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Global stability of prey-t

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paper [15] where a PDE prey-taxis model was derived to interpret the heterogeneous aggregative patterns due to the interactions between individual ladybugs (predators) and aphids (prey) subject to the so-called area-restricted search strategy. In order to put the detailed individual field observations into a meaningful and tractable population-level model, Karevia and Odell [15] treated the prey-taxis as biased random walks which can incorporate micro-scale observations of individuals. Then passing to the continuum limit, they derived a PDE model which, augmented with the predator–prey interaction, can be formulated as:

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u\rho(u,v)\nabla v) + G_1(u,v), \\ v_t = D\Delta v + G_2(u,v), \end{cases} \tag{1.1}$$

where $u = u(x,t)$ denotes the predator density at position x and time $t > 0$ and $v = v(x,t)$ the prey population density; the term $-\nabla \cdot (u\rho(u,v)\nabla v)$ stands for the prey-taxis with a coefficient $\rho(u,v)$ which may depend on the predator or prey density and D is the prey diffusion rate. The functions $G_1(u,v)$ and $G_2(u,v)$ describe the population interactions between the predator and the prey.

Ecological/biological population interactions can be defined as either intra-specific or inter-specific. The former occurs between individuals of the same species, while the later between different species. The predator–prey population interaction, including both intra-specific or inter-specific interactions, possesses the following prototypical form

$$G_1(u,v) = \gamma u F(v) - uh(u),\ G_2(u,v) = f(v) - uF(v)$$

where $uF(v)$ represents the inter-specific interaction, $uh(u)$ and $f(v)$ accounts for the intra-specific interaction. Specifically $F(v)$ is the so-called functional response function accounting for the intake rate of predators as a function of prey density, $h(u)$ is the predator mortality rate function and $f(v)$ is the prey growth function; the parameters $\gamma > 0$ denotes the intrinsic predation rate. The most widely used forms of $F(v)$ in the literature are:

$$\begin{aligned} &F(v) = v \text{ (Lotka–Volterra type or Holling type I)}; \\ &F(v) = \frac{v}{\lambda + v} \text{ (Holling type II)};\ F(v) = \frac{v^m}{\lambda^m + v^m} \text{ (Holling type III)} \end{aligned} \tag{1.2}$$

with constants $\lambda > 0$ and $m > 1$. The predator mortality rate function $h(u)$ is typically of the form

$$h(u) = \theta + \alpha u \tag{1.3}$$

where $\theta > 0$ accounts for the natural death rate and $\alpha \geq 0$ denotes the rate of death resulting from the intra-specific competition (also called density-dependent death, e.g. see [20]). The prey growth function $f(v)$ is usually assumed to be negative for large v due to the limitation of resource (or crowding effect) and typical forms are

$$\begin{aligned} &f(v) = \mu v(1 - v/K) \text{ (Logistic type)}; \\ &f(v) = \mu v(1 - v/K)(v/k - 1) \text{ (Bistable or Allee effect type)} \end{aligned} \tag{1.4}$$

where $\mu > 0$ is the intrinsic growth rate of prey and $K > 0$ is called the carrying capacity and $0 < k < K$. Other type of functional response functions and predator–prey interactions can be found in the excellent surveys [7,24,38].

Without prey-taxis (i.e. the term $\nabla \cdot (u\rho(u, v)\nabla v)$ is ignored), the model (1.1) becomes the well-known diffusive predator–prey system which has been widely studied from numerous perspectives over many years (see [8,39] and references therein). If prey-taxis is included, the system (1.1) becomes a cross-diffusion system which is much more difficult to handle and not many results are available. If $\rho(u, v) = \frac{1}{(1+v)^\sigma} (\sigma = 1, 2)$ or $\rho(u, v) = \chi$ is a constant, the traveling wave solutions of (1.1) in $x \in \mathbb{R}$ have been investigated by Lee et al. [19] for a variety of functional forms $F(v), h(u)$ and $f(v)$. They showed that the incorporation of prey-taxis to the diffusive predator–prey model reduces the effect of the predator on controlling prey spread. In a subsequent work [20] they studied the pattern formation of prey-taxis system (1.1) in a bounded interval with zero Neumann boundary condition. When the prey-taxis model (1.1) is considered in a multi-dimensional bounded domain $\Omega \subset \mathbb{R}^n (n \geq 2)$ with Neumann boundary condition, the first interesting question would be whether the solution blows up, which is interpreted as overcrowding (or outbreak) of populations, since the prototypical taxis model such as Keller–Segel model may blow up in two or higher dimensions (e.g. see [14,45]). Up to date there are a few results available to the prey-taxis model in this direction and we shall recall them below. First if the following Rosenzweig–MacArthur predator–prey model (e.g. see [30])

$$F(v) = \frac{v}{\lambda + v}, \ h(u) = \theta, \ \text{and} \ f(v) = \mu v(1 - v/K), \tag{1.5}$$

is considered and the prey-tactic coefficient $\rho(u, v) = \rho_1(u)$ depends only upon u but is truncated at some number $u_m > 0$ (i.e. $\rho_1(u_m) = 0$ and $\rho_1(u) > 0$ for $0 \leq u < u_m$), Ainseba et al. [1] obtained the global weak solutions of (1.1) with (1.5) for $n \geq 1$ by the Schauder fixed point theorem and duality technique, which was later extended to the global classical solutions by Tao in [34] for $n \leq 3$ via L^p-estimates and Schauder estimates, where the solution bound depends on time. Recently He and Zheng [12] has improved the result of [34] by obtaining the uniform-in-time boundedness of solutions. Note that the truncation assumption used in [1,12,34] for $\rho_1(u)$ is an analogy of the volume-filling effect used in chemotaxis (see [28,41]). Second if $\rho(u, v) = \chi > 0$ is a constant, the existence of non-constant steady states of (1.1) with (1.5) was studied in [21,40] by Hopf bifurcation theorem and index degree theory. Furthermore the numerical solutions of (1.1) with (1.5) were examined in [6] illustrating that the initial condition and the form of $F(v)$ play important roles in the pattern formation of prey-taxis. Recently Wu et al. [46] considered various functional forms of $F(v), h(u)$ and $f(v)$, and showed that the solution is globally bounded if χ is small. The asymptotic behavior of solutions is derived for some particularized predator–prey interactions under certain conditions.

In the existing works as recalled above, either the technical truncation assumption (see [1,12,34]) or smallness assumption (see [46]) was imposed for the prey-taxis coefficient to ensure the global boundedness of solutions. To preclude the population overcrowding (outbreak), these assumptions are often used for the Keller–Segel chemotaxis models which resemble the prey-taxis system in the absence of the predator–prey interaction (see [13,14]). The present situation is somewhat different: the ecologically intrinsic predator–prey interaction has dampening effects. Then a natural question is whether this intrinsic predator–prey interaction itself is sufficient to preclude the population overcrowding without imposing additional conditions like truncation or smallness assumption ? To explore this question, we consider (1.1) with the predator–prey interaction in a two dimensional bounded domain with Neumann boundary conditions. That is we consider the following initial-boundary value problem

$$\begin{cases} u_t = \Delta u - \nabla \cdot (\chi u \nabla v) + \gamma u F(v) - u h(u), & x \in \Omega, \quad t > 0, \\ v_t = D\Delta v - uF(v) + f(v), & x \in \Omega, \quad t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0(x), v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.6}$$

where $\chi > 0$ is referred to as the prey-tactic coefficient measuring the strength of prey-taxis and ν denotes the outward normal vector of $\partial\Omega$. Though the functional forms of $F(v), h(u)$ and $f(v)$ given in (1.2)–(1.4) are typical forms for the predator–prey system, our results in the paper indeed are allowed to cover a wider class of these functions. Specifically in the sequel, we assume that $F(v), h(u)$ and $f(v)$ satisfy the following hypotheses that are fulfilled by the typical examples given in (1.2), (1.3) and (1.4):

(H1) $F(v) \in C^2([0, \infty))$, $F(0) = 0$, $F(v) > 0$ in $(0, \infty)$ and $F'(v) > 0$, $F''(v) \le 0$ on $[0, \infty)$.
(H2) The function $h : [0, \infty) \to (0, \infty)$ is continuously differentiable and there exist two constants $\theta > 0$ and $\alpha \ge 0$ such that $h(u) \ge \theta$ and $h'(u) \ge \alpha$ for any $u \ge 0$.
(H3) The function $f : [0, \infty) \to \mathbb{R}$ is continuously differentiable satisfying $f(0) = 0$, and there exist two constants $\mu, K > 0$ such that $f(v) \le \mu v$ for any $v \ge 0$, $f(K) = 0$ and $f(v) < 0$ for all $v > K$. Moreover the ratio $\frac{f(v)}{F(v)}$ is continuous on $(0, \infty)$ and $\lim_{v \to 0} \frac{f(v)}{F(v)}$ exists.

The first goal of this paper is to show that the solution of the prey-taxis system (1.6) with the predator–prey interaction in two dimensions is globally bounded for any $\chi > 0$. In particular the boundedness results do not require the truncation assumption imposed in [1,12,34] to cut off the prey-taxis or the smallness assumption in [46] to weaken the prey-taxis. This implies that the intrinsic predator–prey interaction is sufficient to preclude the population overcrowding in spite of the aggregation effect of the prey-taxis. The boundedness of solutions is established by first deriving an entropy-like equality based on which the L^2-estimate of u is obtained and the boundedness criterion (see Lemma 3.1) then follows to get L^∞-bound of u. The results are given in the following theorem.

Theorem 1.1 *(B ded e - e e f e c d). Le* $\Omega \subset \mathbb{R}^2$ *be a b ded d a b da a d e e e (H1)–(H3) d. A e* $(u_0, v_0) \in [W^{1,p}(\Omega)]^2$ $u_0, v_0 \ge 0 (\not\equiv 0)$ *a d* $p > 2$. *T e e b e (1.6) a a e ba c a ca* $(u, v) \in C(\bar\Omega \times [0, \infty)) \cap C^{2,1}(\bar\Omega \times (0, \infty))$ *a f*

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{W^{1,\infty}(\Omega)} \le C,$$

e e $C > 0$ *a c a de e de f* t, *a d a c a* $0 < v \le K_0$ *e e*

$$K_0 := \max\{\|v_0\|_L$$

assumption on χ and covering more general response function $F(v)$ in the two dimensional case. However whether the same results hold true for three or higher dimensions remains unknown in the present paper.

In addition to the population overcrowding, another relevant question is whether the interacting predator–prey population will arrive at the coexistence, exclusion or extinction eventually, which is always a central question in population dynamics. It is straightforward to check that the system (1.6) has three homogeneous steady states (u_s, v_s):

$$(u_s, v_s) = \begin{cases} (0,0) \text{ or } (0, K), & \text{if } \gamma F(K) \le \theta, \\ (0,0) \text{ or } (0, K) \text{ or } (u_*, v_*), & \text{if } \gamma F(K) > \theta \end{cases} \tag{1.8}$$

with $u_*, v_* > 0$ determined by the following algebraic equations:

$$u_* = \frac{f(v_*)}{F(v_*)}, \ \gamma F(v_*) = h\left(\frac{f(v_*)}{F(v_*)}\right), \tag{1.9}$$

where $(0,0)$ is the extinction steady state, $(0, K)$ is the prey-only steady state and (u_*, v_*) is the coexistence steady state. The coexistence steady state (u_*, v_*) is determined as follows: first solve for v_* from the second equation of (1.9) and then substitute it into the first equation to get u_*. Given arbitrary functions $F(v)$, $h(u)$ and $f(v)$, the second equation of (1.9) does not guarantee to generate a positive solution v_*, but it usually does for biologically meaningful forms like those given in (1.2)–(1.4). In particular, if $F(v)$, $h(u)$ and $f(v)$ are explicitly given, (u_*, v_*) can often be explicitly found. For example, if $F(v)$ is of Holling type II, $f(v)$ is of logistic type and $h(u)$ is given by (1.3), then one can get $(u_*, v_*) = \left(\frac{\gamma\lambda\mu[(\gamma-\theta)K-\theta\lambda]}{(\gamma-\theta)^2 K}, \frac{\theta\lambda}{\gamma-\theta}\right)$. Next we shall explore the question: which of the above three homogeneous steady states will be eventually attained. This amounts to find the global asymptotical stability of the homogeneous steady states of (1.6). In general, global stability of the cross-diffusion system like chemotaxis or prey-taxis system is difficult and not many approaches are available. Here we manage to use Lyapunov functionals to get the global stability of the homogeneous steady states under certain conditions. Our plan is to first present the global stability results for general functions $F(v)$, $h(u)$ and $f(v)$, and then apply them to some frequently used explicit forms as applications presented in two propositions (see Proposition 1.4 and Proposition 1.6). For the global stability, except the hypotheses (H1)–(H3), we need another hypothesis for the following compound function:

$$\phi(v) = \frac{f(v)}{F(v)} \tag{1.10}$$

as follows:

(H4) The function $\phi(v)$ is continuously differentiable on $(0, \infty)$, $\phi(0) = \lim_{v \to 0} \phi(v) > 0$ and $\phi'(v) < 0$ for any $v \ge 0$.

We remark that the hypothesis (H4) is not stringent, and can be satisfied by many forms given in (1.2) and (1.4) by imposing some conditions on the parameters if needed. For example, if $f(v)$ is of logistic type given in (1.4), then (H4) is automatically satisfied if $F(v)$ is of Holling

Type I and satisfied with $\lambda > K$ if $F(v)$ is of Holling Type II given in (1.2). But (H4) may not be fulfilled by bistable function $f(v)$ or the Holling type III functional response function $F(v)$. In general, if (H4) is not satisfied, pattern formations such as periodic orbits or non-constant steady state may arise (see [47]).

Then our global stability theorem is given as follows:

Theorem 1.3 *(G ba ab). Le e e e (H1)–(H4) a d a m T e em 1.1 d, a d e* (u, v) *be e b a ed T e em 1.1. T e e f e d:*

1. *If e a ame e* θ, γ, K *a f*

$$\gamma F(K) \leq \theta \ (\ ea \ eda \ ),$$

e e $=$ *d e ca e f* $\alpha > 0$, *e e ead a e* $(0, K)$ *ba a m ca ab e. F e m e, e ca d m e c a* $\lambda_i > 0$ *a d* $C_i > 0 (i = 1, 2)$ *a d* $t_0 > 0$ *c a f a* $t > t_0$ *d a*

$$\|u\|_{L^\infty} + \|v - K\|_{L^\infty} \leq C_1 e^{-\lambda_1 t}, \quad f\ \gamma F(K) < \theta,$$

a d

$$\|u\|_{L^\infty} + \|v - K\|_{L^\infty} \leq C_2 (1 + t)^{-\lambda_2}, \quad f\ \gamma F(K) = \theta\ a\ d\ \alpha > 0.$$

2. *If e a ame e* χ, θ, γ, K *a f*

$$\gamma F(K) > \theta \ (\ \ eda \ ),$$

a d e c -e e ce ead a e (u_*, v_*) *e , e* (u_*, v_*) *ba a m ca ab e ded a*

$$\frac{D}{\chi^2} \geq D_c = \frac{u_* F^2(K)}{4\gamma F(v_*) F'(K)} \tag{1.11}$$

e e u_* *a d* v_* *a e de e m ed b (1.9) a d d de e d D a d* χ, *a d* $=$ *d e ca e f* $\|v_0\|_{L^\infty} \leq K$. *F e m e, f* $\alpha > 0$, *e ca d a* $T_0 > 0$ *c a e f deca d f a* $t > T_0$

$$\|u - u_*\|_{L^\infty} + \|v - v_*\|_{L^\infty} \leq C_3 e^{-\lambda_3 t},$$

f me e c a C_3 *a d* λ_3.

We have the several remarks concerning the global stability theorem.

Remark.

- Since the growth of the predator comes from predation, the quantity $\gamma F(K)$ becomes the predation rate of the predator. Hence the results of Theorem 1.3 tell us that if the predation is weak, in the sense of $\gamma F(K) \le \theta$, the prey-only steady state $(0, K)$ will be attained and the predator will go extinct. In this case, the prey-taxis does not play a role in stability. Whilst if the predation is strong, namely $\gamma F(K) > \theta$, the coexistence steady state (u_*, v_*) can be reached if the ratio of prey diffusion to the square of prey-taxis coefficient (namely the quantity $\frac{D}{\chi^2}$) is suitably large. This implies that predation rate, prey diffusion and prey-taxis strength all play a part to reach a coexistence steady state (u_*, v_*) in the predator–prey system with prey-taxis.
- Since u_* and v_* are independent of D and χ, the condition (1.11) in Theorem 1.3 is always achievable by letting D be suitably large or χ be suitably small. Since in the case of weak predation, the global stability of $(0, K)$ is unconditional, Theorem 1.3 implies that global stability of constant steady states will always be achieved and hence no pattern formation arises if the prey diffuses fast or prey-taxis is weak (including no prey-taxis $\chi = 0$). This raises an interesting question as whether the coexistence steady state is stable, or in a further step whether pattern formation (non-constant steady states) is possible, if $\gamma F(K) > \theta$ and $D/\chi^2 < D_c$.
- In the second part of Theorem 1.3, the value of (u_*, v_*) is not explicitly given because the functional forms of $F(v)$, $h(u)$ and $f(v)$ are not specified. Once they are given, the values of (u_*, v_*) and D_c can be explicitly found and hence condition (1.11) can be identified. Below we shall present the applications of Theorem 1.3 to some well-known functional forms in the predator–prey system by specifying (u_*, v_*) and hence the condition (1.11). But we should underline that our results are quite general and applications are not restricted to the examples presented below.

The first example for the application of our results is a widely used class of predator–prey system: the Rosenzweig–MacArthur type given in (1.5).

Proposition 1.4 *(S ab f e R e z e –MacA r eda r– e m e - a).* *Le* $F(v), h(u)$ *a d* $f(v)$ *be e b* *(1.5)* *a d a me* $(u_0, v_0) \in [W^{1,p}(\Omega)]^2$ $u_0, v_0 \ge 0 (\not\equiv 0)$ *a d* $p > 2$. *T e e a -b da a e b em* *(1.6)* *a a e ba c a ca* $\Omega \subset \mathbb{R}^2$ *e f ab e :*

- *If e eda ea , me* $\frac{\gamma K}{\lambda+K} < \theta$, *e e ead a e* $(0, K)$ *ba a m - ca ab e, a d e e e a* $t_0 > 0$ *c a f a* $t > t_0$

$$\|u\|_{L^\infty} + \|v - K\|_{L^\infty} \le C_1 e^{-\lambda_1 t}, \quad f\ a\ t > t_0$$

d f me c a $\lambda_1 > 0$ *a d* $C_1 > 0$.

- *If e eda , me* $\frac{\gamma K}{\lambda+K} > \theta$, *e e em* *(1.6)* *a a e c e e ce ead a e* (u_*, v_*):

$$(u_*, v_*) = \left(\frac{\gamma\lambda\mu[(\gamma-\theta)K - \theta\lambda]}{(\gamma-\theta)^2 K}, \frac{\theta\lambda}{\gamma-\theta} \right),$$

c ba a m ca ab e r ded a $\lambda > K$ *a d*

$$\frac{D}{\chi^2} \geq \frac{\mu K \gamma(\lambda + K)}{4\theta(\gamma - \theta)^2}\left(\frac{\gamma K}{\lambda + K} - \theta\right),$$

e re = d e ca e f $\|v_0\|_{L^\infty} \leq K$.

Remark 1.5. The global stability for the case of weak predation $\frac{\gamma K}{\lambda+K} < \theta$ has been proved in [46, Corollary 5.2]. Here we not only get this result as a consequence of our general theorem, but also derive the exponential convergence rate. Furthermore we obtain the global stability for the case of strong predation: $\frac{\gamma K}{\lambda+K} > \theta$, which was not considered in [46]. We underline that by the general hypothesis (H4), the condition $\lambda > K$ should be imposed in Proposition 1.4 to ensure $\phi'(v) < 0$ for all $v \geq 0$. But from the proof of Lemma 4.2, it is easy to see that in the case of weak predation $\gamma F(K) < \theta$, the condition $\phi' < 0$ for any $v \geq 0$ can be relaxed to $\phi'(K) < 0$ which is naturally satisfied by the function forms in (1.5) (see also [46, Corollary 5.2]). This is why we only give the requirement $\lambda > K$ for the case of strong predation in Proposition 1.4.

The second example to be discussed is when the functional response function is of a Lotka–Volterra (or Holling type I) type, reading as

$$F(v) = v, \; h(u) = \theta + \alpha u, \; \text{and } f(v) = \mu v(1 - \frac{v}{K}). \tag{1.12}$$

For this case, we will have the following results on the global stability of the system with convergence rates.

Proposition 1.6 *(S ab f e L a–V e ra reda re m re - a). Le* $F(v), h(u)$ *a d* $f(v)$ *be e b (1.12) a d a me* $(u_0, v_0) \in [W^{1,p}(\Omega)]^2$ $u_0, v_0 \geq 0 (\not\equiv 0)$ *a d* $p > 2$*. T e e a -b da a e r b m (1.6) a a e ba c a ca* $\Omega \subset \mathbb{R}^2$ *b ec e f ab re :*

- *If e reda ea , .e.* $\gamma K \leq \theta$, *e re = d e ca e f* $\alpha > 0$, *e e ead a e* $(0, K)$ *ba a m ca ab e. F r e m re, e e e a mbe r* $t_0 > 0$ *c a*

$$\|u\|_{L^\infty} + \|v - K\|_{L^\infty} \leq \begin{cases} C_2 e^{-\lambda_1 t}, & f \; \gamma K < \theta \\ C_2(1 + t)^{-\lambda_2}, & f \; \gamma K = \theta \; a \; d \; \alpha > 0 \end{cases}$$

d f r a $t > t_0$ *a d me c a* $C_2 > 0, \lambda_i > 0 (i = 1, 2)$.

- *If e reda r , .e.* $\gamma K > \theta$, *e e m (1.6) a a e c e e ce ead a e* (u_*, v_*):

$$(u_*, v_*) = \left(\frac{\mu(\gamma K - \theta)}{\gamma K + \mu\alpha}, \frac{K(\mu\alpha + \theta)}{\gamma K + \mu\alpha}\right),$$

c ba a m ca ab e f

$$\frac{D}{\chi^2} \geq \frac{\mu K(\gamma K - \theta)}{4\gamma(\alpha\mu + \theta)},$$

e e = , d e ca e f $\|v_0\|_{L^\infty} \leq K$. *F e e, f* $\alpha > 0$, *e e a* $T_0 > 0$ *a e f deca d f a* $t > T_0$ *a d e e c a* C_3 *a d* λ_3:

$$\|u - u_*\|_{L^\infty} + \|v - v_*\|_{L^\infty} \leq C_3 e^{-\lambda_3 t}.$$

2. Preliminaries

In what follows, without confusion, we shall abbreviate $\int_\Omega f dx$ as $\int_\Omega f$ for simplicity. Moreover, we shall use c_i for C_i $(i = 1, 2, 3, \cdots)$ to denote a generic constant which may vary in the context. We first give the existence of local solutions of (1.6), which can be readily proved by the Amann's theorem [3,4] (cf. also [41, Lemma 2.6]).

Lemma 2.1 *(L ca e e ce). Le* $\Omega \subset \mathbb{R}^n (n \geq 2)$ *be a b ded d a b da a d e e e (H1)–(H3) d. A e* $(u_0, v_0) \in [W^{1,p}(\Omega)]^2$ $u_0, v_0 \geq 0(\equiv 0)$ *a d* $p > n$. *T e e e e* $T_{max} > 0$ *c a e b e* (1.6) *a a e c a ca* $(u, v) \in C(\bar{\Omega} \times [0, T_{max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{max}))$ *a f* $u, v > 0$ *f a* $t > 0$. *M e e*

$$if \ T_{max} < \infty, \ then \ \|u(\cdot, t)\|_{L^\infty} \to \infty \ as \ t \nearrow T_{max}.$$

Lemma 2.2. *U de e c d T e e 1.1, e* (u, v) *f* (1.6) *a e*

$$0 < v(x, t) \leq K_0, \ f\ a \ \ x \in \Omega, \ t > 0, \tag{2.1}$$

e e K_0 *de ed b* (1.7)*, a d f e d a*

$$\limsup_{t \to \infty} v(x, t) \leq K \ f\ a \ \ x \in \bar{\Omega}. \tag{2.2}$$

M e e, e e a c a $C > 0$ *de e de f t c a*

$$\|u(\cdot, t)\|_{L^1} \leq C, \ f\ a \ \ t > 0. \tag{2.3}$$

Proof. Using the facts that u, v and $F(v)$ are non-negative, then we have

$$\begin{cases} v_t - \Delta v = -uF(v) + f(v) \leq f(v), & x \in \Omega, \ t > 0, \\ \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, \ t > 0, \\ v(x, 0) = v_0(x), & x \in \Omega. \end{cases} \tag{2.4}$$

Let $v^*(t)$ be the solution of the following ODE problem

$$\begin{cases} \dfrac{dv^*(t)}{dt} = f(v^*(t)), \ t > 0, \\ v^*(0) = \|v_0\|_{L^\infty}. \end{cases} \tag{2.5}$$

Then the hypothesis (H3) yields that $v^{\ast}(t) \le K_0 = \max\{\|v_0\|_{L^\infty}, K\}$. It is clear that $v^{\ast}(t)$ is a super-solution of the following PDE problem

$$\begin{cases} V_t - \Delta V = f(V), & x \in \Omega,\ \ t > 0, \\ \frac{\partial V}{\partial \nu} = 0, & x \in \partial\Omega,\ \ t > 0, \\ V(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{2.6}$$

and hence it holds that

$$0 < V(x,t) \le v^{\ast}(t) \text{ for all } (x,t) \in \bar{\Omega} \times (0,\infty), \tag{2.7}$$

where $V > 0$ results from the strong maximum principle with the fact $f(0) = 0$. Combining (2.4), (2.6) and (2.7), and using the comparison principle, one has

$$0 < v(x,t) \le V(x,t) \le v^{\ast}(t) \le K_0 \ \text{ for all } (x,t) \in \bar{\Omega} \times (0,\infty), \tag{2.8}$$

which gives (2.1). Noting that $f(v) < 0$ for all $v > K$ by hypothesis (H3), we further have from (2.5) that $\limsup_{t\to\infty} v^{\ast}(t) \le K$, which along with (2.8) gives (2.2).

Multiplying the second equation (1.6) by γ and adding the resulting equation into the first equation of (1.6), then integrating the result over $\Omega \times (0,t)$, one has

$$\frac{d}{dt}\left(\int_\Omega u + \gamma \int_\Omega v\right) + \int_\Omega u h(u) = \gamma \int_\Omega f(v) \le \gamma\mu \int_\Omega v,$$

which together with the hypotheses (H2) and (H3) and the fact that $0 < v \le K_0$ gives

$$\frac{d}{dt}\left(\int_\Omega u + \gamma \int_\Omega v\right) + \theta\left(\int_\Omega u + \gamma \int_\Omega v\right) \le (\gamma\mu + \theta\gamma)\int_\Omega v \le (\gamma\mu + \theta\gamma)K_0|\Omega|. \tag{2.9}$$

With the Gronwall's inequality applied to (2.9), we obtain (2.3) and complete the proof of Lemma 2.2. □

Next, we present some basic inequalities which will be used later.

Lemma 2.3 *(Ga a d –N e be e a). Le Ω be a b ded d a $\mathbb{R}^n$ b da . Le $1 \le p, q \le \infty$ a f $(n - kq)p < nq$ f e $k > 0$ a d $r \in (0, p)$. T e , f a $w \in W^{k,q}(\Omega) \cap L^r(\Omega)$, e e e c a c_1 a d c_2 de e d Ω, q, k, r a d n c a*

$$\|w\|_{L^p} \le c_1 \|D^k w\|_{L^q}^{a} \|w\|_{L^r}^{1-a} + c_2 \|w\|_{L^r},$$

e e $a \in (0,1)$ f

$$\frac{1}{p} = a\left(\frac{1}{q} - \frac{k}{n}\right) + (1-a)\frac{1}{r}.$$

We should remark the original Gagliardo–Nirenberg inequality (e.g. see [27]) is stated only for $r \geq 1$, but this condition can be readily relaxed to $r \in (0, p)$ by using the Hölder's inequality (cf. [42, Lemma 3.2]).

Lemma 2.4 *([26]). Le Ω be a b ded d ma $\mathbb{R}^2$ b da a d $w \in W^{1,2}(\Omega)$. T e f a $\varepsilon > 0$, e e a c a $C_\varepsilon > 0$ c a*

$$\|w\|_{L^3}^3 \leq \varepsilon \|\nabla w\|_{L^2}^2 \|w \ln |w|\|_{L^1} + C_\varepsilon (\|w\|_{L^1}^2 \|w \ln |w|\|_{L^1} + \|w\|_{L^1}).$$

Lemma 2.5 *([44]). T e f a e d:*

(i) *S e a $g \in C^2(\mathbb{R})$. T e f a $\psi \in C^2(\bar{\Omega})$ f $\frac{\partial \psi}{\partial \nu} = 0$ $\partial\Omega$, f a*

$$\frac{3}{2}\int_\Omega g'(\psi)|\nabla \psi|^2 \Delta\psi = -\int_\Omega g(\psi)|\Delta \psi|^2 + \int_\Omega g(\psi)|D^2\psi|^2$$
$$- \frac{1}{2}\int_\Omega g''(\psi)|\nabla\psi|^4 - \frac{1}{2}\int_{\partial\Omega} g(\psi)\frac{\partial|\nabla\psi|^2}{\partial\nu}.$$

(ii) *Le $g \in C^1((0, +\infty))$ be e a d e $\mathcal{G}(s) =: \int_1^s \frac{d\sigma}{g(\sigma)}$ f $s > 0$. T e f a e $\psi \in C^2(\bar{\Omega})$ f $\frac{\partial \psi}{\partial \nu} = 0$ $\partial\Omega$, e f e a d :*

$$\int_\Omega \frac{g'(\psi)}{g^3(\psi)}|\nabla\psi|^4 \leq (2 + \sqrt{n})^2 \int_\Omega \frac{g(\psi)}{g'(\psi)}|D^2\mathcal{G}(\psi)|^2.$$

3. Prevention of overcrowding

In this section, we are devoted to proving Theorem 1.1 by deriving some *a* estimate. Motivated by the ideas in [5, lemma 3.2], we first show that the L^∞-boundedness of predator density u can be reduced to proving its L^p-boundedness for $p > \frac{n}{2}$ (see Lemma 3.1 for details). Hence in two-dimensional spaces ($n = 2$), we can immediately obtain the uniform boundedness of solutions provided that there exists a constant $C > 0$ independent of t such that $\|u(\cdot, t)\|_{L^2} \leq C$. To this end, we first derive an entropy-like equality to show the uniform boundedness of $\|u \ln u\|_{L^1}$ and $\|\nabla v\|_{L^2}$ (see Lemma 3.2 and Lemma 3.3), which leads to the boundedness of $\|u(\cdot, t)\|_{L^2}$ in two dimensions based on the argument in [5, Lemma 3.3]. Below we first show the boundedness criterion of solutions of (1.6) below, which is an extension of [5, lemma 3.2] where the growth term in the first equation is uniformly bounded for any $u, v \geq 0$, whereas this is not the case in our model. But the idea of our proof is essentially inspired by [5, lemma 3.2] and we present necessary details of the proof below for clarity.

Lemma 3.1 *(B ded e c e). S e e c d Lemma 2.1 d. Le (u, v) be e f (1.6) de ed ma ma e e ce me e a $[0, T_{max})$. If e e $p > \frac{n}{2}$ a d*

$$\sup_{t\in(0,T_{max})} \|u(\cdot,t)\|_{L^p} \le M_0, \tag{3.1}$$

e e ca d a c a $C>0$ *de e de f t c a*

$$\|u(\cdot,t)\|_{L^\infty} + \|v(\cdot,t)\|_{W^{1,\infty}} \le C \quad f \ a \quad t \in (0,T_{max}). \tag{3.2}$$

Proof. By $\|u(\cdot,t)\|_{L^p} \le M_0$, we claim that

$$\|\nabla v(\cdot,t)\|_{L^r} \le c_1, \text{ for all } t \in (0,T_{max}) \tag{3.3}$$

with

$$r \in \begin{cases} [1, \frac{np}{n-p}), & \text{if} \quad p \le n, \\ [1,\infty], & \text{if} \quad p > n. \end{cases} \tag{3.4}$$

In fact, from the second equation of system (1.6), we know that v solves the following problem

$$v_t = D\Delta v - v + g(u,v) \ \text{ in } \ \Omega, \quad \frac{\partial v}{\partial \nu} = 0, \tag{3.5}$$

where $g(u,v) := v - uF(v) + f(v)$. By the properties of $F(v)$, $f(v)$ and the fact that $0 < v(x,t) \le K_0$ in (2.1), one have

$$\|g(u,v)\|_{L^p} \le c_2(\|u\|_{L^p} + 1) \le c_2(M_0+1) := c_3. \tag{3.6}$$

Then applying the results of [16, Lemma 1] (see also [35, Lemma 1.2]) to the problem (3.5) with (3.6), we obtain (3.3) with (3.4). Without loss of generality, we assume that $\frac{n}{2} < p \le n$ which entails $\frac{np}{n-p} > n$. Then we can find $n < r < \frac{np}{n-p}$ such that (3.3) holds. Now, for each $T \in (0, T_{max})$, we define

$$M(T) := \sup_{t\in(0,T)} \|u(\cdot,t)\|_{L^\infty}, \tag{3.7}$$

which is finite due to the local existence results in Lemma 2.1. Next, we will estimate $M(T)$. Fix $t \in (0,T)$ and let $t_0 = (t-1)_+$. Then using the variation-of-constants formula and noting that $uh(u) \ge 0$, we get

$$u(\cdot,t) \le e^{(t-t_0)\Delta}u(\cdot,t_0) - \chi\int_{t_0}^{t} e^{(t-s)\Delta}\nabla\cdot(u(\cdot,s)\nabla v(\cdot,s))ds + \gamma\int_{t_0}^{t} e^{(t-s)\Delta}u(\cdot,s)F(v(\cdot,s))ds,$$

which implies

$$\begin{aligned}\|u(\cdot,t)\|_{L^\infty} &\le \|e^{(t-t_0)\Delta}u(\cdot,t_0)\|_{L^\infty} + \chi\int_{t_0}^{t}\|e^{(t-s)\Delta}\nabla\cdot(u(\cdot,s)\nabla v(\cdot,s))\|_{L^\infty}ds \\ &\quad + \gamma\int_{t_0}^{t}\|e^{(t-s)\Delta}u(\cdot,s)F(v(\cdot,s))\|_{L^\infty}ds \\ &= I_1 + I_2 + I_3.\end{aligned} \tag{3.8}$$

The argument in [5, Lemma 3.2] has shown that there is a constant $c_4 > 0$ such that

$$I_1 = \|e^{(t-t_0)\Delta}u(\cdot,t_0)\|_{L^\infty} \le c_4. \tag{3.9}$$

Moreover, since $r > n$, we can fix a number $q > n$ satisfying $q \in (\frac{r}{r+1}, r)$. Then by the Hölder inequality, interpolation inequality, (2.3), (3.3) and (3.7), we can find $\delta = \frac{r(q-1)+q}{rq} \in (0,1)$ such that

$$\begin{aligned}\|u(\cdot,s)\nabla v(\cdot,s)\|_{L^q} &\le \|u(\cdot,s)\|_{L^{\frac{rq}{r-q}}}\|\nabla v(\cdot,s)\|_{L^r} \\ &\le \|u(\cdot,s)\|_{L^\infty}^{1-\frac{r-q}{rq}}\|u(\cdot,s)\|_{L^1}^{\frac{r-q}{rq}}\|\nabla v(\cdot,s)\|_{L^r} \\ &\le\end{aligned}$$

$$
\begin{aligned}
I_3 &= \gamma \int_{t_0}^{t} \|e^{(t-s)\Delta}(uF(v) - \overline{uF(v)}) + e^{(t-s)\Delta}\overline{uF(v)}\|_{L^\infty} ds \\
&\leq \gamma \int_{t_0}^{t} \|e^{(t-s)\Delta}(uF(v) - \overline{uF(v)})\|_{L^\infty} ds + \gamma \int_{t_0}^{t} \|e^{(t-s)\Delta}\overline{uF(v)}\|_{L^\infty} ds \\
&\leq \gamma c_9 \int_{t_0}^{t} (t-s)^{-\frac{n}{2p}} \|uF(v) - \overline{uF(v)}\|_{L^p} ds + \gamma \int_{t_0}^{t} c_8 ds \\
&\leq 2\gamma c_9 F(K) \int_{t_0}^{t} (t-s)^{-\frac{n}{2p}} \|u\|_{L^p} ds + \gamma c_8 \\
&\leq c_{10},
\end{aligned}
\tag{3.11}
$$

where we have used (3.1) and the fact that $\int_{t_0}^{t} (t-s)^{-\frac{n}{2p}} ds = \frac{2p}{2p-n}$ due to $p > \frac{n}{2}$. Substituting (3.9), (3.10) and (3.11) into (3.8), we can find a constant $c_{11} > 0$ such that

$$
\|u(\cdot,t)\|_{L^\infty} \leq c_7 M^\delta(T) + c_{11}, \quad \text{for all } \ t \in (0,T),
$$

which implies

$$
M(T) \leq c_7 M^\delta(T) + c_{11}, \quad \text{for all } \ T \in (0, T_{max}). \tag{3.12}
$$

Since $0 < \delta < 1$, from (3.12) one has

$$
M(T) \leq \max\left\{ \left(\frac{c_{11}}{c_7}\right)^{\frac{1}{\delta}}, (2c_7)^{\frac{1}{1-\delta}} \right\}, \quad \text{for all } \ T \in (0, T_{max}),
$$

which implies $\|u(\cdot,t)\|_{L^\infty} \leq c_{14}$ for all $t \in (0, T_{max})$. Furthermore (3.3) with (3.4) yields (3.2). Then the proof of Lemma 3.1 is completed. □

3.1. E e e e ma e

We first derive an entropy-like equality which will be essentially used to

$$[redacted]\left(\frac{1}{\chi}\int_\Omega u\ln u+\frac{1}{2}\int_\Omega\frac{|\square v|^2}{F(v)}\right)+\frac{1}{\chi}\int_\Omega\frac{|\square u|^2}{u}+D\int_\Omega F(v)|D^2\mathcal{F}(v)|^2$$

$$[redacted]\frac{1}{\chi}\int_\Omega(\gamma F(v)-h(u))u\ln u+\frac{1}{\chi}\int_\Omega\left[\gamma uF(v)-uh(u)\right]+\frac{D}{2}\int_{\partial\Omega}\frac{1}{F(v)}\cdot\frac{\partial|\square v|^2}{\partial\nu}$$

$$-\frac{1}{2}\int_\Omega u\frac{F^{\square}(v)}{F(v)}|\square v|^2+\frac{D}{2}\int_\Omega\frac{F^{\square\square}(v)}{F^2(v)}\cdot|\square v|^4+\int_\Omega\frac{|\square v|^2}{F(v)}f^{\square}(v)-\frac{1}{2}\int_\Omega\frac{F^{\square}(v)}{F^2(v)}|\square v|^2 f(v) \tag{3.13}$$

[redacted]*e* $D^2\mathcal{F}$ *de e e ec d- de de a e f* $\mathcal{F}$.

[redacted]**f.** Multiplying the first equation of (1.6) by $1+\ln u$ and integrating the result yields

$$\begin{aligned}&\frac{d}{dt}\int_\Omega u\ln u+\int_\Omega\frac{|\square u|^2}{u}\\&=\chi\int_\Omega\square u\cdot\square v+\int_\Omega(\gamma F(v)-h(u))u\ln u+\int_\Omega\left[\gamma uF(v)-uh(u)\right].\end{aligned} \tag{3.14}$$

[redacted]grating by parts and using the second equation of (1.6), we have the following identity

$$\begin{aligned}[redacted]\int_\Omega\frac{|\square v|^2}{F(v)}&=-\frac{1}{2}\int_\Omega\frac{F^{\square}(v)}{F^2(v)}|\square v|^2 v_t+\int_\Omega\frac{\square v\cdot\square v_t}{F(v)}\\&=\frac{1}{2}\int_\Omega\frac{F^{\square}(v)}{F^2(v)}|\square v|^2 v_t-\int_\Omega\frac{\Delta v}{F(v)}\,(\Omega\ .\ \Omega(\varepsilon)\ .\ \Delta\quad F)\ \Delta\Delta F\chi\Theta\ .\ \Theta\ .\ Tm\Omega(t)T\ \Omega/T\Delta\end{aligned}$$

Letting

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \frac{|\nabla v|^2}{F(v)} + D\int_\Omega F(v)|D^2\mathcal{F}(v)|^2 \\
&= \frac{D}{2}\int_\Omega \frac{F''(v)}{F^2(v)}|\nabla v|^4 + \frac{D}{2}\int_{\partial\Omega}\frac{1}{F(v)}\cdot\frac{\partial|\nabla v|^2}{\partial \nu} - \frac{1}{2}\int_\Omega \frac{F'(v)}{F(v)}|\nabla v|^2 u \qquad (3.21)\\
&\quad + \frac{1}{2}\int_\Omega \frac{F'(v)}{F^2(v)}|\nabla v|^2 f(v) - \int_\Omega \nabla u\cdot\nabla v + \int_\Omega \frac{f'(v)F(v)-F'(v)f(v)}{F^2(v)}|\nabla v|^2.
\end{aligned}$$

The combination of (3.14) and (3.21) yields (3.13). Then the proof is completed. □

Based on the Lemma 3.2, we can obtain the following estimates.

Lemma 3.3. *A me e c d T e em 1.1 a e a ed. T e e e a c a $C > 0$ c a e f (1.6) a e*

$$\|u\ln u\|_{L^1} + \|\nabla v\|_{L^2} \le C.$$

Proof. By the hypotheses (H1), (H2) and the fact $0 < v \le K_0$, we have

$$\begin{aligned}
&\frac{1}{\chi}\int_\Omega (\gamma F(v) - h(u))\, u\ln u + \frac{1}{\chi}\int_\Omega \left[\gamma u F(v) - uh(u)\right] \\
&\le \frac{\gamma F(K)}{\chi}\left(\int_\Omega |u\ln u| + \int_\Omega u\right) - \frac{1}{\chi}\int_{\{0<u<1\}} uh(u)\ln u.
\end{aligned}$$

This, together with the identity (3.13) and the fact that $F''(v) \le 0$ for $v \ge 0$, gives

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{\chi}\int_\Omega u\ln u + \frac{1}{2}\int_\Omega \frac{|\nabla v|^2}{F(v)}\right) + \frac{1}{\chi}\int_\Omega \frac{|\nabla u|^2}{u} + D\int_\Omega F(v)|D^2\mathcal{F}(v)|^2 \\
&\le \frac{\gamma F(K)}{\chi}\left(\int_\Omega |u\ln u| + \int_\Omega u\right) - \frac{1}{\chi}\int_{\{0<u<1\}} uh(u)\ln u + \frac{D}{2}\int_{\partial\Omega}\frac{1}{F(v)}\cdot\frac{\partial|\nabla v|^2}{\partial\nu} \qquad (3.22)\\
&\quad + \int_\Omega \frac{|\nabla v|^2}{F(v)} f'(v) - \frac{1}{2}\int_\Omega \frac{|\nabla v|^2}{F^2(v)} f(v)F'(v).
\end{aligned}$$

From hypothesis (H3), one can show that $\frac{f(v)}{F(v)}$ is bounded for $0 < v \le K_0$. This, along with the facts $h(u)$, $f(v)$, $f'(v)$ and $F'(v)$ are continuous in $[0, \infty)$ (see hypotheses (H1)–(H3)), yields a constant $c_1 > 0$ to update (3.22) as

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{\chi}\int_\Omega u\ln u+\frac{1}{2}\int_\Omega \frac{|\nabla v|^2}{F(v)}\right)+\frac{1}{\chi}\int_\Omega \frac{|\nabla u|^2}{u}+D\int_\Omega F(v)|D^2\mathcal{F}(v)|^2\\
&\le \frac{D}{2}\int_{\partial\Omega}\frac{1}{F(v)}\cdot\frac{\partial|\nabla v|^2}{\partial\nu}+c_1\int_\Omega |u\ln u|+c_1\int_\Omega \frac{|\nabla v|^2}{F(v)}+c_1
\end{aligned}
\tag{3.23}
$$

where the inequality (2.3) has been used. Next, we will estimate the terms on the right hand of (3.23). First, we claim that there exists a constant $c_2>0$ such that

$$
c_2\left(\int_\Omega \frac{|D^2 v|^2}{F(v)}+\int_\Omega \frac{|\nabla v|^4}{F^3(v)}\right)\le D\int_\Omega F(v)|D^2\mathcal{F}(v)|^2. \tag{3.24}
$$

Indeed choosing $\psi=v$ and $g(\psi)=F(v)$ in Lemma 2.5 (ii), one has $\mathcal{G}(v)=\mathcal{F}(v)$ and

$$
\int_\Omega \frac{F'(v)}{F^3(v)}|\nabla v|^4\le (2+\sqrt{2})^2\int_\Omega \frac{F(v)}{F'(v)}|D^2\mathcal{F}(v)|^2,\quad \text{for all } t>0. \tag{3.25}
$$

Since $F'(v)>0$, $F''(v)\le 0$ and $0<v\le K_0$, then $0<F'(K_0)=c_3\le F'(v)\le c_4=F'(0)$. Thus (3.25) gives us that

$$
\int_\Omega \frac{|\nabla v|^4}{F^3(v)}\le \frac{(2+\sqrt{2})^2}{c_3^2}\int_\Omega F(v)|D^2\mathcal{F}(v)|^2 \ \text{ for all } t>0. \tag{3.26}
$$

Furthermore, noting that $(a-b)^2\ge \frac{1}{2}a^2-b^2$ for all $a,b\in\mathbb{R}$, we see that

$$
\begin{aligned}
\int_\Omega F(v)|D^2\mathcal{F}(v)|^2 &= \int_\Omega F(v)\cdot\sum_{k,l=1}^{2}\left|\frac{1}{F(v)}\cdot\frac{\partial^2 v}{\partial x_k\partial x_l}-\frac{F'(v)}{F^2(v)}\cdot\frac{\partial v}{\partial x_k}\cdot\frac{\partial v}{\partial x_l}\right|^2\\
&\ge \int_\Omega \frac{1}{2}F(v)\cdot\sum_{k,l=1}^{2}\left|\frac{1}{F(v)}\cdot\frac{\partial^2 v}{\partial x_k\partial x_l}\right|^2\\
&\quad -\int_\Omega F(v)\cdot\sum_{k,l=1}^{2}\left|\frac{F'(v)}{F^2(v)}\cdot\frac{\partial v}{\partial x_k}\cdot\frac{\partial v}{\partial x_l}\right|^2\\
&=\frac{1}{2}\int_\Omega \frac{|D^2 v|^2}{F(v)}-\int_\Omega \frac{|F'(v)|^2|\nabla v|^4}{F^3(v)},
\end{aligned}
$$

which together with (3.26) gives

$$
\begin{aligned}
\int_\Omega \frac{|D^2 v|^2}{F(v)} &\le 2\left(\int_\Omega F(v)|D^2\mathcal{F}(v)|^2 + c_4^2\int_\Omega \frac{|\nabla v|^4}{F^3(v)}\right)\\
&\le \frac{2(2+\sqrt{2})^2c_4^2 + 2c_3^2}{c_3^2}\int_\Omega F(v)|D^2\mathcal{F}(v)|^2.
\end{aligned}
\tag{3.27}
$$

The combination of (3.26) and (3.27) gives (3.24). Moreover, noting that $|u\ln u| \le c_5 u^{\frac{3}{2}} + c_5$ for some constant $c_5 > 0$, then using the Gagliardo–Nirenberg inequality (see Lemma 2.3) and the inequality (2.3), one can derive that

$$
\frac{1}{\chi}\int_\Omega u\ln u + c_1\int_\Omega |u\ln u| \le \frac{1+c_1\chi}{\chi}\|u\ln u\|_{L^1} \le \frac{1}{\chi}\int_\Omega \frac{|\nabla u|^2}{u} + c_6. \tag{3.28}
$$

Then substituting (3.24) and (3.28) into (3.23) gives

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{\chi}\int_\Omega u\ln u + \frac{1}{2}\int_\Omega \frac{|\nabla v|^2}{F(v)}\right) + \frac{1}{\chi}\int_\Omega u\ln u + c_2\left(\int_\Omega \frac{|D^2 v|^2}{F(v)} + \int_\Omega \frac{|\nabla v|^4}{F^3(v)}\right)\\
&\le \frac{D}{2}\int_{\partial\Omega} \frac{1}{F(v)}\cdot\frac{\partial|\nabla v|^2}{\partial \nu} + c_1\int_\Omega \frac{|\nabla v|^2}{F(v)} + c_7.
\end{aligned}
\tag{3.29}
$$

Next, we shall estimate the first term on the right hand of (3.29). For convenience, we let $\varphi(v) = \frac{|\nabla v|}{F^{\frac{1}{2}}(v)} = \left(\frac{|\nabla v|^2}{F(v)}\right)^{\frac{1}{2}}$. Then by the boundedness of $F'(v)$, it follows that

$$
\begin{aligned}
|\nabla\varphi(v)|^2 &= \frac{F(v)}{4|\nabla v|^2}\cdot\frac{|2F(v)D^2v\cdot\nabla v - F'(v)|\nabla v|^2\nabla v|^2}{F^4(v)}\\
&\le \frac{2|D^2v|^2}{F(v)} + \frac{|F'(v)|^2|\nabla v|^4}{2F^3(v)}\\
&\le c_8\left(\frac{|D^2v|^2}{F(v)} + \frac{|\nabla v|^4}{F^3(v)}\right).
\end{aligned}
\tag{3.30}
$$

To proceed, we recall the following trace inequality [33, Remark 52.9] for any $\varepsilon > 0$:

$$
\|\varphi\|_{L^2(\partial\Omega)} \le \varepsilon\|\nabla\varphi\|_{L^2(\Omega)} + C_\varepsilon\|\varphi\|_{L^2(\Omega)}. \tag{3.31}
$$

Then by the inequality $\frac{\partial|\nabla v|^2}{\partial\nu} \le 2\kappa|\nabla v|^2$ on $\partial\Omega$ for some constant $\kappa > 0$ (see [23, Lemma 4.2]), (3.30) and above trace inequality, we have

$$\begin{aligned}\frac{D}{2}\int_{\partial\Omega}\frac{1}{F(v)}\cdot\frac{\partial|\nabla v|^2}{\partial\nu} \le \kappa D\int_{\partial\Omega}\left(\frac{|\nabla v|}{F^{\frac{1}{2}}(v)}\right)^2 &= \kappa D\|\varphi\|^2_{L^2(\partial\Omega)} \\ &\le \frac{c_2}{2}\int_\Omega\left(\frac{|D^2v|^2}{F(v)}+\frac{|\nabla v|^4}{F^3(v)}\right)+c_9\int_\Omega\frac{|\nabla v|^2}{F(v)}.\end{aligned}\tag{3.32}$$

In virtue of the boundedness of $F(v)$, it follows from the Cauchy–Schwarz inequality that

$$\left(\frac{1}{2}+c_1+c_9\right)\int_\Omega\frac{|\nabla v|^2}{F(v)} = \left(\frac{1}{2}+c_1+c_9\right)\int_\Omega\frac{|\nabla v|^2}{F^{\frac{3}{2}}(v)}F^{\frac{1}{2}}(v) \le \frac{c_2}{2}\int_\Omega\frac{|\nabla v|^4}{F^3(v)}+c_{10}. \tag{3.33}$$

Then adding $\frac{1}{2}\int_\Omega\frac{|\nabla v|^2}{F(v)}$ on both sides of (3.29) and substituting (3.32)–(3.33) into the resulting inequality gives us that

$$\frac{d}{dt}\left(\frac{1}{\chi}\int_\Omega u\ln u+\frac{1}{2}\int_\Omega\frac{|\nabla v|^2}{F(v)}\right)+\left(\frac{1}{\chi}\int_\Omega u\ln u+\frac{1}{2}\int_\Omega\frac{|\nabla v|^2}{F(v)}\right)\le c_{11},$$

which, upon the application of Gronwall's inequality, implies

$$\frac{1}{\chi}\int_\Omega u\ln u+\frac{1}{2}\int_\Omega\frac{|\nabla v|^2}{F(v)}\le c_{12}.$$

This completes the proof by the fact that $-u\ln u\le\frac{1}{e}$ for all $u>0$ and $\frac{1}{F(v)}\ge\frac{1}{F(K_0)}>0$. □

Lemma 3.4. *T e e e a c a $c_1>0$ c a f a t $\in(0,T_{max})$, e f (1.6) a e*

$$\frac{d}{dt}\int_\Omega|\nabla v|^4+D\int_\Omega|\nabla|\nabla v|^2|^2+2D\int_\Omega|\nabla v|^2|D^2v|^2\le c_1\int_\Omega u^2|\nabla v|^2+c_1. \tag{3.34}$$

Proof. We differentiate the second equation of system (1.6) and multiply the result by $2\nabla v$. Then using the identity $\Delta|\nabla v|^2=2\nabla v\cdot\nabla\Delta v+2|D^2v|^2$, we obtain

$$\begin{aligned}(|\nabla v|^2)_t &= 2D\nabla v\cdot\nabla\Delta v-2\nabla v\cdot\nabla(uF(v))+2|\nabla v|^2f'(v)\\ &= D\Delta|\nabla v|^2-2D|D^2v|^2-2\nabla v\cdot\nabla(uF(v))+2|\nabla v|^2f'(v).\end{aligned}$$

Then multiplying above equation by $2|\nabla v|^2$ and using the integration by parts, we have

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega |\nabla v|^4 + 2D\int_\Omega |\nabla|\nabla v|^2|^2 + 4D\int_\Omega |\nabla v|^2|D^2 v|^2 \\
&= 2D\int_{\partial\Omega} |\nabla v|^2\frac{\partial|\nabla v|^2}{\partial\nu}dS - 4\int_\Omega |\nabla v|^2\nabla v\cdot\nabla(uF(v)) + 4\int_\Omega |\nabla v|^4 f'(v) \\
&= 2D\int_{\partial\Omega} |\nabla v|^2\frac{\partial|\nabla v|^2}{\partial\nu}dS + 4\int_\Omega |\nabla v|^4 f'(v) \qquad (3.35)\\
&\quad + 4\int_\Omega uF(v)\Delta v|\nabla v|^2 + 4\int uF(v)\nabla(|\nabla v|^2)\cdot\nabla v \\
&\le 2D\int_{\partial\Omega} |\nabla v|^2\frac{\partial|\nabla v|^2}{\partial\nu}dS + c_2\||\nabla v|^2\|_{L^2}^2 + c_2\int_\Omega u\left(|\Delta v||\nabla v|^2 + |\nabla|\nabla v|^2||\nabla v|\right).
\end{aligned}
$$

With the inequality $\frac{\partial|\nabla v|^2}{\partial\nu} \le 2\kappa|\nabla v|^2$ on $\partial\Omega$ and (3.31) again, we get

$$2D\int_{\partial\Omega} |\nabla v|^2\frac{\partial|\nabla v|^2}{\partial\nu}dS \le 4\kappa D\||\nabla v|^2\|_{L^2(\partial\Omega)}^2 \le \frac{D}{2}\int_\Omega |\nabla|\nabla v|^2|^2 + c_3\||\nabla v|^2\|_{L^2}^2. \qquad (3.36)$$

By the Gagliardo–Nirenberg inequality and the fact $\||\nabla v|^2\|_{L^1} = \|\nabla v\|_{L^2}^2 \le C$ in Lemma 3.3, we can find a constant $\theta_1 = \frac{1}{1+\frac{2}{n}} \in (0,1)$ such that

$$(c_2+c_3)\||\nabla v|^2\|_{L^2}^2 \le c_4\|\nabla|\nabla v|^2\|_{L^2}^{2\theta_1}\||\nabla v|^2\|^{2(1-\theta_1)}$$

$$
\begin{aligned}
& c_2 \int_\Omega u\left(|\Delta v||\nabla v|^2 + \left|\nabla|\nabla v|^2\right||\nabla v|\right) \\
& \le c_2 \sqrt{n} \int_\Omega u|\nabla v|^2|D^2 v| + 2c_2 \int_\Omega u|\nabla v|^2|D^2 v| \\
& \le (\sqrt{n} + 2)c_2 \int_\Omega u|\nabla v|^2|D^2 v| \\
& \le 2D \int_\Omega |\nabla v|^2|D^2 v|^2 + \frac{(2+\sqrt{n})^2 c_2^2}{8D} \int_\Omega u^2|\nabla v|^2 .
\end{aligned}
\tag{3.38}
$$

Substituting (3.37) and (3.38) into (3.35) yields (3.34) and finished the proof. □

By Lemma 3.3, the boundedness of $\|u(\cdot,t)\|_{L^2}$ can be obtained.

Lemma 3.5. *Le e c d T e em 1.1 d. T e e f (1.6) a e*

$$
\|u(\cdot,t)\|_{L^2} \le C, \tag{3.39}
$$

e e $C > 0$ a c a de e de f t.

Proof. Multiplying the first equation of (1.6) by $2u$, integrating the result with respect to x over Ω and using the facts that $0 < F(v) \le F(K_0)$ and $h(u) \ge \theta$, one has

$$
\begin{aligned}
\frac{d}{dt}\int_\Omega u^2 + 2\int_\Omega |\nabla u|^2 &= 2\chi \int_\Omega u\nabla u \cdot \nabla v + 2\gamma \int_\Omega u^2 F(v) - 2\int_\Omega u^2 h(u) \\
&\le \int_\Omega |\nabla u|^2 + \chi^2 \int_\Omega u^2|\nabla v|^2 + 2(\gamma F(K) - \theta)\int_\Omega u^2 .
\end{aligned}
\tag{3.40}
$$

Noticing that the Gagliardo–Nirenberg inequality and Young's inequality, together with Lemma 2.2, can give us that

$$
2(\gamma F(K) - \theta)\int_\Omega u^2 \le c_1(\|\nabla u\|_{L^2}\|u\|_{L^1} + \|u\|_{L^1}^2) \le \frac{1}{2}\|\nabla u\|_{L^2}^2 + c_2,
$$

one has from (3.40) that

$$
\frac{d}{dt}\int_\Omega u^2 + \frac{1}{2}\int_\Omega |\nabla u|^2 \le \chi^2 \int_\Omega u^2|\nabla v|^2 + c_2,
$$

which, combined with Lemma 3.4, yields

$$\frac{d}{dt}\left(\int_\Omega u^2 + \int_\Omega |\nabla v|^4\right) + \frac{1}{2}\int_\Omega |\nabla u|^2 + D\int_\Omega |\nabla|\nabla v|^2|^2 \le c_3 \int_\Omega u^2|\nabla v|^2 + c_4. \tag{3.41}$$

Based on (3.41), using Lemma 2.4, one can readily derive the following inequality (we omit the details for brevity and refer readers to the proof of [5, Lemma 3.3])

$$z'(t) + z(t) \le c_5,$$

where $z(t) = \int_\Omega u^2 + \int_\Omega |\nabla v|^4$. This gives (3.39) by the Gronwall's inequality and concludes the proof. □

3.2. P f f T e e 1.1

Proof of Theorem 1.1. The fact $0 < v \le K_0$ is proved in Lemma 2.2. From Lemma 3.5, we have $\|u(\cdot,t)\|_{L^2} \le c_1$. Then we can apply Lemma 3.1 with $p = 2$ to find a constant $c_2 > 0$ independent of t such that $\|u(\cdot,t)\|_{L^\infty} + \|v(\cdot,t)\|_{W^{1,\infty}} \le c_2$ for all $t \in (0, T_{max})$ in two dimensions ($n = 2$). This along with Lemma 2.1 finishes the proof of Theorem 1.1. □

4. Global stability

In this section, we are devoted to proving the global stability results in Theorem 1.3 via Lyapunov stability and LaSalle's invariant principle under the hypotheses (H1)–(H4). At first we prove a basic result that will be often used in what follows.

4.1. A ba c e a

Lemma 4.1. *Le F a f e c d (H1) a d (u, v) be a f (1.6). De e a f c f me c a $\omega > 0$:*

$$\zeta(v) = \int_\omega^v \frac{F(s) - F(\omega)}{F(s)}ds.$$

T e $\zeta(v)$ a c e f c c a $\zeta(v) \ge 0$. If e f e a e a $v \to \omega$ a $t \to \infty$, e e e a c a $T_0 > 0$ c a f a $t \ge T_0$ d a

$$\frac{F'(\omega)}{4F(\omega)}(v-\omega)^2 \le \zeta(v) = \int_\omega^v \frac{F(s) - F(\omega)}{F(s)}ds \le \frac{F'(\omega)}{F(\omega)}(v-\omega)^2. \tag{4.1}$$

Proof. It is clear that $\zeta(\omega) = \zeta'(\omega) = 0$ and $\zeta''(v) = F(\omega)\frac{F'(v)}{F^2(v)} \ge 0$ thanks to the hypothesis (H1). Then the Taylor's formula applied to $\zeta(v)$ at $v = \omega$ gives

$$\zeta(v) = \frac{1}{2}\zeta''(\tilde{v})(v-\omega)^2 \ge 0$$

where $\tilde{v}$ is between ω and v. Thus the first part of the lemma is proved. Furthermore it follow from the fact $v \to \omega$ as $t \to \infty$ that

$$\lim_{t\to\infty} \frac{\zeta(v)}{(v-\omega)^2} = \frac{1}{2}\lim_{v\to\omega} \zeta''(\tilde{v}) = \frac{1}{2}\lim_{v\to\omega} F(\omega)\frac{F'(v)}{F^2(v)} = \frac{F'(\omega)}{2F(\omega)}$$

which yields a constant $T_0 > 0$ such that for all $t \geq T_0$ it holds that

$$\frac{F'(\omega)}{4F(\omega)}(v-\omega)^2 \leq \zeta(v) \leq \frac{F'(\omega)}{F(\omega)}(v-\omega)^2. \tag{4.2}$$

This concludes the proof of Lemma 4.1. $\square$

4.2. Global stability of the prey-only steady state

In this subsection, we consider the case of weak predation $\gamma F(K) \leq \theta$ for which the system (1.6) has two homogeneous steady states $(0, 0)$ and $(0, K)$ (see also (1.8)). We shall show that the prey-only steady state $(0, K)$ is globally asymptotically stable in this case. Furthermore we shall prove that the exponential stability of the homogeneous steady state $(0, K)$ can be attained if $\gamma F(K) < \theta$, and algebraic decay can be obtained if $\gamma F(K) = \theta$ and $\alpha > 0$. To this end, we employ the following Lyapunov functional:

$$V_1(u(t), v(t)) =: V_1(t)$$

Next, we shall show $\frac{d}{dt}V_1(w) = \frac{d}{dt}V_1(t) \leq 0$ for all $w \in X$ where "=" iff $w = (0, K)$. Differentiating the functional (4.3) with respect to t, one has

$$\begin{aligned}\frac{d}{dt}V_1(t) &= \frac{1}{\gamma}\int_\Omega u_t + \int_\Omega \frac{F(v) - F(K)}{F(v)} v_t \\ &= \frac{1}{\gamma}\int_\Omega (\gamma u F(v) - u h(u)) + \int_\Omega \left(1 - \frac{F(K)}{F(v)}\right)(D\Delta v - uF(v) + f(v)).\end{aligned} \tag{4.6}$$

Applying the integration by parts to the second term on the right hand side of (4.6), after some simple calculations and cancellations, we get

$$\begin{aligned}\frac{d}{dt}V_1(t) = &\frac{1}{\gamma}\int_\Omega u(\gamma F(K) - h(u)) - DF(K)\int_\Omega F'(v)\left|\frac{\nabla v}{F(v)}\right|^2 \\ &+ \int_\Omega \frac{f(v)}{F(v)}(F(v) - F(K)).\end{aligned} \tag{4.7}$$

From assumptions (H3) and (H4), one can easily derive that $\phi(v)(v - K) < 0$ for all $v \neq K$. Hence by the assumption $F'(v) > 0$, we have $\frac{f(v)}{F(v)}(F(v) - F(K)) \leq 0$. By the hypothesis (H2), we can derive $h(u) \geq \theta + \alpha u$ for any $u \geq 0$ and hence $u(\gamma F(K) - h(u)) \leq u(\gamma F(K) - \theta - \alpha u) \leq 0$. This, along with the fact that $F(K) > 0$ and $F'(v) > 0$ yields from (4.7) that $\frac{d}{dt}V_1(w) = \frac{d}{dt}V_1(t) \leq 0$ for all $w \in X$, where $\frac{d}{dt}V_1(w) = 0$ iff $w = (0, K)$ by the hypothesis (H4) and conditions in Lemma 4.2. Then by the LaSalle's invariant principle (see [32, pp. 198–199, Theorem 5.24] or [18, Theorem 3]), the trajectory $w(t; w_0) = (u, v) \to (0, K)$ as $t \to \infty$. This shows that $(0, K)$ is globally asymptotically stable.

We proceed to derive the convergence rate of solutions. Let us first consider the case $\gamma F(K) < \theta$. With the fact that $\phi(K) = \frac{f(K)}{F(K)} = 0$, we use the Taylor's formula to rewrite the third term on the right hand side of (4.7) as

$$\int_\Omega \frac{f(v)}{F(v)}(F(v) - F(K)) = \int_\Omega \phi(v)(F(v) - F(K)) = \int_\Omega \phi'(\eta_1)F'(\eta_2)(v - K)^2$$

where η_1 and η_2 are between v and K. Noticing that $v \to K$ as $t \to \infty$ and

$$\lim_{v \to K}\phi'(\eta_1) = \phi'(K) < 0, \quad \lim_{v \to K}F'(\eta_2) = F'(K) > 0$$

we can find a time $t_1 > 0$ such that for all $t > t_1$ it holds that: $2\phi'(K) < \phi'(\eta_1) < \frac{1}{2}\phi'(K) < 0$ and $0 < \frac{1}{2}F'(K) < F'(\eta_2) < 2F'(K)$. This, along with (4.7), gives that

$$\frac{d}{dt}V_1(t) \le \frac{1}{\gamma}(\gamma F(K)-\theta)\int_\Omega u - \frac{\alpha}{\gamma}\int_\Omega u^2 \\ - DF(K)\int_\Omega F'(v)\left|\frac{\nabla v}{F(v)}\right|^2 + \int_\Omega \phi'(\eta_1)F'(\eta_2)(v-K)^2. \tag{4.8}$$

Therefore there exists a constant $c_1 > 0$ such that

$$\frac{d}{dt}V_1(t) \le -c_1\mathcal{V}_1(t) \ \text{ for all } t > t_1 \tag{4.9}$$

where

$$\mathcal{V}_1(t) := \int_\Omega u + DF(K)\int_\Omega F'(v)\left|\frac{\nabla v}{F(v)}\right|^2 + \int_\Omega (v-K)^2.$$

Next we apply Lemma 4.1 with $\omega = K$ and find a constant $t_2 > 0$ such that

$$c_2(v-K)^2 \le \int_K^v \frac{F(s)-F(K)}{F(s)}ds \le c_3(v-K)^2, \quad \text{for all } \ t \ge t_2 \tag{4.10}$$

holds for some constants $c_2, c_3 > 0$. Then using the definitions of $V_1(t)$ and $\mathcal{V}_1(t)$ with (4.10), one can find a constant $c_4 > 0$ such that $c_4V_1(t) \le \mathcal{V}_1(t)$ for all $t > t_2$, which together with (4.9) and the non-negativity of $V_1(t)$ yields

$$\frac{d}{dt}V_1(t) \le -c_1\mathcal{V}_1(t) \le -c_1c_4V_1(t), \ \text{ for all } \ t > t_0$$

where $t_0 = \max\{t_1, t_2\}$. This gives rise to $V_1(t) \le c_5e^{-c_6t}$ for all $t > t_0$, which, along with (4.3) and (4.10), yields

$$\|u\|_{L^1} + \|v-K\|_{L^2} \le c_7e^{-c_6t}, \quad \text{for all } \ t > t_0. \tag{4.11}$$

Next proceed to derive the decay rates of L^∞-norm. From Theorem 1.1, we know that $\chi u\nabla v$ and $\gamma uF(v) - uh(u)$ is bounded in $L^\infty(\Omega\times(0,\infty))$. Then applying the standard parabolic regularity theory (e.g. see [29, Theorem 1.3] and [37, Lemma 3.2]), from the first equation of system (1.6), we can find a constant $\beta \in (0,1)$ such that

$$\|u\|_{C^{\beta,\frac{\beta}{2}}(\bar\Omega\times[t,t+1])} \le c_8 \quad \text{for all } \ t > 1. \tag{4.12}$$

Moreover, from the second equation of system (1.6), we can use the standard parabolic Schauder theory [17] to obtain

$$\|v\|_{C^{2+\beta,1+\frac{\beta}{2}}(\bar\Omega\times[t,t+1])} \le c_9 \quad \text{for all } \ t > 1. \tag{4.13}$$

Based on (4.12) and (4.13), one can readily get a constant $c_{10} > 0$ (e.g. see [37, Lemma 3.14]) such that

$$\|u\|_{W^{1,\infty}} \leq c_{10}, \text{ for all } t > 1.$$

This, along with (4.11) and the Gagliardo–Nirenberg inequality, yields for all $t > t_0$ that

$$\|u\|_{L^\infty} \leq c_{11}(\|\nabla u\|_{L^\infty}^{\frac{2}{3}} \|u\|^{\frac{1}{3}}$$

$$\|u\|_{L^\infty} + \|v - K\|_{L^\infty} \le c_{25}(1+t)^{-\lambda_2}, \quad \text{for all } \ t > t_0$$

which gives (4.5) and completes the proof. □

4.3. G ba ab f e c -e e ce ead a e

Now we consider the global stability of solutions for the case $\gamma F(K) > \theta$. In this scenario, the system (1.6) has three homogeneous steady states: $(0,0)$, $(0,K)$ and (u_*, v_*), where $u_*, v_* > 0$ and satisfy (1.9). We shall prove the coexistence steady state (u_*, v_*) is globally asymptotically stable under certain conditions by using the following Lyapunov functional:

$$V_2(u(t), v(t)) =: V_2(t) = \frac{1}{\gamma}\int_\Omega \left(u - u_* - u_* \ln\frac{u}{u_*}\right) + \int_\Omega \left(\int_{v_*}^{v} \frac{F(s) - F(v_*)}{F(s)} ds\right). \tag{4.19}$$

The above Lyapunov functional has been used in [48] for the predator–prey system without prey-taxis.

Lemma 4.3. *Le e c d T e e 1.3 d. A e (u, v) e f (1.6) b- a ed T e e 1.1. T e f $\gamma F(K) > \theta$, e c e e ce ead a e (u_*, v_*) ba a ca ab e f*

$$\frac{D}{\chi^2} \ge \frac{u_* F^2(K)}{4\gamma F(v_*) F'(K)}$$

e e u_* a d v_* a e de e ed b (1.9) a d de e de f D a d χ, a d = d e ca e f $\|v_0\|_{L^\infty} \le K$. F e e, f $\alpha > 0$, e e e c a C_3 a d λ_3 a d $t_0 > 0$ c a

$$\|u - u_*\|_{L^\infty} + \|v - v_*\|_{L^\infty} \le C_3 e^{-\lambda_3 t}, \quad f \ a \quad t > t_0. \tag{4.20}$$

Proof. We first show the non-negativity of $V_2(t)$. For this, we define $\psi(z) = z - u_* \ln z$. Then by the Taylor's formula, we have

$$u - u_* - u_* \ln\frac{u}{u_*} = \psi(u) - \psi(u_*) = \frac{\psi''(\xi)}{2}(u - u_*)^2 = \frac{u_*}{2\xi^2}(u - u_*)^2 \tag{4.21}$$

where $\xi \ge 0$ is between u and u_*. This asserts that $u - u_* - u_* \ln\frac{u}{u_*} \ge 0$ due to $u_* > 0$. Furthermore the non-negativity of the second term on the right hand side of (4.19) is guaranteed by the result of Lemma 4.1. This shows that $V_2(t) \ge 0$ for all $u, v > 0$. Next we differentiate $V_2(t)$ with respect to t and use the equations of (1.6) to obtain that

$$\begin{aligned}
\frac{d}{dt}V_2(t) &= \frac{1}{\gamma}\int_\Omega \left(1-\frac{u_*}{u}\right)u_t + \int_\Omega \frac{F(v)-F(v_*)}{F(v)}v_t \\
&= \underbrace{-\frac{u_*}{\gamma}\int_\Omega \left|\frac{\nabla u}{u}\right|^2 - DF(v_*)\int_\Omega F'(v)\left|\frac{\nabla v}{F(v)}\right|^2 + \frac{\chi u_*}{\gamma}\int_\Omega \frac{\nabla u\cdot\nabla v}{u}}_{I_1} \\
&\quad \underbrace{+\frac{1}{\gamma}\int_\Omega \left(1-\frac{u_*}{u}\right)(\gamma u F(v) - uh(u)) + \int_\Omega (F(v)-F(v_*))\left(\frac{f(v)}{F(v)}-u\right)}_{I_2}.
\end{aligned} \tag{4.22}$$

For I_1, we can rewrite it as

$$I_1 = -\int_\Omega \Theta^T A \Theta,\ \ \Theta = \begin{bmatrix}\nabla u\\ \nabla v\end{bmatrix},\ \ A = \begin{bmatrix}\frac{u_*}{\gamma u^2} & -\frac{\chi u_*}{2\gamma u}\\ -\frac{\chi u_*}{2\gamma u} & \frac{DF(v_*)F'(v)}{|F(v)|^2}\end{bmatrix},$$

where Θ^T denotes the transpose of Θ. The basic algebra tells us that the matrix A is non-negative definite and hence $I_1 \le 0$ if and only if

$$\frac{DF(v_*)F'(v)u_*}{\gamma|F(v)|^2u^2} \ge \frac{\chi^2u_*^2}{4\gamma^2u^2} \ \text{ or } \ \frac{D}{\chi^2} \ge \frac{u_*|F(v)|^2}{4\gamma F(v_*)F'(v)}, \tag{4.23}$$

where u_* and v_* do not depend on D and χ, see (1.9).

If $\|v_0\|_{L^\infty} \le K$, then from Lemma 2.2 one has $0 < v(x,t) \le K$. Since $F'(v) > 0$ and $F''(v) \le 0$ in (H1), the condition (4.23) is guaranteed if

$$\frac{D}{\chi^2} \ge \frac{u_*F^2(K)}{4\gamma F(v_*)F'(K)}. \tag{4.24}$$

Next, we consider the case $\|v_0\|_{L^\infty} > K$. Supposing that D and χ satisfy $\frac{D}{\chi^2} > \frac{u_*F^2(K)}{4\gamma F(v_*)F'(K)}$, then there exists a small $\varepsilon_0 > 0$ such that

$$\frac{D}{\chi^2} \ge \frac{u_*F^2(K)}{4\gamma F(v_*)F'(K)} + \varepsilon_0. \tag{4.25}$$

Let $\mathcal{H}(v) = \frac{|F(v)|^2}{F'(v)}$. Then from the hypothesis (H1), it follows that $\mathcal{H}(v) \in C^1([0,\infty))$ and $\mathcal{H}'(v) > 0$, which together with (2.2) gives

$$\limsup_{t\to\infty} \frac{u_*|F(v)|^2}{4\gamma F(v_*)F'(v)} = \frac{u_*}{4\gamma F(v_*)}\limsup_{t\to\infty}\mathcal{H}(v) \le \frac{u_*\mathcal{H}(K)}{4\gamma F(v_*)} = \frac{u_*F^2(K)}{4\gamma F(v_*)F'(K)}. \tag{4.26}$$

Therefore for the $\varepsilon_0 > 0$ chosen in (4.25), there exists $T_* > 0$ such that

$$\frac{u_*|F(v)|^2}{4\gamma F(v_*)F'(v)} \le \frac{u_*F^2(K)}{4\gamma F(v_*)F'(K)} + \varepsilon_0 \ \text{ for } \ (x,t) \in \bar\Omega \times [T_*, \infty). \tag{4.27}$$

The combination of (4.25) and (4.27) gives

$$\frac{u_*|F(v)|^2}{4\gamma F(v_*)F'(v)} \le \frac{D}{\chi^2}, \quad \text{for } (x,t) \in \bar{\Omega} \times [T_*, \infty).$$

In summary, we have shown that if (4.24) is fulfilled where "=" holds in the case of $\|v_0\|_{L^\infty} \le K$, then $I_1 \le 0$ for $t \ge T_*$. Next we estimate I_2 which can be regrouped as

$$\begin{aligned}
I_2 &= \frac{1}{\gamma}\int_\Omega (u-u_*)\Big(\gamma F(v) - h(u)\Big) + \int_\Omega (F(v)-F(v_*))\left(\frac{f(v)}{F(v)} - u\right) \\
&= \int_\Omega (u-u_*)(F(v)-F(v_*)) + \frac{1}{\gamma}\int_\Omega (u-u_*)(\gamma F(v_*) - h(u)) \\
&\quad + \int_\Omega (F(v)-F(v_*))\left(\frac{f(v)}{F(v)} - u\right) \\
&= \frac{1}{\gamma}\int_\Omega (u-u_*)(\gamma F(v_*) - h(u)) + \int_\Omega (F(v)-F(v_*))\left(\frac{f(v)}{F(v)} - u_*\right) \\
&=: M_1 + M_2.
\end{aligned} \tag{4.28}$$

By (1.9), we have $\gamma F(v_*) = h(u_*)$ which,with hypothesis (H2),gives

$$\begin{aligned}
M_1 = \quad & \frac{1}{\gamma}\int (u-u_* \qquad \Omega \\
& \frac{}{\gamma}\int (u-u_* \qquad \Omega \\
& \frac{}{\gamma}\int h'(\xi_1)(u-u_*)^2 \qquad \Omega \\
\le & -\frac{\alpha}{\gamma}\int (u-u_*)^2 \qquad \Omega
\end{aligned}$$

where ξ_2 and ξ_3 are between v and v_*. This, combined with (4.29) and (4.28), yields that $I_2 \le 0$. Now (4.22) can be updated as

$$\frac{d}{dt}V_2(t) = I_1 + I_2 = -\int_\Omega \Theta^T A\Theta - \frac{\alpha}{\gamma}\int_\Omega (u-u_*)^2 + \int_\Omega F'(\xi_2)\phi'(\xi_3)(v-v_*)^2.$$

Hence if (4.24) is satisfied where "=" holds in the case of $\|v_0\|_{L^\infty} \le K$, then $\frac{d}{dt}V_2(t) \le 0$ for $t \ge T_*$. Noticing that $\frac{d}{dt}V_2(t) = 0$ iff $(u, v) = (u_*, v_*)$ and (u, v) is bounded for all $0 < t < T_*$, then (u_*, v_*) is globally asymptotically stable by the LaSalle's invariant principle following the similar argument as in the proof of Lemma 4.2. Thus the first part of Lemma 4.3 is proved.

Next we proceed to show the decay rate (4.20). Since $u \to u_*$ and $v \to v_*$ as $t \to \infty$, by the hypotheses (H1) and (H4), we can find a $T_1 > 0$ such that for all $t > T_1$ it holds that

$$0 < \frac{1}{2}F'(v_*) \le F'(\xi_2) \le 2F'(v_*),\ \phi'(v_*) \le \phi'(\xi_3) \le \frac{1}{2}\phi'(v_*) < 0.$$

This, together with the facts that the matrix A is non-negative definite and $\alpha > 0$, yields a constant $C > 0$ such that

$$\frac{d}{dt}V_2(t) \le -C\int_\Omega [(u-u_*)^2 + (v-v_*)^2],\ \text{for all } t > T_1. \tag{4.30}$$

Applying (4.21) with the fact $u \to u_*$ and hence $\xi \to u_*$ as $t \to \infty$, one can find a $T_2 > 0$ so that for all $t > T_2$ the following inequality holds:

$$\frac{1}{4u_*}\int_\Omega (u-u_*)^2 \le \int_\Omega \left(u - u_* - u_*\ln\frac{u}{u_*}\right) \le \frac{1}{u_*}\int_\Omega (u-u_*)^2. \tag{4.31}$$

Furthermore, we employ Lemma 4.1 with $\omega = v_*$ to find a $T_3 > 0$ so that for all $t > T_3$

$$c_1(v-v_*)^2 \le \int_{v_*}^{v} \frac{F(s)-F(v_*)}{F(s)}ds \le c_2(v-v_*)^2 \tag{4.32}$$

holds for some constants $c_1, c_2 > 0$. Thus by the definition of $V_2(t)$ and the estimates (4.31)–(4.32), we can find two positive constants c_3 and c_4 such that for all $t > t_0 = \max\{T_1, T_2, T_3\}$

$$c_3(\|u-u_*\|_{L^2}^2 + \|v-v_*\|_{L^2}^2) \le V_2(t) \le c_4(\|u-u_*\|_{L^2}^2 + \|v-v_*\|_{L^2}^2).$$

This, along with (4.30), gives a constant $c_3 > 0$ such that

$$\frac{d}{dt}V_2(t) \le -c_5V_2(t),\ \text{for all } t > t_0$$

which, upon the application of Gronwall's inequality, yields the following decay with constants $c_6, c_7 > 0$

$$\|u - u_*\|_{L^2}^2 + \|v - v_*\|_{L^2}^2 \leq c_6 e^{-c_7 t}.$$

Then by the same argument as deriving (4.14) and (4.15), we get (4.20) and complete the proof of Lemma 4.3. □

4.4. Proof of Theorem 1.3

Theorem 1.3 is clearly a direct consequence of Lemma 4.2 and Lemma 4.3.

5. Summary

In this paper, we consider the prey-taxis model with a class of predator–prey population interactions which covers many existing well-known examples like those in (1.2)–(1.4). Our main results consist of two parts: global boundedness (see Theorem 1.1) and global stability (see Theorem 1.3). In the first part, we show that the global solutions of the prey-taxis system subject to zero Neumann boundary conditions in two dimensional domain are uniformly bounded in time for any prey-tactic coefficient $\chi > 0$ thanks to the intrinsic predator–prey population interaction. This result improves the previous one obtained in [1,12,34,46] where either the truncation condition or smallness assumption was imposed on χ to preclude the blow-up of solutions. Our first result implies that the intrinsic (or density-dependent) predator–prey interaction suffices to avoid 0.o Td(resu of environment in the prey-taxis system (e.g. changing $f(v)$ in (1.6) to $f(x,v)$) can lead to the existence of non-constant steady states. Furthermore in an open wild landscape, animals are not confined within a closed (or isolated) area, and hence a non-Neumann boundary conditions may be more suitable. Therefore the study of prey-taxis systems with Dirichlet or Robin boundary conditions would be a very inspiring question to explore.

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