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Large time behavior of the full attraction–repulsion Keller–Segel system in the whole space

microglia, chemoattractant and chemorepellent respectively. Due to the lack of Lyapunov functional, the mathematical studies on the ARKS chemotaxis model are much harder than the chemotaxis model with single attractive signaling chemical v (i.e., $\xi = 0$ in (1.1)) or with single repulsive signaling chemical w (i.e., $\chi = 0$ in (1.1)). There is no pattern formation for the chemotaxis model (1.1) in the case of $\xi = 0$ or $\chi = 0$, due to the fact that the solution will blow up in higher dimensions ($n \geq 2$) in the former case (see [1–6] and reference therein) and converge to a unique constant stationary state in the latter case [7,8].

Cells can interact with several chemoattractive or chemorepulsive chemicals to generate complex biological patterns [9–13]. For the ARKS chemotaxis model (1.1) under homogeneous Neumann boundary conditions in a bounded domain $\Omega \subset \mathrm{R}^n$ with smooth boundary, the global solvability, boundedness, blow-up, critical mass phenomenon, pattern formation, existence of non-trivial stationary solutions and asymptotic behavior of the system for various ranges of parameter values are studied [14–22]. Due to the complexity of the stationary state of (1.1) with $\beta \neq \delta$, to our knowledge, there is no result on the large time behavior of system (1.1) in the case of $\beta \neq \delta$ with large initial data in bounded domain with homogeneous Neumann boundary conditions. If the domain $\Omega = \mathrm{R}^n (n \geq 2)$ and repulsion dominates (i.e., $\xi\gamma > \chi\alpha$), Shi and Wang [23] established the boundedness and decay rate of solution to system (1.1) with $\tau = 0$. However, the solution behavior was left as an open problem for the case repulsion cancels attraction (i.e., $\xi\gamma = \chi\alpha$) and $\tau = 1$.

In this paper, we will study the large time behavior of solutions to the system (1.1) with $\tau = 1$ for the whole space (i.e., $\Omega = \mathrm{R}^n$) in the case of repulsion cancels attraction (i.e., $\xi\gamma = \chi\alpha$). Precisely speaking, when $\tau = 1$ and $\xi\gamma = \chi\alpha$ we show that the uniformly-in-time bounded of solution will decays to zero as $t \to \infty$ and behaves like the heat kernel with large initial data in two or three dimensions. The main results are stated as follows.

Theorem 1.1 (*Global Existence Theorem*). *Let $n = 2$ or 3 and assume that $0 \leq u_0 \in L^1(\mathrm{R}^n) \cap L^\infty(\mathrm{R}^n)$ and $0 \leq (v_0, w_0) \in W^{1,p}(\mathrm{R}^n)(p > n)$. If $\xi\gamma = \chi\alpha$ (repulsion cancels attraction), then there exists a unique triple (u, v, w) of nonnegative functions in $C^\infty(\mathrm{R}^n \times (0, \infty); \mathrm{R}^3)$ which solves (1.1) with $\tau = 1$ classically. Furthermore, there exists a constant $C > 0$ independent of t such that*

$$\|u\|_{L^\infty} \leq C. \tag{1.2}$$

Theorem 1.2 (*Decay Rate and Asymptotic Profiles*). *If the conditions in Theorem 1.1 are satisfied, then for all $1 < p \leq \infty$, we have:*

(i) *u has the following decay rate and asymptotic profiles*

$$\sup_{t>0}(1+t)^{\frac{n}{2}(1-\frac{1}{p})}\|u\|_{L^p(\mathbb{R}^n)} < \infty \quad \text{and} \quad \lim_{t\to\infty} t^{\frac{n}{2}(1-\frac{1}{p})}\left\|u(t) - \int_{\mathbb{R}^n} u_0 dy G(t)\right\|_{L^p(\mathbb{R}^n)} = 0, \tag{1.3}$$

where $G(t) = G(x,t) = (4\pi t)^{-\frac{n}{2}}\exp\left(-\frac{x^2}{4t}\right)$ is the heat kernel.

(ii) *v, w have the same decay rate and asymptotic profiles as u.*

Remark 1.1. In Theorems 1.1 and 1.2, the results are also hold for the case of $\tau = 0$.

2. Proof of Theorem 1.1

When $\tau = 0$, the local existence and regularity of the solution to the problem (1.1) were proved in [23]. In fact, the ideas of the proofs are also valid for the case $\tau = 1$. We omit it for simplicity.

Lemma 2.1 (*Local Existence*). *If $0 \leq u_0 \in L^1(\mathrm{R}^n) \cap L^\infty(\mathrm{R}^n)$ and $0 \leq (v_0, w_0) \in W^{1,p}(\mathrm{R}^n)(p > n)$, then there exist a $T_{max} \in (0, +\infty)$ and a unique triple (u, v, w) of nonnegative functions from $C^\infty(\mathrm{R}^n \times (0, T_{max}]; \mathrm{R}^3)$*

solving (1.1) *with* $\tau = 1$ *classically in* $(0, T_{max}) \times \mathbb{R}^n$. *Moreover*

$$\text{if } T_{max} < +\infty, \text{ then } \|u(\cdot,t)\|_{L^\infty} \to \infty, \quad \text{as } t \to T_{max}. \tag{2.1}$$

Before proving our main results, we recall two lemmas which are concerned with the estimates of the heat kernel.

Lemma 2.2 ([24]). *If* $1 \le q \le p \le \infty$, *then for* $f \in L^q(\mathbb{R}^n)$, *we have*

$$\|G * f\|_{L^p(\mathbb{R}^n)} \le C t^{-\frac{n}{2}\cdot\left(\frac{1}{q}-\frac{1}{p}\right)} \|f\|_{L^q(\mathbb{R}^n)} \tag{2.2}$$

and

$$\|\nabla G * f\|_{L^p(\mathbb{R}^n)} \le C t^{-\frac{n}{2}\cdot\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{1}{2}} \|f\|_{L^q(\mathbb{R}^n)}, \tag{2.3}$$

where C *is a positive constant depending only on* n, p *and* q.

Lemma 2.3 ([24]). *Let* $1 \le q \le p \le \infty$, $\frac{1}{q} - \frac{1}{p} < \frac{1}{n}$ *and suppose that* z *is solution of the Cauchy problem*

$$\begin{cases} z_t = \Delta z - b_0 z + a_0 f, & x \in \mathbb{R}^n,\ t > 0, \\ z(x,0) = z_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{2.4}$$

with $z_0 \in W^{1,p}(\mathbb{R}^n)$. *If* $f \in L^\infty(0,\infty; L^q(\mathbb{R}^n))$, *then*

$$\|z(t)\|_{L^p(\mathbb{R}^n)} \le e^{-b_0 t}\|z_0\|_{L^p(\mathbb{R}^n)} + C \cdot \Gamma(\gamma) \sup_{0<s<t} \|f\|_{L^q(\mathbb{R}^n)} \tag{2.5}$$

and

$$\|\nabla z(t)\|_{L^p(\mathbb{R}^n)} \le e^{-b_0 t}\|\nabla z_0\|_{L^p(\mathbb{R}^n)} + C \cdot \Gamma(\tilde{\gamma}) \sup_{0<s<t} \|f\|_{L^q(\mathbb{R}^n)}, \tag{2.6}$$

for $[t, \infty)$, *where* C *is a constant independent of* p, $\Gamma(\cdot)$ *is the gamma function, and* $\gamma = 1 - \left(\frac{1}{q} - \frac{1}{p}\right)\cdot\frac{n}{2}$, $\tilde{\gamma} = \frac{1}{2} - \left(\frac{1}{q} - \frac{1}{p}\right)\cdot\frac{n}{2}$.

As in [19], we set $s := \xi w - \chi v$. With $\xi\gamma = \chi\alpha$ and $\tau = 1$, then the system (1.1) can be transformed into

$$\begin{cases} u_t = \Delta u + \nabla\cdot(u\nabla s), & x \in \mathbb{R}^n,\ t > 0, \\ s_t = \Delta s - \delta s + \chi(\beta - \delta)v, & x \in \mathbb{R}^n,\ t > 0, \\ v_t = \Delta v + \alpha u - \beta v, & x \in \mathbb{R}^n, t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad s(x,0) = \xi w_0(x) - \chi v_0(x) := s_0(x), & x \in \mathbb{R}^n. \end{cases} \tag{2.7}$$

Thus we have the following lemma which is the cornerstone of our mathematical analysis of system (2.7).

Lemma 2.4. *Letting* $\xi\gamma = \chi\alpha$ *and* $n = 2$ *or* 3, *then there exists a constant* $C > 0$ *such that*

$$\|\nabla s\|_{L^\infty} \le C. \tag{2.8}$$

Proof. First, we consider the case of $n = 2$. From the first equation of (1.1), one has $u \in L^1(\mathbb{R}^2)$. Then we can apply Lemma 2.3 to the third equation of (2.7) to obtain $\|\nabla v\|_{L^{\frac{6}{5}}(\mathbb{R}^2)} \le c_1$, which together with Sobolev inequality (see [25, Proposition 2.2] on page 4) gives

$$\|v\|_{L^3(\mathbb{R}^2)} \le c_2 \|\nabla v\|_{L^{\frac{6}{5}}(\mathbb{R}^2)} \le c_1 c_2. \tag{2.9}$$

Similarly, via (2.9) and applying Lemma 2.3 to the second equation of (2.7), one has (2.8). Next, we consider the case of $n = 3$. Similarly, applying Lemma 2.3 to the third equation of (2.7) and according to

$\|u\|_{L^1(\mathbb{R}^3)} \le c_3$, one obtain $\|\nabla v\|_{L^{\frac{12}{11}}(\mathbb{R}^3)} \le c_4$ and then

$$\|v\|_{L^{\frac{12}{7}}(\mathbb{R}^3)} \le c_5\|\nabla v\|_{L^{\frac{12}{11}}(\mathbb{R}^3)} \le c_4c_5, \tag{2.10}$$

which implies

$$\|\nabla s\|_{L^4(\mathbb{R}^3)} \le c_6, \tag{2.11}$$

by applying Lemma 2.3 to the second equation of (2.7). Multiplying the first equation of (2.7) by u and integrating with respect to x over R^3, we end up with

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} u^2dx + \int_{\mathbb{R}^3}|\nabla u|^2dx &= -\int_{\mathbb{R}^3} u\nabla u\cdot\nabla s dx\\ &\le \frac{1}{2}\|\nabla u\|^2_{L^2(\mathbb{R}^3)} + \frac{1}{2}\|u\|^2_{L^4(\mathbb{R}^3)}\|\nabla s\|^2_{L^4(\mathbb{R}^3)}.\end{aligned} \tag{2.12}$$

Substituting (2.11) into (2.12) and using the Gagliardo–Nirenberg inequality and Young's inequality, we get

$$\frac{d}{dt}\int_{\mathbb{R}^3} u^2dx + \int_{\mathbb{R}^3}|\nabla u|^2dx \le c_6^2\|u\|^2_{L^4(\mathbb{R}^3)} \le c_7\|\nabla u\|^{\frac{9}{5}}_{L^2}\|u\|^{\frac{1}{5}}_{L^1(\mathbb{R}^3)} \le \frac{1}{2}\|\nabla u\|^2_{L^2(\mathbb{R}^3)} + c_8,$$

which yields

$$\frac{d}{dt}\|u\|^2_{L^2(\mathbb{R}^3)} + \frac{1}{2}\|\nabla u\|^2_{L^2(\mathbb{R}^3)} \le c_8. \tag{2.13}$$

By using the Gagliardo–Nirenberg inequality, it follows that

$$\|u\|^2_{L^2} \le c_9\|\nabla u\|^{\frac{6}{5}}_{L^2(\mathbb{R}^3)}\|u\|^{\frac{4}{5}}_{L^1(\mathbb{R}^3)} \le \frac{1}{2}\|\nabla u\|^2_{L^2(\mathbb{R}^3)} + c_{10}. \tag{2.14}$$

Substituting (2.14) into (2.13), one has

$$\frac{d}{dt}\|u\|^2_{L^2(\mathbb{R}^3)} + \|u\|^2_{L^2(\mathbb{R}^3)} \le c_8 + c_{10}, \tag{2.15}$$

which together with the Gronwall's inequality gives

$$\|u\|_{L^2(\mathbb{R}^3)} \le c_{11}. \tag{2.16}$$

Applying Lemma 2.3 to the third equation of (2.7), and from (2.16) we have

$$\|\nabla v\|_{L^{\frac{12}{7}}(\mathbb{R}^3)} \le c_{12},$$

which implies

$$\|v\|_{L^4(\mathbb{R}^3)} \le c_{13}\|\nabla v\|_{L^{\frac{12}{7}}(\mathbb{R}^3)} \le c_{12}c_{13} = c_{14}. \tag{2.17}$$

Applying Lemma 2.3 to the second equation of (2.7) and via (2.17), one has (2.8). Hence, we complete the proof of Lemma 2.4. □

Proof of Theorem 1.1. It follows from (2.8) and the well-known Moser–Alikakos iteration procedure (cf. [23, 26]) that there exists a constant $c_3 > 0$ such that

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \le c_3 \quad \text{for all } t \in (0,T) \text{ and } n = 2,3. \tag{2.18}$$

Theorem 1.1 is an immediate consequence of (2.18) and Lemma 2.1. □

3. Proof of Theorem 1.2

In this section, we use a similar argument as in [24,27] to study the decay rate and asymptotic profiles of solutions to the ARKS model (1.1) with large initial data.

Proof of Theorem 1.2. Introducing the scalings

$$u_\varepsilon = \varepsilon^n u(\varepsilon x, \varepsilon^2 t), \qquad v_\varepsilon = v(\varepsilon x, \varepsilon^2 t), \qquad w_\varepsilon = w(\varepsilon x, \varepsilon^2 t), \tag{3.1}$$

and from the first equation of (1.1), we have

$$u_\varepsilon(x,t) = G * u_{0,\varepsilon} + \int_0^t \nabla G(t-s,\cdot) * (\chi u_\varepsilon \nabla v_\varepsilon - \xi u_\varepsilon \nabla w_\varepsilon)(s) ds, \tag{3.2}$$

where $u_{0,\varepsilon} = \varepsilon^n u_0(\varepsilon x)$. Assuming that $1 < p < 3$ and applying Lemma 2.2 to (3.2), we obtain

$$\begin{aligned} \|u_\varepsilon(t)\|_{L^p(\mathbb{R}^n)} &\le c_1 t^{-\frac{n}{2}(1-\frac{1}{p})} \|u_0\|_{L^1(\mathbb{R}^n)} \\ &\quad + c_2 \int_0^t (t-s)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})-\frac{1}{2}} (\|u_\varepsilon \nabla v_\varepsilon\|_{L^q(\mathbb{R}^n)} + \|u_\varepsilon \nabla w_\varepsilon\|_{L^q(\mathbb{R}^n)}) ds. \end{aligned} \tag{3.3}$$

Letting $\frac{1}{q} = \frac{1}{p} + \frac{1}{\kappa}$ and choosing $0 < \eta = -\frac{n}{2}\left(\frac{1}{q} - \frac{1}{p}\right) + \frac{1}{2} = \frac{1}{2} - \frac{n}{2\kappa} \le \frac{1}{2}$, from (3.3) we have

$$\begin{aligned} \|u_\varepsilon(t)\|_{L^p(\mathbb{R}^n)} &\le c_1 t^{-\frac{n}{2}(1-\frac{1}{p})} \|u_0\|_{L^1(\mathbb{R}^n)} + c_2 \int_0^t (t-s)^{\eta-1} \|u_\varepsilon\|_{L^p(\mathbb{R}^n)} (\|\nabla v_\varepsilon\|_{L^\kappa(\mathbb{R}^n)} + \|\nabla w_\varepsilon\|_{L^\kappa(\mathbb{R}^n)}) ds \\ &\le c_3 t^{-\frac{n}{2}(1-\frac{1}{p})} + c_4 \varepsilon^{1-\frac{n}{\kappa}} \int_0^t (t-s)^{\eta-1} \|u_\varepsilon\|_{L^p(\mathbb{R}^n)} ds \\ &= c_3 t^{-\frac{n}{2}(1-\frac{1}{p})} + c_4 \varepsilon^{2\eta} \int_0^t (t-s)^{\eta-1} \|u_\varepsilon\|_{L^p(\mathbb{R}^n)} ds, \end{aligned} \tag{3.4}$$

where we have used the scaling (3.1) and the boundedness of $\|\nabla v\|_{L^\kappa(\mathbb{R}^n)}$ and $\|\nabla w\|_{L^\kappa(\mathbb{R}^n)}$ to obtain

$$c_2(\|\nabla v_\varepsilon\|_{L^\kappa(\mathbb{R}^n)} + \|\nabla w_\varepsilon\|_{L^\kappa(\mathbb{R}^n)}) = c_2 \varepsilon^{1-\frac{n}{\kappa}} (\|\nabla v\|_{L^\kappa(\mathbb{R}^n)} + \|\nabla w\|_{L^\kappa(\mathbb{R}^n)}) \le c_4 \varepsilon^{1-\frac{n}{\kappa}}.$$

Applying the singular Gronwall's inequality (see [28, Lemma 7.1.1]) to (3.4) and letting $0 < t \le 1$, we get

$$\|u_\varepsilon(t)\|_{L^p(\mathbb{R}^n)} \le c_3 t^{-\frac{n}{2}(1-\frac{1}{p})} + c_5 \varepsilon^{2\eta} \int_0^t (t-s)^{\eta-1} s^{-\frac{n}{2}(1-\frac{1}{p})} ds \le c_6 (1 + \varepsilon^{2\eta}) t^{-\frac{n}{2}(1-\frac{1}{p})}, \tag{3.5}$$

where we have used the assumptions $1 < p < 3$ and $0 < \eta \le \frac{1}{2}$ to derive

$$\begin{aligned} \int_0^t (t-s)^{\eta-1} s^{-\frac{n}{2}(1-\frac{1}{p})} ds &= \int_0^{\frac{t}{2}} (t-s)^{\eta-1} s^{-\frac{n}{2}(1-\frac{1}{p})} ds + \int_{\frac{t}{2}}^t (t-s)^{\eta-1} s^{-\frac{n}{2}(1-\frac{1}{p})} ds \\ &\le \left(\frac{t}{2}\right)^{\eta-1} \int_0^{\frac{t}{2}} s^{-\frac{n}{2}(1-\frac{1}{p})} ds + \left(\frac{t}{2}\right)^{-\frac{n}{2}(1-\frac{1}{p})} \int_{\frac{t}{2}}^t (t-s)^{\eta-1} ds \\ &= \left(\frac{2p\eta + 2p - n(p-1)}{[2p - n(p-1)]\eta}\right) \left(\frac{t}{2}\right)^{-\frac{n}{2}(1-\frac{1}{p})+\eta}. \end{aligned}$$

From the scaling (3.1), we obtain

$$\|u_\varepsilon(t)\|_{L^p(\mathbb{R}^n)} = \varepsilon^{n-\frac{n}{p}} \|u(\varepsilon^2 t)\|_{L^p(\mathbb{R}^n)}. \tag{3.6}$$

Hence substituting (3.6) into (3.5) and choosing $t = 1$, we have

$$\|u(\varepsilon^2)\|_{L^p(\mathbb{R}^n)} \le c_6 (1 + \varepsilon^{2\eta}) \varepsilon^{-n+\frac{n}{p}}. \tag{3.7}$$

Noting that ε is arbitrary, from (3.7) we have

$\|u(t)\|_{L^p(\mathbb{R}^n)} \le c$626 Tf 41.585 87426 a5 4.312 -1.495 Td [(626.7))]TJ0 g 0 G0-334(w)28(e)-92 Td [((6 TJ/F17 9.96

C"626 Tf 41.585 87426 a5 4.312 -1.495 Td [(626.7))]TJ0 g 0 G0-334(w)28(e)-92 Td [((6 TJ/F16 Tf 3.597 0 Td [()]TJ/F83 9.96264Tf 4.312 -1Td [Td [(0]TJ/

where the boundedness of $\|u(t)\|_{L^p(\mathbb{R}^n)}$ is used. Next we will show the decay rate of $\|u\|_{L^\infty(\mathbb{R}^n)}$. From the first equation of (1.1), we have

$$\begin{aligned} u &= G * u_0 + \int_0^t \nabla G(t-s,\cdot) * (\chi u \nabla v - \xi u \nabla w)(s) ds \\ &= G * u_0 + \int_0^{\frac{t}{2}} \nabla G(t-s,\cdot) * (\chi u \nabla v - \xi u \nabla w)(s) ds + \int_{\frac{t}{2}}^t \nabla G(t-s,\cdot) * (\chi u \nabla v - \xi u \nabla w)(s) ds \\ &= G * u_0 + I_1 + I_2. \end{aligned} \tag{3.9}$$

Via the Hölder inequality and (3.8), we estimate the term I_1 as follows

$$\begin{aligned} \|I_1\|_{L^\infty(\mathbb{R}^n)} &\leq c_7 \int_0^{\frac{t}{2}} (t-s)^{-\frac{n}{2}-\frac{1}{2}} \|\chi u \nabla v - \xi u \nabla w\|_{L^1(\mathbb{R}^n)} ds \\ &\leq c_8 \int_0^{\frac{t}{2}} (t-s)^{-\frac{n}{2}-\frac{1}{2}} \|u\|_{L^p} \left(\|\nabla v\|_{L^{\frac{p}{p-1}}(\mathbb{R}^n)} + \|\nabla w\|_{L^{\frac{p}{p-1}}(\mathbb{R}^n)} \right) ds \\ &\leq c_9 t^{-\frac{n}{2}-\frac{1}{2}} \int_0^{\frac{t}{2}} s^{-\frac{n}{2}\left(1-\frac{1}{p}\right)+\eta} ds \\ &\leq c_{10} t^{-\frac{n}{2}+\frac{1}{2}-\frac{n}{2}\left(1-\frac{1}{p}\right)+\eta}, \end{aligned} \tag{3.10}$$

where $0 < \eta < \frac{1}{2}$. If ρ satisfies $0 < \rho = \frac{n}{2}\left(\frac{n-1}{n} - \frac{1}{p}\right) - \eta$ and $0 < \eta < \frac{n}{2}\left(\frac{n-1}{n} - \frac{1}{p}\right)$, then we can rewrite the estimate $\|I_1\|_{L^\infty}$ as follows

$$\|I_1\|_{L^\infty(\mathbb{R}^n)} \leq c_{10} t^{-\frac{n}{2}-\rho}, \qquad 0 < \rho < \frac{n}{2}\left(\frac{n-1}{n} - \frac{1}{p}\right). \tag{3.11}$$

Using the Hölder inequality, from (3.8) and the boundedness of $\|u\|_{L^\infty(\mathbb{R}^n)}$, we have

$$\|u\|_{L^q(\mathbb{R}^n)} \leq \|u\|_{L^p(\mathbb{R}^n)}^{\frac{p}{q}} \|u\|_{L^\infty(\mathbb{R}^n)}^{1-\frac{p}{q}} \leq c_{11}(1+t)^{-\frac{np}{2q}\left(1-\frac{1}{p}\right)+\frac{p\eta}{q}}. \tag{3.12}$$

Applying Lemma 2.3 to the second and third equation of (1.1), one has

$$\|\nabla v\|_{L^\infty(\mathbb{R}^n)} + \|\nabla w\|_{L^\infty(\mathbb{R}^n)} \leq c_{12}(1+t)^{-\frac{np}{2q}\left(1-\frac{1}{p}\right)+\eta\frac{p}{q}}. \tag{3.13}$$

Using (3.12) and (3.13), we have

$$\begin{aligned} \|I_2\|_{L^\infty(\mathbb{R}^n)} &\leq c_{13} \int_{\frac{t}{2}}^t (t-s)^{-\frac{n}{2q}-\frac{1}{2}} \|\chi u \nabla v - \xi u \nabla w\|_{L^q(\mathbb{R}^n)} ds \\ &\leq c_{14} \int_{\frac{t}{2}}^t (t-s)^{-\frac{n}{2q}-\frac{1}{2}} \|u\|_{L^q(\mathbb{R}^n)} (\|\nabla v\|_{L^\infty} + \|\nabla w\|_{L^\infty(\mathbb{R}^n)}) ds \\ &\leq c_{15} \int_{\frac{t}{2}}^t (t-s)^{-\frac{n}{2q}-\frac{1}{2}} (1+s)^{-\frac{np}{q}\left(1-\frac{1}{p}\right)+\frac{2p\eta}{q}} ds \\ &\leq c_{16} t^{-\frac{n}{2}-\phi(p,q,n)+\frac{2p\eta}{q}}, \end{aligned} \tag{3.14}$$

where $\varphi(p,q,n) = \frac{np}{q} - \frac{n}{2q} - \frac{n+1}{2}$. When $n = 2$ and $p = q$, we have $\varphi(p,q,2) = \frac{1}{2} - \frac{1}{q} > 0$. When $n = 3$, we can choose $\frac{5}{2} < p < 3$ and $q > 3$ such that $\varphi(p,q,3) > 0$. Hence letting $\eta > 0$ sufficiently small, one has $\varphi(p,q,n) - \frac{2p\eta}{q} > 0$, which implies that there exists a constant $\varrho > 0$ such that

$$\|I_2\|_{L^\infty(\mathbb{R}^n)} \leq c_{17} t^{-\frac{n}{2}-} . \tag{3.15}$$

Substituting (3.11) and (3.15) into (3.9), we have $\|u\|_{L^\infty(\mathbb{R}^n)} \le c_{18} t^{-\frac{n}{2}}$, which together with $u \in L^1(\mathrm{R}^n)$ gives

$$\|u\|_{L^p(\mathbb{R}^n)} \le \|u\|_{L^1(\mathbb{R}^n)}^{\frac{1}{p}} \|u\|_{L^\infty(\mathbb{R}^n)}^{1-\frac{1}{p}} \le c_{19} t^{-\frac{n}{2}(1-\frac{1}{p})}, \quad \text{for all } 1 < p \le \infty. \tag{3.16}$$

Now, we consider the asymptotic profiles of u. First, the combination of (3.9), (3.11) and (3.15) gives

$$\lim_{t\to\infty} t^{\frac{n}{2}} \|u(t) - G * u_0\|_{L^\infty(\mathbb{R}^n)} = 0. \tag{3.17}$$

Thus we have

$$\lim_{t\to\infty} t^{\frac{n}{2}} \left\| u(t) - \int_{\mathbb{R}^n} u_0(y)dyG(t) \right\|_{L^\infty(\mathbb{R}^n)} = 0, \tag{3.18}$$

by noting that $\lim_{t\to\infty} \left\| G * u_0 - \int_{\mathbb{R}^n} u_0(y)dyG(t) \right\|_{L^\infty(\mathbb{R}^n)} = 0$. For $1 < p < \infty$, using the Hölder inequality, we have

$$\begin{aligned} \left\| u(t) - \int_{\mathbb{R}^n} u_0(y)dyG(t) \right\|_{L^p(\mathbb{R}^n)} &\le \left\| u(t) - \int_{\mathbb{R}^n} u_0(y)dyG(t) \right\|_{L^1(\mathbb{R}^n)}^{\frac{1}{p}} \left\| u(t) - \int_{\mathbb{R}^n} u_0(y)dyG(t) \right\|_{L^\infty(\mathbb{R}^n)}^{1-\frac{1}{p}} \\ &\le c_{21} \left\| u(t) - \int_{\mathbb{R}^n} u_0(y)dyG(t) \right\|_{L^\infty(\mathbb{R}^n)}^{1-\frac{1}{p}} \le c_{22} t^{-\frac{n}{2}(1-\frac{1}{p})}, \end{aligned} \tag{3.19}$$

which together with (3.16) complete the proof of Theorem 1.2(i). Applying Lemma 2.3 to the second equations of (1.1) and via (3.16), for all $1 < p \le \infty$ we can derive that $\|v\|_{L^p} \le c_{20} t^{-\frac{n}{2}(1-\frac{1}{p})}$. For the asymptotic behavior of v, we give derivation as follows. Rewrite v as

$$\begin{aligned} v(t) &= e^{-\beta t} G * v_0 + \alpha \int_0^t e^{-\beta(t-s)} G(t-s,\cdot) * u(s)ds \\ &= G * u_0 + e^{-\beta t} G * (v_0 - u_0) + \alpha \int_0^{\frac{t}{2}} e^{-\beta(t-s)} G(t-s,\cdot) * (u(s) - G(s) * u_0)ds \\ &\quad + \alpha \int_{\frac{t}{2}}^t e^{-\beta(t-s)} G(t-s,\cdot) * (u(s) - G(s) * u_0)ds \\ &= G * u_0 + e^{-\beta t} G * (v_0 - u_0) + J_1(t) + J_2(t). \end{aligned} \tag{3.20}$$

From (2.2), we have $\|e^{-\beta t} G * (v_0 - u_0)\|_{L^\infty(\mathbb{R}^n)} \le e^{-\beta t} \|u_0 - v_0\|_{L^\infty(\mathbb{R}^n)}$ and

$$\begin{aligned} \|J_1(t)\|_{L^\infty(\mathbb{R}^n)} &\le \alpha \int_0^{\frac{t}{2}} e^{-\beta(t-s)} \|G(t-s,\cdot) * (u(s) - G(s) * u_0)\|_{L^\infty(\mathbb{R}^n)} ds \\ &\le c_{23} \int_0^{\frac{t}{2}} e^{-\beta(t-s)} ds \le c_{24} e^{-\frac{\beta t}{2}}. \end{aligned} \tag{3.21}$$

Combining of (2.2), (3.10), (3.11) and (3.15) leads

$$\begin{aligned} \|J_2(t)\|_{L^\infty(\mathbb{R}^n)} &\le \alpha \int_{\frac{t}{2}}^t e^{-\beta(t-s)} \|u(s) - G(s) * u_0\|_{L^\infty(\mathbb{R}^n)} ds \le \int_{\frac{t}{2}}^t e^{-\beta(t-s)} (\|I_1\|_{L^\infty(\mathbb{R}^n)} + \|I_2\|_{L^\infty(\mathbb{R}^n)}) ds \\ &\le c_{25} (1+t)^{-\frac{n}{2}-} , \end{aligned} \tag{3.22}$$

where $\varrho > 0$. Substituting (3.21) and (3.22) into (3.20), one has

$$\|v(t) - G * u_0\|_{L^\infty} \le c_{27} (1+t)^{-\frac{n}{2}-} . \tag{3.23}$$

This gives the asymptotic profiles for v. Furthermore, we can use the same argument as v to obtain the decay rate and asymptotic profiles of w. Hereto the proof of Theorem 1.2 is completed. □

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