aligned} & \|u_{xx}(t)\|_{L^2}^2 + \|(v_{xxx}, w_{xxx})(t)\|_{L^2}^2 \\ & \leq C (\|u_{xx}(s)\|_{L^2}^2 + \|(v_{xxx}, w_{xxx})(s)\|_{L^2}^2 + \|u_0\|_{H^1}^2 + \|(v_0, w_0)\|_{H^2}^2), \quad 0 < s < t. \end{aligned} \quad (4.10)$$

Furthermore, the integration of (4.10) with respect to s over $(0, t)$ gives

$$\begin{aligned} & t (\|u_{xx}(t)\|_{L^2}^2 + \|(v_{xxx}, w_{xxx})(t)\|_{L^2}^2) \\ & \leq C \int_0^t (\|u_{xx}(s)\|_{L^2}^2 + \|(v_{xxx}, w_{xxx})(s)\|_{L^2}^2) ds + Ct (\|u_0\|_{H^1}^2 + \|(v_0, w_0)\|_{H^2}^2), \quad 0 < s < t. \end{aligned} \quad (4.11)$$

Applying (3.13) to (4.11) yields that

$$\|u_{xx}(t)\|_{L^2}^2 + \|(v_{xxx}, w_{xxx})(t)\|_{L^2}^2 \leq C \left(\frac{1}{t} + 1 \right) (\|u_0\|_{H^1}^2 + \|(v_0, w_0)\|_{H^2}^2). \quad (4.12)$$

The combination of Lemmas 3.4, 3.5, and (4.12) gives

$$\begin{aligned} & \|u(t)\|_{H^2}^2 + \|(v, w)(t)\|_{H^3}^2 \\ & \leq C \left(\frac{1}{t} + 1 \right) (\|u_0\|_{H^1}^2 + \|(v_0, w_0)\|_{H^2}^2) + C (\|u_0\|_{H^1}^2 + \|(v_0, w_0)\|_{H^2}^2) \\ & \leq C \left(\frac{1}{t} + 1 \right) (\|u_0\|_{H^1}^2 + \|(v_0, w_0)\|_{H^2}^2). \end{aligned}$$

Then, we complete the proof of Proposition 4.2. $\square$

As a consequence of Proposition 4.2, we have the following result.

Proposition 4.3

For each bounded ball $B_r = \{(u_0, v_0, w_0) \in \mathcal{X} \mid \|u_0\|_{H^1}^2 + \|(v_0, w_0)\|_{H^2}^2 \leq r\}$, there exists a time t_r depending on B_r such that for any $U_0 \in B_r$, it has that

$$\sup_{t \geq t_r} \sup_{U_0 \in B_r} \|S(t)U_0\|_{H^2 \times H^3 \times H^3} \leq C,$$

where $C > 0$ is a constant.

4.2. Proof of Theorem 2.2

In this section, we are devoted to proving Theorem 2.2. First, we present a result in [11].

Lemma 4.4 ([11])

Assume that for some subset $\mathcal{B} \subset \mathcal{X}$, $\mathcal{B} \neq \emptyset$, and for some $t_0 > 0$, the set $\cup_{t \geq t_0} S(t)\mathcal{B}$ is relatively compact in $\mathcal{X}$. Then, $I(\mathcal{B})$ is nonempty, compact, and invariant.

We are now in a position to prove Theorem 2.2.

Proof of Theorem 2.2

Define the set

$$\mathcal{B} = \{(u, v, w) \in H^2(I) \times H^3(I) \times H^3(I) \mid \|u(t)\|_{H^2}^2 + \|(v, w)(t)\|_{H^3}^2 \leq C\} \cap \mathcal{X},$$

where C is a constant appearing in Proposition 4.3. By the Sobolev imbedding theorem, it follows that $\mathcal{B}$ is a compact subset of $\mathcal{X}$. From Proposition 4.3, we know that for any bounded subset $B_r \subset \mathcal{X}$, there is a time t_r such that $\cup_{t \geq t_r} S(t)B_r \subset \mathcal{B}$. Hence, $\mathcal{B}$ is a compact absorbing set for $(\{S(t)_{t \geq 0}\}, \mathcal{X})$. Using [11, Theorem 1.1], we conclude that $\mathcal{A} = I(\mathcal{B})$ is a global attractor of the dynamical system $(\{S(t)_{t \geq 0}\}, \mathcal{X})$. By Lemma 4.4, this global attractor is nonempty, compact, and invariant in $\mathcal{X}$. Then, the proof of Theorem 2.2 is completed. $\square$

5. Convergence to stationary solution

In this section, we are devoted to proving Theorem 2.3. If $\beta = \delta$ and $\xi\gamma - \chi\alpha < 0$, we set

$$s := \chi v - \xi w. \quad (5.1)$$

Substituting (5.1) into (1.2), we have

$$\begin{cases} u_t = u_{xx} - (us_x)_x, & x \in I, t > 0, \\ s_t = s_{xx} + (\chi\alpha - \xi\gamma)u - \beta s, & x \in I, t > 0, \\ u_x = s_x = 0, & x \in \partial I, t > 0, \\ u(x, 0) = u_0(x), s(x, 0) = \chi v_0(x) - \xi w_0(x) := s_0(x), & x \in I. \end{cases} \quad (5.2)$$

Because of the conservation of cell mass (2.1), the corresponding stationary problem of system (5.2) is

$$\begin{cases} 0 = U_{xx} - (US_x)_x, & x \in I, \\ 0 = S_{xx} + (\chi\alpha - \xi\gamma)U - \beta S, & x \in I, \\ U_x = S_x = 0, & x \in \partial I, \\ \int_I U(x) = M, & x \in I. \end{cases} \quad (5.3)$$

Notice that the nonconstant stationary steady state solution (U, S) of (5.3) have been established in [10, Proposition 2.3] when $\xi\gamma - \chi\alpha < 0$. By Theorem 2.1 and the Minkowski inequality, we have the following estimates on the solution of (5.2).

Lemma 5.1

Let $u_0 \in H^1(I)$, $(v_0, w_0) \in [H^2(I)]^2$. Then, problem (5.2) has a global classical solution $(u, s) \in [C^0(\bar{I} \times [0, \infty); \mathbb{R}^2)]^2 \cap [C^{2,1}(\bar{I} \times (0, \infty); \mathbb{R}^2)]^2$ such that

$$\|u(t)\|_{H^1}^2 + \|s(t)\|_{H^2}^2 \leq C. \quad (5.4)$$

It is well known (e.g., see [3]) that if $s \geq 0$, the system (5.2) has a Lyapunov function

$$E(u, s) = \int_I \left\{ u \ln u + \frac{1}{2(\chi\alpha - \xi\gamma)} (s_x^2 + \beta s^2) - us \right\} dx \quad (5.5)$$

satisfying

$$\frac{d}{dt}E(u(t), s(t)) + \int_I u [(\ln u - s)_x]^2 dx + \frac{1}{\chi\alpha - \xi\gamma} \int_I s_t^2 dx = 0. \quad (5.6)$$

However, we should underline that the initial condition $s_0(x) = \chi v_0(x) - \xi w_0(x)$ may be negative in principle, and hence, the nonnegativity of the solution component s cannot be guaranteed. Fortunately, the second and third terms of (5.6) do not depend on the sign of s , and hence, (5.5) is still a Lyapunov function of (5.2) for any $s \in \mathbb{R}$. However, the sign of s will affect the lower bound of the Lyapunov function. Because in one dimension, the solution (u, s) is uniformly bounded in time, we can easily find a lower bound for the Lyapunov function (5.5) as given next.

Lemma 5.2

For $(u_0, s_0) \in H^1(I) \times H^2(I)$, the Lyapunov function (5.5) satisfies

$$E(u, s) \geq -C - \frac{|I|}{e} \text{ for any } t > 0$$

where C is a positive constant.

Proof

Employing (2.1), (5.4), and Sobolev embedding $H^1 \hookrightarrow L^\infty$, we have

$$\int_I u s dx \leq \|s\|_{L^\infty} \|u_0\|_{L^1} \leq C. \quad (5.7)$$

Substituting (5.7) into (5.5), and using $u \ln u \geq -\frac{1}{e}$ for all $u > 0$, we obtain that

$$E(u, s) \geq -C - \frac{|I|}{e}$$

$$E(u, s) - E_\infty \leq C_0 \left\{ \int_I \left(u [(\ln u - s)_x]^2 + \frac{1}{\chi\alpha - \xi\gamma} s_t^2 \right) dx \right\}^{\frac{1}{2(1-\theta)}}, \quad (5.11)$$

where $\theta \in (0, \frac{1}{2})$ and t_* is a time such that when $t \geq t_*$ one has

$$\|u(t) - U\|_{L^2} + \|s(t) - S\|_{H^1} < \varepsilon.$$

Proof of Theorem 2.3

The convergence of the solution (u, s) to (U, S) follows from the results of [14] directly. Next, we derive the convergence rate announced in Theorem 2.3 based on an idea of [15–17]. First, note that

$$E(u(t), s(t)) - E_\infty = \int_t^\infty \left\{ \int_I \left(u [(\ln u - s)_x]^2 + \frac{1}{\chi\alpha - \xi\gamma} s_t^2 \right) dx \right\} d\tau \geq 0.$$

Combining (5.6) and (5.11), we have for any $t \geq t_*$

$$\frac{d}{dt}(E(u(t), s(t)) - E_\infty) + \frac{1}{C_0^{2(1-\theta)}} (E(u(t), s(t)) - E_\infty)^{2(1-\theta)} \leq 0, \quad (5.12)$$

and

$$\frac{d}{dt}(E(u(t), s(t)) - E_\infty) + \frac{\theta}{C_0^{1-\theta}} \left\{ \int_I \left(u [(\ln u - s)_x]^2 + \frac{1}{\chi\alpha - \xi\gamma} s_t^2 \right) dx \right\}^{1-\theta} \leq 0. \quad (5.13)$$

The inequality (5.12) entails that

$$E(u(t), s(t)) - E_\infty \leq C(1 + t)$$

Multiplying the first equation in (5.19) by $\phi, A_0^{-1}\phi$ and $A_0^{-1}\phi_t$, respectively, and integrating by parts, then applying the Young's inequality (3.7), we end up with

$$\frac{1}{2} \frac{d}{dt} \|\phi\|_{L^2}^2 + \|\phi_x\|_{L^2}^2 \leq \varepsilon \|\phi_x\|_{L^2}^2 + C_{\uparrow} (\|\psi_x\|_{L^2}^2 + \|\phi\|_{L^2}^2), \quad (5.20)$$

$$\frac{1}{2} \frac{d}{dt} \|A_0^{-1/2} \phi\|_{L^2}^2 + \|\phi\|_{L^2}^2 \leq \varepsilon (\|\phi\|_{L^2}^2 + \|\psi_x\|_{L^2}^2) + C_{\uparrow} \|\partial_x A_0^{-1} \phi\|_{L^2}^2, \quad (5.21)$$

and

$$\frac{1}{2} \frac{d}{dt} \|\phi\|_{L^2}^2 + \|A_0^{-1/2} \phi_t\|_{L^2}^2 \leq \varepsilon \|A_0^{-1/2} \phi_t\|_{L^2}^2 + C_{\uparrow} (\|\psi_x\|_{L^2}^2 + \|\phi\|_{L^2}^2) \quad (5.22)$$

where the boundedness of u and S_x have been used. Integrating the second equation in (5.2) multiplied by $\psi, -\psi_{xx}$ and ψ_t , respectively, we have by the Young's inequality (3.7)

$$\frac{1}{2} \frac{d}{dt} \|\psi\|_{L^2}^2 + \|\psi_x\|_{L^2}^2 + \beta \|\psi\|_{L^2}^2 = \langle (\chi\alpha - \xi\gamma)\phi, \psi \rangle \leq \varepsilon \|\psi\|_{H^1} + C_{\uparrow} \|\phi\|_{H^1}, \quad (5.23)$$

$$\frac{1}{2} \frac{d}{dt} \|\psi_x\|_{L^2}^2 + \|\psi_{xx}\|_{L^2}^2 + \beta \|\psi_x\|_{L^2}^2 \leq \varepsilon \|\psi_x\|_{L^2}^2 + C_{\uparrow} \|\phi_x\|_{L^2}^2, \quad (5.24)$$

and

$$\frac{1}{2} \frac{d}{dt} (\|\psi_x\|_{L^2}^2 + \beta \|\psi\|_{L^2}^2) + \|\psi_t\|_{L^2}^2 \leq \varepsilon \|\psi_t\|_{L^2}^2 + C_{\uparrow} \|\phi\|_{L^2}^2. \quad (5.25)$$

Differentiating (5.19) with respect to t and noticing that $\phi_t = u_t$, we have

$$\begin{cases} \phi_{tt} = \phi_{xxt} - (\phi_t \psi_x + u \psi_{xt} + \phi_t S_x)_x, \\ \psi_{tt} = \psi_{xxt} + (\chi\alpha - \xi\gamma)\phi_t - \beta \psi_t. \end{cases} \quad (5.26)$$

Multiplying the first equation in (5.26) by $A_0^{-1}\phi_t$ and noticing that $\partial_x A_0^{-1}\phi_t = \phi_x - u\psi_x - \phi S_x = 0$ for $x \in \partial I$, we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|A_0^{-1/2} \phi_t\|_{L^2}^2 + \|\phi_t\|_{L^2}^2 &= \langle \phi_t \psi_x + u \psi_{xt} + \phi_t S_x, \partial_x A_0^{-1} \phi_t \rangle \\ &= \langle \phi_t \psi_x + \phi_t S_x, \partial_x A_0^{-1} \phi_t \rangle + \langle u \psi_{xt}, \partial_x A_0^{-1} \phi_t \rangle \\ &\leq \varepsilon \|\phi_t\|_{L^2}^2 + C_{\uparrow} \|\partial_x A_0^{-1} \phi_t\|_{L^2}^2 - \langle u \psi_t, \phi_t \rangle - \langle u_x \psi_t, \partial_x A_0^{-1} \phi_t \rangle \\ &\leq \varepsilon \|\phi_t\|_{L^2}^2 + C_{\uparrow} (\|\psi_t\|_{L^2}^2 + \|\partial_x A_0^{-1} \phi_t\|_{L^2}^2) \end{aligned} \quad (5.27)$$

where the boundedness of ψ_x, S_x, u and u_x has been used. We multiply the second equation in (5.25) by ψ_t and integrate over I to obtain

$$\frac{1}{2} \frac{d}{dt} \|\psi_t\|_{L^2}^2 + \|\psi_{xt}\|_{L^2}^2 + \beta \|\psi_t\|_{L^2}^2 \leq \varepsilon \|\psi_t\|_{L^2}^2 + C_{\uparrow} \|\phi_t\|_{L^2}^2. \quad (5.28)$$

Set

$$y(t) = \|A_0^{-1/2} \phi\|_{L^2}^2 + \|\phi\|_{L^2}^2 + \|A_0^{-1/2} \phi_t\|_{L^2}^2 + \|\psi_x\|_{L^2}^2 + \|\psi\|_{L^2}^2 + \|\psi_t\|_{L^2}^2.$$

Noting $\|\partial_x A_0^{-1} \phi\|_{L^2}^2 = \|A_0^{-1/2} \phi\|_{L^2}^2 \leq C \|\phi\|_{H^1}^2$, and letting ε small, we deduce from inequalities (5.20)–(5.25) and (5.27) and (5.28) that

$$\frac{d}{dt} y(t) + ky(t) \leq C (\|\phi\|_{H^1}^2 + \|\psi\|_{L^2}^2) \leq C(1+t)^{-2-1-2},$$

for some $k > 0$, where (5.17) and (5.18) have been used. Solving the aforementioned inequality yields that

$$y(t) \leq y(0)e^{-kt} + Ce^{-kt} \int_0^t e^{ks} (1+s)^{-2-1-2} ds. \quad (5.29)$$

Note that the last term of (5.29) can be estimated as

$$\begin{aligned} \int_0^t e^{ks} (1+s)^{-2-1-2} ds &= \int_0^{t/2} e^{ks} (1+s)^{-2-1-2} ds + \int_{t/2}^t e^{ks} (1+s)^{-2-1-2} ds \\ &\leq e^{\frac{k}{2}t} \int_0^{t/2} (1+s)^{-2-1-2} ds + (1+t/2)^{-2-1-2} \int_{t/2}^t e^{ks} ds \\ &\leq Ce^{\frac{k}{2}t} (1+t/2)^{-2-1-2+1} + \frac{1}{k} e^{kt} (1+t/2)^{-2-1-2}. \end{aligned}$$

Then, substituting the aforementioned inequality into (5.29) gives rise to

$$y(t) \leq C(1+t)^{-2-1-2} \quad (5.30)$$

which implies that

$$\|u(t) - U\|_{L^2} + \|s(t) - S\|_{H^1} \leq C(1+t)^{-1-2}. \quad (5.31)$$

To derive the decay rates for v and w , we subtract (2.2) from (1.2) and obtain that

$$\begin{cases} (v - V)_t = (v - V)_{xx} + \alpha(u - U) - \beta(v - V), \\ (w - W)_t = (w - W)_{xx} + \gamma(u - U) - \delta(w - W). \end{cases} \quad (5.32)$$

By the Duhamel principle, $v(t) - V$ can be represented in the form of

$$v(t) - V = e^{-tA^+} (v_0 - V) + \alpha \int_0^t e^{-t-sA^+} (u(s) - U) ds. \quad (5.33)$$

Noting that $\int_I (u(s) - U) ds = 0$, which allows us to use the inequality $\|e^{-tA} f\|_{L^p} \leq C\|f\|_{L^p}$ for any $f \in L^p$ such that $\int_I f dx = 0$, we have

$$\begin{aligned} \|v(t) - V\|_{L^2} &\leq C e^{-t} + C \int_0^t e^{-t-s} \|u(s) - U\|_{L^2} ds \\ &\leq C e^{-t} + C \int_0^t e^{-t-s} (1+s)^{-1-2} ds \\ &\leq C \left(e^{-t} + \int_0^t e^{-s} (1+t-s)^{-1-2} ds \right) \\ &\leq C(1+t)^{-1-2} \end{aligned} \quad (5.34)$$

where we have used the inequality

$$\int_0^t (1+t-s)^{-\kappa} e^{-s} ds \leq C(1+t)^{-\kappa} \quad \text{for any } \kappa, \rho > 0$$

which was proved in [21, Lemma 4.4]. Similarly, we can prove that the convergence of w satisfies

$$\|w(t) - W\|_{L^2} \leq C(1+t)^{-1-2}.$$

By the first equation of (5.19) and the classic elliptic regularity theory, using (5.30), we have

$$\begin{aligned} \|u_x - U_x\|_{L^2}^2 &= \|\phi_x\|_{L^2}^2 = \|A_0^{-1/2} \phi\|_{L^2}^2 \\ &\leq C \left(\|A_0^{-1/2} \phi_t\|_{L^2}^2 + \|A_0^{-1/2} (u\psi_x + \phi S_x)\|_{L^2}^2 \right) \\ &\leq C \left(\|A_0^{-1/2} \phi_t\|_{L^2}^2 + \|u\psi_x + \phi S_x\|_{L^2}^2 \right) \\ &\leq Cy(t) \leq C(1+t)^{-2-1-2}. \end{aligned} \quad (5.35)$$

The combination of (5.31) and (5.35) gives that

$$\|u(t) - U\|_{H^1} \leq C(1+t)^{-1-2}. \quad (5.36)$$

Letting $\varphi = v_x - V_x$ and using the first equation of (5.32), one has

$$\varphi_t = \varphi_{xx} + \alpha(u_x - U_x) - \beta\varphi.$$

Applying the same procedure to φ as was carried out to $v - V$, and using (5.36), we have a $K > 0$ such that

$$\begin{aligned} \|\varphi\|_{L^2} &\leq C e^{-Kt} + C \int_0^t e^{-K(t-s)} \|u_x(s) - U_x\|_{L^2} ds \\ &\leq C e^{-Kt} + C \int_0^t e^{-K(t-s)} (1+s)^{-1-2} ds \\ &\leq C(1+t)^{-1-2}. \end{aligned} \quad (5.37)$$

Collecting (5.34) and (5.37), one has

$$\|v - V\|_{H^1} \leq C(1 + t)^{-1-2}.$$

Applying the same procedure to w , we have

$$\|w - W\|_{H^1} \leq C(1 + t)^{-1-2}.$$

Thus, the proof of Theorem 2.3 is completed. □

Acknowledgements

The authors thank Yanyan Zhang for the helpful discussion on the last section of this paper. The research of Z.A. Wang was supported in part by the Hong Kong RGC General Research Fund No. 502711.

References

1. Liu J, Wang ZA. Classical solutions and steady states of an attraction–repulsion chemotaxis model in one dimension. *Journal of Biological Dynamics* 2012; **6**:31–41.
2. Keller EF, Segel LA. Initiation of slime mold aggregation viewed as an instability. *Journal of Theoretical Biology* 1970; **26**(3):399–415.