2.1. **Local existence and basic estimates.** Based on the classical Amann's theorem [3, 4] and the Schauder fixed point theorem along with the parabolic regularity theory [11, 27], we can establish the local existence of solutions for (2) as follows.

**Lemma 2.1** (Local existence). *Let the assumption (H0) hold and assume* $(u_0; v_0; w_0) \in [W^{1,p}(\Omega)]^3$ *with* $u_0; v_0; w_0 \geq 0$ *for some* $p > 2$. *Then there exists* $T_{max} \in (0; \infty]$ *such that the problem* (2) *has a unique nonnegative classical solution satisfying*

$$(u; v; w) \in [C^0(\bar{\Omega} \times [0; T_{max})) \cap C^{2,1}(\bar{\Omega} \times (0; T_{max}))]^3:$$

*Moreover, it holds that if* $T_{max} < \infty$, *then*

$$\|u(\cdot; t)\|_{W^{1,p}} + \|v(\cdot; t)\|_{W^{1,p}} + \|w(\cdot; t)\|_{L^\infty} \to \infty \text{ as } t \nearrow T_{\max} \text{ for some } p > 2:$$

**Lemma 2.2.** *Let* $(u; v; w)$ *be the solution obtained in Lemma 2.1, then it holds that*

$$\|u(\cdot; t)\|_{L^\infty} \leq M \quad \text{for all } t \in (0; T_{max}); \tag{8}$$

*where* $M = \max\{\|u_0\|_{L^\infty}; 1\}$.

*Proof.* We can use the similar arguments as in [12, Lemma 2.2], we omit the details of the proof of (8) for simplification. □

**Lemma 2.3** ([31]). *Let* $T > 0$ *and* $= \min\{1; \frac{T}{2}\}$. *Suppose that the nonnegative functions* $y \in C^1([0; T))$ *and* $f \in L^1_{loc}([0; T))$ *satisfy*

$$y'(t) + ay(t) \leq f(t); \quad t \in (0; T);$$

*and*

$$\int_t^{t+\tau} f(s)ds \leq b; \quad t \in (0; T - ):$$

*then it holds that*

$$y(t) \leq y(0) + 2b + \frac{b}{a}; \quad t \in (0; T):$$

2.2. **Boundedness of** $\|v\|_{L^\infty}$. In this subsection, we shall establish the boundedness of $\|v\|_{L^\infty}$. To this end, we first show that

**Lemma 2.4.** *Let* $(u; v; w)$ *be the solution of* (2) *obtained in Lemma 2.1. Then there exist positive constants* $M_1$ *and* $M_2$ *such that*

$$\|v(\cdot; t)\|_{L^1} \leq M_1 \quad \text{for all } t \in (0; T_{max}); \tag{9}$$

*and*

$$\int_t^{t+\tau} \int_\Omega v^2 \leq M_2 \quad \text{for all } t \in (0; \tilde{T}_{max}); \tag{10}$$

*where*

$$0 < < \min\{1; \frac{T_{\max}}{2}\} \text{ and } \tilde{T}_{\max} = \begin{cases} T_{\max} - ; & \text{if } T_{\max} < \infty; \\ \infty; & \text{if } T_{\max} = \infty: \end{cases} \tag{11}$$

*Proof.* Integrating the second equation of (2) in $\Omega$ and using (8), one can get

$$\frac{d}{dt}\int_\Omega v = \int_\Omega uv - \int_\Omega v^2 - \int_\Omega (b_2 vw + v) \leq M \int_\Omega v - \int_\Omega v^2: \tag{12}$$

Using the Young's inequality, from (12), we have

$$\frac{d}{dt}\int_\Omega v + \int_\Omega v + \int_\Omega v^2 \leq (M+1)\int_\Omega v \leq \frac{1}{2}\int_\Omega v^2 + \frac{(M+1)^2}{2}|\Omega|: \tag{13}$$

Then we apply Lemma 2.3 into (13) to get

$$\int_\Omega v \le \int_\Omega v_0 + \frac{(M+1)^2}{2_1}|\Omega| =: M_1;$$

which gives (9). Integrating (13) from $t$ to $t+$ , we can obtain

$$\int_t^{t+\tau}\int_\Omega v^2 \le \frac{2M_1}{1} =: M_2;$$

and (10) follows. Hence the proof of Lemma 2.4 is completed. □

**Lemma 2.5.** *Suppose the assumptions in Lemma 2.1 hold, then the solution of (2) satisfies*

$$\|\nabla u\|_{L^2} \le M_3 \quad \text{for all } t \in (0; T_{max}); \tag{14}$$

*and*

$$\int_t^{t+\tau}\int_\Omega |\Delta u|^2 \le M_4 \quad \text{for all } t \in (0; T_{max}); \tag{15}$$

*where $M_3$ and $M_4$ are positive constants independent of $t$.*

*Proof.* Multiplying the first equation of (2) by $-\Delta u$ and using (8) as well as the Young's inequality, we derive that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u|^2 + d_1\int_\Omega |\Delta u|^2 &= -\int_\Omega u(1-u)\Delta u + b_1\int_\Omega uv\Delta u\\ &\le M(1+M)\int_\Omega |\Delta u| + b_1 M\int_\Omega v|\Delta u|\\ &\le \frac{d_1}{2}\int_\Omega |\Delta u|^2 + \frac{b_1^2M^2}{d_1}\int_\Omega v^2 + \frac{M^2(1+M)^2|\Omega|}{d_1};\end{aligned}$$

which yields that

$$\frac{d}{dt}\int_\Omega |\nabla u|^2 + d_1\int_\Omega |\Delta u|^2 \le \frac{2b_1^2M^2}{d_1}\int_\Omega v^2 + \frac{2M^2(1+M)^2|\Omega|}{d_1}. \tag{16}$$

On the other hand, using the integration by parts and the Young's inequality, one obtains that

$$\int_\Omega |\nabla u|^2 = -\int_\Omega u\Delta u \le M\int_\Omega |\Delta u| \le \frac{d_1}{4}\int_\Omega |\Delta u|^2 + \frac{M^2|\Omega|}{d_1}. \tag{17}$$

Then we substitute (17) into (16) to derive that

$$\begin{aligned}&\frac{d}{dt}\int_\Omega |\nabla u|^2 + \int_\Omega |\nabla u|^2 + \frac{3d_1}{4}\int_\Omega |\Delta u|^2\\ &\le \frac{2b_1^2M^2}{d_1}\int_\Omega v^2 + \frac{M^2+2M^2(1+M)^2}{d_1}|\Omega|.\end{aligned} \tag{18}$$

Applying the Lemma 2.3 and (10), we can get (14) by letting

$$M_3 := \left\|\nabla u_0\|_{L^2}^2 + \frac{6b_1^2M^2M_2}{d_1} + \frac{3}{d_1}(M^2+2M^2(1+M)^2)|\Omega|\right\|^{\frac{1}{2}}.$$

Integrating (18) from $t$ to $t+$ and then using (10) as well as (14), one can derive that

$$\int_t^{t+\tau}\int_\Omega |\Delta u|^2 \le \frac{8b_1^2M^2M_2}{3d_1^2} + \frac{4(M^2+M^2(1+M)^2)}{3d_1^2}|\Omega| + \frac{4M_3^2}{3d_1};$$

which gives (15). Then we complete the proof of Lemma 2.5. □

**Lemma 2.6.** *Let $(u;v;w)$ be the solution of (2) obtained in Lemma 2.1. Then it holds that*

$$\|v(\cdot;t)\|_{L^2} \le M_5 \quad \text{for all } t \in (0;T_{max}); \tag{19}$$

*where $M_5 > 0$ is a constant independent of $t$.*

*Proof.* Multiplying the second equation of (2) by $v$ and integrating the result with respect to $x$ over $\Omega$, we end up with

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega v^2 + b_2\int_\Omega v^2 w + {}_1\int_\Omega v^2 + {}_1\int_\Omega v^3 \\
&= \int_\Omega \nabla\cdot({}_1(u)\nabla v)v - \int_\Omega \nabla\cdot({}_1(u)v\nabla u)v + \int_\Omega uv^2 \\
&= -\int_\Omega {}_1(u)|\nabla v|^2 + \int_\Omega {}_1(u)v\nabla u\nabla v + \int_\Omega uv^2
\end{aligned}$$

which together with (4) and the Young's inequality gives

$$\frac{1}{2}\frac{d}{dt}\int_\Omega v^2 + {}_1\int_\Omega |\nabla v|^2 \le \frac{1}{2}\int_\Omega |\nabla v|^2 + \frac{{}_2^2}{2\,{}_1}\int_\Omega v^2|\nabla u|^2 + M\int_\Omega v^2. \tag{20}$$

Applying the Gagliardo-Nirenberg inequality in two dimensions, there exists positive constants $c_1;c_2$ depending only on $\Omega$ such that

$$\begin{aligned}
&\|v\|_{L^4}^2 \le c_1(\|\nabla v\|_{L^2}\|v\|_{L^2} + \|v\|_{L^2}^2); \\
&\|\nabla u\|_{L^4}^2 \le c_2(\|\Delta u\|_{L^2}\|\nabla u\|_{L^2} + \|\nabla u\|_{L^2}^2).
\end{aligned} \tag{21}$$

Then using the Hölder inequality in (20) and (21), we can derive that

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega v^2 + {}_1\int_\Omega |\nabla v|^2 \\
&\le \frac{{}_2^2}{{}_1}\int_\Omega v^2|\nabla u|^2 + 2M\int_\Omega v^2 \\
&\le \frac{{}_2^2}{{}_1}\int_\Omega v^4{}^{\frac{1}{2}}\int_\Omega |\nabla u|^4{}^{\frac{1}{2}} + 2M\int_\Omega v^2 \\
&\le \frac{{}_2^2}{{}_1}c_1c_2(\|\nabla v\|_{L^2}\|v\|_{L^2} + \|v\|_{L^2}^2)(\|\Delta u\|_{L^2}\|\nabla u\|_{L^2} + \|\nabla u\|_{L^2}^2) + 2M\|v\|_{L^2}^2 \\
&\le \frac{{}_2^2c_1c_2M_3}{{}_1}\|\nabla v\|_{L^2}\|v\|_{L^2}\|\Delta u\|_{L^2} + \frac{{}_2^2c_1c_2M_3}{{}_1}\|v\|_{L^2}^2\|\Delta u\|_{L^2} \\
&\quad + \frac{{}_2^2c_1c_2M_3^2}{{}_1}\|\nabla v\|_{L^2}\|v\|_{L^2} + \frac{{}_2^2c_1c_2M_3^2}{{}_1}\|v\|_{L^2}^2 + 2M\|v\|_{L^2}^2 \\
&\le {}_1\|\nabla v\|_{L^2}^2 + c_3\|v\|_{L^2}^2\|\Delta u\|_{L^2}^2 + c_4\|v\|_{L^2}^2;
\end{aligned}$$

which gives

$$\frac{d}{dt}\|v\|_{L^2}^2 \le c_3\|v\|_{L^2}^2\|\Delta u\|_{L^2}^2 + c_4\|v\|_{L^2}^2; \tag{22}$$

where $c_3 = \frac{\kappa_2^4c_1^2c_2^2M_3^2}{2\kappa_1^3} + \frac{\kappa_2^2c_1c_2M_3}{2\kappa_1}$ and $c_4 = \frac{\kappa_2^4c_1^2c_2^2M_3^4}{2\kappa_1^3} + \frac{\kappa_2^2c_1c_2M_3}{2\kappa_1} + \frac{\kappa_2^2c_1c_2M_3^2}{\kappa_1} + 2M$.

On the other hand, for any fixed constant $t_0 \in [(t - \ )_+;t)$ with $t \in (0;T_{max})$, using Lemma 2.4 and Lemma 2.5, we deduce that

$$\|v(\cdot;t_0)\|_{L^2}^2 \le \max\{\|v_0\|_{L^2}^2; M_2\} \quad \text{and} \quad \int_{t_0}^{t_0+\tau}\int_\Omega |\Delta u|^2 \le M_4.$$

Then applying the Grönwall inequality into (22), we can obtain

$$\begin{aligned}\|v(\cdot;t)\|_{L^2}^2 &\le \|v(\cdot;t_0)\|_{L^2}^2 \cdot e^{c_3\int_{t_0}^t \int |\Delta u|^2+c_4} \\ &\le \max\left\{\|v_0\|_{L^2}^2; \frac{M_2}{\quad}\right\} \cdot e^{c_3 M_4+c_4};\end{aligned} \tag{23}$$

which gives (19) with $M_5 := \max\left\{\|v_0\|_{L^2}; (\frac{2M_2}{\tau})^{\frac{1}{2}}\right\} \cdot e^{\frac{c_3 M_4+c_4}{2}}$. □

**Lemma 2.7.** *The solution $(u;v;w)$ of (2) obtained in Lemma 2.1 satisfies*

$$\|\nabla u(\cdot;t)\|_{L^4} \le M_6 \quad \text{for all } t \in (0;T_{max}); \tag{24}$$

*where $M_6 > 0$ is a constant independent of $t$.*

*Proof.* The first equation of (2) can be rewritten as

$$u_t - d_1\Delta u = u(1-u) - b_1 uv =: f_1(x;t); \tag{25}$$

and obviously $\|f_1(\cdot;t)\|_{L^2} \le M(1+M)|\Omega|^{\frac{1}{2}} + b_1 M M_5$. Then applying the variant-of-constants formula to (25) and using the well-known semigroup estimates under homogeneous Neumann boundary (see [32]), one has

$$\begin{aligned}\|\nabla u(\cdot;t)\|_{L^4} &\le \|\nabla e^{(d_1\Delta-1)t}u_0\|_{L^4} + \int_0^t \|\nabla e^{(d_1\Delta-1)(t-s)} f_1\|_{L^4} ds \\ &\le {}_1\|\nabla u_0\|_{L^4} + {}_2\int_0^t (1+(t-s)^{-\frac{3}{4}})e^{-d_1\lambda_1(t-s)}\|f_1\|_{L^2} ds \\ &\le {}_1\|\nabla u_0\|_{L^4} + \frac{2}{d_1\ {}_1}(M(1+M)|\Omega|^{\frac{1}{2}} + b_1 M M_5)\quad 1+\Gamma\quad \frac{1}{4}\quad d_1^{\frac{3}{4}}\ {}^{\frac{3}{4}}_{1}\quad ;\end{aligned}$$

which gives (24), where ${}_1$; ${}_2$ are constants and $\Gamma(\cdot)$ represents the Gamma function defined by $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$. Then Lemma 2.7 is proved. □

**Lemma 2.8.** *Let $(u;v;w)$ be the solution of (2) obtained in Lemma 2.1, then it holds*

$$\|v(\cdot;t)\|_{L^3} \le M_7 \quad \text{for all } t \in (0;T_{max}); \tag{26}$$

*where $M_7 > 0$ is a constant independent of $t$.*

*Proof.* We multiply the second equation of (2) by $v^2$ and integrate the result with respect to $x$ over $\Omega$ to obtain

$$\begin{aligned}&\frac{1}{3}\frac{d}{dt}\int_\Omega v^3 + 2\int_\Omega {}_1(u)v|\nabla v|^2 + \int_\Omega {}_1 v^4 + \int_\Omega {}_1 v^3 \\ &= 2\int_\Omega {}_1(u)v^2\nabla u\cdot\nabla v + \int_\Omega uv^3 - b_2\int_\Omega wv^3;\end{aligned}$$

which together with the facts ${}_1(u) \ge {}_1$ and $|{}_1(u)| \le {}_2$ gives

$$\frac{1}{3}\frac{d}{dt}\int_\Omega v^3 + 2\int_\Omega {}_1 v|\nabla v|^2 + \int_\Omega {}_1 v^4 \le 2\int_\Omega {}_2 v^2|\nabla u||\nabla v| + M\int_\Omega v^3; \tag{27}$$

By the Hölder inequality and (24), we have

$$\begin{aligned} 2\int_\Omega v^2|\nabla u||\nabla v| &\le \int_\Omega v|\nabla v|^2 + \frac{2}{2}\int_\Omega v^3|\nabla u|^2 \\ &\le \int_\Omega v|\nabla v|^2 + \frac{2}{2}\Big(\int_\Omega v^6\Big)^{\frac12}\Big(\int_\Omega |\nabla u|^4\Big)^{\frac12} \\ &\le \int_\Omega v|\nabla v|^2 + \frac{2M_6^2}{2}\int_\Omega v^6 . \end{aligned} \tag{28}$$

Then we substitute (28) into (27) and utilize the fact $\|\nabla v^{\frac32}\|_{L^2}^2 = \frac94\int_\Omega v|\nabla v|^2$ to obtain

$$\frac{d}{dt}\int_\Omega v^3 + \frac{4}{3}\|\nabla v^{\frac32}\|_{L^2}^2 + 3\int_\Omega v^4 \le \frac{3\,2M_6^2}{2}\|v\|_{L^6}^3 + 3M\int_\Omega v^3. \tag{29}$$

On the other hand, using the Gagliardo-Nirenberg inequality, the Young's inequality and the fact that $\|v^{\frac32}\|_{L^{\frac43}} = \|v\|_{L^2}^{\frac32} \le M_5^{\frac32}$, we can derive that

$$\begin{aligned} \frac{3\,2M_6^2}{2}\|v\|_{L^6}^3 &= \frac{3\,2M_6^2}{2}\|v^{\frac32}\|_{L^4}^2 \\ &\le \frac{3\,2M_6^2c_1}{2}\big(\|\nabla v^{\frac32}\|_{L^2}^{\frac43}\|v^{\frac32}\|_{L^{\frac43}}^{\frac23} + \|v^{\frac32}\|_{L^{\frac43}}^2\big) \\ &\le \frac{4}{3}\|\nabla v^{\frac32}\|_{L^2}^2 + \frac{27}{8}\frac{6}{2}\frac{M_6^6c_1^3M_5^3}{1} + \frac{3\,2M_6^2c_1M_5^3}{2}; \end{aligned}$$

which substituted into (29) gives

$$\begin{aligned} \frac{d}{dt}\int_\Omega v^3 + \int_\Omega v^3 + 3\int_\Omega v^4 &\le (3M+1)\int_\Omega v^3 + c_1 \\ &\le 3\int_\Omega v^4 + c_1 + c_2; \end{aligned}$$

and hence

$$\frac{d}{dt}\int_\Omega v^3 + \int_\Omega v^3 \le c_1 + c_2; \tag{30}$$

where $c_1 = \frac{27\kappa_2^6M_6^6c_1^3M_5^3}{8\kappa_1^5} + \frac{3\kappa_2^2M_6^2c_1M_5^3}{\kappa_1}$ and $c_2 = 3^{-1}4^{-3}\,{}_1^{-3}(3M+1)^4|\Omega|$. Applying the Grönwall inequality to (30), we get (26) by letting $M_7 := (\|v_0\|_{L^3}^3 + c_1 + c_2)^{\frac13}$. Then we complete the proof of Lemma 2.8. □

**Lemma 2.9.** *Suppose the assumptions in Lemma 2.1 hold, then the solution of (2) satisfies*

$$\|\nabla u(\cdot;t)\|_{L^\infty} \le M_8 \quad \text{for all } t\in(0;T_{max}); \tag{31}$$

*and*

$$\|v(\cdot;t)\|_{L^\infty} \le M_9 \quad \text{for all } t\in(0;T_{max}); \tag{32}$$

*where $M_8 > 0$ and $M_9 > 0$ are constants independent of $t$.*

*Proof.* Applying the variation-of-constants formula, from the first equation of (2

which gives

$$\nabla u(\cdot;t) = \nabla e^{d_1 t\Delta} u_0 + \int_0^t \nabla e^{d_1(t-s)\Delta}[u(1-u) - b_1 uv]ds. \tag{33}$$

Since $\|u(\cdot;t)\|_{L^\infty} \le M$ and $\|v(\cdot;t)\|_{L^3} \le M_7$, one has

$$\|u(1-u) - b_1 uv\|_{L^3} \le \|u(1-u)\|_{L^3}$$

where $c_3 = c_2 + 1$. Now we want to let $\int_\Omega v^p$ be controlled by $\int_\Omega v^{p-2}|\nabla v|^2$ and $\int_\Omega v^{\frac{p}{2}}$. In order to obtain this, we introduce the inequality below:

$$\|f\|_{L^2}^2 \le \varepsilon\|\nabla f\|_{L^2}^2 + c_4(1+\varepsilon^{-1})\|f\|_{L^1}^2 \text{ for any } \varepsilon > 0;$$

where the constant $c_4 > 0$ depends only on $\Omega$. Here taking $f = v^{\frac{p}{2}}$ and $\varepsilon = \frac{2\kappa_1}{c_2 p^2}$, we get

$$\begin{aligned} & c_3 p(p-1)\int_\Omega v^p \\ & \le c_3 p(p-1)\frac{2}{c_2 p^2}\int_\Omega |\nabla v^{\frac{p}{2}}|^2 + c_3 c_4 p(p-1)\left(1+\frac{c_2 p^2}{2\kappa_1}\right)\left(\int_\Omega v^{\frac{p}{2}}\right)^2 \\ & \le \frac{\kappa_1 p(p-1)}{2}\int_\Omega v^{p-2}|\nabla v|^2 + c_5 p(p-1)(1+p^2)\left(\int_\Omega v^{\frac{p}{2}}\right)^2; \end{aligned} \tag{39}$$

where $c_5 = c_3 c_4 \cdot \max\{1; \frac{c_2}{2\kappa_1}\}$. Substituting (39) into (38), multiplying $e^{p(p-1)t}$ on both sides of (38) and noting the fact $(1+p^2) \le (1+p)^2$, we have

$$\frac{d}{dt}\left(e^{p(p-1)t}\int_\Omega v^p\right) \le c_5(1+p)^2 p(p-1)e^{p(p-1)t}\left(\int_\Omega v^{\frac{p}{2}}\right)^2; \tag{40}$$

which integrated from 0 to $t$ gives

$$\begin{aligned} \int_\Omega v^p &\le \int_\Omega v_0^p + c_5(1+p)^2 \sup_{t\in(0,T_{max})}\left(\int_\Omega v^{\frac{p}{2}}\right)^2 \\ &\le c_6(1+p)^2 \cdot \max\left\{\|v_0\|_{L^\infty}^p; \sup_{t\in(0,T_{max})}\left(\int_\Omega v^{\frac{p}{2}}\right)^2\right\}; \end{aligned} \tag{41}$$

that is

$$\|v(\cdot;t)\|_{L^p} \le c_6^{\frac{1}{p}}(1+p)^{\frac{2}{p}} \cdot \max\{\|v_0\|_{L^\infty}; \sup_{t\in(0,T_{max})}\|v\|_{L^{\frac{p}{2}}}\}; \tag{42}$$

where $c_6 = c_5 + |\Omega|$. Noticing that $c_6^{\frac{1}{p}}(1+p)^{\frac{2}{p}} > 1$, we can rewrite (42) as

$$\max\{\|v_0\|_{L^\infty}; \sup_{t\in(0,T_{max})}\|v\|_{L^p}\} \le c_6^{\frac{1}{p}}(1+p)^{\frac{2}{p}} \cdot \max\{\|v_0\|_{L^\infty}; \sup_{t\in(0,T_{max})}\|v\|_{L^{\frac{p}{2}}}\}: \tag{43}$$

Denoting $H(p) := \max\{\|v_0\|_{L^\infty}; \sup_{t\in(0,T_{max})}\|v(\cdot;t)\|_{L^p}\}$ and taking $p = 2^j$; $j \in \mathbb{Z}$, from (43) we have

$$H(2^j) \le c_6^{1-2^{-j}} \cdot \prod_{i=1}^{j}(1+2^i)^{2^{1-i}} \cdot H(1): \tag{44}$$

Meanwhile, it is obvious that $\ln(1+x) \le \sqrt{x}$ holds for all $x \ge 0$, so

$$\ln\prod_{i=1}^{j}(1+2^i)^{2^{-i}} = \sum_{i=1}^{j}\frac{\ln(1+2^i)}{2^i} \le \sum_{i=1}^{j}\left(\frac{1}{\sqrt{2}}\right)^i < 4;$$

and hence

$$\prod_{i=1}^{j}(1+2^i)^{2^{1-i}} \le e^8: \tag{45}$$

Combing (45) with (44) and using the definition of $H(p)$, we have

$$\|v(\cdot;t)\|_{L^\infty} \le \lim_{j\to\infty} H(2^j) \le c_6 e^8 H(1) \le c_6 e^8 \cdot \max\{\|v_0\|_{L^\infty}; M_1\}:$$

Then (32) is proved with $M_9 := c_6 e^8 \cdot \max\{\|v_0\|_{L^\infty}; M_1\}$. □

2.3. **Boundedness of $\|w\|_{L^\infty}$.** Now we are in position to prove the boundedness of $\|w\|_{L^\infty}$. To this end, we first improve the regularity of $u$ as follows.

**Lemma 2.10.** *Let the conditions in Theorem 1.1 hold, then for all $p \geq 1$, the solution $(u; v; w)$ of (2) satisfies*

$$\int_t^{t+\tau} \|D^2 u\|_{L^p}^p \leq M_{10} \quad \text{for all } t \in (0; T_{max}); \tag{46}$$

*where $M_{10} > 0$ is some constant independent of $t$.*

*Proof.* Using the fact $\|v(\cdot; t)\|_{L^\infty} \leq M_9$ in (32), we can use the similar arguments as in [10, Lemma 2.4] to obtain (46). The details of proof are omitted for convenience. □

**Lemma 2.11.** *Let $(u; v; w)$ be the solution obtained in Lemma 2.1, then there exists a constant $M_{11} > 0$ independent of $t$ such that*

$$\int_t^{t+\tau} \int_\Omega w^2 \leq M_{11} \quad \text{for all } t \in (0; T_{max}): \tag{47}$$

*Proof.* Integrating the third equation of (2) in $\Omega$, one has

$$\frac{d}{dt} \int_\Omega w + \int_\Omega w + \_2 \int_\Omega w^2 \leq (M_9 + 1) \int_\Omega w \leq \frac{2}{2} \int_\Omega w^2 + \frac{(M_9 + 1)^2 |\Omega|}{2 \_2}; \tag{48}$$

which combined with the Grönwall inequality implies

$$\int_\Omega w \leq \int_\Omega w_0 + \frac{(M_9 + 1)^2 |\Omega|}{2 \_2}:$$

Then we can integrate (48) from $t$ to $t +$ to obtain

$$\int_t^{t+\tau} \int_\Omega w^2 \leq \frac{2\|w_0\|_{L^1}}{2} + \frac{(M_9 + 1)^2 |\Omega|}{\frac{2}{2}}$$

which gives (47). □

**Lemma 2.12.** *Assume the conditions in Lemma 2.1 hold, then the solution of (2) satisfies*

$$\|\nabla v(\cdot; t)\|_{L^2} \leq M_{12} \quad \text{for all } t \in (0; T_{max}); \tag{49}$$

*and*

$$\int_t^{t+\tau} \int_\Omega |\Delta v|^2 \leq M_{13} \quad \text{for all } t \in (0; T_{max}); \tag{50}$$

*where $M_{12}$ and $M_{13}$ are positive constants independent of $t$.*

*Proof.* Multiplying the second equation of (2) by $-\Delta v$ and integrating the result with respect to $x$, we have

$$\frac{1}{2} \frac{d}{dt} \int_\Omega |\nabla v|^2 + \int_\Omega \_1(u) |\Delta v|^2 = I_1 + I_2; \tag{51}$$

where

$$I_1 := -\int_\Omega \nabla \_1(u) \cdot \nabla v \Delta v + \int_\Omega \nabla \cdot (\_1(u) v \nabla u) \Delta v$$

and

$$I_2 := -\int_\Omega (uv - b_2 vw - \_1 v - \_1 v^2) \Delta v:$$

From the assumptions in (H0), we have $(\ _1;\ \ _1)(\cdot) \in [C^2([0;\infty))]^2$, and hence there exist positive constants $`_1;`_2$ such that

$$|\ '_1(\cdot)| \le `_1 \quad \text{and} \quad |\ '_1(\cdot)| \le `_2: \tag{52}$$

Noting the facts (4) and (52), and then using the Young's inequality, we can derive that

$$\begin{aligned} I_1 &= -\int_\Omega \ '_1(u)\nabla u\cdot\nabla v\Delta v + \int_\Omega \ '_1(u)\nabla u\cdot v\nabla u\Delta v \\ &\quad + \int_\Omega \ _1(u)\nabla u\cdot\nabla v\Delta v + \int_\Omega \ _1(u)v\Delta v\Delta u \\ &\le (`_1+\ _2)M_8\int_\Omega |\nabla v||\Delta v| + `_2M_8^2M_9\int_\Omega |\Delta v| + \ _2M_9\int_\Omega |\Delta u||\Delta v| \\ &\le \frac{5\ _1}{8}\int_\Omega |\Delta v|^2 + \frac{(`_1+\ _2)^2M_8^2}{1}\int_\Omega |\nabla v|^2 \\ &\quad + \frac{2`_2^2M_8^4M_9^2}{1}|\Omega| + \frac{\ _2^2M_9^2}{1}\int_\Omega |\Delta u|^2; \end{aligned} \tag{53}$$

and

$$\begin{aligned} I_2 &= -\int_\Omega uv\Delta v + b_2\int_\Omega vw\Delta v + \ _1\int_\Omega v\Delta v + \ _1\int_\Omega v^2\Delta v \\ &\le (M\,M_9+\ _1M_9+\ _1M_9^2)\int_\Omega |\Delta v| + b_2M_9\int_\Omega w|\Delta v| \\ &\le \frac{1}{4}\int_\Omega |\Delta v|^2 + \frac{2b_2^2M_9^2}{1}\int_\Omega w^2 + \frac{2(M\,M_9+\ _1M_9+\ _1M_9^2)^2}{1}|\Omega|: \end{aligned} \tag{54}$$

Then substituting (53) and (54) into (51), we have

$$\begin{aligned} &\frac{d}{dt}\int_\Omega |\nabla v|^2 + \int_\Omega |\nabla v|^2 + \frac{1}{4}\int_\Omega |\Delta v|^2 \\ &\le \frac{2(`_1+\ _2)^2M_8^2}{1}+1\int_\Omega |\nabla v|^2 + \frac{2\ _2^2M_9^2}{1}\int_\Omega |\Delta u|^2 \\ &\quad + \frac{4b_2^2M_9^2}{1}\int_\Omega w^2 + \frac{4l_2^2M_8^4M_9^2}{1}|\Omega| + \frac{4(M\,M_9+\ _1M_9+\ _1M_9^2)^2}{1}|\Omega|: \end{aligned} \tag{55}$$

Meanwhile, using the Hölder inequality, one obtains that

$$\begin{aligned} \frac{2(`_1+\ _2)^2M_8^2}{1}+1\int_\Omega |\nabla v|^2 &= -\frac{2(l_1+\ _2)^2M_8^2}{1}+1\int_\Omega v\Delta v \\ &\le M_9\left(\frac{2(l_1+\ _2)^2M_8^2}{1}+1\right)\int_\Omega |\Delta v| \\ &\le \frac{1}{8}\int_\Omega |\Delta v|^2 + \frac{2M_9^2}{1}\left(\frac{2(`_1+\ _2)^2M_8^2}{1}+1\right)^2|\Omega|; \end{aligned}$$

which substituting into (55) gives

$$\begin{aligned} &\frac{d}{dt}\int_\Omega |\nabla v|^2 + \int_\Omega |\nabla v|^2 + \frac{1}{8}\int_\Omega |\Delta v|^2 \\ &\le \frac{2\ _2^2M_9^2}{1}\int_\Omega |\Delta u|^2 + \frac{4b_2^2M_9^2}{1}\int_\Omega w^2 + c_1; \end{aligned} \tag{56}$$

where

$$c_1 = \frac{4l_2^2 M_8^4 M_9^2}{1}|\Omega| + \frac{4(M M_9 + \ _1M_9 + \ _1M_9^2)^2}{1}|\Omega| + \frac{2M_9^2}{1}\ \frac{2(l_1 + \ _2)^2 M_8^2}{1} + 1\ ^2\ |\Omega|:$$

Applying Lemma 2.3 again and using Lemma 2.5 as well as Lemma 2.11, one can derive (49). Then integrating (56) from $t$ to $t + \ $, we can also directly obtain (50). □

**Lemma 2.13.** *Let $(u; v; w)$ be the solution obtained in Lemma 2.1. Then it hold that*

$$\|w(\cdot; t)\|_{L^2} \le M_{14} \quad \text{for all } t \in (0; T_{max}); \tag{57}$$

*where $M_{14} > 0$ is a constant independent of $t$.*

*Proof.* Multiplying the third equation of (2) by $w$ and integrating the result with respect to $x$, then using the Young's inequality and (4), we derive

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega w^2 + \ _3\int_\Omega |\nabla w|^2 &\le \ _4\int_\Omega w|\nabla v|\cdot|\nabla w| + M_9\int_\Omega w^2\\ &\le \frac{3}{2}\ _\Omega\int|\nabla w|^2 + \frac{\frac{2}{4}}{2\ _3}\int_\Omega w^2|\nabla v|^2 + M_9\int_\Omega w^2;\end{aligned}$$

which gives

$$\frac{d}{dt}\int_\Omega w^2 + \ _3\int_\Omega |\nabla w|^2 \le \frac{\frac{2}{4}}{3}\int_\Omega w^2|\nabla v|^2 + 2M_9\int_\Omega w^2: \tag{58}$$

Then using the Hölder inequality and the Gagliardo-Nirenberg inequality as in (21), we can deduce from (58) that

$$\begin{aligned}&\frac{d}{dt}\int_\Omega w^2 + \ _3\int_\Omega |\nabla w|^2\\ &\le \frac{\frac{2}{4}}{3}\int_\Omega w^2|\nabla v|^2 + 2M_9\int_\Omega w^2\\ &\le \frac{\frac{2}{4}}{3}\int_\Omega w^4\ ^{\frac{1}{2}}\int_\Omega |\nabla v|^4\ ^{\frac{1}{2}} + 2M_9\int_\Omega w^2\\ &\le \frac{\frac{2}{4}}{3}c_1c_2(\|\nabla w\|_{L^2}\|w\|_{L^2} + \|w\|_{L^2}^2)(\|\Delta v\|_{L^2}\|\nabla v\|_{L^2} + \|\nabla v\|_{L^2}^2) + 2M_9\|w\|_{L^2}^2\\ &\le \frac{\frac{2}{4}c_1c_2M_{12}}{3}\|\nabla w\|_{L^2}\|w\|_{L^2}\|\Delta v\|_{L^2} + \frac{\frac{2}{4}c_1c_2M_{12}}{3}\|w\|_{L^2}^2\|\Delta v\|_{L^2}\\ &\quad + \frac{\frac{2}{4}c_1c_2M_{12}^2}{3}\|\nabla w\|_{L^2}\|w\|_{L^2} + \frac{\frac{2}{4}c_1c_2M_{12}^2}{3}\|w\|_{L^2}^2 + 2M_9\|w\|_{L^2}^2\\ &\le \ _3\|\nabla w\|_{L^2}^2 + c_3\|w\|_{L^2}^2\|\Delta v\|_{L^2}^2 + c_4\|w\|_{L^2}^2;\end{aligned}$$

which gives

$$\frac{d}{dt}\|w\|_{L^2}^2 \le c_3\|w\|_{L^2}^2\|\Delta v\|_{L^2}^2 + c_4\|w\|_{L^2}^2; \tag{59}$$

where $c_3 = \frac{\kappa_4^4 c_1^2 c_2^2 M_{12}^2}{2\kappa_3^3} + \frac{\kappa_4^2 c_1 c_2 M_{12}}{2\kappa_3}$ and $c_4 = \frac{\kappa_4^4 c_1^2 c_2^2 M_{12}^4}{2\kappa_3^3} + \frac{\kappa_4^2 c_1 c_2 M_{12}}{2\kappa_3} + \frac{\kappa_4^2 c_1 c_2 M_{12}^2}{\kappa_3} + 2M_9$. Then thanks to (47) and (50), we can use the similar methods as in (23) to obtain (57) from (59). □

To prove Lemma 2.16, we shall obtain the $L^p$-energy estimates of $w$ as in Lemma 2.14 and $L^{2p}$-energy estimates of $\nabla v$ as in Lemma 2.15 separately.

**Lemma 2.14.** *Let $(u;v;w)$ be the solution of (2), $p \geq 2$, then for all $t \in (0;T_{max})$ it holds that*

$$\frac{d}{dt}\int_\Omega w^p + p\ _2\int_\Omega w^p + \frac{\ _3(p-1)}{p}\int_\Omega |\nabla w^{\frac{p}{2}}|^2 \leq \frac{\ ^2_4p(p-1)}{2\ _3}\int_\Omega w^p|\nabla v|^2 + C; \quad (60)$$

*where $C > 0$ is a positive constant independent of $t$.*

*Proof.* We multiply the third equation of (2) by $w^{p-1}(p \geq 3)$ and integrate the result with respect to $x$ to derive that

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_\Omega w^p + \ _2\int_\Omega w^p + \ _3(p-1)\int_\Omega w^{p-2}|\nabla w|^2 \\
&\leq \ _4(p-1)\int_\Omega w^{p-1}|\nabla v|\cdot|\nabla w| + M_9\int_\Omega w^p \qquad (61)\\
&\leq \frac{\ _3(p-1)}{2}\int_\Omega w^{p-2}|\nabla w|^2 + \frac{\ ^2_4(p-1)}{2\ _3}\int_\Omega w^p|\nabla v|^2 + M_9\int_\Omega w^p.
\end{aligned}$$

Noting the fact that $\int_\Omega w^{p-2}|\nabla w|^2 = \frac{4}{p^2}\int_\Omega |\nabla w^{\frac{p}{2}}|^2$, we can rewrite (61) as

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega w^p + p\ _2\int_\Omega w^p + \frac{2\ _3(p-1)}{p}\int_\Omega |\nabla w^{\frac{p}{2}}|^2 \\
&\leq \frac{\ ^2_4p(p-1)}{2\ _3}\int_\Omega w^p|\nabla v|^2 + pM_9\int_\Omega w^p. \qquad (62)
\end{aligned}$$

Using the Gagliardo-Nirenberg inequality along with the facts $\int_\Omega w^p = \|w^{\frac{p}{2}}\|^2_{L^2}$ and $\|w^{\frac{p}{2}}\|_{L^{\frac{4}{p}}} = \|w\|^{\frac{p}{2}}_{L^2} \leq M^{\frac{p}{2}}_{14}$, we can estimate (62) as follows:

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega w^p + p\ _2\int_\Omega w^p + \frac{2\ _3(p-1)}{p}\int_\Omega |\nabla w^{\frac{p}{2}}|^2 \\
&\leq \frac{\ ^2_4p(p-1)}{2\ _3}\int_\Omega w^p|\nabla v|^2 + c_1(\|\nabla w^{\frac{p}{2}}\|^{2\theta}_{L^2}\|w^{\frac{p}{2}}\|^{2(1-\theta)}_{L^{\frac{4}{p}}} + \|w^{\frac{p}{2}}\|^2_{L^{\frac{4}{p}}}) \\
&\leq \frac{\ ^2_4p(p-1)}{2\ _3}\int_\Omega w^p|\nabla v|^2 + \frac{\ _3(p-1)}{p}\int_\Omega |\nabla w^{\frac{p}{2}}|^2 + c_2;
\end{aligned}$$

where $\ = 1 - \frac{2}{p}$ and hence comes (60). □

**Lemma 2.15.** *Let $(u;v;w)$ be the solution of (2) obtained in Lemma 2.1 and $p \geq 2$, then*

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega |\nabla v|^{2p} + \int_\Omega |\nabla v|^{2p} + \frac{2\ _1(p-1)}{p}\int_\Omega |\nabla|\nabla v|^p|^2 \\
&\leq C\int_\Omega w^2|\nabla v|^{2p-2} + \int_\Omega |D^2u|^{2p} \quad + C \qquad (63)
\end{aligned}$$

*holds for $t \in (0;T_{max})$, where $C > 0$ is a positive constant independent of $t$.*

*Proof.* From the second equation of (2), we obtain

$$\begin{aligned}
&\frac{1}{2p}\frac{d}{dt}\int_\Omega |\nabla v|^{2p} \\
&= \int_\Omega |\nabla v|^{2(p-1)}\nabla v\cdot\nabla v_t \\
&= \int_\Omega |\nabla v|^{2(p-1)}\nabla v\cdot\Delta(\gamma_1(u)\nabla v) - \int_\Omega |\nabla v|^{2(p-1)}\nabla v\cdot\Delta(\gamma_1(u)v\nabla u) \\
&\quad + \int_\Omega |\nabla v|^{2(p-1)}\nabla v\cdot\nabla(uv - b_2vw - \gamma_1 v - \gamma_1 v^2) \\
&=: J_1 + J_2 + J_3.
\end{aligned} \tag{64}$$

By integrating by parts and the fact that $\nabla\Delta v\cdot\nabla v = \frac{1}{2}\Delta|\nabla v|^2 - |D^2 v|^2$, $J_1$ can be rewritten as

$$\begin{aligned}
J_1 &= \int_\Omega |\nabla v|^{2(p-1)}\nabla v\cdot\Delta(\gamma_1(u)\nabla v) \\
&= -\int_\Omega \nabla\cdot(|\nabla v|^{2(p-1)}\nabla v)\nabla\cdot(\gamma_1(u)\nabla v) \\
&= -\frac{p-1}{2}\int_\Omega \gamma_1(u)|\nabla v|^{2(p-2)}|\nabla|\nabla v|^2|^2 - \frac{1}{2}\int_\Omega \gamma_1'(u)|\nabla v|^{2(p-1)}\nabla|\nabla v|^2\cdot\nabla u \\
&\quad + \frac{1}{2}\int_{\partial\Omega}\gamma_1(u)|\nabla v|^{2(p-1)}\frac{\partial|\nabla v|^2}{\partial}dS - \int_\Omega \gamma_1(u)|\nabla v|^{2(p-1)}|D^2 v|^2 \\
&\quad - (p-1)\int_\Omega \gamma_1'(u)|\nabla v|^{2(p-2)}\nabla|\nabla v|^2\cdot\nabla v\nabla u\cdot\nabla v.
\end{aligned}$$

To estimate $J_1$, we introduce the trace inequality as follows [24]:

$$\int_{\partial\Omega}|\nabla f|^{2p-2}\cdot\partial_\nu|\nabla f|^2 dS \le \varepsilon\int_\Omega |\nabla f|^{2p-2}\cdot|\nabla|\nabla f|^2|^2 + C_\varepsilon\int_\Omega |\nabla f|^{2p} \quad \text{for any } \varepsilon > 0. \tag{65}$$

Thanks to (65) and (4), we can find a constant $c_1 > 0$ such that

$$\begin{aligned}
J_1 &\le -\frac{\gamma_1(p-1)}{2}\int_\Omega |\nabla v|^{2(p-2)}|\nabla|\nabla v|^2|^2 - \gamma_1\int_\Omega |\nabla v|^{2(p-1)}|D^2 v|^2 \\
&\quad + c_1\int_\Omega |\nabla v|^{2p} + \lambda_1 M_8\left(p-\frac{1}{2}\right)\int_\Omega |\nabla v|^{2(p-1)}|\nabla|\nabla v|^2|.
\end{aligned} \tag{66}$$

Noting the fact that

$$\begin{aligned}
\nabla\cdot(\gamma_1(u)v\nabla u) &= \gamma_1'(u)\nabla u v\cdot\nabla u + \gamma_1(u)\nabla v\cdot\nabla u + \gamma_1(u)v\Delta u \\
&\le \lambda_2 M_8^2 M_9 + \gamma_2 M_8|\nabla v| + \gamma_2 M_9|\Delta u| \\
&\le c_2(1 + |\Delta u| + |\nabla v|)
\end{aligned}$$

with $c_2 = \max\{\grave{}_2 M_8^2 M_9;\ {}_2M_8;\ {}_2M_9\}$, we can estimate $J_2$ as follows:

$$\begin{aligned} J_2 &= \int_\Omega \nabla(|\nabla v|^{2(p-1)}\nabla v)\cdot\nabla({}_1(u)v\nabla u) \\ &= \int_\Omega \nabla|\nabla v|^{2(p-1)}\cdot\nabla v\nabla\cdot({}_1(u)v\nabla u) + \int_\Omega |\nabla v|^{2(p-1)}\Delta v\nabla\cdot({}_1(u)v\nabla u) \\ &\le c_2(p-1)\int_\Omega |\nabla v|^{2p-3}|\nabla|\nabla v|^2|\,(1+|\Delta u|+|\nabla v|) \\ &\quad + c_2\int_\Omega |\nabla v|^{2(p-1)}|\Delta v|(1+|\Delta u|+|\nabla v|). \end{aligned} \tag{67}$$

Similarly, using the fact $|\Delta u| \le \sqrt{n}|D^2u| = \sqrt{2}|D^2u|$, $J_3$ can be estimated as follows:

$$\begin{aligned} J_3 &= \int_\Omega |\nabla v|^{2(p-1)}\nabla v\cdot\nabla(uv - b_2vw - {}_1v - {}_1v^2) \\ &= -\int_\Omega \nabla\cdot(|\nabla v|^{2(p-1)}\nabla v)(uv - b_2vw - {}_1v - {}_1v^2) \\ &\le (M + {}_1 + {}_1M_9)M_9(p-1)\int_\Omega |\nabla v|^{2p-3}|\nabla|\nabla v|^2| \\ &\quad + b_2M_9(p-1)\int_\Omega w|\nabla v|^{2p-3}|\nabla|\nabla v|^2| \\ &\quad + (M + {}_1 + {}_1M_9)M_9\sqrt{2}\int_\Omega |\nabla v|^{2(p-1)}|D^2v| \\ &\quad + b_2M_9\sqrt{2}\int_\Omega w|\nabla v|^{2(p-1)}|D^2v|. \end{aligned} \tag{68}$$

Substituting (66), (67), (68) into (64), and using the Young's inequality, we can deduce that

$$\begin{aligned} &\frac{d}{dt}\int_\Omega |\nabla v|^{2p} + {}_1p(p-1)\int_\Omega |\nabla v|^{2p-4}|\nabla|\nabla v|^2|^2 + 2\,{}_1p\int_\Omega |\nabla v|^{2p-2}|D^2v|^2 \\ &\le c_3\int_\Omega |\nabla v|^{2p-2}|\nabla|\nabla v|^2| + c_3\int_\Omega |\nabla v|^{2p-3}|\nabla|\nabla v|^2|(1+w) \\ &\quad + c_3\int_\Omega |\nabla v|^{2p-3}|\nabla|\nabla v|^2||D^2u| + c_3\int_\Omega |\nabla v|^{2p-1}|D^2v| \\ &\quad + c_3\int_\Omega |\nabla v|^{2p-2}|D^2v|(1+w) + c_3\int_\Omega |\nabla v|^{2p-2}|D^2v||D^2u| + 2pc_1\int_\Omega |\nabla v|^{2p} \\ &\le \frac{{}_1p(p-1)}{4}\int_\Omega |\nabla v|^{2p-4}|\nabla|\nabla v|^2|^2 + {}_1p\int_\Omega |\nabla v|^{2p-2}|D^2v|^2 \\ &\quad + c_4\int_\Omega |\nabla v|^{2p} + c_4\int_\Omega |\nabla v|^{2p-2}w^2 + c_4\int_\Omega |D^2u|^{2p} + c_5; \end{aligned} \tag{69}$$

which gives

$$\begin{aligned} &\frac{d}{dt}\int_\Omega |\nabla v|^{2p} + \frac{3\ {}_1p(p-1)}{4}\int_\Omega |\nabla v|^{2p-4}|\nabla|\nabla v|^2|^2 + {}_1p\int_\Omega |\nabla v|^{2p-2}|D^2v|^2 \\ &\le c_4\int_\Omega |\nabla v|^{2p} + c_4\int_\Omega |\nabla v|^{2p-2}w^2 + c_4\int_\Omega |D^2u|^{2p} + c_5. \end{aligned} \tag{70}$$

On the other hand, we can check that

$$\begin{aligned}
&(2+c_4)\int_\Omega |\nabla v|^{2p} \\
&= (2+c_4)\int_\Omega |\nabla v|^{2p-2}\nabla v\cdot\nabla v \\
&= -(2+c_4)(p-1)\int_\Omega v|\nabla v|^{2p-4}\nabla v\cdot\nabla|\nabla v|^2 - (2+c_4)\int_\Omega v|\nabla v|^{2p-2}\Delta v \\
&\le \frac{{}_1p(p-1)}{4}\int_\Omega |\nabla v|^{2p-4}|\nabla|\nabla v|^2|^2 + \frac{M_9^2(2+c_4)^2(p-1)}{{}_1p}\int_\Omega |\nabla v|^{2p-2} \\
&\quad + \frac{{}_1p}{4}\int_\Omega |\nabla v|^{2p-2}|D^2v|^2 + \frac{2M_9^2(2+c_4)^2}{{}_1p}\int_\Omega |\nabla v|^{2p-2} \\
&\le \frac{{}_1p(p-1)}{4}\int_\Omega |\nabla v|^{2p-4}|\nabla|\nabla v|^2|^2 + \frac{{}_1p}{4}\int_\Omega |\nabla v|^{2p-2}|D^2v|^2 + \int_\Omega |\nabla v|^{2p} + c_6;
\end{aligned}$$

and hence

$$(1+c_4)\int_\Omega |\nabla v|^{2p} \le \frac{{}_1p(p-1)}{4}\int_\Omega |\nabla v|^{2p-4}|\nabla|\nabla v|^2|^2 + \frac{{}_1p}{4}\int_\Omega |\nabla v|^{2p-2}|D^2v|^2 + c_6; \tag{71}$$

Then substituting (71) into (70), and using the fact that

$$\frac{4}{p^2}|\nabla|\nabla v|^p|^2 = |\nabla v|^{2p-4}|\nabla|\nabla v|^2|^2;$$

we finally get (63). □

**Lemma 2.16** (Coupled energy estimate). *Assume the conditions in Lemma 2.1 hold, then for all $p \ge 2$, there exist a constant $M_{15} > 0$ independent of $t$, such that*

$$\int_\Omega w^p + \int_\Omega |\nabla v|^{2p} \le M_{15} \quad \text{for all } t \in (0; T_{max}). \tag{72}$$

*Proof.* We add the results of Lemma 2.14 and Lemma 2.15 together to obtain

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega w^p + |\nabla v|^{2p} + p_2\int_\Omega w^p + \int_\Omega |\nabla v|^{2p} + \frac{{}_3(p-1)}{p}\int_\Omega |\nabla w^{\frac{p}{2}}|^2 \\
&\quad + \frac{2\ {}_1(p-1)}{p}\int_\Omega |\nabla|\nabla v|^p|^2 \\
&\le \frac{{}_4^2p(p-1)}{2\ {}_3}\int_\Omega w^p|\nabla v|^2 + c_1\int_\Omega w^2|\nabla v|^{2p-2} + c_1\int_\Omega |D^2u|^{2p} + c_2;
\end{aligned} \tag{73}$$

Using the Hölder inequality, one has

$$\frac{{}_4^2p(p-1)}{2\ {}_3}\int_\Omega w^p|\nabla v|^2 \le \frac{{}_4^2p(p-1)}{2\ {}_3}\left(\int_\Omega w^{p+2}\right)^{\frac{p}{p+2}}\left(\int_\Omega |\nabla v|^{p+2}\right)^{\frac{2}{p+2}}; \tag{74}$$

and

$$c_1\int_\Omega w^2|\nabla v|^{2p-2} \le c_1\left(\int_\Omega w^{p+2}\right)^{\frac{2}{p+2}}\left(\int_\Omega |\nabla v|^{\frac{(2p-2)(p+2)}{p}}\right)^{\frac{p}{p+2}}; \tag{75}$$

Using the Gagliardo-Nirenberg inequality as in (21) and the fact that $\|w\|_{L^2} \le M_{14}$ as well as $\|\nabla v\|_{L^2} \le M_{12}$, we can obtain

$$\begin{aligned}\int_\Omega w^{p+2} = \|w^{\frac{p}{2}}\|^{\frac{2(p+2)}{p}}_{L^{\frac{2(p+2)}{p}}} &\le c_3 \quad \|\nabla w^{\frac{p}{2}}\|^2_{L^2}\|w^{\frac{p}{2}}\|^{\frac{4}{p}}_{L^{\frac{4}{p}}} + \|w^{\frac{p}{2}}\|^{\frac{2(p+2)}{p}}_{L^{\frac{4}{p}}} \\ &\le c_3 M_{14}^2\|\nabla w^{\frac{p}{2}}\|^2_{L^2} + c_3 M_{14}^{p+2};\end{aligned} \tag{76}$$

and

$$\begin{aligned}\int_\Omega |\nabla v|^{p+2} = \||\nabla v|^p\|^{\frac{(p+2)}{p}}_{L^{\frac{(p+2)}{p}}} &\le c_4 \quad \|\nabla|\nabla v|^p\|_{L^2}\||\nabla v|^p\|^{\frac{2}{p}}_{L^{\frac{2}{p}}} + \||\nabla v|^p\|^{\frac{p+2}{p}}_{L^{\frac{2}{p}}} \\ &\le c_4 M_{12}^2\|\nabla|\nabla v|^p\|_{L^2} + c_4 M_{12}^{p+2};\end{aligned} \tag{77}$$

as well as

$$\begin{aligned}\int_\Omega |\nabla v|^{\frac{2(p-1)(p+2)}{p}} &= \||\nabla v|^p\|^{\frac{2(p-1)(p+2)}{p^2}}_{L^{\frac{2(p-1)(p+2)}{p^2}}} \\ &\le c_5 \quad \|\nabla|\nabla v|^p\|^{\frac{2(p^2-2)}{p^2}}_{L^2}\||\nabla v|^p\|^{\frac{2}{p}}_{L^{\frac{2}{p}}} + \||\nabla v|^p\|^{\frac{2(p-1)(p+2)}{p^2}}_{L^{\frac{2}{p}}} \\ &\le c_5 M_{12}^2\|\nabla|\nabla v|^p\|^{\frac{2(p^2-2)}{p^2}}_{L^2} + c_5 M_{12}^{\frac{2(p-1)(p+2)}{p}};\end{aligned} \tag{78}$$

Using (76), (77) and (78), we can derive from (74) and (75) that

$$\begin{aligned}&\frac{2}{4}p(p-1)\int_\Omega w^p|\nabla v|^2 + c_1\int_\Omega w^2|\nabla v|^{2p-2} \\ &\le c_6 \quad \|\nabla w^{\frac{p}{2}}\|^{\frac{2p}{p+2}}_{L^2} + 1 \quad \|\nabla|\nabla v|^p\|^{\frac{2}{p+2}}_{L^2} + 1 \\ &\quad + c_6 \quad \|\nabla w^{\frac{p}{2}}\|^{\frac{4}{p+2}}_{L^2} + 1 \quad \|\nabla|\nabla v|^p\|^{\frac{2(p^2-2)}{p(p+2)}}_{L^2} + 1 \\ &\le \frac{{}_3(p-1)}{p}\|\nabla w^{\frac{p}{2}}\|^2_{L^2} + \frac{2\ {}_1(p-1)}{p}\|\nabla|\nabla v|^p\|^2_{L^2} + c_6;\end{aligned} \tag{79}$$

where we have used the Young's inequality with the fact that $\frac{p}{p+2} + \frac{1}{p+2} < 1$ and $\frac{2}{p+2} + \frac{p^2-2}{p(p+2)} < 1$. Then we substitute (79) into (73) to derive that

$$\frac{d}{dt}\int_\Omega w^p + \int_\Omega |\nabla v|^{2p} + \int_\Omega w^p + \int_\Omega |\nabla v|^{2p} \le c_7\int_\Omega |D^2 u|^{2p} + c_8; \tag{80}$$

Applying Lemma 2.3 to (80) and using Lemma 2.10, we obtain (72) directly. Hence the proof of Lemma 2.16 is completed. □

**Lemma 2.17.** *Suppose the hypotheses in Lemma 2.1 hold, then the solution of (2) satisfies*

$$\|w(\cdot;t)\|_{L^\infty} \le M_{16} \quad \text{for} \quad t \in (0;T_{max}); \tag{81}$$

*where $M_{16} > 0$ is a constant independent of $t$.*

*Proof.* From Lemma 2.16, we can find a constant $c_1 > 0$ such that

$$\|w(\cdot;t)\|_{L^5} + \|w(\cdot;t)\|_{L^{10}} + \|\nabla v(\cdot;t)\|_{L^{10}} \le c_1;$$

and hence it follows that

$$\| \ {}_2(v)w\nabla v\|_{L^5} \le \ {}_4\|w\|_{L^{10}}\|\nabla v\|_{L^{10}} \le c_1^2\ {}_4 \tag{82}$$

and

$$\|vw - {}_2w - {}_2w^2\|_{L^5} \le (M_9 + {}_2)c_1 + {}_2c_1^2. \tag{83}$$

Rewriting the third equation of $w$ as follows:

$$w_t = \nabla \cdot ( {}_2(v)\nabla w) - \nabla \cdot ( {}_2(v)w\nabla v) + vw - {}_2w - {}_2w^2. \tag{84}$$

Then applying the similar Moser iteration as in [26, Lemma A.1] to (84), and using the fact (82) and (83), we have (81). Then we complete the proof of Lemma 2.17. □

*Proof of Theorem 1.1.* Applying Lemma 2.2, Lemma 2.9 and Lemma 2.16, for $p >$ 2, we can find a positive constant $C$ such that

$$\|u(\cdot,t)\|_{W^{1,p}} + \|v(\cdot,t)\|_{W^{1,p}} + \|w(\cdot,t)\|_{L^\infty} \le C,$$

which combined with Lemma 2.1 proves Theorem 1.1. □

3. **Global stabilization: Proof of Theorem 1.3.** In this section, we shall study the global stability of constant steady states $(u_s, v_s, w_s)$ satisfying (5). Before that, we first introduce the following lemmas.

**Lemma 3.1** ([5]). *If $h : [1,\infty) \to \mathbb{R}$ is a uniformly continuous function and $\lim_{t\to\infty} \int_1^t h(s)ds$ exist, then $\lim_{t\to\infty} h(t) = 0$.*

**Lemma 3.2.** *Let $(u,v,w)$ be the unique global bounded classical solution of (2) as stated in Theorem 1.1. Then for any fixed $0 <$ $<1$, there exist a constant $C > 0$ such that*

$$\begin{aligned}\|u(\cdot,t)\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar\Omega\times[1,\infty))} + \|v(\cdot,t)\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar\Omega\times[1,\infty))} \\ + \|w(\cdot,t)\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar\Omega\times[1,\infty))} \le C.\end{aligned} \tag{85}$$

*Proof.* Noting the results obtained in Theorem 1.1 and using the similar arguments as in [15, Lemma 4.2], we can obtain (85) directly. We omit the details of proof for convenience. □

3.1. **Global stabilization of prey-only steady state.** One can check that if ${}_1 \ge 1$, there exist only two possible steady states $(0,0,0)$ and $(1,0,0)$. However, after some calculations, we can derive that $(0,0,0)$ is linearly unstable. Hence we only need to focus on studying the global stability of constant steady state $(1,0,0)$. In fact, we can derive the following results.

**Lemma 3.3.** *If ${}_1 \ge 1$, then $(u,v,w)$ obtained in Theorem 1.1 converges to $(1,0,0)$ in $L^\infty$-norm. Then convergence rate is exponential if ${}_1 > 1$ and is algebraical if ${}_1 = 1$.*

*Proof.* We construct the following energy functional:

$$\mathcal{E}_1(t) = \int_\Omega (u - 1 - \ln u) + b_1 \int_\Omega v + b_1 b_2 \int_\Omega w.$$

We can check that $\mathcal{E}_1(t) \ge 0$ for any $t > 0$. In fact, letting $(z) = z - z^* \ln z$ and using Taylor's expansion, we have

$$\begin{aligned} z - z^* - z^* \ln\frac{z}{z^*} &= (z) - (z^*) = '(z^*)(z - z^*) + \frac{''( )}{2}(z - z^*)^2 \\ &= \frac{z^*}{2\ {}^2}(z - z^*)^2, \end{aligned} \tag{86}$$

where $\quad = z + (1 - )z^*$ is between $z$ and $z^*$ and $0 < \quad < 1$. Letting $z = u$ and $z^* = 1$ in (86), we can derive that

$$\int_\Omega (u - 1 - \ln u) = \frac{1}{2} \int_\Omega \frac{(u-1)^2}{\tilde{u}^2} \geq 0 \tag{87}$$

where $\tilde{u}$ is between $u$ and 1. Then $\mathcal{E}_1(t) \geq 0$ due to (87) and the fact $v; w > 0$ for all $t > 0$. From (2) and using the definition of $\mathcal{E}_1(t)$, we obtain

$$\begin{aligned} \frac{d}{dt}\mathcal{E}_1(t) = &- d_1 \int_\Omega \frac{|\nabla u|^2}{u^2} - \int_\Omega (u-1)^2 - b_1(\ _1 - 1) \int_\Omega v \\ &- \ _1 b_1 \int_\Omega v^2 - b_1 b_2 \ _2 \int_\Omega w - \ _2 b_1 b_2 \int_\Omega w^2; \end{aligned} \tag{88}$$

which gives a constant $\quad > 0$ such that

$$\frac{d}{dt}\mathcal{E}_1(t) + \quad \mathcal{F}_1(t) \leq 0; \tag{89}$$

where

$$\mathcal{F}_1(t) = \begin{cases} \int_\Omega (u-1)^2 + \int_\Omega v^2 + \int_\Omega w^2; & \text{if } \ _1 = 1; \\ \int_\Omega (u-1)^2 + \int_\Omega v + \int_\Omega w; & \text{if } \ _1 > 1. \end{cases}$$

Integrating (89) over $(1; \infty)$ and using the fact $\mathcal{E}_1(t) \geq 0$ we derive that

$$\int_1^\infty \mathcal{F}_1 \leq \mathcal{E}_1(1)$$

where $_1 = 2$ if $_1 = 1$ and $_1 = 1$ if $_1 > 1$. Then substituting (96) into (89) gives

$$\frac{d}{dt}\mathcal{E}_1(t) + \ c_1\mathcal{E}_1^{\sigma_1}(t) \le 0; \quad t \ge t_1;$$

which gives for all $t \ge t_1$ that

$$\mathcal{E}_1(t) \le \begin{cases} \frac{\mathcal{E}(t_1)}{\alpha c_1 \mathcal{E}(t_1)(t-t_1)+1}; & \text{if } _1 = 1; \\ \mathcal{E}_1(t_1)e^{-\alpha c_1(t-t_1)}; & \text{if } _1 > 1: \end{cases} \tag{97}$$

Then noting the facts $\mathcal{F}_1(t) \le c_2\mathcal{E}_1(t)$ in (96) and (97) as well as the definition of $\mathcal{F}_1(t)$, and then using (91) and the Gagliardo-Nirenberg inequality, we obtain the convergence results in Lemma 3.3. □

3.2. **Global stabilization of semi-coexistence steady state.** If $_1 < 1$ and $_2(\ _1 + b_1) + \ _1 \ge 1$, the system (2) has three constant steady state $(0;0;0)$, $(1;0;0)$ and

$$(u_1^*; v_1^*; 0) = \ \frac{\ _1 + b_1\ _1}{\ _1 + b_1}; \ \frac{1 - \ _1}{\ _1 + b_1}; \ 0 \ :$$

In this subsection, we shall show that the semi-coexistence steady state $(u_1^*; v_1^*; 0)$ under certain conditions. More precisely, it holds that

**Lemma 3.4.** *Assume that* $_1 < 1$ *and* $_2(\ _1 + b_1) + \ _1 \ge 1$. *If* (6) *holds, then it holds that*

$$\lim_{t\to\infty}(\|u - u_1^*\|_{L^\infty} + \|v - v_1^*\|_{L^\infty} + \|w\|_{L^\infty}) = 0: \tag{98}$$

*Moreover, the convergence rate is exponential if* $_2(\ _1 + b_1) + \ _1 > 1$, *and is algebraical if* $_2(\ _1 + b_1) + \ _1 = 1$.

*Proof.* We introduce the following energy functional:

$$\mathcal{E}_2(t) = \mathcal{I}_{21}(t) + b_1\mathcal{I}_{22}(t) + b_1b_2\mathcal{I}_{23}(t); \tag{99}$$

where

$$\begin{aligned} \mathcal{I}_{21}(t) &= \int_\Omega u - u_1^* - u_1^* \ln\frac{u_1^*}{u} \ ; \\ \mathcal{I}_{22}(t) &= \int_\Omega v - v_1^* - v_1^* \ln\frac{v_1^*}{v} \ ; \\ \mathcal{I}_{23}(t) &= \int_\Omega w: \end{aligned} \tag{100}$$

Using (86), we can easily check that $\mathcal{E}_2(t) \ge 0$ for all $u; v; w > 0$ and " = " holds if and only if $(u; v; w) = (u_1^*; v_1^*; 0)$. Then using (2) and (99), we derive

$$\begin{aligned} \frac{d}{dt}\mathcal{E}_2(t) &= -\int_\Omega d_1u_1^*\frac{|\nabla u|^2}{u^2} - b_1v_1^*\int_\Omega \ _1(u)\frac{|\nabla v|^2}{v^2} + b_1v_1^*\int_\Omega u \ _1(u)\frac{\nabla u\cdot\nabla v}{uv} \\ &\quad - \int_\Omega (u-u_1^*)^2 - \int_\Omega \ _1b_1(v-v_1^*)^2 - \int_\Omega \ _2b_1b_2w^2 + b_1b_2(v_1^* - \ _2)\int_\Omega w \\ &= -\int_\Omega XAX^T - \int_\Omega (u-u_1^*)^2 - \int_\Omega \ _1b_1(v-v_1^*)^2 - \int_\Omega \ _2b_1b_2w^2 \\ &\quad - \frac{b_1b_2[\ _2(\ _1+b_1) + \ _1 - 1]}{\ _1+b_1}\int_\Omega w; \end{aligned} \tag{101}$$

where $X = (\frac{\nabla u}{u}; \frac{\nabla v}{v})$ and

$$A = \begin{pmatrix} d_1 u_1^* & -\frac{b_1 v_1^* u \chi_1(u)}{2} \\ -\frac{b_1 v_1^* u \chi_1(u)}{2} & b_1 v_1^* \ _1(u) \end{pmatrix} :$$

If (6) holds, it is easy to check that $A$ is nonnegative definite and hence $-\int_\Omega XAX^T \le 0$. Then from (101), we can find a constant $\ > 0$ such that

$$\frac{d}{dt}\mathcal{E}_2(t) + \ \mathcal{F}_2(t) \le 0; \tag{102}$$

where

$$\mathcal{F}_2(t) = \begin{cases} \int_\Omega (u-u_1^*)^2 + \int_\Omega (v-v_1^*)^2 + \int_\Omega w^2; & \text{if } \ _2(\ _1 + b_1) + \ _1 = 1; \\ \int_\Omega (u-u_1^*)^2 + \ \int_\Omega (v-v_1^*)^2 + \ \int_\Omega w; & \text{if } \ _2(\ _1 + b_1) + \ _1 > 1: \end{cases}$$

Using similar arguments as in Lemma 3.3, we can obtain (98) and

$$c_1 \mathcal{E}_2^{\sigma_2}(t) \le \mathcal{F}_2(t) \le c_2 \mathcal{E}_2(t); \text{ for some } t \ge t_2 > 0 \tag{103}$$

with

$$_2 = \begin{cases} 2; & \text{if } \ _2(\ _1 + b_1) + \ _1 = 1; \\ 1; & \text{if } \ _2(\ _1 + b_1) + \ _1 > 1: \end{cases}$$

Then the combination of (102) and (103) gives the convergence rate in $L^2$-norm, which together with Lemma 3.2 and the Gagliardo-Nirenberg inequality give the convergence rate in $L^\infty$-norm. □

3.3. **Global stabilization of coexistence steady state.** If $\ _1 < 1$ and $\ _2(\ _1 + b_1) + \ _1 < 1$, the coexistence steady state $(u_2^*; v_2^*; w_2^*)$ exists, in this subsection, we shall study the global stabilization of the coexistence steady state under certain conditions.

**Lemma 3.5.** *Let $(u; v; w)$ be the solution of (2) obtained Theorem 1.1. Assume that $\ _1 < 1$ and $\ _2(\ _1 + b_1) + \ _1 < 1$. Then if (7) holds, we have*

$$\|u - u_2^*\|_{L^\infty} + \|v - v_2^*\|_{L^\infty} + \|w - w_2^*\|_{L^\infty} \le Ce^{-\lambda t}; \tag{104}$$

*where $C$ and $\ $ are two positive constants independent of $t$.*

*Proof.* We introduce the following energy functional:

$$\mathcal{E}_3(t) = \mathcal{I}_{31}(t) + b_1 \mathcal{I}_{32}(t) + b_1 b_2 \mathcal{I}_{33}(t);$$

where

$$\mathcal{I}_{31}(t) = \int_\Omega u - u_2^* - u_2^* \ln \frac{u_2^*}{u} \ ; \qquad \mathcal{I}_{32}(t) = \int_\Omega v - v_2^* - v_2^* \ln \frac{v_2^*}{v} \ ;$$
$$\mathcal{I}_{33}(t) = \int_\Omega w - w_2^* - w_2^* \ln \frac{w_2^*}{w} \ :$$

Using (86) again, we can derive that $\mathcal{E}_3(t) \ge 0$ for all $u; v; w > 0$ and " = " holds if and only if $(u; v; w) = (u_2^*; v_2^*; w_2^*)$. Moreover, using (2) and the definition of $\mathcal{E}_3(t)$, we can derive that

$$\begin{aligned} \frac{d}{dt}\mathcal{E}_3(t) = &- \int_\Omega Y B Y^T - \int_\Omega (u-u_2^*)^2 - \ _1 b_1 \int_\Omega (v-v_2^*)^2 \\ &- \ _2 b_1 b_2 \int_\Omega (w-w_2^*)^2; \end{aligned} \tag{105}$$

where $Y = \frac{\nabla u}{u}; \frac{\nabla v}{v}; \frac{\nabla w}{w}$ and

$$B = \begin{pmatrix} d_1 u_2^* & -\frac{b_1 v_2^* u \chi_1(u)}{2} & 0 \\ -\frac{b_1 v_2^* u \chi_1(u)}{2} & b_1 v_2^* \ _1(u) & -\frac{b_1 b_2 w_2^* v \chi_2(v)}{2} \\ 0 & -\frac{b_1 b_2 w_2^* v \chi_2(v)}{2} & b_1 b_2 w_2^* \ _2(v) \end{pmatrix}.$$

If (7) holds, it is easy to check that $B$ is nonnegative definite, and so we have

$$\frac{d}{dt}\mathcal{E}_3(t) + \quad \mathcal{F}_3(t) \le 0; \tag{106}$$

where $= \min\{1; \ _1 b_1; \ _2 b_1 b_2\}$ and

$$\mathcal{F}_3(t) = \int_\Omega (u - u_2^*)^2 + \int_\Omega (v - v_2^*)^2 + \int_\Omega (w - w_2^*)^2:$$

Then using the similar arguments as in Lemma 3.3, we can derive that

$$\lim_{t\to\infty} (\|u - u_2^*\|_{L^\infty} + \|v - v_2^*\|_{L^\infty} + \|w - w_2^*\|_{L^\infty}) = 0:$$

and moreover there exists some $t_3 > 0$ such that

$$c_1 \mathcal{E}_3(t) \le \mathcal{F}_3(t) \le c_2 \mathcal{E}_3(t) \quad \text{for } t \ge t_3; \tag{107}$$

which substituted into (106) gives

$$\frac{d}{dt}\mathcal{E}_3(t) + \quad c_1 \mathcal{E}_3(t) \le 0 \quad \text{for } t \ge t_3: \tag{108}$$

Applying the Grönwall inequality to (108), we get

$$\mathcal{E}_3(t) \le \mathcal{E}_3(t_3) e^{-\alpha c_1 (t - t_3)}; \quad t \ge t_3;$$

which combined with (107) implies

$$\int_\Omega (u - u_2^*)^2 + \int_\Omega (v - v_2^*)^2 + \int_\Omega (w - w_2^*)^2 = \mathcal{F}_3(t) \le c_3 e^{-c_4 t}; \quad t \ge t_3: \tag{109}$$

Then using Lemma 3.2 and the Gagliardo-Nirenberg inequality, from (109), we can derive (104). Then the proof of Lemma 3.5 is completed. □

*Proof of Theorem 1.3.* Combining Lemma 3.3, Lemma 3.4 with Lemma 3.5, we finally prove Theorem 1.3. □

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