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Abstract. - : $u_t = \Delta(\gamma(v)u) + \mu u(1-u); v_t = \Delta v + u - v$, $\Omega \subset \mathbb{R}^2$ N , $\gamma(v) \in C^3(0,\infty)), \gamma(v) > 0, \gamma'(v) < 0$ $v \geq 0$, $_{v\to\infty} \gamma(v) = 0$, $_{v\to\infty} \frac{\gamma'(v)}{\gamma(v)}$. : - - - . ffi - ff $_{v\to\infty} \gamma(v) = 0$. I , $\gamma(v)$, L^∞- v - - - . F , $\mu > \frac{K_0}{16}$ $K_0 = _{0\leq v\leq\infty} \frac{|\gamma'(v)|^2}{\gamma(v)}$, $(1,1)$, , . F $\mu > 0$, .

Key words. - , , ,

AMS subject classifications. 35A01, 35B40, 35B44, 35K57, 35Q92, 92C17

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1. I

. I - ff - 11 . I , K 21, C 5 . I , ff . H ” *E. coli* 14 (AHL) AHL , AHL ,

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14 , fi 6 6

$$
\begin{cases} u_t = \Delta(\gamma(v)u) + \mu u(1 - u/K), \\ v_t = d\Delta v + \alpha u - \beta v, \end{cases} \tag{1.1}
$$

$u(x,t)$ $v(x,t)$ AHL, fi K μ, ff $\gamma(v)$ ff AHL ( ) $\gamma'(v) < 0$.

I L fi , (1.1)

$$
\begin{cases} u_t = \nabla \cdot (\gamma(v)\nabla u + u\gamma'(v)\nabla v) + \mu u(1 - u/K), \\ v_t = d\Delta v + \alpha u - \beta v, \end{cases} \tag{1.2}
$$

K 10 ff / , 10 38 . ff , , (4). F ,

$$
\begin{cases} u_t = \nabla \cdot (\nabla u - \chi u \nabla v) + \mu u(1 - u/K), \\ v_t = d\Delta v + \alpha u - \beta v, \end{cases} \tag{1.3}
$$

$\Omega \subset \mathbb{R}^n (n \geq 2)$ N , $\mu_* \geq 0$ ($n = 2$) $\mu > \mu_*$, $n \geq 2$ (37, 16, 31, 30), ff $\mu = 0$ fi (7, 33, 4). H , (. .,) . I , $\mu u(1-u/K)$ (1.3) $\lambda u - \mu u^\zeta$ $\lambda \geq 0$ $1 < \zeta < \frac{3}{2} + \frac{1}{2n-2}$, 32 fi n- $(n \geq 5)$ fi . B , (1.3) fi , (. ., μ_*); 19, 34, 3 . M , 30, 13 , 18, 35 , 22 .

(1.2) ff v $\gamma(v)$. H

(. ., $\gamma(v)$) K ff . F (1.1), $\gamma(v)$, (1.1) 6 $\gamma(v)$ 24 . (1.1) $\gamma(v)$ N :

$$
\begin{cases}
u_t = \Delta(\gamma(v)u) + \mu u(1-u), & x \in \Omega, \quad t > 0,\\
v_t = \Delta v + u - v, & x \in \Omega, \quad t > 0,\\
\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\\
u(x,0) = u_0(x), v(x,0) = v_0(x), & x \in \Omega,
\end{cases}
\tag{1.4}
$$

$\Omega \subset \mathbb{R}^2$ ν $\partial\Omega$. I (1.4), $d = \alpha = \beta = K = 1$ fi . A , $_{v \in 0,\infty}$ $\gamma(v) = 0$ $\gamma'(v) < 0$, . (1.4). fi $\mu = 0$ (,). F , 39 , $\gamma(v)$:

$$
\gamma(v) = \frac{c_0}{v^k}, \ c_0 > 0, k > 0. \tag{1.5}
$$

(1.4) v , $\gamma(v)$ fi . H ff . I 39 , K $\gamma(v)$ ffi $c_0 > 0$, (1.4) $\mu = 0$. F $\gamma(v)$ (. ., $\gamma_1 \leq \gamma(v) \leq \gamma_2$ $v \geq 0$, γ_1, γ_2), 29 (1.4) $\mu = 0$. , (1.4) $\mu > 0$ 6, 24 . (1.4) $\mu > 0$ 6 . 39 29 , ff (1.4). F fi , $\gamma(v)$:

(H) $\gamma(v) \in C^3(0,\infty)), \gamma(v) > 0$, $\gamma'(v) < 0$ $0,\infty)$, $_{v\to\infty} \gamma(v) = 0$, $_{v\to\infty} \frac{\gamma'(v)}{\gamma(v)}$.

$\gamma(v)$ (H),

$$\gamma(v) = be^{-av}, \quad \gamma(v) = \frac{a}{(1+e^{bv})^m}, \quad \gamma(v) = \frac{a}{(1+bv)^m}, \quad \gamma(v) = 1 - \frac{v}{\sqrt{1+v^2}},$$

a, b, m . I H 14 fi (H). $\gamma(v) \to 0$ $v \to \infty$, L^∞- v (1.4). $\gamma(v)$ (. ., - ff) ff . L^2- u (3.1) - - v (1.4), . M u (3.2). F fi L . O .

1.1. *Let Ω be a bounded domain in $\mathbb{R}^2$ with smooth boundary and the hypothesis* (H) *holds. Suppose that $(u_0, v_0) \in W^{1,\infty}(\Omega)^2$ with $u_0, v_0 \geq 0 (\not\equiv 0)$. Then the problem* (1.4) *has a unique nonnegative global classical solution $(u,v) \in C^0(0,\infty) \times \bar{\Omega}) \cap C^{2,1}((0,\infty) \times \bar{\Omega}) \cap L^\infty_{loc}(0,\infty); W^{1,\infty}(\Omega))^2$ satisfying*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} \leq C \quad t > 0, \tag{1.6}$$

where $C > 0$ is a constant independent of t. Furthermore, if $\mu > \frac{K_0}{16}$ with $K_0 = {}_{0 \leq v \leq \infty} \frac{|\gamma'(v)|^2}{\gamma(v)}$, the constant steady state $(1,1)$ is globally asymptotically stable in the sense that

$$\lim_{t\to\infty} \|u(\cdot,t) - 1\|_{L^\infty(\Omega)} + \|v(\cdot,t) - 1\|_{L^\infty(\Omega)}\big) = 0.$$

(H) $\gamma(v)$ (1.5). I , 39 29 , $\mu > 0$. F $\mu > 0$, (1.4) , ff μ.

2. : L . I , $\int_\Omega f dx$ $\int_\Omega f$ $\|f\|_{L^2(\Omega)}$ $\|f\|_{L^2}$. M , $c_i (i = 1, 2, 3, \ldots)$ fi (1.4) fi 26, L 2.1 . I , (2, 36, 29, 39) .

L 2.1 ( ). *Let Ω be a bounded domain in $\mathbb{R}^2$ with smooth boundary and the hypothesis (H) holds. Assume $(u_0, v_0) \in W^{1,\infty}(\Omega)^2$ with $u_0, v_0 \geq 0 (\not\equiv 0)$. Then there exists $T_{max} \in (0, \infty$ such that the problem* (1.4) *has a unique classical solution*

$$(u,v) \in C^0(0, T_{max}) \times \bar{\Omega}) \cap C^{2,1}((0,T_{max}) \times \bar{\Omega}) \cap L^\infty_{loc}(0, T_{max}); W^{1,\infty}(\Omega))^2$$

satisfying $u, v \geq 0$ *for all* $t > 0$. *Moreover, we have*

$$\text{(2.1)} \qquad \textit{either} \quad T_{max} = \infty \ \textit{or} \ \lim_{t \nearrow T_{max}} (\|u(\cdot,t)\|_{L^\infty} + \|v(\cdot,t)\|_{W^{1,\infty}}) = \infty.$$

Proof.

( ) *Existence.* D fi $R := \|u_0\|_{L^\infty} + \|v_0\|_{W^{1,\infty}} + 1$. R $T \in (0,1)$ fi , B $X := C^0(\bar{\Omega} \times 0, T)$:

$$S_T = \{u \in X : \|u(\cdot,t)\|_{L^\infty} \leq R \quad \Omega \times (0,T)\}$$

fi $\Phi : S_T \to S_T$ $u \in S_T$, $\Phi(u) = u$, u

$$\text{(2.2)} \qquad \begin{cases} u_t = \\ \\ \qquad s \end{cases}$$

$$\|u(\cdot,t)\|_{L^\infty} \le \|u_0\|_{L^\infty} + c_3 t^{\frac{\theta_2}{2}} \quad t\in(0,T),$$

$\|v(\cdot,t)\|_{L^\infty(\Omega\times(0,T))} \le R$.

$$\|u(\cdot,t)\|_{L^\infty(\Omega\times(0,T))} \le R$$

fi

$$T = t_0 < \left\{ \left(\frac{1}{c_1}\right)^{\frac{2}{\theta_1}}, \left(\frac{1}{c_3}\right)^{\frac{2}{\theta_2}} \right\}.$$

H $u \in S_T$ Φ S_T ffi T.

N , Φ $\Phi(S_T)$ X. L $u_i \in X$ $u_i = \Phi(u_i)$ $i = 1, 2,$ v_i

$$\text{(2.8)} \quad \begin{cases} v_{it} - \Delta v_i + v_i = u_i, & x\in\Omega, \quad t\in(0,T),\\ \frac{\partial v_i}{\partial \nu} = 0, & x\in\partial\Omega, \quad t\in(0,T),\\ v_i(x,0) = v_0(x), & x\in\Omega. \end{cases}$$

(2.7) 17 , $u_i, v_i \in C^{2+\theta_3, 1+\frac{\theta_3}{2}}(\bar\Omega \times \eta, T)$ $\eta \in (0,T)$ $\theta_3 \in (0,1)$. L $w = \Phi(u_1) - \Phi(u_2) = u_1 - u_2,$ (2.2)

(2.9)

$$\begin{cases} w_t = \nabla\cdot(\bar\gamma(v_1)\nabla w) + \bar\gamma'(v_1)\nabla v_1\cdot\nabla w + f_1(x,t)w + f_2(x,t), & x\in\Omega, \quad t\in(0,T),\\ \frac{\partial w}{\partial \nu} = 0, & x\in\partial\Omega, \quad t\in(0,T),\\ w(x,0) = 0, & x\in\Omega, \end{cases}$$

$$f_1(x,t) = \bar\gamma''(v_1)|\nabla v_1|^2 + \bar\gamma'(v_1)\Delta v_1 + \mu(1 - u_1)$$

$$\begin{aligned} f_2(x,t) = &\nabla\cdot(\ \gamma(v_1) - \gamma(v_2)\ \nabla u_2) + \nabla\cdot\ (\bar\gamma'(v_1) - \bar\gamma'(v_2))u_2\nabla v_1 \\ &+ \nabla\cdot\ \bar\gamma'(v_2)u_2(\nabla v_1 - \nabla v_2)\ + \mu u_2(u_2 - u_1). \end{aligned}$$

N $f_1(x,t), \bar\gamma'(v_1)\nabla v_1 \in L^\infty(\Omega\times(0,T))$ - L^p- (2.9),

$$\text{(2.10)} \quad \begin{aligned} \|w\|_{W^{2,1,p}(\Omega\times(0,T))} &\le c_4\|f_2\|_{L^p(\Omega\times(0,T))} \\ &\le c\ \ \|v_1 - v_2\|_{W^{2,p}(\Omega\times(0,T))} + \|u_1 - u_2\|_{L^p(\Omega\times(0,T))}\big) \\ &\le c_6\|u_1 - u_2\|_{C^0(\bar\Omega\times\ 0,T\ )}, \end{aligned}$$

(2.8)

$$\|v_1 - v_2\|_{W^{2,p}(\Omega\times(0,T))} \le c\ \|u_1 - u_2\|_{L^p(\Omega\times(0,T))} \le c\ \|u_1 - u_2\|_{C^0(\bar\Omega\times\ 0,T\ )}.$$

D , p, $\theta_4 \in (0,1)$

$$\|w\|_{C^0(\bar\Omega\times\ 0,T\ )} \le c_9\|w\|_{C^{\theta_4,\frac{\theta_4}{2}}(\bar\Omega\times\ 0,T\ )} \le c_{10}\|w\|_{W^{2,1,p}(\Omega\times(0,T))},$$

, (2.10),

$$\|\Phi(u_1) - \Phi(u_2)\|_{C^0(\bar\Omega\times 0,T)} = \|w\|_{C^0(\bar\Omega\times 0,T)} \le c_6 c_{10} \|u_1 - u_2\|_{C^0(\bar\Omega\times 0,T)},$$

Φ . M , , $u \in S_T$, $c_{11} > 0$ $\|\Phi(u)\|_{C^{\theta_5,\frac{\theta_5}{2}}(\bar\Omega\times 0,T)} \le c_{11}$ $\theta \in (0,1)$, $\Phi(S_T)$ X . fi , Φ fi X u. F u u, v (2.3). u, v . H $0 \le v \le R$ $\bar\gamma(v) = \gamma(v)$ (2.4), fi (u,v) (1.4). (2.1) T $\|u_0\|_{L^\infty}$ $\|v_0\|_{W^{1,\infty}}$.

( ) *Uniqueness.* 31, 30, 8 , . F $T > 0$, (u_1, v_1) (u_2, v_2) (1.4) $\Omega \times (0,T)$. fi $T_0 \in (0,T)$ U V fi

$$U := u_1 - u_2 \qquad V := v_1 - v_2$$

, ,

$$U(0,x) = 0 \qquad V(0,x) = 0. \tag{2.11}$$

$(u_i, v_i)(i = 1,2)$ $\gamma(v)$ (H),

$$\|u_1\|_{W^{1,\infty}} + \|u_2\|_{W^{1,\infty}} + \|v_1\|_{W^{1,\infty}} + \|v_2\|_{W^{1,\infty}} \le c_{12} \tag{2.12}$$

$$\gamma(v_1) \ge c_{13} > 0, \quad |\gamma'(v_1)| + |\gamma'(v_2)| \le c_{14}, \quad |\gamma(v_1) - \gamma(v_2)| + |\gamma'(v_1) - \gamma'(v_2)| \le c_1\ |V| \tag{2.13}$$

$t \in (0, T_0)$. M , (1.4),

$$\begin{aligned} U_t &= \nabla\cdot(\gamma(v_1)\nabla U) + \nabla\cdot\ (\gamma(v_1) - \gamma(v_2))\nabla u_2 \\ &\quad + \nabla\cdot\ \gamma'(v_1)u_1\nabla v_1 - \gamma'(v_2)u_2\nabla v_2\ + \mu(1 - u_1 - u_2)U \end{aligned} \tag{2.14}$$

$$V_t = \Delta V + U - V. \tag{2.15}$$

M (2.14) U, Ω ,

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega U^2 + \int_\Omega \gamma(v_1)|\nabla U|^2 &= -\int_\Omega (\gamma(v_1) - \gamma(v_2))\nabla u_2 \cdot \nabla U + \mu\int_\Omega (1 - u_1 - u_2)U^2 \\ &\quad - \int_\Omega (\gamma'(v_1)u_1\nabla v_1 - \gamma'(v_2)u_2\nabla v_2)\cdot\nabla U \\ &= I_1 + I_2 + I_3, \end{aligned}$$

, $\gamma(v_1) \ge c_{13} > 0$ (2.13),

$$\frac{1}{2}\frac{d}{dt}\int_\Omega U^2 + c_{13}\int_\Omega |\nabla U|^2 \le I_1 + I_2 + I_3 \quad t \in (0,T). \tag{2.16}$$

N , H , , , (2.12), (2.13), $t \in (0,T)$

$$I_1 = -\int_\Omega (\gamma(v_1) - \gamma(v_2))\nabla u_2 \cdot \nabla U$$
$$\le c_{12}\int_\Omega |\gamma(v_1) - \gamma(v_2)||\nabla U|$$
$$\le c_{12}c_1\int_\Omega |V||\nabla U| \tag{2.17}$$
$$\le \frac{c_{13}}{8}\|\nabla U\|_{L^2}^2 + \frac{2c_{12}^2c_1^2}{c_{13}}\|V\|_{L^2}^2$$

(2.18)

$$I_2 = \mu\int_\Omega (1 - u_1 - u_2)U^2 \le \mu(1 + \|u_1\|_{L^\infty} + \|u_2\|_{L^\infty})\|U\|_{L^2}^2 \le \mu(1 + c_{12})\|U\|_{L^2}^2$$

(2.19)

$$I_3 = -\int_\Omega (\gamma'(v_1)u_1\nabla v_1 - \gamma'(v_2)u_2\nabla v_2)\cdot\nabla U$$
$$= -\int_\Omega \gamma'(v_1)u_1\nabla V\cdot\nabla U - \int_\Omega (\gamma'(v_1) - \gamma'(v_2))u_1\nabla v_2\cdot\nabla U - \int_\Omega \gamma'(v_2)U\nabla v_2\cdot\nabla U$$
$$\le c_{12}c_{14}\int_\Omega |\nabla V||\nabla U| + c_{12}^2c_1\int_\Omega |V||\nabla U| + c_{12}c_{14}\int_\Omega |U||\nabla U|$$
$$\le \frac{3c_{13}}{8}\|\nabla U\|_{L^2}^2 + \frac{2c_{12}^2c_{14}^2}{c_{13}}(\|U\|_{L^2}^2 + \|\nabla V\|_{L^2}^2) + \frac{2c_{12}^4c_1^2}{c_{13}}\|V\|_{L^2}^2.$$

(2.17) (2.19) (2.16), $t \in (0,T)$

$\frac{d}{dt}$ T / T| /F∞ ∞T{/ T /|| T| /F ∞T{/ / ∈ / T∈/∈ T| /F∞ ∞T

, G (2.11), $U \equiv 0$ $V \equiv 0$ $(0,T)$ $T_0 \in (0,T)$ $\square$

N , (1.4).

L 2.2. *Let (u,v) be the solution of system* (1.4). *Then there exist two positive constants m_* and C such that*

(2.22)

$$\frac{d}{dt}\int_\Omega |\nabla v|^2 + \int_\Omega |\Delta v|^2 + 2\int_\Omega |\nabla v|^2 \le \int_\Omega u^2 \quad t \in (0, T_{max}). \tag{2.29}$$

M (2.29) μ, (2.25), $t \in (0, T_{max})$

$$\frac{d}{dt}\left(\int_\Omega u + \mu\int_\Omega |\nabla v|^2\right) + \int_\Omega u + 2\mu\int_\Omega |\nabla v|^2 + \mu\int_\Omega |\Delta v|^2 \le (\mu+1)m_*. \tag{2.30}$$

L $y(t) := \int_\Omega u + \mu\int_\Omega |\nabla v|^2$. (2.30)

$$y'(t) + y(t) \le (\mu+1)m_* \quad t \in (0, T_{max}),$$

, G , (2.27). F , (2.28) (2.30) $(t, t+\tau)$. ∎

L 2.4 (12). *Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary. Let $T \in (0, \infty$ and suppose that $z \in C^0(\bar\Omega \times 0, T)) \cap C^{2,1}(\bar\Omega \times (0, T))$ is a solution of*

$$\begin{cases} z_t = \Delta z - z + g, \quad x \in \Omega 5 \\ \end{cases}$$

3. B . I (1.4).

$c_1 := \frac{32\mu|\Omega|}{2}$. O ,

$$\gamma^{\frac12}(v)\nabla u = \nabla(\gamma^{\frac12}(v)u) - \frac12 \frac{\gamma'(v)}{\gamma^{\frac12}(v)} u\nabla v \quad t \in (0, T_{max}),$$

, $|X-Y|^2 \ge \frac12 X^2 - Y^2$,

(3.4)
$$\gamma(v)|\nabla u|^2 = |\gamma^{\frac12}(v)\nabla u|^2 = \left|\nabla(\gamma^{\frac12}(v)u) - \frac12\frac{\gamma'(v)}{\gamma^{\frac12}(v)}u\nabla v\right|^2$$
$$\ge \frac12|\nabla(\gamma^{\frac12}(v)u)|^2 - \frac14\frac{|\gamma'(v)|^2}{\gamma(v)}u^2|\nabla v|^2 \quad t \in (0, T_{max}).$$

(3.4) (3.3)

(3.5)
$$\frac{d}{dt}\ _\Omega u^2 + \frac12\ _\Omega |\nabla(\gamma^{\frac12}(v)u)|^2 + \mu\ _\Omega u^3 \le \frac54\ _\Omega \frac{|\gamma'(v)|^2}{\gamma(v)}u^2|\nabla v|^2 + c_1 \quad t \in (0, T_{max}).$$

F (H), fi $K_1 > 0$

$$\frac{|\gamma'(v)|}{\gamma(v)} \le K_1 \quad v \ge 0 \quad t \in (0, T_{max}). \tag{3.6}$$

(3.6) H , , (3.5)

(3.7)
$$\frac{d}{dt}\ _\Omega u^2 + \frac12\ _\Omega |\nabla(\gamma^{\frac12}(v)u)|^2 + \mu\ _\Omega u^3$$
$$\le \frac54\ _\Omega \frac{|\gamma'(v)|^2}{\gamma^2(v)}(\gamma^{\frac12}(v)u)^2|\nabla v|^2 + c_1$$
$$\le \frac{5K_1^2}{4}\ _\Omega |\gamma^{\frac12}(v)u|^2|\nabla v|^2 + c_1$$
$$\le \frac{5K_1^2}{4}\left(\ _\Omega |\gamma^{\frac12}(v)u|^4\right)^{\frac12}\left(\ _\Omega |\nabla v|^4\right)^{\frac12} + c_1 \quad t \in (0, T_{max}).$$

O , G N $\gamma(v) \le \gamma(0) = c_2$ $\gamma(v) \in C^3(\ 0,\infty))$

(3.8)
$$\left(\ _\Omega |\gamma^{\frac12}(v)u|^4\right)^{\frac12} = \|\gamma^{\frac12}(v)u\|_{L^4}^2$$
$$\le c_3\left(\|\nabla(\gamma^{\frac12}(v)u)\|_{L^2}\|\gamma^{\frac12}(v)u\|_{L^2} + \|\gamma^{\frac12}(v)u\|_{L^2}^2\right)$$
$$\le c_4(\|\nabla(\gamma^{\frac12}(v)u)\|_{L^2}\|u\|_{L^2} + \|u\|_{L^2}^2) \quad t \in (0, T_{max}),$$

$c_4 := (c_2^{\frac12} + c_2)c_3$. L 2.5 $\|\nabla v\|_{L^2} \le c$ ((2.27)),

(3.9)
$$\left(\ _\Omega |\nabla v|^4\right)^{\frac12} = \|\nabla v\|_{L^4}^2 \le c_6\ \ \|\Delta v\|_{L^2}\|\nabla v\|_{L^2} + \|\nabla v\|_{L^2}^2)$$
$$\le c\ (\|\Delta v\|_{L^2} + 1) \quad t \in (0, T_{max})$$

c := $(c + c^2)c_6$. C (3.8) (3.9) ' ,

$$\frac{5K_1^2}{4}\left({}_{\Omega}|\gamma^{\frac{1}{2}}(v)u|^4\right)^{\frac{1}{2}}\left({}_{\Omega}|\nabla v|^4\right)^{\frac{1}{2}}$$

$$\begin{aligned}
&\le \frac{5K_1^2c_4c}{4}\left(\|\nabla(\gamma^{\frac{1}{2}}(v)u)\|_{L^2}\|u\|_{L^2}+\|u\|_{L^2}^2\right)(\|\Delta v\|_{L^2}+1)\\
&= \frac{5K_1^2c_4c}{4}\|\nabla(\gamma^{\frac{1}{2}}(v)u)\|_{L^2}\|u\|_{L^2}\|\Delta v\|_{L^2}+\frac{5K_1^2c_4c}{4}\|\nabla(\gamma^{\frac{1}{2}}(v)u)\|_{L^2}\|u\|_{L^2}\\
&\quad+\frac{5K_1^2c_4c}{4}\|u\|_{L^2}^2\|\Delta v\|_{L^2}+\frac{5K_1^2c_4c}{4}\|u\|_{L^2}^2\\
&\le \frac{1}{2}\|\nabla(\gamma^{\frac{1}{2}}(v)u)\|_{L^2}^2+c\ \|u\|_{L^2}^2+c_9\|u\|_{L^2}^2\|\Delta v\|_{L^2}^2 \quad t\in(0,T_{max}),
\end{aligned}\tag{3.10}$$

c := $\frac{(\ K_1^2c_4c_7+2)^2}{16}$ $c_9 := \frac{2\ K_1^4c_4^2c_7^2}{}$. (3.10) (3.7),
$c\ \|u\|_{L^2}^2 \le \mu\int_\Omega u^3 + \frac{4c_8^3|\Omega|}{2\ \mu^2}$ $c_{10} := c_1 + \frac{4c_8^3|\Omega|}{2\ \mu^2}$,

$$\frac{d}{dt}\|u\|_{L^2}^2 \le c_9\|u\|_{L^2}^2\|\Delta v\|_{L^2}^2 + c_{10} \quad t\in(0,T_{max}).\tag{3.11}$$

N , (2.23) (2.28) (3.1) (3.11). I , $t \in (0,T_{max})$ $t \in (0,\tau)$ $t \ge \tau$ $\tau =$ $\{1, \frac{1}{2}T_{max}\}$, (2.23) fi $t_0 = t_0(t) \in ((t-\tau)_+, t)$ $t_0 \ge 0$

$$ {}_{\Omega} u^2(x,t_0) \le c_{11}.\tag{3.12}$$

O , (2.28) L 2.3, fi $c_{12} > 0$

$$ {}_{t_0}^{t_0+\tau} {}_{\Omega} |\Delta v(x,s)|^2 \le c_{12} \quad t_0 \in (0,T_{max}).\tag{3.13}$$

(3.11) (t_0,t), (3.12) (3.13) $t \le$ $t_0 + \tau \le t_0 + 1$,

$$\begin{aligned}
\|u(\cdot,t)\|_{L^2}^2 &\le \|u(\cdot,t_0)\|_{L^2}^2 \cdot e^{c_9\int_{t_0}^t \|\Delta v(\cdot,s)\|_{L^2}^2 ds} + c_{10} {}_{t_0}^{t} e^{c_9\int_s^t \|\Delta v(\cdot,\sigma)\|_{L^2}^2 d\sigma} ds\\
&\le c_{11}e^{c_9c_{12}} + c_{10}e^{c_9c_{12}} \quad t\in(0,T_{max}),
\end{aligned}$$

(3.1) ∎

3.2. L^∞**-** . I , $\|u(\cdot,t)\|_{L^\infty}$. , fi - - $\|v(\cdot,t)\|_{L^\infty}$ $\|u(\cdot,t)\|_{L^2}$, L 3.1 u $L^\infty((0,T_{max});L^p(\Omega))$ $p \ge 2$, $\|\nabla v(\cdot,t)\|_{L^\infty}$ L 2.4. , $\|u(\cdot,t)\|_{L^\infty}$ - M (1 25, 39, 27).

L 3.2. *Suppose that the conditions in Lemma* 3.1 *hold. Then the solution of system* (1.4) *satisfies*

$$\|u(\cdot,t)\|_{L^\infty} \le C \quad t \in (0,T_{max}),\tag{3.14}$$

where the constant $C > 0$ *independent of* t.

Proof. A L 2.4 $\|u(\cdot,t)\|_{L^2} \le c_1$ L 3.1,
(1.4)

$$\|v(\cdot,t)\|_{W^{1,4}} \le c_2 \quad t \in (0, T_{max}), \tag{3.15}$$

, ,

$$\|v(\cdot,t)\|_{L^\infty} \le K_* \quad t \in (0, T_{max}). \tag{3.16}$$

(H) (3.16),

$$\gamma(v) \ge \gamma(K_*) > 0 \qquad |\gamma'(v)| \le c_3 \quad t \in (0, T_{max}). \tag{3.17}$$

N , u^{p-1} $p \ge 2$ fi (1.4),

(3.20) (3.19), $c_6 := c_4p + \frac{2\gamma(K_*)}{p}\left(\frac{c_1c_5}{2\gamma(K_*)}\right)^p + c\ c_1^p$,

$$\frac{d}{dt}\int_\Omega u^p + p\mu\int_\Omega u^p \le c_6 \quad t \in (0, T_{max}),$$

, G

(3.21) $\|u(\cdot,t)\|_{L^p}^p \le e^{-p\mu t}\|u_0\|_{L^p}^p + \frac{c_6}{p\mu}(1-e^{-p\mu t}) \le \|u_0\|_{L^p}^p + \frac{c_6}{p\mu} \quad t \in (0, T_{max})$.

$p = 4$ (3.21) L 2.4, fi $c > 0$ p $\|\nabla v(\cdot,t)\|_{L^\infty} \le c$. M (1 25, 27, 39), (3.14). H L 3.2 . ∎

H ,

L 3.3. *Let Ω be a bounded domain in $\mathbb{R}^2$ with smooth boundary and the hypothesis (H) holds. Assume $(u_0, v_0) \in W^{1,\infty}(\Omega)^2$ with $u_0, v_0 \ge 0(\not\equiv 0)$. Then the problem* (1.4) *has a unique global classical solution*

$$(u,v) \in C^0(0,\infty) \times \bar\Omega) \cap C^{2,1}((0,\infty)\times\bar\Omega) \cap L^\infty_{loc}(0,\infty), W^{1,\infty}(\Omega))^2$$

satisfying (1.6).

Proof. F L 3.2 L 2.4, $\|u(\cdot,t)\|_{L^\infty} + \|v(\cdot,t)\|_{W^{1,\infty}} \le c_1$. L 2.1, L 3.3 . ∎

4. L . F (1.3) $\mu_* > 0$ $\mu > \mu_*$, 19, 34, 3 . (1.4) (1.3). B 3, 28 , (1.4) L

(4.1) $$K_0 = \sup_{0\le v\le\infty}\frac{|\gamma'(v)|^2}{\gamma(v)}.$$

N fi K_0 (H). N $\mu > \frac{K_0}{16}$, (1, 1) L^∞- , (1.4) $\mu > 0$.

4.1. C . M 28, 9 , L

L 4.1. *Let (u,v) be the solution of system* (1.4) *obtained in Lemma* 3.3 *and K_0 defined by* (4.1). *Suppose that*

(4.2) $$\mu > \frac{K_0}{16},$$

then there exist two positive constants δ, α_0 such that for all $t > 0$, the nonnegative function

(4.3) $$\mathcal{E}(t) := \int_\Omega (u - 1 - \ln u) + \frac{\delta}{2}\int_\Omega (v-1)^2$$

satisfies

$$\mathcal{E}'(t) \le -\mathcal{F}(t) \qquad t \in (0, T_{max}), \tag{4.4}$$

where

$$\mathcal{F}(t) := \alpha_0 \cdot \left\{ {}_{\Omega}(u-1)^2 + {}_{\Omega}(v-1)^2 \right\}.$$

Proof. M fi (1.4) $\frac{u-1}{u}$ Ω,

$$\begin{aligned} \frac{d}{dt}{}_{\Omega}(u - 1 - \ u) &= - {}_{\Omega}\nabla\Big(\frac{u-1}{u}\Big)\cdot\ \gamma(v)\nabla u + \gamma'(v)u\nabla v\ - \mu {}_{\Omega}(u-1)^2 \\ &= - {}_{\Omega}\gamma(v)\frac{|\nabla u|^2}{u^2} - {}_{\Omega}\gamma'(v)\frac{\nabla u\cdot\nabla v}{u} - \mu {}_{\Omega}(u-1)^2 \end{aligned} \tag{4.5}$$

$t \in (0, T_{max})$. (1.4) $v - 1$

$$\frac{1}{2}\frac{d}{dt}{}_{\Omega}(v-1)^2 = - {}_{\Omega}|\nabla v|^2 - {}_{\Omega}(v-1)^2 + {}_{\Omega}(u-1)(v-1) \qquad t \in (0, T_{max}). \tag{4.6}$$

N (4.6) δ (4.5), $t \in (0, T_{max})$

$$\begin{aligned} &\frac{d}{dt}{}_{\Omega}\ (u - 1 - \ u) + \frac{\delta}{2}(v-1)^2 \\ &= - {}_{\Omega}\Big(\gamma(v)\frac{|\nabla u|^2}{u^2} + \delta|\nabla v|^2 + \gamma'(v)\frac{\nabla u\cdot\nabla v}{u}\Big) \\ &\quad + {}_{\Omega} -\mu(u-1)^2 - \delta(v-1)^2 + \delta(u-1)(v-1) \\ &=: J_1 + J_2. \end{aligned} \tag{4.7}$$

F J_1,

$$J_1 = -\Theta_1^T A_1 \Theta_1,\ \ \Theta_1 = \begin{bmatrix}\nabla u\\ \nabla v\end{bmatrix},\quad A_1 = \begin{bmatrix}\frac{\gamma(v)}{u^2} & \frac{\gamma'(v)}{2u}\\ \frac{\gamma'(v)}{2u} & \delta\end{bmatrix},$$

Θ_1^T Θ_1. O A_1 fi , , $J_1 \le 0$

$$\delta \ge {}_{0\le v\le\infty}\frac{|\gamma'(v)|^2}{4\gamma(v)} = \frac{K_0}{4}. \tag{4.8}$$

, J_2

$$J_2 = -\Theta_2^T A_2 \Theta_2,\ \ \Theta_2 = \begin{bmatrix}u-1\\ v-1\end{bmatrix},\quad A_2 = \begin{bmatrix}\mu & -\frac{\delta}{2}\\ -\frac{\delta}{2} & \delta\end{bmatrix}.$$

H J_2 fi

$$\mu > \frac{\delta}{4}. \tag{4.9}$$

(4.2), fi δ fi (4.8)
(4.9) A_1 fi A_2 fi ,
(4.7) fi $\mathcal{E}(t)$ fi (4.3), fi $\alpha_0 > 0$
(4.4)

F $\mathcal{E}(t)$. I , $\phi(u) := u - 1 -$ $u, u > 0$.
$\phi'(u) = 1 - \frac{1}{u}$ $\phi''(u) = \frac{1}{u^2} > 0$ $u > 0$, ,
, $\phi(1) = \phi'(1) = 0$. $_{u>0}\,\phi(u) = \phi$

O , $\mathcal{E}(t)$ L 4.1, (4.4)

$$\int_1^{\infty} \mathcal{F}(s)ds \leq \mathcal{E}(1) - \mathcal{E}(t) \leq \mathcal{E}(1) < \infty.$$

fi $\mathcal{F}(t)$ L 4.1,

$$\int_1^{\infty} \int_\Omega (u-1)^2 + (v-1)^2 < \infty. \tag{4.16}$$

F (4.16), $j \to \infty$, $t_j \to \infty$, ,

$$\int_{t_j}^{t_j+T_1} \int_\Omega (u(x,t)-1)^2 dxdt \leq \int_{t_j}^{\infty} \int_\Omega (u(x,t)-1)^2 dxdt \to 0 \quad j \to \infty,$$

, , (4.15). (4.12) (4.10) . I , (4.11) $\int_1^{\infty} \int_\Omega (v-1)^2 < \infty$. H L 4.2 . ∎

Proof of Theorem 1.1. 1.1 L 3.3 4.2. ∎

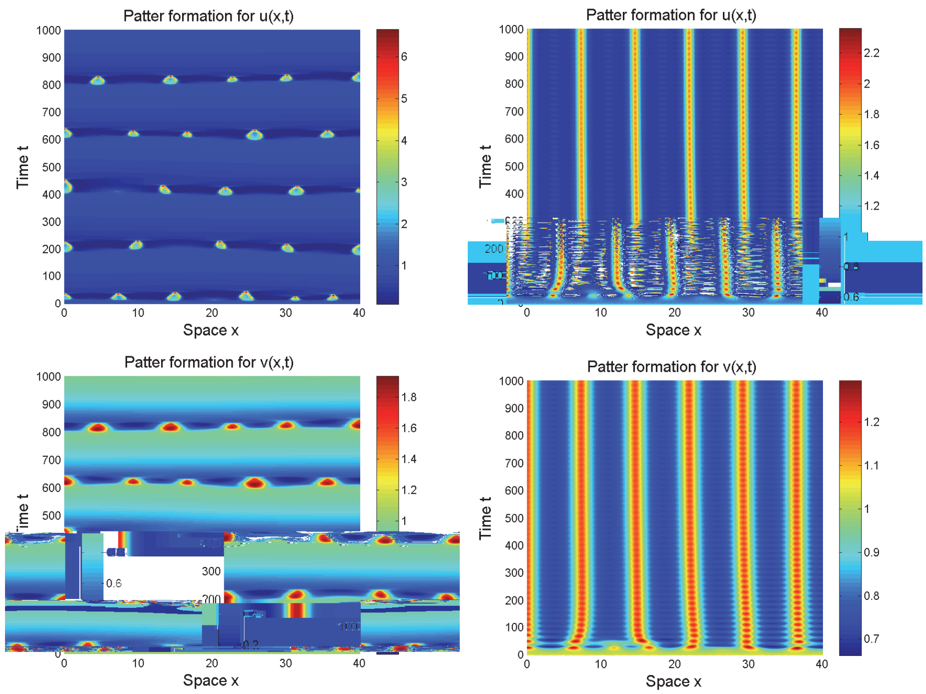
5. [redacted] .

5.1. N [redacted] [redacted]**.** I , - ff (1.4) - 6 . , (1,1) . O 29, 39 - . , . I , . I , (1.4) - ff - ( ) - (. 21). fi . , fi (1.4) ff D, , $\gamma(v) = D$,

$$\begin{cases} u_t = D\Delta u + \mu u(1-u), & x \in \Omega,\ t > 0, \\ v_t = \Delta v + u - v, & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), v(x,0) = v_0(x), & x \in \Omega. \end{cases} \tag{5.1}$$

fi (5.1) - F -K $u = 1$ (., 15). ( ) L 4.2, (1,1) (5.1). (5.1) . N (1.4) ffi $\gamma(v)$ v

$$\begin{cases} u_t = \mu u(1-u), \\ v_t = u - v. \end{cases} \tag{5.2}$$



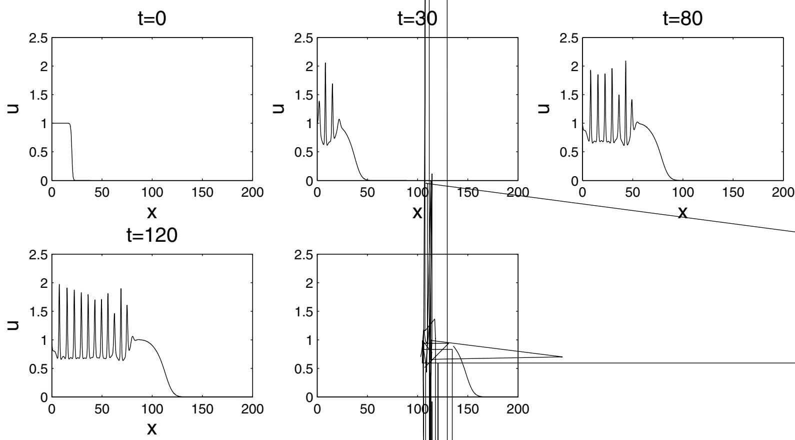
( ) $\mu = 0.02$ ( ) $\mu = 0.2$

F . 1. *Numerical simulation of spatio-temporal pattern formation of* (1.4) *in the interval* 0, 20 , *where the initial data* (u_0, v_0) *are chosen as a small random perturbation of the constant steady state* (1, 1).

fi ff (1.4) $\Omega = 0, l = 0, 20$, (1.4). (u_0, v_0) (1, 1), fi $\gamma(v)$

$$\gamma(v) = \frac{1}{1 + e^{\ (v-1)}}. \tag{5.9}$$

$\gamma'(v$


t=0
t=30
t=80
t=120
u
x
0
50
100
150
200
0
0.5
1
1.5
2
2.5

. F $\mu = 0$ F 2 3. H , fi ff

5.2. D . N (1.4) . B fi fi ( ). 1.1 $\mu > \mu^0 = \frac{K_0}{16}$, $K_0 = \ {}_{0\le v\le\infty} \frac{|\gamma'(v)|^2}{\gamma(v)}$. F $\mu > 0$, F 1, . H (1.4) ff , . (20). F $\mu^0 = \frac{K_0}{16}$ / , (5.9) $\mu^0 = \frac{K_0}{16} = 0.59$ $\mu_0 = 0.5$ $\mu^0 = \frac{K_0}{16}$ / . H fi . E $\mu > 0$, F 1 ff μ. H .

(1.1). N (1.1) F 2 3. ff μ. I 6, 14 . H (1.1) ff , , . E .

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