DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Volume 38, Number 7, July 2018

doi:10.3934/dcds.2018155

pp. 3595{3616

BOUNDEDNESS AND LARGE TIME BEHAVIOR IN A TWO-DIMENSIONAL KELLER-SEGEL-NAVIER-STOKES SYSTEM WITH SIGNAL-DEPENDENT DIFFUSION AND SENSITIVITY

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(Communicated by Jose A. Carrillo)

Abstract. This paper is concerned with the following Keller-Segel-Navier-Stokes system with signal-dependent diffusion and sensitivity

$$\begin{cases} n_t + u \cdot \nabla n = \nabla \cdot (d(c)\nabla n) - \nabla \cdot (\chi(c) n \nabla c) + an - bn^2, & x \in \Omega,\ t > 0, \\ c_t + u \cdot \nabla c = \Delta c + n - c, & x \in \Omega,\ t > 0, \\ u_t + u \cdot \nabla u = \Delta u - \nabla P + n\nabla \phi, & x \in \Omega,\ t > 0, \\ \nabla \cdot u = 0 & x \in \Omega,\ t > 0, \end{cases} \tag{}$$

in a bounded smooth domain $\Omega \subset \mathbb{R}^2$ with homogeneous Neumann boundary conditions, where $a \geq 0$ and $b > 0$ are constants, and the functions $d(c)$ and $\chi(c)$ satisfy the following assumptions:

$(d(c), \chi(c)) \in [C^2([0, 1))]^2$ with $d(c), \chi(c) > 0$ for all

tissue homeostasis, wound healing, as well as nding food and forming the multicellular body of protozoa [12]. The rst mathematical model describing chemotaxis at population level was proposed by Keller and Segel [25], which was reduced to the following system under the quasilinear-steady-state assumptions for the chemical reaction

$$\begin{cases} n_t = r\ (D(n;c)r\ n)\ \ r\ \ (S(n;c)r\ c); \\ c_t = \ \ c + n\ \ c; \end{cases} \tag{1}$$

where $n = n(x;t)$ denotes the cell density and $c = c(x;t)$ stands for the concentration of chemical substance. $D(n;c) > 0$ is a measure of the vigor of the random motion of the individual cell and $S(n;c) > 0$ is a measure of the strength of the in uence of chemical concentration. The theoretical studies of system (1) have been done for di erent $D(n;c)$ and $S(n;c)$. And the most extensively studied case is $D(n;c) = D_0$ and $S(n;c) = \ _0 n$ with constants $D_0 > 0$ and $\ _0 > 0$:

$$\begin{cases} n_t = D_0\ \ n\ \ \ _0 r\ \ (nr\ c); \\ c_t = \ \ c + n\ \ c; \end{cases}$$

which is called the classical Keller-Segel chemotaxis model [6] and whose solution behaviors strongly depend on the space dimensions (No blow-up in 1-D [19, 33], critical mass blow-up in 2-D [17, 23, 31, 32, 36] and, generic blow-up in 3-D [45, 47], see also the review articles [1, 20, 21]). Since the nite time blow-up of solutions limits the applicability of model to real world. Based on the biological inspiration or mathematical motivation, di erent mechanisms have been proposed to prevent nite time blow-up and exhibit the spatial pattern formations [18]. Among these, the cell self-interaction was considered as the important role, which a ect both cell random motion and chemotactic movement [34] and can be described as

$$\begin{cases} n_t = r\ \ (\ (n)r\ n)\ \ \ _0 r\ \ (n\ \ (n)r\ c); & x\ 2\ \ ;t > 0; \\ c_t = \ \ c + n\ \ c; & x\ 2\ \ ;t > 0; \end{cases} \tag{2}$$

where the di usion coe cient (n) and chemotactic sensitivity function (n) both depend on the cell density. For the non- ux initial-boundary value problem of system (2) in smoothly bounded domain $\mathbb{R}^N$ ($N\ \ 2$), it has been proved that the blow-up or boundedness of solution depends on the values of, which is the ratio of $\frac{(n)}{(n)}\ \ n$ at large values of n (see the review papers [1, 20, 21] and recent progress in [7, 8, 22, 37, 44, 50] for details).

Since the chemical is secreted by the cells themselves, when cells get crowded during chemotactic movement, the chemical concentration also increases. Hence it is nature to consider the e ects of the local interactions between cells and chemicals such as

$$\begin{cases} n_t = r\ \ (d(c)r\ n)\ \ r\ \ \ (\ (c)nr\ c); & x\ 2\ \ ;\ t > 0; \\ c_t = \ \ c + n\ \ c; & x\ 2\ \ ;\ t > 0; \end{cases} \tag{3}$$

where the di usion function depends on the chemical concentration c. However, to our knowledge, few mathematical results are available for the cell-chemical interaction chemotaxis system (3) except for the special case $(c) = \ \ d'(c) > 0$. In fact, if $(c) = \ \ d'(c) > 0$, the rst equation of system (3) can be rewritten as $n_t = (\ \ d(c)n)$, which together with the second equation was proposed in [14] to describe the stripe pattern driven by the density-suppressed motility. In this case, Yoon and Kim [51] considered a particular form of $d(c) = \frac{c_0}{c^k}; c_0 > 0; k > 0$ and showed the existence of global classical solution with uniform-in-time bound for

any dimensions if $c_0 > 0$ is small. Tao and Winkler [42] recently established the existence of global classical solution in two dimensions and global weak solutions in three dimensions by assuming that $d(c)$ has a positive lower and upper bound (i.e. $0 < d_1 \quad d(c) \quad d_2$ for all $c \quad 0$, where $d_1; d_2$ are two constants). Furthermore, if the rst equation of system (3) was replaced by $n_t = (\ d(c)n) + \ n(1 \quad n)$ (i.e., cell has growth), Jin *et al.* [24] studied the global dynamics and patterns (like aggregation and wavefronts) without small assumption as imposed in [51] and the lower-upper bound assumption in [42].

In the above cell self-interaction chemotaxis model (2) or cell-chemical interaction chemotaxis model (3), the relevant in uence of the cells or chemical on the surrounding habitat was ignored. However, some experiment evidence and numerical simulation indicated that liquid environment play an important role on the chemotactic motion of cells [5, 30, 43]. In this paper, we will study the following cell-chemical interaction chemotaxis model in the uid environment

$$\begin{cases} n_t + u \ r \ n = r \quad (d(c)r \ n) \quad r \quad (\ (c)nr \ c) + an \quad bn^2; & x \ 2 \quad ; \ t > 0; \\ c_t + u \ r \ c = \quad c + n \quad c; & x \ 2 \quad ; \ t > 0; \\ u_t + \ u \ r \ u = \quad u \ r \quad P + nr \ ; & x \ 2 \quad ; \ t > 0; \\ r \quad u = 0 & x \ 2 \quad ; \ t > 0; \\ \frac{@n}{@} = \frac{@c}{@} = u = 0 & x \ 2 \ @ \ ; t > 0; \\ n(x; 0) = n_0(x); \quad c(x; 0) = c_0(x); \quad u(x; 0) = u_0(x); & x \ 2 \quad ; \end{cases} \tag{4}$$

where $\mathbb{R}^2$ is a bounded domain with smooth boundary, $u = u(x; t)$ and $P = P(x; t)$ represent the velocity and the associated pressure of the uid ow, respectively. Suppose that the given potential function $= (x)$ satis es

$$(x) \ 2 \ W^{1;\infty}( ) \ ; \tag{5}$$

and $d(c); \ (c) > 0$ satisfy the assumptions

(H1) $(d(c); \ (c)) \ 2 \ [C^2([0; 1 \ ))]^2; d'(c) < 0$ on $[0; 1 \ )$ and $\lim_{c \to \infty} d(c) = 0$.

(H2) $\lim_{c \to \infty} \frac{d^0(c)}{d(c)}$ and $\lim_{c \to \infty} \frac{(c)}{d(c)}$ exist.

For simplicity, we assume the initial data satisfy

$$\begin{cases} n_0 \ 2 \ C^0( ) \ ; n_0 \quad 0 \text{ in } \quad \text{and } \quad n_0 \ 6 \ 0; \\ c_0 \ 2 \ W^{1;\infty}( ) \ ; c_0 \quad 0 \text{ in } \quad \text{and} \\ u_0 \ 2 \ D(A \ ); \quad \text{for some } \quad 2 \ (\frac{1}{2}; 1); \end{cases} \tag{6}$$

where $A := \ P$ denotes the realization of the Stokes operator in $L^2( )$, de ned on its domain $D(A) = W^{2;2}( ) \ \setminus W_0^{1;2}( ) \ \setminus L^2( )$ with $L^2 := f \quad 2 \ L^2( ) \ jr \quad = 0$ in $D'( ) \ g$, and with P denoting the Helmholtz projection of $L^2( )$ onto its closed subspace $L^2( )$.

Before stating our main results, we rst recall some relevant results: If $d(c) = d_0$ and $(c) = \ _0$ are positive constants, Espejo and Suzuki [13] proved the global existence of certain weak solutions of system (4) in two dimensions with $a = 0$. After that, Tao and Winkler [40] investigated the global existence and large time behavior of classical solutions of the system (4) with a given external force $g = g(x; t)$ in the Navier-Stokes uid equation, and the results have been extended to three dimensions with Stokes uid and the condition $b \quad 23$ [41]. On the other hand, if the zero-term $n \quad c$ is replaced by the absorption term $nf(c)$ in the second equation of system (4)(see [43] for details), Lorz [29] established the local

existence of solution with $a = 0$ and $b = 0$. Furthermore, the global existence and large time behavior of solution were studied extensively with linear di usion and porous medium di usion (e.g. see [3, 4, 9, 10, 11, 28, 38, 48, 46, 49] and the review article [1] for details). However, if the di usion function $d(c)$ depends on the chemical concentration c, to our knowledge, no much rigorous results for the system (4) have been available to date. The purpose of this paper is to establish the boundedness and large time behavior of solutions to system (4). These are not so trivial due to the following two di cult points. The rst di culty is the possibility of degeneracy for di usion since $d(c) \to 0$ as $c \to 1$, which may cause many di culties to establish the global dynamics of solutions. The second di culty comes from the uid described by Navier-Stokes equation. To overcome this problems, we will use $d(c)$ as a weight function and some techniques developed recently in [40] to obtain the global existence of classical solutions. Moreover, the large time behavior of solutions will be established by constructing Lyapunov functional. Our main results are stated as follows.

Theorem 1.1. *Let be a bounded domain in $\mathbb{R}^2$ with smooth boundary and $d(c)$; (c) satisfy the hypotheses (H1) and (H2). Assume satisfies* (5)*, and suppose the initial data $n_0; c_0; u_0$ satisfy* (6)*. Then the problem* (4) *has a unique nonnegative global classical solution*

$$\begin{cases} n \in C^0([0,1) \times \Omega) \cap C^{2,1}((0,1) \times \Omega), \\ c \in C^0([0,1) \times \Omega) \cap C^{2,1}((0,1) \times \Omega), \\ u \in C^0([0,1) \times \Omega) \cap C^{2,1}((0,1) \times \Omega) \text{ and} \\ P \in C^{1,0}(\Omega \times (0,1)) \end{cases}$$

satisfying

$$\|n(\cdot,t)\|_{L^1(\Omega)} + \|c(\cdot,t)\|_{L^1(\Omega)} + \|u(\cdot,t)\|_{L^1(\Omega)} \le C_1; \tag{7}$$

where $C_1 > 0$ is a constant independent of t. Moreover, if $b > \frac{K_0}{16}$ with $K_0 = \max_{0 \le c \le \infty} \frac{|\chi(c)|^2}{d(c)}$, then the solution of (4) *satisfies*

$$\|n(\cdot,t) - \frac{a}{b}\|_{L^1(\Omega)} \to 0,\ \|c(\cdot,t) - \frac{a}{b}\|_{L^1(\Omega)} \to 0 \text{ and } \|u(\cdot,t)\|_{L^1(\Omega)} \to 0 \text{ as } t \to 1 : \tag{8}$$

Remark 1. The results in Theorem 1.1 are new even in the uid-free system (4) with $u = 0$. Indeed, in our results, we do not require the special structure $\chi(c) = d'(c)$, which plays an important role in [24, 42, 51].

Remark 2. In fact, with the boundedness results obtained in Theorem 1.1, using the similar arguments as in [40], we can further obtain the following results on the large time behavior of solutions: If $a = 0$, then for any $b > 0$ the solution of (4) satisfies

$$\|n(\cdot,t)\|_{L^1(\Omega)} \to 0,\ \|c(\cdot,t)\|_{L^1(\Omega)} \to 0 \text{ and } \|u(\cdot,t)\|_{L^1(\Omega)} \to 0 \text{ as } t \to 1 : \tag{9}$$

2. Local existence and preliminaries. In what follows, without confusion, we shall abbreviate $\int_\Omega f dx$ as $\int f$ for simplicity. Moreover, we shall use c_i for $C_i (i = 1,2,3,\ldots)$ to denote a generic constant which may vary in the context. We rst give the existence of local solutions of (4) by Schauder xed point and the standard parabolic regularity theory.

Lemma 2.1 (Local existence) *Let be a bounded domain in* $\mathbb{R}^2$ *with smooth boundary and* $d(c)$; (c) *satisfy the hypotheses (H1) and (H2). Assume satis es*

semigroup[15], one has

$$\begin{aligned} \|u(\cdot,t)\|_{L^2} \le \|A^{-\alpha} e^{-tA}\| \|u_0\|_{L^2} + c_1 \int_0^t (t-s)^{-\alpha} \|(\mu+\cdot)\mu(\cdot,s)\|_{L^2} ds \\ + c_1 \int_0^t (t-s)^{-\alpha} \|\eta(\cdot,s)\cdot\|_{L^2} ds. \end{aligned} \tag{15}$$

Due to $\in (\frac{1}{2}, 1)$, there exists a constant $c_2 > 0$ such that

$$\|(\mu+\cdot)\mu(\cdot,s)\|_{L^2} \le \|\mu(\cdot,s)\|_{L^1} + \|\cdot\,\mu(\cdot,s)\|_{L^2} \le c_2 \|\mu(\cdot,s)\|_{L^2}^2 + c_2 R^2. \tag{16}$$

On the other hand, using the boundedness of $\|\cdot\|_{L^1}$ and $\|\eta(\cdot,s)\|_{L^2}$, one has

$$\|\eta(\cdot,s)\cdot\|_{L^2} \le \|\eta(\cdot,s)\|_{L^2} \|\cdot\|_{L^1} \le c_3(R). \tag{17}$$

Substituting (16) and (17) into (15), we have

$$\|u(\cdot,t)\|_{L^2} \le \|A^{-\alpha}\| \|u_0\|_{L^2} + c_4(R) T^{1-\alpha} \tag{18}$$

and hence

$$\|u(\cdot,t)\|_{L^1} \le \|A^{-\alpha}\| \|u(\cdot,t)\|_{L^2} \le c_5(R)(1 + T^{1-\alpha}). \tag{19}$$

Similarly, we apply the variation-of-constant formula to (12) to obtain

$$c(\cdot,t) = e^{t(\Delta-1)} c_0 + \int_0^t e^{(t-s)(\Delta-1)} [\eta(\cdot,s) - u(\cdot,s)\cdot\nabla c(\cdot,s)] ds. \tag{20}$$

Then using the semigroup estimates and (19), from (20) one has

$$\begin{aligned} \|c(\cdot,t)\|_{W^{1,1}} &\le \|c_0\|_{W^{1,1}} + c_6 \int_0^t [1 + (t-s)^{-\frac{1}{2}}] (\|\eta(\cdot,s)\|_{L^1}) ds \\ &\quad + c_6 \int_0^t [1 + (t-s)^{-\frac{1}{2}}] (\|u(\cdot,s)\|_{L^1} \|\nabla c(\cdot,s)\|_{L^1}) ds \\ &\le \|c_0\|_{W^{1,1}} + c_7(R)(1 + T^{1-\alpha}) \int_0^t [1 + (t-s)^{-\frac{1}{2}}] ds \\ &\le \|c_0\|_{W^{1,1}} + c_8(R)(T^{\frac{1}{2}} + T^{\frac{3}{2}-\alpha} + T + T^{2-\alpha}); \end{aligned} \tag{21}$$

which gives $\|c(\cdot,t)\|_{W^{1,1}} \le C(T)$ for all $t \in (0,T)$. Then noting the fact $d'(c) < 0$ in (H1) and $\|c(\cdot,t)\|_{L^1} \le C(T)$, we derive $d(c)$ has positive lower bound, which implies the equation in (11) is uniformly parabolic. Then noting the fact $\chi(c)\nabla c \in L^\infty(\Omega \times (0,T))$ and $a - b\eta \in L^\infty(\Omega \times (0,T))$ and using the parabolic regularity estimate ([27, Theorem V1.1]), one has

$$\|\eta(\cdot,t)\|_{C^{\theta,\frac{\theta}{2}}(\Omega \times [0,T])} \le c_8(R); \tag{22}$$

for some $\theta \in (0,1)$ and $c_9(R) > 0$. From (22), we have

$$\|\eta(\cdot,t) - \eta_0\|_{L^1} \le c_9(R) T^{\frac{\theta}{2}};$$

and thus

$$\|\eta(\cdot,t)\|_{L^1} \le \|\eta_0\|_{L^1} + c_9(R) T^{\frac{\theta}{2}}. \tag{23}$$

Hence, combining (18), (21) and (23) and choosing

$$T = T_0 < \min\left\{ \left(\frac{1}{3c_4(R)}\right)^{\frac{1}{1-\alpha}}, \left(\frac{1}{12c_8(R)}\right)^2, \left(\frac{1}{3c_9(R)}\right)^{\frac{2}{\theta}} \right\},$$

we have

$$\|\eta(\cdot,t)\|_{L^1} + \|c(\cdot,t)\|_{W^{1,1}} + \|A^{\alpha} u(\cdot,t)\|_{L^2} \le R.$$

Then we prove that $Z \ 2 \ S_T$ and hence maps S_T into itself. Moreover, one can show the map is compact in X. Using the Schauder xed point, we can conclude that there exists a $Z \ 2 \ S_T$ such that $(\ Z) = Z$, the uniqueness can be proved by using the similar arguments as in [24, 46].

(ii) *Regularity and non-negativity.* Using the standard bootstrap arguments involving the regularity theories for parabolic equations and the stokes semigroup, we obtain the smooth properties listed in (10). The solution may be prolonged in the interval $[0; T_{max})$ with either $T_{max} = 1$ or $T_{max} < 1$ in the case

$$kn(\ ;t)k_{L^1} + kc(\ ;t)k_{W^{1,1}} + kA \ u(\ ;t)k_{L^2} \ ! \ 1 \quad ; \text{as } t \ \% \ T_{max} :$$

At last, the non-negativity of solutions n and c follows from the classical maximum principle. □

Next, we present some basic boundedness properties based on the integration in the rst equation of system (4).

Lemma 2.2. *There exist two positive constants m_* and C_1 such that the solution of system (4) satis es*

$$n \quad m_* := \max \left\{ \int n_0; \frac{a|\ |}{b} \right\}; \quad \text{for all} \quad t \ 2 \ (0; T_{max}); \tag{24}$$

and

$$\int_t^{t+} \int n^2 \quad C_1; \quad \text{for all} \quad t \ 2 \ (0; \ b_{max}); \tag{25}$$

where

$$:= \min \left\{ 1; \frac{1}{6} T_{max} \right\} \quad \text{and} \quad \ b_{max} := \begin{cases} T_{max} & ; \quad \text{if} \quad T_{max} < 1 ; \\ 1 & \text{if} \quad T_{max} = 1 : \end{cases} \tag{26}$$

Proof. Integrating the rst equation of system (4) over , and using the fact $\int_{\partial} u \ r \ n = \int nr \ u = 0$, we have

$$\frac{d}{dt} \int n + b \int n^2 = a \int n: \tag{27}$$

On the other hand, using the Cauchy-Schwarz inequality, we have $\int n^2 \quad \frac{1}{|\ |} \left(\int n \right)^2$, which combined with (27) gives

$$\frac{d}{dt} \int n \quad a \int n \quad \frac{b}{|\ |} \left(\int n \right)^2 : \tag{28}$$

Then applying the ODE comparison to (28), we obtain (24). Integrating (27) over $(t; t + \ )$ and using (24), one has (25). □

With the estimates in Lemmas 2.2 in hand, we can derive some basic time-independent of c and u by using the similar argument as in [40, Lemmas 3.4-3.7].

Lemma 2.3. *There exists a constant $C_3 > 0$ such that*

$$ku(\ ;t)k_{W^{1,2}} \quad C_3 \text{ for all } t \ 2 \ (0; T_{max}) \tag{29}$$

and

$$kr \ c(\ ;t)k_{L^2} \quad C_3 \text{ for all } t \ 2 \ (0; T_{max}) \tag{30}$$

as well as

$$\int_t^{t+} \int |\ c|^2 \quad C_3 \quad \text{for all} \quad t \ 2 \ (0; \ b_{max}) \tag{31}$$

where and $\bar{p}$ are defined by (26).

Proof. The estimates in (29)-(31) can be proved by using the similar argument as in [40, Lemmas 3.4-3.7], we omit the details of proof for convenience. □

Next, we show some basic properties which will be used later.

Lemma 2.4 ([26]). *Let be a bounded domain in* $\mathbb{R}^n$ *with smooth boundary. Assume there is a constant* $C_4 > 0$ *such that*

$$\|g(\cdot,t)\|_{L^p} \leq C_4, \text{ for all } t \in (0,T).$$

If $z_0 \in W^{1,\infty}(\Omega)$, *then there exists a constant* $C_5 > 0$ *such that for every* $t \in (0,T)$ *the solution of the problem*

$$z_t = \Delta z - z + g \text{ in } \Omega, \quad \frac{\partial z}{\partial \nu} = 0 \text{ on } \partial\Omega$$

satisfies

$$\|z(\cdot,t)\|_{W^{1,r}} \leq C_5$$

with

$$r \in \begin{cases} [1, \frac{np}{n-p}), & \text{if } p \leq n, \\ [1, 1], & \text{if } p > n. \end{cases}$$

Lemma 2.5. *Let be a bounded domain in* $\mathbb{R}^2$ *with smooth boundary. Then, for any* $\psi \in W^{2,2}(\Omega)$ *satisfying* $\frac{\partial \psi}{\partial \nu}|_{\partial\Omega} = 0$, *there exists a positive constant* C_6 *depending only on such that*

$$\|\nabla \psi\|_{L^4} \leq C_6(\|\psi\|_{L^2}^{\frac{1}{2}} \|\Delta \psi\|_{L^2}^{\frac{1}{2}} + \|\nabla \psi\|_{L^2}). \tag{32}$$

Proof. Using Gagliardo-Nirenberg inequality, we have

$$\|\nabla \psi\|_{L^4} \leq c_1 \|D\nabla \psi\|_{L^2}^{\frac{1}{2}} \|\nabla \psi\|_{L^2}^{\frac{1}{2}} + c_2 \|\nabla \psi\|_{L^2}; \tag{33}$$

where $|D\nabla \psi| = \Big(\sum_{|i|=1} |D^i \nabla \psi|^2\Big)^{\frac{1}{2}}$ and i is a multi-index of order. On the other hand, one can check that

$$\|D\nabla \psi\|_{L^2} \leq c_3 \|\nabla \psi\|_{H^1}. \tag{34}$$

Moreover, under the homogeneous Neumann boundary condition (i.e. $\frac{\partial \psi}{\partial \nu}|_{\partial\Omega} = 0$), it follows from [2, Lemma 1] that $\|\nabla \psi\|_{H^1} \leq c_4 \|\Delta \psi\|_{L^2}$, which applied to (34) gives

$$\|D\nabla \psi\|_{L^2} \leq c_3 c_4 \|\Delta \psi\|_{L^2}. \tag{35}$$

Then substituting (35) into (33), one derives (32) directly. □

3. Proof of Theorem 1.1. In this section, we are devoted to proving Theorem 1.1.

3.1. Boundedness. First, we establish the global boundedness of solutions. Due to the possible degeneracy ($\lim_{c\to\infty} d(c) = 0$), we can not directly used the di usive dissipation as in [40] to obtain the boundedness of $\|n(\cdot,t)\|_{L^p}$ ($p > 1$) (see [40, Lemma 3.8]). To overcome the di cult, we will use the ideas as in [24] and employ the L^2 energy estimate directly by using $d(c)$ as a weight function to pick up the advantage of di usive dissipation to absorb the cross-di usion and get the following Gronwall type inequality:

$$\frac{d}{dt}\|n\|_{L^2}^2 \quad c_1\|n\|_{L^2}^2\|c\|_{L^2}^2 \quad \|c\|_{L^2}^2 + c_2;$$

which will lead to the uniform-in-time bound of $\|n(\cdot,t)\|_{L^2}$ by using the boundedness of $\int_t^{t+} \int n^2$ and $\int_t^{t+} \int |\nabla c|^2$ established in Lemma 2.2 and Lemma 2.3. When $\|n(\cdot,t)\|_{L^2} \le c_3$ holds, we can show $\|c(\cdot,t)\|_{L^1}$ is uniformly bounded based on the parabolic regularity (see Lemma 2.4), which prevents the possibility of degeneracy. Then we use the a bootstrap argument same as chemotaxis- uid model with linear di usion [40] to prove the existence of global classical solutions with uniform-in-time bound.

3.1.1. *L^2-estimate.* In this subsection, we establish the boundedness of $\|n(\cdot,t)\|_{L^2}$.

Lemma 3.1. *Let be a bounded domain in $\mathbb{R}^2$ with smooth boundary and the hypotheses (H1)-(H2) hold. Then there exists a constant $C_7 > 0$ such that*

$$\|n(\cdot,t)\|_{L^2} \le C_7, \text{ for all } t \in (0, T_{max}): \tag{36}$$

Proof. We multiply the rst equation of system (4) by n and integrate the result by parts, and then use the Hölder inequality and the Young's inequality to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int n^2 + \int d(c)\nabla n\cdot\nabla n + b\int n^3 \\
&= \int (c)n\nabla n \cdot \nabla c + a\int n^2 \\
&\le \int d(c)\nabla n\cdot\nabla n^{\frac{1}{2}} \int \frac{|\nabla (c)|^2}{d(c)} n^2\nabla c\cdot\nabla c^{\frac{1}{2}} + a\|\nabla n\|_{L^1}^{\frac{1}{3}} \int n^3 \Big)^{\frac{2}{3}} \\
&\le \frac{1}{2}\int d(c)\nabla n\cdot\nabla n + \frac{1}{2}\int \frac{|\nabla (c)|^2}{d(c)} n^2\nabla c\cdot\nabla c + \frac{b}{2}\int n^3 + c_1;
\end{aligned}$$

which gives

$$\frac{d}{dt}\int n^2 + \int d(c)\nabla n\cdot\nabla n + b\int n^3 \le \int \frac{|\nabla (c)|^2}{d(c)} n^2\nabla c\cdot\nabla c + 2c_1. \tag{37}$$

On the other hand, using the fact

$$d^{\frac{1}{2}}(c)\nabla n = \nabla (d^{\frac{1}{2}}(c)n) \quad \frac{1}{2}\frac{d'(c)}{d^{\frac{1}{2}}(c)} n\nabla c;$$

and the inequality $|X \quad Y|^2 \quad \frac{1}{2}|X|^2 \quad |Y|^2$, we derive

$$\begin{aligned}
d(c)\nabla n\cdot\nabla n = |d^{\frac{1}{2}}(c)\nabla n|^2 = &\quad \nabla (d^{\frac{1}{2}}(c)n) \quad \frac{1}{2}\frac{d'(c)}{d^{\frac{1}{2}}(c)} n\nabla c \Big|^2 \\
&\frac{1}{2}|\nabla (d^{\frac{1}{2}}(c)n)|^2 \quad \frac{1}{4}\frac{|d'(c)|^2}{d(c)} n^2\nabla c\cdot\nabla c.
\end{aligned} \tag{38}$$

Substituting (38) into (37), one has

$$\begin{aligned}\frac{d}{dt}&\int n^2+\frac{1}{2}\int jr\ (d^{\frac{1}{2}}(c)n)j^2+b\int n^3\\ &\frac{1}{4}\int\frac{jd'(c)j^2}{d(c)}n^2jr\ cj^2+\int\frac{j\ (c)j^2}{d(c)}n^2jr\ cj^2+2c_1:\end{aligned}\tag{39}$$

The assumptions in (H1) and (H2) imply that there exist two constants $K_1;K_2>0$ such that

$$\frac{jd'(c)j}{d(c)}\quad K_1 \text{ and } \frac{(c)}{d(c)}\quad K_2;\quad \text{for all } c\quad 0:\tag{40}$$

Using (40) and the Holder inequality, we derive from (39) that

$$\begin{aligned}\frac{d}{dt}&\int n^2+\frac{1}{2}\int jr\ (d^{\frac{1}{2}}(c)n)j^2+\int n^3\\ &\frac{1}{4}\int\frac{jd'(c)j^2}{d^2(c)}(d^{\frac{1}{2}}(c)n)^2jr\ cj^2+\int\frac{j\ (c)j^2}{d^2(c)}(d^{\frac{1}{2}}(c)n)^2jr\ cj^2+2c_1\\ &\frac{K_1^2+4K_2^2}{4}\int jd^{\frac{1}{2}}(c)nj^2jr\ cj^2+2c_1\\ &\frac{K_1^2+4K_2^2}{4}\int jd^{\frac{1}{2}}(c)nj^4{}^{\frac{1}{2}}\int jr\ cj^4{}^{\frac{1}{2}}+2c_1\\ =\ &K_3\int jd^{\frac{1}{2}}(c)nj^4{}^{\frac{1}{2}}\int jr\ cj^4{}^{\frac{1}{2}}+2c_1:\end{aligned}\tag{41}$$

Applying the Gagliardo-Nirenberg inequality and using the fact $d(c)\quad d(0)=c_2$, one has

$$\begin{aligned}\int jd^{\frac{1}{2}}(c)nj^4{}^{\frac{1}{2}}=\ &kd^{\frac{1}{2}}(c)nk_{L^4}^2\quad c_3\ kr\ (d^{\frac{1}{2}}(c)n)k_{L^2}kd^{\frac{1}{2}}(c)nk_{L^2}+kd^{\frac{1}{2}}(c)nk_{L^2}^2\\ &c_4\ kr\ (d^{\frac{1}{2}}(c)n)k_{L^2}knk_{L^2}+knk_{L^2}^2\ :\end{aligned}\tag{42}$$

On the other hand, using Lemma 2.5 and noting the fact$kr\ ck_{L^2}\quad c_5$ (see (30)), we obtain

$$\begin{aligned}\int jr\ cj^4{}^{\frac{1}{2}}=\ &kr\ ck_{L^4}^2\quad c_6\ k\ \ ck_{L^2}kr\ ck_{L^2}+kr\ ck_{L^2}^2\\ &c_7(k\ \ ck_{L^2}+1):\end{aligned}\tag{43}$$

Combining (42) and (43), and using the Young's inequality, one derive

$$\begin{aligned}K_3&\int jd^{\frac{1}{2}}(c)nj^4{}^{\frac{1}{2}}\int jr\ cj^4{}^{\frac{1}{2}}\\ &K_3c_4c_7\ kr\ (d^{\frac{1}{2}}(c)n)k_{L^2}knk_{L^2}+kuk_{L^2}^2\ \ (k\ \ ck_{L^2}+1)\\ =\ &K_3c_4c_7kr\ (d^{\frac{1}{2}}(c)n)k_{L^2}knk_{L^2}k\ \ ck_{L^2}+K_3c_4c_7kr\ (d^{\frac{1}{2}}(c)n)k_{L^2}knk_{L^2}\\ &+K_3c_4c_7knk_{L^2}^2k\ \ ck_{L^2}+K_3c_4c_7knk_{L^2}^2\\ &\frac{1}{2}kr\ (d^{\frac{1}{2}}(c)n)k_{L^2}^2+c_8knk_{L^2}^2+c_9knk_{L^2}^2k\ \ ck_{L^2}^2:\end{aligned}\tag{44}$$

Substituting (44) into (41), and using the estimate $c_8 k n k_{L^2}^2 \quad \mathrm{R} \quad n^3 + c_{10}$, we obtain

$$\frac{d}{dt} k n k_{L^2}^2 \quad c_9 k n k_{L^2}^2 k \quad c k_{L^2}^2 + c_{11}: \tag{45}$$

Next, we shall use (25) and (31) to obtain (36) based on the inequality (45). In fact, for arbitrary $t \; 2 \; (0; T_{max})$ and in both cases $t \; 2 \; (0; \;)$ and t with $= \min \; 1; \frac{1}{6} T_{max}$, from (25) we can nd $t_0 = t_0(t) \; 2 \; ((t \quad)_+ ; t)$ such that $t_0 \quad 0$ and

$$\mathrm{Z} \quad n^2(x; t_0) \quad c_{12}: \tag{46}$$

On the other hand, using (31) in Lemma 2.3, we can nd a constant $c_{13} > 0$ such that

$$\mathrm{Z}_{t_0 +} \quad \mathrm{Z} \quad j \quad c(x; s) j^2 \quad c_{13} \quad \text{for all} \quad t_0 \; 2 \; (0; T_{max}): \tag{47}$$

Then integrating (45) over $(t_0; t)$, and using (46) and (47) and noting that $t \quad t_0 + \quad t_0 + 1$, we derive

$$k n(\; ; t) k_{L^2}^2 \quad k \; n(\; ; t_0) k_{L^2}^2 \quad e^{c_9 \int_{t_0}^t \| \quad c(\cdot; s) \|_{L^2}^2 ds} + c_{11} \quad \mathrm{Z}_{t_0}^{t} \quad e^{c_9 \int_s^t \| \quad c(\cdot; \quad) \|_{L^2}^2 d} \; ds$$

$$c_{12} e^{c_9 c_{13}} + c_{11} e^{c_9 c_{13}};$$

which yields (36). Then the proof of Lemma 3.1 is completed. □

3.1.2. *L^∞-estimate.* In this subsection, we will show the boundedness of $k n(\; ; t) k_{L^1}$. Before that, we rst improve the regularity of $t \; 2 \; (0; T_{max})$.

to show

$$\begin{aligned} kA\ \ u(\ ;t)k_{L^2}\ \ k\ A\ \ e^{-(t-t_0)A}u(\ ;t_0)k_{L^2} + c_2\int_{t_0}^{t}(t\quad s)^{-}\ kn(\ ;s)r\quad k_{L^2}ds \\ + c_3\int_{t_0}^{t}(t\quad s)^{-}\ k(u\ \ r\ )u(\ ;s)k_{L^2}ds \\ I_1 + I_2 + I_3 \end{aligned} \tag{51}$$

If $t_0 =$, we have

$$I_1 = ke^{(t-t_0)A}A\ \ u(\ ;t_0)k_{L^2}\quad c_4kA\ \ u(\ ;\ )k_{L^2}\quad c_5: \tag{52}$$

On the other hand if $t_0 >$, then $t_0 = t$ 1, from (29) one has

$$I_1\quad c_5(t\quad t_0)^{-}\ ku(\ ;t_0)k_{L^2} = c_5ku(\ ;t_0)k_{L^2}\quad c_6: \tag{53}$$

Next, we estimate I_2 as follows. Noting the boundedness of $kr\quad k_{L^1}\quad c_7$ and $kn(\ ;t)k_{L^2}\quad c_8$

$$I_2\quad c_3c_7c_8\int_{t_0}^{t}(t\quad s)^{-}\ ds\quad \frac{c_3c_7c_8}{1}: \tag{54}$$

At last, we estimate the term I_3. In fact, we can choose $2\ (\frac{1}{2};\ )$ to obtain

$$\begin{aligned} k(u(\ ;s)\ \ r\ )u(\ ;s)k_{L^2}\ \ k\ u(\ ;s)k_{L^1}\ kr\ u(\ ;s)k_{L^2} \\ c_9kA\ \ u(\ ;s)k_{L^2}kr\ u(\ ;s)k_{L^2} \\ c_{10}kA\ \ u(\ ;s)k_{L^2}kA^{\frac{1}{2}}u(\ ;s)k_{L^2}^{1-}\ kr\ u(\ ;s)k_{L^2} \\ c_{10}kA\ \ u(\ ;s)k_{L^2}kr\ u(\ ;s)k_{L^2}^{2-} \end{aligned} \tag{55}$$

where we have used $kA^{\frac{1}{2}}u(\ ;s)k_{L^2} = kr\ u(\ ;s)k_{L^2}$ and the interpolation to nd a $= \frac{2\ -1}{2\ -1}\ 2\ (0;1)$ such that

$$kA\ \ u(\ ;s)k_{L^2}\quad c_{11}kA\ \ u(\ ;s)k_{L^2}kA^{\frac{1}{2}}u(\ ;s)k_{L^2}^{1-} = c_{11}kA\ \ u(\ ;s)k_{L^2}kr\ u(\ ;s)k_{L^2}^{1-}\ :$$

Using the fact $kr\ u(\ ;s)k_{L^2}\quad c_{12}$ in Lemma 2.3 and the de nition of $M(T)$, from (55) we have

$$\begin{aligned} I_3\quad c_3c_{10}c_{12}^{2-}\ M\ \ (T)\int_{t_0}^{t}(t\quad s)^{-}\ ds \\ \frac{c_3c_{10}c_{12}^{2-}}{1}M\ \ (T) \\ = c_{13}M\ \ (T): \end{aligned} \tag{56}$$

Substituting (52), (53), (54) and (56) into (51), one obtain

$$M(T)\quad c_5 + c_6 + \frac{c_3c_7c_8}{1} + c_{13}M\ \ (T) = c_{14} + c_{13}M\ \ (T);$$

which gives

$$M(T)\quad \max\{2c_{14};(2c_{13})^{\frac{1}{1\ \theta}}\}: \tag{57}$$

Then using the de nition of $M(T)$, from (57) we obtain (48).

On the other hand, choosing $2\ (\frac{2}{3};1)$, and using (48) and the fact $D(A\ ) \not\subset W^{1;3}(\ )$ for $> \frac{2}{3}$(see [15, p.201] or [48, p.479]), we have

$$ku(\ ;t)k_{W^{1,3}}\quad c_{15}kA\ \ uk_{L^2}\quad c_{16}; \tag{58}$$

which yields (49). Then the proof of this lemma is completed. □

Next, we show the boundedness of $\|c(\cdot,t)\|_{L^1}$ to exclude the possibility of degeneracy.

Lemma 3.3. *Suppose the conditions in Lemma 3.1 hold. Then there exists a constant* $K_* > 0$ *such that*

$$\|c(\cdot,t)\|_{L^1} \geq K_* \tag{59}$$

and

$$d(c) \geq d(K_*) > 0. \tag{60}$$

Proof. Noting the boundedness of $\|u(\cdot,t)\|_{L^2}$ in (29) and the estimate (58), then using the Gagliardo-Nirenberg inequality we obtain

$$\|u(\cdot,t)\|_{L^1} \leq \|u(\cdot,t)\|_{W^{1,3}}^{\frac{3}{4}} \|u(\cdot,t)\|_{L^2}^{\frac{1}{4}} \leq c_1. \tag{61}$$

Let $g(\cdot,t) := n(\cdot,t) - u(\cdot,t) - r\,c(\cdot,t)$, then the combination of (30), (36) and (61) gives

$$\|g(\cdot,t)\|_{L^2} \leq \|n(\cdot,t)\|_{L^2} + \|u(\cdot,t)\|_{L^2} + r\,\|c(\cdot,t)\|_{L^2} \leq c_2 + \|u(\cdot,t)\|_{L^1}), c$$

Substituting (67) and (68) into (66), using the Hölder inequality and (63), we obtain

$$\frac{1}{3}\frac{d}{dt}\int_{\Omega} n^3 + \frac{4d(K_*)}{9}\int_{\Omega} jr\ n^{\frac{3}{2}} j^2 + \int_{\Omega} n^3 \quad \frac{c_1^2}{d(K_*)}\int_{\Omega} n^3 jr\ cj^2 + c_2$$
$$\frac{c_1^2}{d(K_*)} knk_{L^6}^3 kr\ ck_{L^4}^2 + c_2 \qquad (69)$$
$$\frac{c_3 c_1^2}{d(K_*)} knk_{L^6}^3 + c_2:$$

Using the Gagliardo-Nirenberg inequality and noting the fact $kn^{\frac{3}{2}} k_{L^{\frac{4}{3}}} = knk_{L^2}^{\frac{3}{2}}$ c_4, one derives

$$\frac{c_3 c_1^2}{d(K_*)} knk_{L^6}^3 = \frac{c_3 c_1^2}{}$$

$_{)}k2$

from (73) we obtain

$$\frac{1}{p}\frac{d}{dt}\int n^p + (p-1)d(K_*)\int n^{p-2}|\nabla n|^2 + b\int n^{p+1}$$
$$\int c_1c_2(p-1)\int n^{p-1}|\nabla n| + a\int n^p$$
$$c_1c_2(p-1)\frac{d(K_*)}{2c_1c_2}\int n^{p-2}|\nabla n|^2 + \frac{c_1c_2}{2d(K_*)}\int n^p + a\int u^p$$
$$\frac{(p-1)d(K_*)}{2}\int n^{p-2}|\nabla n|^2 + \frac{c_1^2c_2^2+2ad(K_*)}{2d(K_*)}(p-1)\int n^p;$$

which along with the facts $a \le a(p-1)$ and

$$\frac{p(p-1)d(K_*)}{2}\int n^{p-2}|\nabla n|^2 = \frac{2(p-1)d(K_*)}{p}\int |\nabla n^{\frac{p}{2}}|^2$$

gives

$$\frac{d}{dt}\int n^p + p(p-1)\int n^p + \frac{2(p-1)d(K_*)}{p}\int |\nabla n^{\frac{p}{2}}|^2 \le c_3p(p-1)\int n^p \quad (74)$$

for all $t \in (0, T_{max})$ and $p \ge 2$, where $c_3 = \frac{c_1^2c_2^2+2ad(K_*)}{2d(K_*)}+1$. Next, we will show that the last term on the right-hand side of (74) can be controlled by $\varepsilon\int |\nabla n^{\frac{p}{2}}|^2$ and $\int n^{\frac{p}{2}}{}^2$ with some small ε. Recalling the following estimates in two dimensions

$$\|f\|_{L^2}^2 \le \varepsilon\|\nabla f\|_{L^2}^2 + C(1+\varepsilon^{-1})\|f\|_{L^1}^2 \text{ for any } \varepsilon > 0;$$

and choosing $f = n^{\frac{p}{2}}$ and $\varepsilon = \frac{2d(K_*)}{c_3p^2}$, we obtain

$$c_3p(p-1)\int n^p \le \frac{2(p-1)d(K_*)}{p}\int |\nabla n^{\frac{p}{2}}|^2 + c_4p(p-1)(1+p^2)\int n^{\frac{p}{2}}{}^2. \quad (75)$$

Substituting (75) into (74) and noting that $(1+p^2) \le (1+p)^2$, one has

$$\frac{d}{dt}\int n^p + p(p-1)\int n^p \le c_4p(p-1)(1+p^2)\int n^{\frac{p}{2}}{}^2$$
$$c_4p(p-1)(1+p)^2\int n^{\frac{p}{2}}{}^2;$$

which gives

$$\frac{d}{dt}\left(e^{p(p-1)t}\int n^p\right) \le c_4e^{p(p-1)t}p(p-1)(1+p)^2\int n^{\frac{p}{2}}{}^2. \quad (76)$$

We integrate (76) over $[0, t]$ for $0 < t < T_{max}$ to obtain

$$\int n^p(x,t) \le \int n_0^p + c_4(1+p)^2\sup_{0\le t\le T_{max}}\int n^{\frac{p}{2}}(x,t)^2. \quad (77)$$

Define

$$N(p) := \max\left\{\|n_0\|_{L^1}; \sup_{0\le t\le T_{max}}\int n^p(x,t)\right\}^{\frac{1}{p}}. \quad (78)$$

Then, from (77) and (78) we can find a constant $c_5 = c_4 + |\Omega|$ such that

$$N(p) \le [c_5(1+p)^2]^{\frac{1}{p}}N\left(\frac{p}{2}\right) \text{ for all } p \ge 2.$$

Taking $p = 2^j; j = 1; 2; \dots;$ one derives

$$\begin{aligned} N(2^j) &\le c_5^{2^j}(1+2^j)^{2^j\cdot 2}N(2^{j-1}) \\ &\vdots \\ &\le c_5^{2^j+\cdots+2^1}(1+2^j)^{2^1\cdots j}\cdots(1+2)^1 N(1) \\ &\le c_5[2^{j2^1}(2^{-j}+1)^{2^1\cdots j}]\cdots[2(2^{-1}+1)^1]N(1) \\ &\le c_5 2^{2[j2^j+(j-1)2^{(j-1)}+\cdots+2^1]}\, 2^{2(2^j+2^{(j-1)}+\cdots+2^1)}N(1) \\ &\le c_5 2^6 N(1): \end{aligned}$$

Letting $j \to 1$ and noting the fact $\|n(\cdot,t)\|_{L^1} \le m_* := \max\{\|n_0\|_{L^1}; j \ge j_0\}$ in (24), we have

$$\|n(\cdot,t)\|_{L^1} \le c_5 2^6 N(1) \le c_5 2^6 \max\{\|n_0\|_{L^1}, \|n_0\|_{L^1}; j \ge j_0\} \le c_6;$$

which gives (72). □

Then we obtain the following results on the existence of global classical solution.

Lemma 3.6 (Boundedness of solution). *Let the conditions in Theorem 1.1 hold. Then the system* (4) *has a unique global classical solution satisfying* (7).

Proof. Combining Lemma 3.2, Lemma 3.4 and Lemma 3.5, we obtain two constants $\alpha \in (\frac{1}{2}, 1)$ and $C_{12} > 0$ such that

$$\|n(\cdot,t)\|_{L^1} + \|c(\cdot,t)\|_{W^{1,1}} + \|A\nabla u(\cdot,t)\|_{L^2} \le C_{12};$$

which, together with Lemma 2.1, leads to Lemma 3.6. □

3.2. Large time behavior. In this subsection, we will study the large time behavior of solution for system (4). Suppose that

$$K_0 = \max_{0\le c\le\infty} \frac{j\chi(c)j^2}{d(c)}: \tag{79}$$

We shall show that if $b > \frac{K_0}{16}$, the constant stationary state $(\frac{a}{b}, \frac{a}{b}, 0)$ is globally asymptotically stable. Motivated by the idea in [24, 39], we will construct a Lyapunov functional to study the convergence of solutions.

Lemma 3.7. *Let* $(n; c; n; P)$ *be the solution of system* (4) *obtained in Lemma 3.6 and* K_0 *defined by* (79). *Suppose that*

$$b > \frac{K_0}{16}; \tag{80}$$

then there exist positive constants $; \ _1; \ _2$ *such that for all* $t > 0$, *the nonnegative function*

$$E(t) := \int \left(n - n_* - n_* \ln \frac{n}{n_*} \right) + \frac{}{2} \int (c - n_*)^2$$

satisfies

$$E'(t) \le F(t); \tag{81}$$

where $n_* = \frac{a}{b}$ *and*

$$F(t) := {}_1 \left\{ \int (n - n_*)^2 + \int (c - n_*)^2 \right\} + {}_2 \left\{ \int d(c)\frac{jr\ nj^2}{n^2} + \int jr\ cj^2 \right\}:$$

Proof. Multiplying the first equation of system (4) by $\frac{n-n_*}{n}$ with $n_* = \frac{a}{b}$ and integrating the result over Ω, we have

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega (n - n_* - n_* \ln \frac{n}{n_*}) \\
&= -\int_\Omega r\,\frac{n-n_*}{n}\,[d(c)r\,n - \gamma(c)nr\,c] - b\int_\Omega (n-n_*)^2 \qquad (82)\\
&= -\int_\Omega d(c)\frac{jr\,nj^2}{n^2} + \int_\Omega \gamma(c)\frac{r\,n - r\,c}{n} - b\int_\Omega (n-n_*)^2.
\end{aligned}$$

We multiply the second equation of system (4) by $c - n_*$ and integrate the result to obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (c-n_*)^2 = -\int_\Omega jr\,cj^2 - \int_\Omega (c-n_*)^2 + \int_\Omega (n-n_*)(c-n_*). \qquad (83)$$

Letting $\varepsilon > 0$, then multiplying (83) by ε and adding the result to (3.10), we derive

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \Big[(n - n_* - n_* \ln \frac{n}{n_*}) + \frac{\varepsilon}{2}(c-n_*)^2\Big] \\
&= \underbrace{-\int_\Omega d(c)\frac{jr\,nj^2}{n^2} - \varepsilon\int_\Omega jr\,cj^2 + \int_\Omega \gamma(c)\frac{r\,n - r\,c}{n}}_{I_1} \\
&\quad \underbrace{-b\int_\Omega (n-n_*)^2 - \varepsilon\int_\Omega (c-n_*)^2 + \varepsilon\int_\Omega (n-n_*)(c-n_*)}_{I_2}.
\end{aligned} \qquad (84)$$

First, I_1 can be rewritten as

$$I_1 = -\int_\Omega {}^T_1 A_1\, {}_1; \quad {}_1 = \begin{pmatrix} r\,n \\ r\,c \end{pmatrix}; \quad A_1 = \begin{pmatrix} \frac{d(c)}{n^2} & -\frac{\gamma(c)}{2n} \\ -\frac{\gamma(c)}{2n} & \varepsilon \end{pmatrix};$$

where ${}^T_1$ denotes the transpose of ${}_1$. One can check that A_1 is positive definite and hence $I_1 < 0$ if and only if

$$\varepsilon > \max_{0\le c\le\infty} \frac{j\,\gamma(c)j^2}{4d(c)} = \frac{K_0}{4}. \qquad (85)$$

Similarly, we can also rewrite I_2 as

$$I_2 = -\int_\Omega {}^T_2 A_2\, {}_2; \quad {}_2 = \begin{pmatrix} n - n_* \\ c - n_* \end{pmatrix}; \quad A_2 = \begin{pmatrix} b & -\frac{\varepsilon}{2} \\ -\frac{\varepsilon}{2} & \varepsilon \end{pmatrix};$$

Hence I_2 is positive definite if and only if

$$b > \frac{\varepsilon}{4}. \qquad (86)$$

Then under the assumption (80), we can always find a positive constant ε such that (85) and (86) hold. Since A_1 and A_2 are positive definite, there exists two positive constant ${}_1, {}_2$ such that (81) holds.

Next, we will show the non-negativity of $E(t)$. In fact, let $(n) := n - n_* - n_* \ln \frac{n}{n}; n > 0$, one can check that $'(n) = 1 - \frac{n}{n}$ and $''(n) = \frac{1}{n^2}$

where $u^{\wedge}$ is between n_* and u, which implies $E(t) \quad 0$. □

Lemma 3.8. *Let the conditions in Lemma 3.10 hold. Then the solution of system (4) satisfies*

$$kn(\ ;t) \quad n_*k_{L^1} \ ! \ 0; \quad \text{as} \quad t \ ! \ 1 \tag{87}$$

and

$$kc(\ ;t) \quad n_*k_{L^1} \ ! \ 0; \quad \text{as} \quad t \ ! \ 1 \quad : \tag{88}$$

Proof. In fact, (87) follows directly, if we have

$$kn(\ ;t) \quad n_*k_{C^0} \ ! \ 0; \quad \text{as} \quad t \ ! \ 1 \quad : \tag{89}$$

Next, we will show (89) by the argument of contradiction based on some ideas in [39, 24]. In fact, suppose that (89) is wrong, then for some $c_1 > 0$, there exist some sequences $(x_j)_{j\in\mathbb{N}}$ and $(t_j)_{j\in\mathbb{N}}$ $(0;1)$ satisfying $t_j \ ! \ 1$ as $j \ ! \ 1$ such that

$$jn(x_j;t_j) \quad n_*j \quad c_1; \quad \text{for all} \quad j \ 2 \ \mathbb{N}:$$

From the parabolic regularity theory (e.g. see [35, Theorem 1.3] and [39, Lemma 3.2]), we have

$$knk_{C^{\sigma,\frac{\sigma}{2}}(\quad\times[t;t+1])} + kck_{C^{2+\sigma,1+\frac{\sigma}{2}}(\quad\times[t;t+1])} \quad C; \text{ for all } t \quad 1$$

Hence $n \quad n_*$ is uniformly continuous in $(1;1)$, and there exists $r > 0$ and $T_1 > 0$ such that for any $j \ 2 \ \mathbb{N}$,

$$jn(x;t) \quad n_*j \quad \frac{c_1}{2} \quad \text{for all} \quad x \ 2 \ B_r(x_j) \setminus \quad \text{and} \quad t \ 2 \ (t_j;t_j+T_1): \tag{90}$$

Due to the smoothness of Ω, we can get a constant $c_2 > 0$ such that

$$jB_r(x_j) \setminus \quad j \quad c_2; \quad \text{for all } x_j \ 2 \quad : \tag{91}$$

Using (90) and (91), then for all $j \ 2 \ \mathbb{N}$, one derive

$$\begin{aligned}
&\int_{t_j}^{t_j+T_1}\int \quad jn(x;t) \quad n_*j^2dxdt \\
&\int_{t_j}^{t_j+T_1}\int_{B_r(x_j)\cap} \quad jn(x;t) \quad n_*j^2dxdt \\
&\int_{t_j}^{t_j+T_1} \quad jB_r(x_j) \setminus \quad j \quad \frac{c_1}{2}\ ^2 dt \\
&\frac{c_1^2c_2T_1}{4}:
\end{aligned} \tag{92}$$

Since $E(t)$ is nonnegative by Lemma 3.10, it follows (81) that

$$\int_1^{\infty} F(s)ds \quad E(1) \quad E(t) \quad E(1) < 1 :$$

Using the definition of $F(t)$ in Lemma 3.10, one obtain

$$\int_1^{\infty}\int \left[(n \quad n_*)^2 + (c \quad n_*)^2\right] < 1 : \tag{93}$$

From (93), we derive that as $j \ ! \ 1$, $t_j \ ! \ 1$

$$\int_{t_j}^{t_j+T_1}\int (n(x;t) \quad n_*)^2dxdt \quad \int_{t_j}^{\infty}\int (n(x;t) \quad n_*)^2dxdt \ ! \ 0 \ \text{ as } j \ ! \ 1 \ ;$$

which however contradict (92). Then (89) and (87) hold by the argument of contradiction. Similarly, we can derive (88) by noting the fact $\int_1^{\infty}\int (c \quad n_*)^2 < 1$. Hence the proof of Lemma 3.8 is completed. □

Next, we will show the convergence of u as follows.

Lemma 3.9. *Let the conditions in Lemma 3.10 hold. Then it holds that*

$$\|u(\cdot,t)\|_{L^1} \to 0 \quad \text{as} \quad t \to \infty. \tag{94}$$

Proof. Multiplying the third equation of (4) by u, and using the fact $u|_{\partial\Omega} = 0$ and $\nabla u = 0$, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2 + \int_\Omega |\nabla u|^2 = \int_\Omega \nabla n \, u. \tag{95}$$

On the other hand, we use the Poincaré inequality and the fact $u|_{\partial\Omega} = 0$ to find a constant $c_1 > 0$ such that

$$\|u\|_{L^2} \le c_1 \|\nabla u\|_{L^2}; \tag{96}$$

which, together with the boundedness of χ and the Young's inequality, gives

$$\int_\Omega |\nabla n \, u \, \chi| \le \|\nabla n\|_{L^1} \|\nabla n\|_{L^2} \|u\|_{L^2} \le \frac{1}{2}\int_\Omega |\nabla u|^2 + c_2 \int_\Omega |\nabla n|^2. \tag{97}$$

Substituting (97) into (95), and using (96) again we obtain

$$\frac{d}{dt}\int_\Omega |u|^2 + c_3 \int_\Omega |u|^2 \le 2c_2 \int_\Omega |\nabla n|^2; \tag{98}$$

where $c_3 = \frac{1}{c_1}$. Using (81) and the definition of $F(t)$, we can find a constant $c_4 > 0$ such that

$$\int_1^\infty \int_\Omega d(c)\frac{|\nabla n|^2}{n^2} \le c_4;$$

which, combined with the facts $d(c) \le K_*$ in (81)$_1$

Lemma 3.10 (Large time behavior). *Suppose the conditions in Lemma 3.10 hold. Then the solution of (4) satisfies*

$$\|n(\cdot,t) - \frac{a}{b}\|_{L^1} \to 0,\ \|c(\cdot,t) - \frac{a}{b}\|_{L^1} \to 0 \text{ and } \|u(\cdot,t)\|_{L^1} \to 0 \text{ as } t \to \infty.$$

Proof. Lemma 3.10 is a consequence of Lemma 3.8 and Lemma 3.9. □

3.3. Proof of Theorem 1.1.

Proof of Theorem 1.1. Combining Lemma 3.6 and Lemma 2.1, we obtain the existence of global classical solution satisfying (7). Furthermore, the large time behavior of solution is proved in Lemma 3.10. Hence we finish the proof of Theorem 1.1. □

Acknowledgments. The author is grateful to two referees for their valuable comments, which greatly improved the exposition of the paper.

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