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...ct

$$
\begin{cases}
u_t = d_1 \Delta u + u(1-u) - b_1 uv, \ldots \\
v_t = d_2 \Delta v - \nabla \cdot (\xi v \nabla u) + \ldots \\
w_t = \Delta w - \nabla \cdot (\chi w \nabla v) \ldots
\end{cases}
$$

... $\Omega \subset \mathbb{R}^2$ ...

MS... 5A01; 35B40; 3... 57; 35 92; 92 17

Key...s: ... ; B ...

*

E-m... resses: ...

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1. Introduction and main results

A ... [33]. ... [12], ...

$$\begin{cases} u_t = u(1-u) - f_1(u,v)v, & t>0, \\ v_t = f_1(u,v)v - f_2(v,w)w - \theta_1 v, & t>0, \\ w_t = f_2(v,w)w - \theta_2 w, & t>0, \end{cases} \tag{1.1}$$

$(u,v,w) := (u,v,w)(t)$ ... $t>0$. ... $f_i (i=1,2)$ ... $f_i (i=1,2)$ ... $f_i(y,z) = \frac{a_i}{b_i+y}$ ... $a_i, b_i > 0$), ... (1.1) ... [12,22,24,26], ... [25], ... [32]. $f_i (i=1,2)$ ... $f_i(y,z) = \frac{y}{y+m_i z}$ ... $m_i > 0$), ... (1.1) ... [13]. $f_i (i=1,2)$ B ... An ... $f_i(y,z) = \frac{y}{1+A_i y + B_i z}$ ... $A_i, B_i > 0$), ... (1.1) ... [30,46]. ... [8,31].

A ... (1.1) ... (1.1) ...

$$f_1(u,v) = u, \quad f_2(v,w) = v. \tag{1.2}$$

$$\begin{cases} u_t = d_1\Delta u + u(1-u) - b_1 uv, & x\in\Omega, t>0\\ v_t = d_2\Delta v - \nabla\cdot(\xi v\nabla u) + uv - b_2 vw - \theta_1 v, & x\in\Omega, t>0\\ w_t = \Delta w - \nabla\cdot(\chi w\nabla v) + vw - \theta_2 w, & x\in\Omega, t>0,\\ \partial_\nu u = \partial_\nu v = \partial_\nu w = 0, & x\in\partial\Omega, t>0,\\ u(x,0)=u_0(x), v(x,0)=v_0(x), w(x,0)=w_0(x), & x\in\Omega, \end{cases} \tag{1.3}$$

, n n n n. n n :

- Ω - n n n $\mathbb{R}^2$ n n ;
- u, v, w - n , n n , ;
- d_1, d_2 - n n n , ;
- ξ, χ - - n ;
- b_1, b_2 - n n n n , ;
- θ_1, θ_2 - n n , .

n , .

$$(u_s, v_s, w_s) = \begin{cases} (0,0,0) \quad (1,0,0), & \theta_1 \geq 1, \\ (0,0,0) \quad (1,0,0) \quad (\theta_1, \frac{1-\theta_1}{b_1}, 0), & \theta_1 < 1 \quad \theta_1 + b_1\theta_2 \geq 1, \\ (0,0,0) \quad (1,0,0) \quad (\theta_1, \frac{1-\theta_1}{b_1}, 0) \quad (u_*, v_*, w_*) & \theta_1 < 1 \quad \theta_1 + b_1\theta_2 < 1, \end{cases}$$

$$(u_*, v_*, w_*) = \left(1 - b_1\theta_2, \theta_2, \frac{1 - b_1\theta_2 - \theta_1}{b_2}\right). \tag{1.4}$$

$(0,0,0)$ … $(1,0,0)$ … $(\theta_1, \frac{1-\theta_1}{b_1}, 0)$ … (u_*, v_*, w_*) … $\|v\|_{L^\infty}$ … 1.1 … $\chi > 0$ (… 3.5). …

Theorem 1.2 *(Global stabilization). Assume the conditions in Theorem 1.1 hold. Let (u, v, w) be the solution of (1.3) obtained in Theorem 1.1 and let $K = \quad \{1, \|u_0\|_{L^\infty}\}$. Then the following results hold true.*

- *If $\theta_1 > 1$, the steady state $(1, 0, 0)$ is globally asymptotically stable;*
- *If $\theta_1 < 1$ and $\theta_1 + b_1\theta_2 > 1$, the steady state $(\theta_1, \frac{1-\theta_1}{b_1}, 0)$ is globally asymptotically stable provided*

$$\xi^2 < \frac{4d_1d_2\theta_1}{(1-\theta_1)K^2}.$$

- *If $\theta_1 < 1$ and $\theta_1 + b_1\theta_2 < 1$, the steady state (u_*, v_*, w_*) defined by (1.4) is globally asymptotically stable provided*

$$\xi^2 < \frac{4d_1d_2u_*}{b_1K^2v_*} \quad \text{and} \quad \chi^2 < \frac{4d_1d_2u_*v_* - \xi^2 b_1 K^2 v_*^2}{b_2 d_1 u_* w_* \|v\|_{L^\infty}^2}, \tag{1.5}$$

where $\|v\|_{L^\infty}$ depends on b_1, b_2, θ_1 but is independent of χ.

… (1.3) … 21 … (… 1,2,6,10,17,19,27,36,38 41,44,45 …). … (1.3) … L^∞- … w, … $\|\nabla v\|_{L^\infty}$ … w … u. … L^∞- … ∇v … $\|v \ln v\|_{L^1} + \|\nabla u\|_{L^2}$ … $\|v\|_{L^\infty}$ …

... [37] ... ([17]) ... [16, 18]. ... $\|\nabla v\|_{L^\infty}$, ... [7,15] ... u (... 3.6) ... [15], ... u_0 ... (e.g., $u_0 \in W^{2,\infty}(\Omega)$). ... 1.2 ... [11] ... [20]. ... [11,20] ... (1.3) ... (...) v and w. [20] ... L^2- ... v and w, ... (1.3) ... v and w, ... [20] ... (1.3).

... 2, ... 3, ... 1.1. ... 4, ... 1.2.

2. Local existence and preliminaries

... $\int_\Omega f\,dx$ ... $\int_\Omega f$ ... c_i and K_i $(i = 1, 2, \dots)$ ... χ and t. ... (1.3) ... Amann's ([3,4]). ...

Lemma 2.1 *(Local existence). Let the assumptions in Theorem 1.1 hold. Then there exists a constant $T_{max} \in (0, \infty]$ such that the problem (1.3) has a unique classical solution*

$$(u, v, w) \in [C^0(\bar{\Omega} \times [0, T_{max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{max}))]^3$$

satisfying $u, v, w > 0$ for all $t > 0$. Moreover,

$$if \ \ T_{max} < \infty, \ then \ \limsup_{t \nearrow T_{max}} \Big(\|u(\cdot,t)\|_{W^{1,\infty}} + \|v(\cdot,t)\|_{W^{1,\infty}} + \|w(\cdot,t)\|_{L^\infty} \Big) = \infty.$$

Lemma 2.2. *Let the assumptions in Lemma 2.1 hold. Then the solution of (1.3) satisfies*

$$\|u(\cdot,t)\|_{L^\infty} \le K, \qquad t \in (0, T_{max}), \tag{2.1}$$

where $K = \max\{1, \|u_0\|_{L^\infty}\}$.

Proof. ... (2.1) ... [17, ... 2.2]. □

Lemma 2.3. *Suppose the assumptions in Lemma 2.1 hold. Then the solution of (1.3) satisfies*

$$\|v(\cdot,t)\|_{L^1} + \|w(\cdot,t)\|_{L^1} \le K_1, \qquad t \in (0, T_{max}), \tag{2.2}$$

where $K_1 > 0$ is a constant independent of t, ξ and χ.

Proof. n n (1.3), n

$$\frac{d}{dt}\int_\Omega (u + b_1 v + b_1 b_2 w) = \int_\Omega u(1-u) - b_1\theta_1 \int_\Omega v - b_1 b_2 \theta_2 \int_\Omega w. \tag{2.3}$$

n n n

$$2\int_\Omega u \le \int_\Omega u^2 + |\Omega|,$$

n (2.3)

$$\frac{d}{dt}\int_\Omega (u + b_1 v + b_1 b_2 w) + \int_\Omega u + b_1\theta_1 \int_\Omega v + b_1 b_2 \theta_2 \int_\Omega w \le |\Omega|. \tag{2.4}$$

n $\sigma_1 =$ n$\{1, \theta_1, \theta_2\}$, (2.4),

$$\frac{d}{dt}\int_\Omega (u + b_1 v + b_1 b_2 w) + \sigma_1 \int_\Omega (u + b_1 v + b_1 b_2 w) \le |\Omega|,$$

n n (2.2). □

n n n n 35, 3.4 .

Lemma 2.4. *Let $T > 0$, $\tau \in (0, T)$, $a > 0$ and $b > 0$. Suppose that $y : [0, T) \to [0, \infty)$ is absolutely continuous and fulfills*

$$y'(t) + ay(t) \le h(t) \quad \textit{for all } t \in (0, T),$$

with some nonnegative function $h \in L^1_{loc}([0, T))$ satisfying $\int_t^{t+\tau} h(t) \le b$ for all $t \in [0, T - \tau)$. Then

$$y(t) \le \left\{ y(0) + b, \frac{b}{a\tau} + 2b \right\} \quad \textit{for all } t \in (0, T).$$

Lemma 2.5 *([29]). Let Ω be a bounded domain in $\mathbb{R}^2$ with smooth boundary and $f \in W^{1,2}(\Omega)$. Then for any $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that*

$$\|f\|_{L^3}^3 \le \varepsilon \|\nabla f\|_{L^2}^2 \|f \; n|f|\|_{L^1} + C_\varepsilon (\|f\|_{L^1}^2 \|f \; n|f|\|_{L^1} + \|f\|_{L^1}).$$

Lemma 2.6 *([28]). Assume that Ω is a bounded domain, and let $g \in C^2(\bar{\Omega})$ satisfy $\frac{\partial g}{\partial \nu} = 0$ on $\partial\Omega$. Then we have*

$$\frac{\partial |\nabla g|^2}{\partial \nu} \leq 2\kappa |\nabla g|^2,$$

where $\kappa = \kappa(\Omega)$ is an upper bound of the curvatures of $\partial\Omega$.

Lemma 2.7. *Let $\phi \in C^2(\bar{\Omega})$ be a positive function satisfying $\frac{\partial \phi}{\partial \nu} = 0$ on $\partial\Omega$. Then there exists a constant $\kappa_1 > 0$ such that*

$$\kappa_1 \left(\int_\Omega \frac{|D^2\phi|^2}{\phi} + \int_\Omega \frac{|\nabla\phi|^4}{\phi^3} \right) \leq \int_\Omega \phi |D^2 \ln \phi|^2. \tag{2.5}$$

Proof. n 43 ,

$$\int_\Omega \frac{|\nabla\phi|^4}{\phi^3} \leq (2+\sqrt{2})^2 \int_\Omega \phi |D^2 \ln \phi|^2. \tag{2.6}$$

n , n n n n nn n n n $\nabla\phi \cdot \nu = 0$ n $\partial\Omega$, n , n n n $\nabla|\nabla\phi|^2 = 2D^2\phi \cdot \nabla\phi$, n

$$\begin{aligned}
\int_\Omega \frac{|\nabla\phi|^4}{\phi^3} &= \int_\Omega |\nabla \ln\phi|^2 \nabla \ln\phi \cdot \nabla\phi \\
&= -\int_\Omega \phi \nabla|\nabla \ln\phi|^2 \cdot \nabla \ln\phi - \int_\Omega \phi |\nabla \ln\phi|^2 \Delta \ln\phi \\
&= -2\int_\Omega \frac{(D^2 \ln\phi \cdot \nabla\phi)\cdot\nabla\phi}{\phi} - \int_\Omega \frac{|\nabla\phi|^2 \Delta \ln\phi}{\phi} \\
&\leq 2\left(\int_\Omega \phi |D^2 \ln\phi|^2\right)^{\frac{1}{2}} \left(\int_\Omega \frac{|\nabla\phi|^4}{\phi^3}\right)^{\frac{1}{2}} + \left(\int_\Omega \phi |\Delta \ln\phi|^2\right)^{\frac{1}{2}} \left(\int_\Omega \frac{|\nabla\phi|^4}{\phi^3}\right)^{\frac{1}{2}} \\
&\leq (2+\sqrt{2}) \left(\int_\Omega \phi |D^2 \ln\phi|^2\right)^{\frac{1}{2}} \left(\int_\Omega \frac{|\nabla\phi|^4}{\phi^3}\right)^{\frac{1}{2}},
\end{aligned}$$

(2.6).
n n , n $(a-b)^2 \geq \frac{1}{2}a^2 - b^2$ $a, b \in \mathbb{R}$, n

$$\int_\Omega \phi |D^2 \ln\phi|^2 = \int_\Omega \phi \sum_{k,l=1}^{2} \left(\frac{1}{\phi} \cdot \frac{\partial^2\phi}{\partial x_k \partial x_l} - \frac{1}{\phi^2} \cdot \frac{\partial\phi}{\partial x_k} \cdot \frac{\partial\phi}{\partial x_l} \right)^2$$

$$\geq \frac{1}{2}\int_\Omega \phi \sum_{k,l=1}^{2}\left|\frac{1}{\phi}\cdot\frac{\partial^2\phi}{\partial x_k \partial x_l}\right|^2 - \int_\Omega \phi\left|\frac{1}{\phi^2}\cdot\frac{\partial\phi}{\partial x_k}\cdot\frac{\partial\phi}{\partial x_l}\right|^2$$
$$= \frac{1}{2}\int_\Omega |D^2\phi|^2$$

n , n 2.8 ( ) , n $z \in L^q(\Omega)$ $1 \le q < \infty$ n $C_0^\infty(\Omega)$ n n $L^q(\Omega)(1 \le q < \infty)$ (42).

3. Proof of Theorem 1.1

n n, , n n , n n n n .

3.1. Boundedness of $\|v(\cdot,t)\|_{L^\infty}$

, n n $\|v(\cdot,t)\|_{L^\infty}$ n n .

Lemma 3.1. *Let (u,v,w) be the solution obtained in Lemma 2.1. Then it holds that*

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{2}\int_\Omega \frac{|\nabla u|^2}{u} + \frac{b_1}{\xi}\int_\Omega v\ln v\right) + d_1\int_\Omega u|D^2\ln u|^2 + \frac{b_1 d_2}{\xi}\int_\Omega \frac{|\nabla v|^2}{v}\\
&\le \frac{d_1}{2}\int_{\partial\Omega} \frac{\partial|\nabla u|^2}{\partial \nu}\frac{1}{u}dS + \frac{1}{2}\int_\Omega \frac{|\nabla u|^2}{u} + \frac{b_1(K+\theta_1)}{\xi}\int_\Omega |v\ln v| + \frac{K_1 b_1}{\xi}\left(\frac{b_2}{e} + K\right),
\end{aligned} \tag{3.1}$$

for all $t \in (0, T_{max})$, where K and K_1 are presented in (2.1) and (2.2), respectively.

Proof. , , n (1.3) $-\frac{\Delta u}{u}$ n n . , n , n n

$$\begin{aligned}
-\int_\Omega \frac{u_t}{u}\Delta u + d_1\int_\Omega \frac{|\Delta u|^2}{u} &= \int_\Omega (u-1)\Delta u + b_1\int_\Omega v\Delta u\\
&= -\int_\Omega |\nabla u|^2 - b_1\int_\Omega \nabla u\cdot\nabla v, \qquad t\in(0,T_{max}).
\end{aligned} \tag{3.2}$$

n

$$\begin{aligned}
-\int_\Omega \frac{u_t}{u}\Delta u &= \int_\Omega \nabla u\cdot\nabla(\ln u)_t\\
&= \int_\Omega \nabla u\cdot\left(\frac{\nabla u}{u}\right)_t\\
&= \frac{d}{dt}\int_\Omega \frac{|\nabla u|^2}{u} - \frac{1}{2}\int_\Omega \frac{(|\nabla u|^2)_t}{u}\\
&= \frac{1}{2}\frac{d}{dt}\int_\Omega \frac{|\nabla u|^2}{u} - \frac{1}{2}\int_\Omega \frac{|\nabla u|^2}{u^2}u
\end{aligned}$$

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \frac{|\nabla u|^2}{u} + d_1 \int_\Omega \frac{|\Delta u|^2}{u} + \int_\Omega |\nabla u|^2 = \frac{1}{2}\int_\Omega \frac{|\nabla u|^2}{u^2} u_t - b_1 \int_\Omega \nabla u \cdot \nabla v. \tag{3.4}$$

n n (1.3) n, $t \in (0, T_{max})$ n

$$\begin{aligned}\frac{1}{2}\int_\Omega \frac{|\nabla u|^2}{u^2} u_t &= \frac{d_1}{2}\int_\Omega \frac{|\nabla u|^2}{u^2}\Delta u + \frac{1}{2}\int_\Omega \frac{|\nabla u|^2(1-u)}{u} - \frac{b_1}{2}\int_\Omega \frac{|\nabla u|^2 v}{u} \\ &= \frac{d_1}{2}\int_\Omega \frac{|\nabla u|^2}{u^2}\Delta u + \frac{1}{2}\int_\Omega \frac{|\nabla u|^2}{u} - \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{b_1}{2}\int_\Omega \frac{|\nabla u|^2 v}{u},\end{aligned}$$

n (3.4)

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega \frac{|\nabla u|^2}{u} + d_1\int_\Omega \frac{|\Delta u|^2}{u} + \frac{3}{2}\int_\Omega |\nabla u|^2 + \frac{b_1}{2}\int_\Omega \frac{|\nabla u|^2 v}{u} \\ &= \frac{d_1}{2}\int_\Omega \frac{|\nabla u|^2}{u^2}\Delta u - b_1 \int_\Omega \nabla u \cdot \nabla v + \frac{1}{2}\int_\Omega \frac{|\nabla u|^2}{u} \qquad t \in (0, T_{max}).\end{aligned} \tag{3.5}$$

n n n n $\nabla u \cdot \nabla \Delta u =$

$$\begin{aligned}\int_\Omega u|D^2\ln u|^2 &= \int_\Omega \frac{|D^2u|^2}{u} + \int_\Omega \frac{|\nabla u|^4}{u^3} - 2\int_\Omega \frac{1}{u^2}\left(D^2u\cdot\nabla u\right)\cdot\nabla u\\ &= \int_\Omega \frac{|D^2u|^2}{u} + \int_\Omega \frac{|\nabla u|^4}{u^3} - \int_\Omega \frac{1}{u^2}\nabla(|\nabla u|^2)\cdot\nabla u\\ &= \int_\Omega \frac{|D^2u|^2}{u} + \int_\Omega \frac{|\nabla u|^4}{u^3} + \left(\int_\Omega \frac{|\nabla u|^2}{u^2}\Delta u - 2\int_\Omega \frac{|\nabla u|^4}{u^3}\right)\\ &= \int_\Omega \frac{|D^2u|^2}{u} - \int_\Omega \frac{|\nabla u|^4}{u^3} + \int_\Omega \frac{|\nabla u|^2}{u^2}\Delta u \qquad t\in(0,T_{max}).\end{aligned}\tag{3.7}$$

n n n (3.6) n (3.7)

$$\frac{d_1}{2}\int_\Omega \frac{|\nabla u|^2}{u^2}\Delta u = d_1\int_\Omega \frac{|\Delta u|^2}{u} + \frac{d_1}{2}\int_{\partial\Omega}\frac{\partial|\nabla u|^2}{\partial\nu}\frac{1}{u}dS - d_1\int_\Omega u|D^2\ln u|^2 \tag{3.8}$$

$t\in(0,T_{max})$, n (3.5)

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega \frac{|\nabla u|^2}{u} + \frac{3}{2}\int_\Omega |\nabla u|^2 + d_1\int_\Omega u|D^2\ln u|^2 + \frac{b_1}{2}\int_\Omega \frac{|\nabla u|^2 v}{u}\\ &= \frac{d_1}{2}\int_{\partial\Omega}\frac{\partial|\nabla u|^2}{\partial\nu}\frac{1}{u}dS + \frac{1}{2}\int_\Omega \frac{|\nabla u|^2}{u} - b_1\int_\Omega \nabla u\cdot\nabla v \qquad t\in(0,T_{max}).\end{aligned}\tag{3.9}$$

n n n (1.3) $\frac{b_1(\ln v+1)}{\xi}$ n n n , $t\in(0,T_{max})$ n

$$\begin{aligned}&\frac{b_1}{\xi}\frac{d}{dt}\int_\Omega v\ln v + \frac{b_1 d_2}{\xi}\int_\Omega \frac{|\nabla v|^2}{v}\\ =&b_1\int_\Omega \nabla u\cdot\nabla v + \frac{b_1}{\xi}\int_\Omega uv(\ln v+1) - \frac{b_1\theta_1}{\xi}\int_\Omega v(\ln v+1) - \frac{b_1 b_2}{\xi}\int_\Omega wv(\ln v+1).\end{aligned}\tag{3.10}$$

n n (3.9) n (3.10), n n $0<u\le K$, $\|v(\cdot,t)\|_{L^1}\le K_1$, $\|w(\cdot,t)\|_{L^1}\le K_1$ n $v\ln v\ge -\frac{1}{e}$ $v\ge 0$, n

$$\begin{aligned}&\frac{d}{dt}\left(\frac{1}{2}\int_\Omega \frac{|\nabla u|^2}{u} + \frac{b_1}{\xi}\int_\Omega v\ln v\right) + \frac{3}{2}\int_\Omega |\nabla u|^2\\ &+ d_1\int_\Omega u|D^2\ln u|^2 + \frac{b_1}{2}\int_\Omega \frac{|\nabla u|^2 v}{u} + \frac{b_1 d_2}{\xi}\int_\Omega \frac{|\nabla v|^2}{v}\end{aligned}$$

$$
\begin{aligned}
&= \frac{d_1}{2}\int_{\partial\Omega} \frac{\partial|\nabla u|^2}{\partial\nu}\frac{1}{u}dS + \frac{1}{2}\int_{\Omega}\frac{|\nabla u|^2}{u} + \frac{b_1}{\xi}\int_{\Omega} uv(\ln v+1) \\
&\quad - \frac{b_1\theta_1}{\xi}\int_{\Omega} v(\ln v+1) - \frac{b_1 b_2}{\xi}\int_{\Omega} wv(\ln v+1) \\
&\le \frac{d_1}{2}\int_{\partial\Omega} \frac{\partial|\nabla u|^2}{\partial\nu}\frac{1}{u}dS + \frac{1}{2}\int_{\Omega}\frac{|\nabla u|^2}{u} + \frac{b_1(K+\theta_1)}{\xi}\int_{\Omega}|v\ln v| + \frac{K_1 b_1 b_2}{e\xi} + \frac{b_1 K K_1}{\xi}
\end{aligned}
$$

$t \in (0, T_{max})$, (3.1). n 3.1. □

Lemma 3.2. *Let the assumptions in Lemma 2.1 hold and (u, v, w) be the solution of (1.3). Then one has*

$$\|v\ln v(\cdot,t)\|_{L^1} + \|\nabla u(\cdot,t)\|_{L^2} \le K_2, \qquad t \in (0, T_{max}) \tag{3.11}$$

and

$$\int_t^{t+\tau}\int_{\Omega}\frac{|\nabla v|^2}{v} + \int_t^{t+\tau}\int_{\Omega}|D^2 u|^2 \le K_3 \qquad t \in (0, \tilde{T}_{max}), \tag{3.12}$$

where K_2 and K_3 are positive constants independent of χ and

$$\tau := \min\left\{1, \frac{1}{2}T_{max}\right\} \quad \text{n} \quad \tilde{T}_{max} := \begin{cases} T_{max} - \tau, & T_{max} < \infty, \\ \infty, & T_{max} = \infty. \end{cases} \tag{3.13}$$

Proof. n 2.7, n n n n n $c_1 = d_1\kappa_1$

$$d_1\int_{\Omega} u|D^2\ln u|^2 \ge c_1\left(\int_{\Omega}\frac{|D^2 u|^2}{u} + \int_{\Omega}\frac{|\nabla u|^4}{u^3}\right) \qquad t \in (0, T_{max}),$$

n (3.1)

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{2}\int_{\Omega}\frac{|\nabla u|^2}{u} + \frac{b_1}{\xi}\int_{\Omega} v\ln v\right) + c_1\left(\int_{\Omega}\frac{|D^2 u|^2}{u} + \frac{|\nabla u|^4}{u^3}\right) + \frac{b_1 d_2}{\xi}\int_{\Omega}\frac{|\nabla v|^2}{v} \\
&\le \frac{d_1}{2}\int_{\partial\Omega}\frac{\partial|\nabla u|^2}{\partial\nu}\frac{1}{u}dS + \frac{1}{2}\int_{\Omega}\frac{|\nabla u|^2}{u} + \frac{b_1(K+\theta_1)}{\xi}\int_{\Omega}|v\ln v| + \frac{b_1 K_1}{\xi}\left(\frac{b_2}{e} + K\right)
\end{aligned}
\tag{3.14}
$$

$t \in (0, T_{max})$. n 2.6 n n n (34, 52.9)

$$\|z\|_{L^2(\partial\Omega)} \le \varepsilon\|\nabla z\|_{L^2(\Omega)} + C_\varepsilon\|z\|_{L^2(\Omega)} \qquad \text{n} \quad \varepsilon > 0, \tag{3.15}$$

n , - n ,

$$\frac{d_1}{2}\int_{\partial\Omega}\frac{\partial|\nabla u|^2}{\partial\nu}\frac{1}{u}dS \le \kappa d_1\int_{\partial\Omega}\frac{|\nabla u|^2}{u}dS = 4\kappa d_1\|\nabla u^{\frac12}\|^2_{L^2(\partial\Omega)}$$
$$\le \frac{c_1}{2}\int_\Omega\left(\frac{|D^2u|^2}{u}+\frac{|\nabla u|^4}{u^3}\right)+c_2\int_\Omega\frac{|\nabla u|^2}{u} \qquad t\in(0,T_{max}),$$

, n (3.14),

$$\begin{aligned}&\frac{d}{dt}\left\{\frac12\int_\Omega\frac{|\nabla u|^2}{u}+\frac{b_1}{\xi}\int_\Omega v\ln v\right\}+\frac{c_1}{2}\int_\Omega\left(\frac{|D^2u|^2}{u}+\frac{|\nabla u|^4}{u^3}\right)+\frac{b_1d_2}{\xi}\int_\Omega\frac{|\nabla v|^2}{v}\\&\le\frac{1+2c_2}{2}\int_\Omega\frac{|\nabla u|^2}{u}+\frac{b_1(K+\theta_1)}{\xi}\int_\Omega|v\ln v|+\frac{b_1K_1}{\xi}\left(\frac{b_2}{e}+K\right)\qquad t\in(0,T_{max}).\end{aligned}\tag{3.16}$$

n n n , n n $\|u(\cdot,t)\|_{L^\infty}\le K$, n

$$\begin{aligned}\left(\frac12+\frac{1+2c_2}{2}\right)\int_\Omega\frac{|\nabla u|^2}{u}&\le(1+c_2)\left(\int_\Omega\frac{|\nabla u|^4}{u^3}\right)^{\frac12}\left(\int_\Omega u\right)^{\frac12}\\&\le\frac{c_1}{4}\int_\Omega\frac{|\nabla u|^4}{u^3}+\frac{(1+c_2)^2K|\Omega|}{c_1}\qquad t\in(0,T_{max}).\end{aligned}\tag{3.17}$$

n n , n - n n , n $\|v^{\frac12}\|_{L^2}=\|v\|_{L^1}^{\frac12}\le K_1^{\frac12}$ (2.3), n

$$\begin{aligned}\frac{b_1}{\xi}\int_\Omega v\ln v+\frac{b_1(K+\theta_1)}{\xi}\int_\Omega|v\ln v|&\le\frac{b_1(1+K+\theta_1)}{\xi}\int_\Omega|v\ln v|\\&\le\frac{c_3}{\xi}\|v^{\frac12}\|^3_{L^3}+\frac{c_3}{\xi}\\&\le\frac{c_4}{\xi}(\|\nabla v^{\frac12}\|_{L^2}\|v^{\frac12}\|^2_{L^2}+\|v^{\frac12}\|^3_{L^2})\\&\le\frac{c_4K_1}{\xi}\|\nabla v^{\frac12}\|_{L^2}+\frac{c_4K_1^{\frac32}}{\xi}\\&\le\frac{b_1d_2}{\xi}\|\nabla v^{\frac12}\|^2_{L^2}+\frac{c_5}{\xi}\qquad t\in(0,T_{max}).\end{aligned}\tag{3.18}$$

n n (3.17) n (3.18) n (3.16), n $t\in(0,T_{max})$

$$\frac{d}{dt}\left(\frac{1}{2}\int_\Omega\left(\frac{|\nabla u|^2}{u}+\frac{b_1}{\xi}\int_\Omega v\ln v\right)\right)+\left(\frac{1}{2}\int_\Omega\frac{|\nabla u|^2}{u}+\frac{b_1}{\xi}\int_\Omega v\ln v\right)$$
$$+\frac{c_1}{4}\int_\Omega\left(\frac{|D^2u|^2}{u}+\frac{|\nabla u|^4}{u^3}\right)+\frac{3b_1d_2}{4\xi}\int_\Omega\frac{|\nabla v|^2}{v}\le c_6, \tag{3.19}$$

$c_6 := \frac{(1+c_2)^2K|\Omega|}{c_1}+\frac{c_5}{\xi}+\frac{b_1K_1}{\xi}(\frac{b_2}{e}+K)$. (3.19),

$$\frac{1}{2}\int_\Omega\frac{|\nabla u|^2}{u}+\frac{b_1}{\xi}\int_\Omega v\ln v\le c_7\left(1+\frac{1}{\xi}\right)\qquad t\in(0,T_{max})$$

(3.11) $0<u\le K$ $v\ln v\ge -\frac{1}{e}$ $v\ge 0$. $0<u\le K$, (3.19) $(t,t+\tau)$ (3.11)

$$\frac{c_1}{4}\int_t^{t+\tau}\int_\Omega\left(\frac{|D^2u|^2}{u}+\frac{|\nabla u|^4}{u^3}\right)+\frac{3b_1d_2}{4\xi}\int_t^{t+\tau}\int_\Omega\frac{|\nabla v|^2}{v}\le c_8\left(1+\frac{1}{\xi}\right)\qquad t\in(0,T_{max}),$$

(3.12). □

3.2

Lemma 3.3. *Let the assumptions in Lemma 2.1 hold and (u,v,w) be the solution of (1.3). Then it holds that*

$$\|v(\cdot,t)\|_{L^2}+\|\nabla u(\cdot,t)\|_{L^4}\le K_4\qquad t\in(0,T_{max}), \tag{3.20}$$

where $K_4>0$ is a constant independent of χ and t.

Proof. (1.3) $2\nabla u$, $\Delta|\nabla u|^2=2\nabla u\cdot\nabla\Delta u+2|D^2u|^2$

$$\begin{aligned}(|\nabla u|^2)_t&=2d_1\nabla u\cdot\nabla\Delta u-2b_1\nabla u\cdot\nabla(uv)+2|\nabla u|^2(1-2u)\\&=d_1\Delta|\nabla u|^2-2d_1|D^2u|^2-2b_1\nabla u\cdot\nabla(uv)+2|\nabla u|^2(1-2u)\end{aligned} \tag{3.21}$$

$t\in(0,T_{max})$. (3.21) $2|\nabla u|^2$ $0<u\le K$,

$$\frac{d}{dt}\int_\Omega |\nabla u|^4 + 2d_1\int_\Omega |\nabla|\nabla u|^2|^2 + 4d_1\int_\Omega |\nabla u|^2|D^2u|^2$$
$$= 2d_1\int_{\partial\Omega} |\nabla u|^2\frac{\partial|\nabla u|^2}{\partial\nu}dS$$

$$
\begin{aligned}
&= 4(\sqrt{2}+2)b_1K\int_\Omega v|\nabla u|^2|D^2u| \\
&\le 2d_1\int_\Omega |\nabla u|^2|D^2u|^2 + \frac{2(2+\sqrt{2})^2b_1^2K^2}{d_1}\int_\Omega v^2|\nabla u|^2 \qquad t\in(0,T_{max}).
\end{aligned}
$$

n (3.25) n (3.26) n (3.22), n

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega |\nabla u|^4 + \int_\Omega |\nabla u|^4 + d_1\int_\Omega |\nabla|\nabla u|^2|^2 + 2d_1\int_\Omega |\nabla u|^2|D^2u|^2 \\
&\le \frac{2(2+\sqrt{2})^2b_1^2K^2}{d_1}\int_\Omega v^2|\nabla u|^2 + c_3 \qquad t\in(0,T_{max}).
\end{aligned}
\tag{3.27}
$$

n (1.3) v, n n , n n n n $\|u(\cdot,t)\|_{L^\infty}\le K$, n $t\in(0,T_{max})$

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega v^2 + d_2\int_\Omega |\nabla v|^2 + b_2\int_\Omega v^2 w + \theta_1\int_\Omega v^2 &= \xi\int_\Omega v\nabla u\cdot\nabla v + \int_\Omega uv^2 \\
&\le \frac{d_2}{2}\int_\Omega |\nabla v|^2 + \frac{\xi^2}{2d_2}\int_\Omega v^2|\nabla u|^2 + K\int_\Omega v^2,
\end{aligned}
$$

$$
\frac{d}{dt}\int_\Omega v^2 + \int_\Omega v^2 + d_2\int_\Omega |\nabla v|^2 \le \frac{\xi^2}{d_2}\int_\Omega v^2|\nabla u|^2 + (2K+1)\int_\Omega v^2 \qquad t\in(0,T_{max}). \tag{3.28}
$$

n $y(t) := \int_\Omega v^2 + \int_\Omega |\nabla u|^4$, n n n (3.27) n (3.28), n

$$
\begin{aligned}
&y'(t) + y(t) + d_2\int_\Omega |\nabla v|^2 + d_1\int_\Omega |\nabla|\nabla u|^2|^2 + 2d_1\int_\Omega |\nabla u|^2|D^2u|^2 \\
&\le \left(\frac{2(2+\sqrt{2})^2b_1^2K^2d_2+\xi^2d_1}{d_1d_2}\right)\int_\Omega v^2|\nabla u|^2 + (2K+1)\int_\Omega v^2 + c_3 \qquad (3.29)\\
&\le \left(\frac{2(2+\sqrt{2})^2b_1^2K^2d_2+\xi^2d_1}{d_1d_2}\right)\left(\int_\Omega v^3\right)^{\frac{2}{3}}\left(\int_\Omega |\nabla u|^6\right)^{\frac{1}{3}} + (2K+1)|\Omega|^{\frac{1}{3}}\left(\int_\Omega v^3\right)^{\frac{2}{3}} + c_3 \\
&\le c_4\|v\|_{L^3}^2(\|\nabla u\|_{L^6}^2+1) + c_3 \qquad t\in(0,T_{max}),
\end{aligned}
$$

$c_4 = \frac{2(2+\sqrt{2})^2b_1^2K^2d_2+\xi^2d_1}{d_1d_2} + (2K+1)|\Omega|^{\frac{1}{3}}$.

n n n n (3.11), n

$$\begin{aligned}\|\nabla u\|_{L^6}^2 = \||\nabla u|^2\|_{L^3} &\le c_5 \|\nabla|\nabla u|^2\|_{L^2}^{\frac{2}{3}} \||\nabla u|^2\|_{L^1}^{\frac{1}{3}} + c_5\||\nabla u|^2\|_{L^1} \\ &\le c_5 K_2^{\frac{2}{3}} \|\nabla|\nabla u|^2\|_{L^2}^{\frac{2}{3}} + c_5 K_2^2 \qquad t \in (0, T_{max}).\end{aligned} \tag{3.30}$$

n n n n (3.30), n

$$\begin{aligned}c_4\|v\|_{L^3}^2(\|\nabla u\|_{L^6}^2 + 1) &\le c_4 c_5 K_2^{\frac{2}{3}} \|v\|_{L^3}^2 \|\nabla|\nabla u|^2\|_{L^2}^{\frac{2}{3}} + c_4(c_5 K_2^2 + 1)\|v\|_{L^3}^2 \\ &\le d_1\|\nabla|\nabla u|^2\|_{L^2}^2 + c_6\|v\|_{L^3}^3 + c_7 \qquad t \in (0, T_{max}),\end{aligned} \tag{3.31}$$

c_6 n c_7 n n n n n χ n t.
n n , n $\|v$ n $v\|_{L^1} \le K_2$ (3.2), 2.3 n 2.5,

$$c_6\|v\|_{L^3}^3 \le d_2\|\nabla v\|_{L^2}^2 + c_8 \qquad t \in (0, T_{max}). \tag{3.32}$$

n n (3.31) n (3.32) n (3.29), n

$$y'(t) + y(t) \le c_3 + c_7 + c_8 \qquad t \in (0, T_{max}),$$

n n (3.20) n -
3.3. □

Lemma 3.4. *Let the assumptions in Lemma 2.1 hold and (u, v, w) be the solution of (1.3). Then there exists a constant $K_5 > 0$ independent of χ and t such that*

$$\|v(\cdot, t)\|_{L^3} \le K_5 \qquad t \in (0, T_{max}). \tag{3.33}$$

Proof. n n (1.3) v^2, n n Ω. n (2.1) n (3.20), n

$$\begin{aligned}&\frac{1}{3}\frac{d}{dt}\int_\Omega v^3 + 2d_2\int_\Omega v|\nabla v|^2 + \theta_1\int_\Omega v^3 \\ &= 2\xi\int_\Omega v^2\nabla u\cdot\nabla v + \int_\Omega uv^3 - b_2\int_\Omega v^3 w \\ &\le d_2\int_\Omega v|\nabla v|^2 + \frac{\xi^2}{d_2}\int_\Omega v^3|\nabla u|^2 + K\int_\Omega v^3 \\ &\le d_2\int_\Omega v|\nabla v|^2 + \frac{\xi^2}{d_2}\left(\int_\Omega v^6\right)^{\frac{1}{2}}\left(\int_\Omega |\nabla u|^4\right)^{\frac{1}{2}} + K\int_\Omega v^3\end{aligned}$$

$$\le d_2\int_\Omega v|\nabla v|^2 + \frac{K_4^2\xi^2}{d_2}\|v\|_{L^6}^3 + K\|v\|_{L^3}^3 \qquad t\in(0,T_{max}),$$

n

$$\frac{d}{dt}\|v\|_{L^3}^3 + \frac{4d_2}{3}\|\nabla v^{\frac{3}{2}}\|_{L^2}^2 + 3\theta_1\|v\|_{L^3}^3 \le \frac{3K_4^2\xi^2}{d_2}\|v\|_{L^6}^3 + 3K\|v\|_{L^3}^3$$
$$\le \left(\frac{3K_4^2\xi^2}{d_2} + |\Omega|^{\frac{1}{2}}\right)\|v\|_{L^6}^3 \qquad t\in(0,T_{max}). \tag{3.34}$$

n $\|v^{\frac{3}{2}}(\cdot,t)\|_{L^{\frac{4}{3}}} = \|v(\cdot,t)\|_{L^2}^{\frac{3}{2}} \le K_4^{\frac{3}{2}}$, n n - n n , - n n ' n , ,

$$\begin{aligned}\left(\frac{3K_4^2\xi^2}{d_2} + |\Omega|^{\frac{1}{2}}\right)\|v\|_{L^6}^3 &= \left(\frac{3K_4^2\xi^2}{d_2} + |\Omega|^{\frac{1}{2}}\right)\|v^{\frac{3}{2}}\|_{L^4}^2\\ &\le c_1\left(\|\nabla v^{\frac{3}{2}}\|_{L^2}^{\frac{4}{3}}\|v^{\frac{3}{2}}\|_{L^{\frac{4}{3}}}^{\frac{2}{3}} + \|v^{\frac{3}{2}}\|_{L^{\frac{4}{3}}}^2\right)\\ &\le c_2\|\nabla v^{\frac{3}{2}}\|_{L^2}^{\frac{4}{3}} + c_3\\ &\le \frac{4d_2}{3}\|\nabla v^{\frac{3}{2}}\|_{L^2}^2 + c_4 \qquad t\in(0,T_{max}).\end{aligned} \tag{3.35}$$

n n (3.35) n (3.34), n

$$\frac{d}{dt}\|v\|_{L^3}^3 + 3\theta_1\|v\|_{L^3}^3 \le c_4 \qquad t\in(0,T_{max}),$$

(3.33) n ' n . □

Lemma 3.5. *Let the assumptions in Lemma 2.1 hold and* (u,v,w) *be the solution of* (1.3)*. Then it holds that*

$$\|v(\cdot,t)\|_{L^\infty} \le K_6 \qquad t\in(0,T_{max}), \tag{3.36}$$

where K_6 *is a positive constant independent of* χ *and* t*.*

Proof. A n n- - n n n (1.3),

$$u(\cdot,t) = e^{d_1t\Delta}u_0 + \int_0^t e^{d_1(t-s)\Delta}[u(1-u)-b_1uv]ds \qquad t\in(0,T_{max}),$$

n n

$$\nabla u(\cdot,t) = \nabla e^{d_1 t\Delta} u_0 + \int_0^t \nabla e^{d_1(t-s)\Delta}[u(1-u) - b_1 uv]ds \qquad t \in (0, T_{max}). \tag{3.37}$$

n n n $0 < u \le K$ n $\|v(\cdot,t)\|_{L^3} \le K_5$, n all

A n L^p-L^q (2.10), (2.11) n , n (3.40), (3.42)

$$
\begin{aligned}
\|v(\cdot,t)\|_{L^\infty} \le& \|e^{t(d_2\Delta-\theta_1)}v_0\|_{L^\infty} + \xi\int_0^t \|e^{(t-s)(d_2\Delta-\theta_1)}\nabla\cdot(v\nabla u)\|_{L^\infty}ds \\
&+ \int_0^t \|e^{(t-s)(d_2\Delta-\theta_1)}uv\|_{L^\infty}ds \\
\le& \|v_0\|_{L^\infty} + \gamma_4\xi\int_0^t (1+(t-s)^{-\frac{5}{6}})e^{-(\lambda_1 d_2+\theta_1)(t-s)}\|v\nabla u\|_{L^3}ds \\
&+ \gamma_3\int_0^t (1+(t-s)^{-\frac{1}{3}})e^{-\theta_1(t-s)}\|uv\|_{L^3}ds \\
\le& \|v_0\|_{L^\infty} + \gamma_4\xi(c_1+K)K_5\int_0^\infty (1+(t-s)^{-\frac{5}{6}})e^{-\theta_1(t-s)}ds \\
&+ \gamma_3(c_1+K)K_5\int_0^\infty (1+(t-s)^{-\frac{1}{3}})e^{-\theta_1(t-s)}ds \\
\le& \|v_0\|_{L^\infty} + (c_1+K)K_5\left(\frac{\gamma_4}{\theta_1}\xi(1+\theta_1^{\frac{5}{6}}\Gamma(1/6)) + \frac{\gamma_3}{\theta_1}(1+\theta_1^{\frac{1}{3}}\Gamma(2/3))\right)
\end{aligned}
$$

$t \in (0, T_{max})$, (3.36). 3.5. □

3.2. Boundedness of $\|w(\cdot,t)\|_{L^\infty}$

n , n, , n n $\|w(\cdot,t)\|_{L^\infty}$. n , n n 15, 2.4 n 7, 2.5 , n n n. , , u .

Lemma 3.6. *Let the assumptions in Lemma 2.1 hold and (u, v, w) be the solution of the system (1.3). Then there exists a constant $K_7 > 0$ independent of t such that and for all $p > 1$*

$$
\int_t^{t+\tau} \|D^2u\|_{L^p}^p \le K_7 \qquad t \in (0, \tilde{T}_{max}) \tag{3.43}
$$

and

$$
\int_\tau^t e^{-p(t-s)}\|\Delta u\|_{L^p}^p \le K_7 \qquad t \in (\tau, T_{max}), \tag{3.44}
$$

where τ and $\tilde{T}_{max}$ are defined in (3.13).

Proof. (1.3)

$$u_t - d_1 \Delta u + u = F(x,t)$$

$F(x,t) := u(2-u) - b_1 uv$. B 2.2 3.5, $\|F(x,t)\|_{L^\infty}$ $u_0 \in W^{2,\infty}(\Omega)$, (3.43) 15, 2.4 . $u(x,\tau) \in C^2(\bar{\Omega})$ $\tau \in (0, T_{max})$ 2.1. (3.44) (7, 2.5). □

Lemma 3.7. *Let the assumptions in Lemma 2.1 hold and* (u, v, w) *be the solution of* (1.3). *Then it holds that*

$$\|w \ln w(\cdot,t)\|_{L^1} \le K_8 \qquad t \in (0, T_{max}), \tag{3.45}$$

where $K_8 > 0$ *is a constant independent of* t.

Proof. (1.3) $-\frac{\Delta v}{v}$,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \frac{|\nabla v|^2}{v} + d_2 \int_\Omega \frac{|\Delta v|^2}{v} \\
&= \frac{1}{2}\int_\Omega \frac{|\nabla v|^2}{v^2} v_t + \xi \int_\Omega \nabla\cdot(v\nabla u)\frac{\Delta v}{v} - \int_\Omega u\Delta v - b_2 \int_\Omega \nabla v\cdot\nabla w \\
&= \frac{1}{2}\int_\Omega \frac{|\nabla v|^2}{v^2} v_t + \xi \int_\Omega \frac{\nabla v\cdot\nabla u\Delta v}{v} + \xi \int_\Omega \Delta u\cdot\Delta v - \int_\Omega u\Delta v - b_2 \int_\Omega \nabla v\cdot\nabla w
\end{aligned} \tag{3.46}$$

$t \in (0, T_{max})$. (1.3) , $t \in (0, T_{max})$

$$\begin{aligned}
\frac{1}{2}\int_\Omega \frac{|\nabla v|^2}{v^2} v_t =& \frac{d_2}{2}\int_\Omega \frac{|\nabla v|^2}{v^2}\Delta v - \frac{\xi}{2}\int_\Omega \frac{|\nabla v|^2}{v^2}\nabla\cdot(v\nabla u) + \frac{1}{2}\int_\Omega \frac{|\nabla v|^2 u}{v} \\
&- \frac{b_2}{2}\int_\Omega \frac{|\nabla v|^2 w}{v} - \frac{\theta_1}{2}\int_\Omega \frac{|\nabla v|^2}{v} \\
=& \frac{d_2}{2}\int_\Omega \frac{|\nabla v|^2}{v^2}\Delta v - \frac{\xi}{2}\int_\Omega \frac{|\nabla v|^2\nabla v\cdot\nabla u}{v^2} - \frac{\xi}{2}\int_\Omega \frac{|\nabla v|^2\Delta u}{v} \\
&+ \frac{1}{2}\int_\Omega \frac{|\nabla v|^2 u}{v} - \frac{b_2}{2}\int_\Omega \frac{|\nabla v|^2 w}{v} - \frac{\theta_1}{2}\int_\Omega \frac{|\nabla v|^2}{v}.
\end{aligned} \tag{3.47}$$

$\nabla v\cdot\nu = 0$ $\partial\Omega$, (3.6), (3.7) (3.8), $t \in (0, T_{max})$

$$\frac{d_2}{2}\int_\Omega \frac{|\nabla v|^2}{v^2}\Delta v = d_2\int_\Omega \frac{|\Delta v|^2}{v} + \frac{d_2}{2}\int_{\partial\Omega}\frac{\partial|\nabla v|^2}{\partial\nu}\frac{1}{v}dS - d_2\int_\Omega v|D^2 \ln v|^2. \tag{3.48}$$

, n 2.7, n n n n $c_1=\kappa_1 d_2>0$,

$$c_1\left(\int_\Omega \frac{|D^2 v|^2}{v} + \int_\Omega \frac{|\nabla v|^4}{v^3}\right) \le d_2\int_\Omega v|D^2 \ln v|^2 \qquad t\in(0,T_{max}). \tag{3.49}$$

n n (3.47), (3.48) n (3.49) n (3.46), n

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \frac{|\nabla v|^2}{v} + c_1\left(\int_\Omega \frac{|D^2 v|^2}{v} + \int_\Omega \frac{|\nabla v|^4}{v^3}\right) + \frac{b_2}{2}\int_\Omega \frac{|\nabla v|^2 w}{v} + \frac{\theta_1}{2}\int_\Omega \frac{|\nabla v|^2}{v}\\
\le& \frac{d_2}{2}\int_{\partial\Omega}\frac{\partial|\nabla v|^2}{\partial\nu}\frac{1}{v}dS - \frac{\xi}{2}\int_\Omega \frac{|\nabla v|^2\nabla v\cdot\nabla u}{v^2} - \frac{\xi}{2}\int_\Omega \frac{|\nabla v|^2\Delta u}{v} + \frac{1}{2}\int_\Omega \frac{|\nabla v|^2 u}{v}\\
&+\xi\int_\Omega \frac{\nabla v\cdot\nabla u\Delta v}{v} + \xi\int_\Omega \Delta u\cdot\Delta v - \int_\Omega u\Delta v - b_2\int_\Omega \nabla v\cdot\nabla w \qquad t\in(0,T_{max}).
\end{aligned}\tag{3.50}$$

n n (1.3) $\frac{b_2}{\chi}(\ln w+1)$, n n n , $t\in(0,T_{max})$ n

$$\frac{b_2}{\chi}\frac{d}{dt}\int_\Omega w\ln w + \frac{b_2}{\chi}\int_\Omega \frac{|\nabla w|^2}{w} = b_2\int_\Omega \nabla v\cdot\nabla w + \frac{b_2}{\chi}\int_\Omega vw(\ln w+1) - \frac{b_2\theta_2}{\chi}\int_\Omega w(\ln w+1). \tag{3.51}$$

n n (3.51) (3.50), n n $\|w(\cdot,t)\|_{L^1}\le K_1$ ((2.2)), $\|u\|_{L^\infty}+\|\nabla u\|_{L^\infty}+\|v\|_{L^\infty}\le c_2$ ((2.1), (3.36) n (3.39)), n

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{b_2}{\chi}\int_\Omega w\ln w + \frac{1}{2}\int_\Omega \frac{|\nabla v|^2}{v}\right) + \left(\frac{b_2}{\chi}\int_\Omega w\ln w + \frac{1}{2}\int_\Omega \frac{|\nabla v|^2}{v}\right)\\
&+c_1\left(\int_\Omega \frac{|D^2 v|^2}{v} + \int_\Omega \frac{|\nabla v|^4}{v^3}\right) + \frac{b_2}{\chi}\int_\Omega \frac{|\nabla w|^2}{w}\\
\le& \frac{d_2}{2}\int_{\partial\Omega}\frac{\partial|\nabla v|^2}{\partial\nu}\frac{1}{v}dS + \frac{\xi(c_2+1)}{2}\int_\Omega \frac{|\nabla v|^3}{v^2} + \frac{\xi}{2}\int_\Omega \frac{|\nabla v|^2|\Delta u|}{v} + \frac{c_2+1}{2}\int_\Omega \frac{|\nabla v|^2}{v}\\
&+\xi c_2\int_\Omega \frac{|\nabla v||\Delta v|}{v} + \xi\int_\Omega |\Delta u||\Delta v| + c_2\int_\Omega |\Delta v|\\
&+\frac{b_2(c_2+1+2\theta_2)}{\chi}
\end{aligned}$$

( ) / 3 1 576 0 0 7.57 7.5716 58.908 314.13 (2) 53.92 1 40 96 0 0 n 2.6 n n , (3.15) , n ' n , , $t \in (0, T_{max})$ n n

$$\frac{d_2}{2}\int_{\partial\Omega}\frac{\partial|\nabla v|^2}{\partial\nu}\frac{1}{v}dS \le \kappa d_2\int_{\partial\Omega}\frac{|\nabla v|^2}{v}dS = 4\kappa d_2\|\nabla v^{\frac{1}{2}}\|^2_{L^2(\partial\Omega)}$$
$$\le \frac{c_1}{8}\int_\Omega\left(\frac{|D^2v|^2}{v}+\frac{|\nabla v|^4}{v^3}\right)+c_3\int_\Omega\frac{|\nabla v|^2}{v} \tag{3.53}$$
$$\le \frac{c_1}{6}\int_\Omega\left(\frac{|D^2v|^2}{v}+\frac{|\nabla v|^4}{v^3}\right)+c_4.$$

, n , n ' n , n , n n $\|v\|_{L^\infty}$

$$\frac{\xi c_2}{2}\int_\Omega\frac{|\nabla v|^3}{v^2} \le \frac{c_1}{8}\int_\Omega\frac{|\nabla v|^4}{v^3}+c_5\int_\Omega\frac{|\nabla v|^2}{v} \le \frac{c_1}{6}\int_\Omega\frac{|\nabla v|^4}{v^3}+c_6 \qquad t\in(0,T_{max})$$

n

$$\frac{\xi}{2}\int_\Omega\frac{|\nabla v|^2|\Delta u|}{v}+\frac{c_2+1}{2}\int_\Omega\frac{|\nabla v|^2}{v}+\xi c_2\int_\Omega\frac{|\nabla v||\Delta v|}{v}+\xi\int_\Omega|\Delta u||\Delta v|+c_2\int_\Omega|\Delta v|$$
$$\le \frac{c_1}{6}\int_\Omega\left(\frac{|D^2v|^2}{v}+\frac{|\nabla v|^4}{v^3}\right)+c_7\int_\Omega\left(v|\Delta u|^2+v\right)$$
$$\le \frac{c_1}{6}\int_\Omega\left(\frac{|D^2v|^2}{v}+\frac{|\nabla v|^4}{v^3}\right)+c_8\int_\Omega|D^2u|^2+c_9 \qquad t\in(0,T_{max}).$$

A , n - n n

$$\frac{b_2(c_2+1+\theta_2)}{\chi}\int_\Omega|w\ \mathrm{n}\,w| \le c_{10}\int_\Omega w^{\frac{3}{2}}+c_{11} \le \frac{b_2}{\chi}\int_\Omega\frac{|\nabla w|^2}{w}+c_{12} \qquad t\in(0,T_{max}). \tag{3.54}$$

n , n (3.53)-(3.54) n (3.52), $t\in(0,T_{max})$ n

$$\frac{d}{dt}\left(\frac{b_2}{\chi}\int_\Omega w\ \mathrm{n}\,w+\frac{1}{2}\int_\Omega\frac{|\nabla v|^2}{v}\right)+\left(\frac{b_2}{\chi}\int_\Omega w\ \mathrm{n}\,w+\frac{1}{2}\int_\Omega\frac{|\nabla v|^2}{v}\right)\le c_8\int_\Omega|D^2u|^2+c_{13},$$

(3.12) n 2.4

$$\frac{b_2}{\chi}\int_\Omega w\ \mathrm{n}\,w+\frac{1}{2}\int_\Omega\frac{|\nabla v|^2}{v}\le c_{14} \qquad t\in(0,T_{max}),$$

n n

$$\int_\Omega w \ln w \le c_{15} \qquad t \in (0, T_{max}),$$

(3.45) n n $w \ln w \ge -\frac{1}{e}$ $w \ge 0$. □

$\int_\Omega (w^2 + |\nabla v|^4)$. n n $\|w(\cdot,t)\|_{L^2}$ n n

Lemma 3.8. *Let the assumptions in Lemma 2.1 hold and (u, v, w) be the solution of (1.3). Then there is a constant $K_9 > 0$ such that*

$$\|w(\cdot,t)\|_{L^2} + \|\nabla v(\cdot,t)\|_{L^4} \le K_9, \qquad t \in (0, T_{max}). \tag{3.55}$$

Proof. n n (1.3) w, n n n n (3.36), n

$$\frac{1}{2}\frac{d}{dt}\int_\Omega w^2 + \int_\Omega |\nabla w|^2 + \theta_2 \int_\Omega w^2 = \chi \int_\Omega w \nabla v \cdot \nabla w + \int_\Omega v w^2 \le \frac{1}{2}\int_\Omega |\nabla w|^2$$
$$+ \frac{\chi^2}{2}\int_\Omega w^2 |\nabla v|^2 + K_6 \int_\Omega w^2 \qquad t \in (0, T_{max}),$$

$$\frac{d}{dt}\int_\Omega w^2 + \int_\Omega |\nabla w|^2 + \int_\Omega w^2 \le \chi^2 \int_\Omega w^2 |\nabla v|^2 + (2K_6 + 1)\int_\Omega w^2 \qquad t \in (0, T_{max}). \tag{3.56}$$

n (3.45), (2.2) n n 2.5, n $(2K_6+1)\int_\Omega w^2 \le \frac{1}{2}\int_\Omega |\nabla w|^2 + c_1$, n (3.56)

$$\frac{d}{dt}\int_\Omega w^2 + \frac{1}{2}\int_\Omega |\nabla w|^2 + \int_\Omega w^2 \le \chi^2 \int_\Omega w^2 |\nabla v|^2 + c_1 \qquad t \in (0, T_{max}). \tag{3.57}$$

n n n n n (1.3) n $\nabla \Delta v \cdot \nabla v = \frac{1}{2}\Delta |\nabla v|^2 - |D^2 v|^2$, n

$$\frac{1}{4}\frac{d}{dt}\int_\Omega |\nabla v|^4 = \int_\Omega |\nabla v|^2 \nabla v \cdot \nabla v_t$$
$$= d_2 \int_\Omega |\nabla v|^2 \nabla v \cdot \nabla \Delta v - \xi \int_\Omega |\nabla v|^2 \nabla v \cdot \nabla(\nabla \cdot (v \nabla u)) + \int_\Omega \nabla(uv - b_2 vw - \theta_1 v) \cdot \nabla v |\nabla v|^2$$

$$
\begin{aligned}
&= \frac{d_2}{2}\int_{\partial\Omega} |\nabla v|^2 \frac{\partial |\nabla v|^2}{\partial \nu} dS - \frac{d_2}{2}\int_\Omega |\nabla|\nabla v|^2|^2 - d_2 \int_\Omega |\nabla v|^2 |D^2 v|^2 \\
&\quad - \xi \int_\Omega |\nabla v|^2 \nabla v \cdot \nabla(\nabla\cdot(v\nabla u)) - \int_\Omega (b_2 w + \theta_1)|\nabla v|^4 + \int_\Omega u|\nabla v|^4 \\
&\quad + \int_\Omega v(\nabla u - b_2 \nabla w)\cdot \nabla v |\nabla v|^2 \qquad t \in (0, T_{max}),
\end{aligned}
$$

, (3.36) n (3.39),

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega |\nabla v|^4 + 2d_2 \int_\Omega |\nabla|\nabla v|^2|^2 + 4d_2 \int_\Omega |\nabla v|^2 |D^2 v|^2 + 4\int_\Omega (b_2 w + \theta_1)|\nabla v|^4 \\
=&2d_2 \int_{\partial\Omega} |\nabla v|^2 \frac{\partial |\nabla v|^2}{\partial \nu} dS - 4\xi \int_\Omega |\nabla v|^2 \nabla v \cdot \nabla(\nabla\cdot(v\nabla u)) \\
&+ 4\int_\Omega v(\nabla u - b_2\nabla w)\cdot\nabla v|\nabla v|^2 + 4\int_\Omega u|\nabla v|^4 \\
\le&2d_2 \int_{\partial\Omega} |\nabla v|^2 \frac{\partial |\nabla v|^2}{\partial \nu} dS - 4\xi \int_\Omega |\nabla v|^2 \nabla v \cdot \nabla(\nabla\cdot(v\nabla u)) \\
&+ c_1 \int_\Omega |\nabla w||\nabla v|^3 + c_1 \int_\Omega |\nabla v|^3 + 4K \int_\Omega |\nabla v|^4 \qquad t \in (0, T_{max}).
\end{aligned}
\tag{3.58}
$$

n 2.6 n n , (3.15) n

$$
2d_2 \int_{\partial\Omega} |\nabla v|^2 \frac{\partial |\nabla v|^2}{\partial \nu} dS \le 4\kappa d_2 \int_{\partial\Omega} |\nabla v|^4 dS \le \frac{d_2}{2}\int_\Omega |\nabla|\nabla v|^2|^2 + c_2 \int_\Omega |\nabla v|^4 \tag{3.59}
$$

$t \in (0, T_{max})$. A n , n n $\|u(\cdot,t)\|_{L^\infty}$, $\|v(\cdot,t)\|_{L^\infty}$ n $\|\nabla u(\cdot,t)\|_{L^\infty}$, , n ' n , n ' n , , $t \in (0, T_{max})$ n n n n (3.58)

$$
\begin{aligned}
&-4\xi \int_\Omega |\nabla v|^2 \nabla v\cdot\nabla(\nabla\cdot(v\nabla u)) \\
&= 4\xi \int_\Omega \nabla|\nabla v|^2 \cdot \nabla v \nabla\cdot(v\nabla u) + 4\xi \int_\Omega |\nabla v|^2 \Delta v \nabla\cdot(v\nabla u) \\
&\le c_3 \int_\Omega |\nabla v||\nabla|\nabla v|^2|(|\nabla v| + |\Delta u|) + c_3 \int_\Omega |\nabla v|^2 |D^2 v|(|\nabla v| + |\Delta u|)
\end{aligned}
\tag{3.60}
$$

$$\le d_2\int_\Omega |\nabla|\nabla v|^2|^2 + 2d_2\int_\Omega |\nabla v|^2|D^2v|^2 + \frac{c_3^2}{d_2}\int_\Omega |\nabla v|^4 + \frac{c_3^2}{d_2}\int_\Omega |\nabla v|^2|\Delta u|^2$$
$$\le d_2\int_\Omega |\nabla|\nabla v|^2|^2 + 2d_2\int_\Omega |\nabla v|^2|D^2v|^2 + c_4\int_\Omega |\nabla v|^4 + c_4\int_\Omega |D^2u|^4.$$

n (3.59) n (3.60) n (3.58), n n n ’ n , n

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega |\nabla v|^4 + \frac{d_2}{2}\int_\Omega |\nabla|\nabla v|^2|^2 + 2d_2\int_\Omega |\nabla v|^2|D^2v|^2\\
&\le (4K + c_2 + c_4)\int_\Omega |\nabla v|^4 + c_1\int_\Omega |\nabla v|^3 + c_4\int_\Omega |D^2u|^4 + c_1\int_\Omega |\nabla v|^3|\nabla w| \\
&\le c_5\int_\Omega |\nabla v|^4 + c_4\int_\Omega |D^2u|^4 + c_1\int_\Omega |\nabla v|^3|\nabla w| + c_6 \qquad t\in(0,T_{max}).
\end{aligned} \tag{3.61}$$

n $\|v(\cdot,t)\|_{L^\infty} \le K_6$, n n n n ’ n , n

$$\begin{aligned}
(c_5+2)\int_\Omega |\nabla v|^4 &= (c_5+2)\int_\Omega |\nabla v|^2\nabla v\cdot\nabla v\\
&= -(c_5+2)\int_\Omega v\nabla|\nabla v|^2\cdot\nabla v - (c_5+2)\int_\Omega v|\nabla v|^2\Delta v\\
&\le (c_5+2)K_6\int_\Omega |\nabla|\nabla v|^2||\nabla v| + (c_5+2)K_6\sqrt{2}\int_\Omega |\nabla v|^2|D^2v|\\
&\le \frac{d_2}{4}\int_\Omega |\nabla|\nabla v|^2|^2 + \frac{d_2}{2}\int_\Omega |\nabla v|^2|D^2v|^2 + c_7\int_\Omega |\nabla v|^2\\
&\le \frac{d_2}{4}\int_\Omega |\nabla|\nabla v|^2|^2 + \frac{d_2}{2}\int_\Omega |\nabla v|^2|D^2v|^2\\
&\quad + \int_\Omega |\nabla v|^4 + \frac{c_7^2}{4} \qquad t\in(0,T_{max}),
\end{aligned}$$

$$(c_5+1)\int_\Omega |\nabla v|^4 \le \frac{d_2}{4}\int_\Omega |\nabla|\nabla v|^2|^2 + \frac{d_2}{2}\int_\Omega |\nabla v|^2|D^2v|^2 + \frac{c_7^2}{4} \qquad t\in(0,T_{max}). \tag{3.62}$$

, n n , n

$$\begin{aligned}
\int_\Omega |\nabla v|^6 &= \int_\Omega |\nabla v|^4 \nabla v \cdot \nabla v \\
&= -2\int_\Omega v|\nabla v|^2 \nabla|\nabla v|^2 \cdot \nabla v - \int_\Omega v|\nabla v|^4 \Delta v \\
&\le 2K_6 \int_\Omega |\nabla v|^3 |\nabla|\nabla v|^2| + \sqrt{2}K_6 \int_\Omega |\nabla v|^4 |D^2 v| \\
&\le \frac{3}{8}\int_\Omega |\nabla v|^6 + 4K_6^2 \left(\int_\Omega |\nabla|\nabla v|^2|^2 + \int_\Omega |\nabla v|^2 |D^2 v|^2\right) \qquad t \in (0, T_{max}),
\end{aligned}$$

$$\int_\Omega |\nabla|\nabla v|^2|^2 + \int_\Omega |\nabla v|^2 |D^2 v|^2 \ge c_8 \int_\Omega |\nabla v|^6 \qquad t \in (0, T_{max}). \tag{3.63}$$

, , n (3.62) n (3.63) n (3.61), n , n $|\Delta u| \le \sqrt{2}|D^2 u|$, $t \in (0, T_{max})$ n

$$\begin{aligned}
\frac{d}{dt}\int_\Omega |\nabla v|^4 + \int_\Omega |\nabla v|^4 + \frac{c_8 d_2}{4}\int_\Omega |\nabla v|^6 &\le c_4 \int_\Omega |D^2 u|^4 + c_1 \int_\Omega |\nabla v|^3 |\nabla w| + c_9 \\
&\le c_4 \int_\Omega |D^2 u|^4 + \frac{c_8 d_2}{8}\int_\Omega |\nabla v|^6 + c_{10}\int_\Omega |\nabla w|^2 + c_{11},
\end{aligned}$$

$t \in (0, T_{max})$

$$\frac{d}{dt}\int_\Omega |\nabla v|^4 + \int_\Omega |\nabla v|^4 + \frac{c_8 d_2}{8}\int_\Omega |\nabla v|^6 \le c_4 \int_\Omega |D^2 u|^4 + c_{10}\int_\Omega |\nabla w|^2 + c_{11}. \tag{3.64}$$

, n (3.57) $4c_{10}$ n n (3.64),

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \left(4c_{10} w^2 + |\nabla v|^4\right) + \int_\Omega \left(4c_{10} w^2 + |\nabla v|^4\right) + c_{10}\int_\Omega |\nabla w|^2 + \frac{c_8 d_2}{8}\int_\Omega |\nabla v|^6 \\
&\le 4c_{10}\chi^2 \int_\Omega w^2 |\nabla v|^2 + c_4 \int_\Omega |D^2 u|^4 + c_{12} \\
&\le \frac{c_8 d_2}{8}\int_\Omega |\nabla v|^6 + c_{13}\int_\Omega w^3 + c_4 \int_\Omega |D^2 u|^4 + c_{12} \qquad t \in (0, T_{max}),
\end{aligned}$$

$$\frac{d}{dt}\int_\Omega\left(4c_{10}w^2+|\nabla v|^4\right)+\int_\Omega\left(4c_{10}w^2+|\nabla v|^4\right)+c_{10}\int_\Omega|\nabla w|^2 \le c_{13}\int_\Omega w^3+c_4\int_\Omega|D^2u|^4+c_{12} \quad t\in(0,T_{max}). \tag{3.65}$$

n n , n 2.5, (2.2) n (3.45), n n

$$c_{13}\int_\Omega w^3\le c_{10}\int_\Omega|\nabla w|^2+c_{13} \quad t\in(0,T_{max}). \tag{3.66}$$

n n (3.66) n (3.65),

$$\frac{d}{dt}\int_\Omega\left(4c_{10}w^2+|\nabla v|^4\right)+\int_\Omega\left(4c_{10}w^2+|\nabla v|^4\right)\le c_4\int_\Omega|D^2u|^4+c_{14} \quad t\in(0,T_{max}). \tag{3.67}$$

n $\int_t^{t+\tau}\int_\Omega|D^2u|^4\le c_{15}$ ((3.43)), n (3.55) n 2.4 (3.67). n 3.8 . □

Lemma 3.9. *Let the assumptions in Lemma 2.1 hold and (u,v,w) be the solution of (1.3). Then we can find a constant $K_{10}>0$ independent of t such that*

$$\|w(\cdot,t)\|_{L^3}\le K_{10} \quad t\in(0,T_{max}). \tag{3.68}$$

Proof. n n (1.3) w^2, n n n ,

$$\frac{1}{3}\frac{d}{dt}\int_\Omega w^3+2\int_\Omega w|\nabla w|^2+\theta_2\int_\Omega w^3=2\chi\int_\Omega w^2\nabla v\cdot\nabla w+\int_\Omega vw^3 \quad t\in(0,T_{max}). \tag{3.69}$$

n n n n n n $\|v(\cdot,t)\|_{L^\infty}\le K_6$, $t\in(0,T_{max})$ n

$$\begin{aligned}2\chi\int_\Omega w^2\nabla v\cdot\nabla w+\int_\Omega vw^3&\le 2\chi\int_\Omega w^2|\nabla v||\nabla w|+K_6\int_\Omega w^3\\&\le\int_\Omega w|\nabla w|^2+\chi^2\int_\Omega w^3|\nabla v|^2+K_6\int_\Omega w^3.\end{aligned} \tag{3.70}$$

(3.70) n (3.69) n

$$\frac{d}{dt}\int_\Omega w^3+3\int_\Omega w|\nabla w|^2+\int_\Omega w^3\le 3\chi^2\int_\Omega w^3|\nabla v|^2+(3K_6+1)\int_\Omega w^3 \tag{3.71}$$

$t\in(0,T_{max})$. n n - n n n n n , n n n $\|w(\cdot,t)\|_{L^2}+\|\nabla v\|_{L^4}\le K_9$ ((3.55)), n

$$
\begin{aligned}
3\chi^2 \int_\Omega w^3|\nabla v|^2 &\le 3\chi^2 \left(\int_\Omega w^6\right)^{\frac{1}{2}} \left(\int_\Omega |\nabla v|^4\right)^{\frac{1}{2}} \\
&\le 3\chi^2 K_9^2 \|w^{\frac{3}{2}}\|_{L^4}^2 \\
&\le 3\chi^2 K_9^2 c_1 (\|\nabla w^{\frac{3}{2}}\|_{L^2}^{\frac{4}{3}} \|w^{\frac{3}{2}}\|_{L^{\frac{4}{3}}}^{\frac{2}{3}} + \|w^{\frac{3}{2}}\|_{L^{\frac{4}{3}}}^2) \\
&\le \int_\Omega w|\nabla w|^2 + c_2 \qquad t \in (0, T_{max}),
\end{aligned} \tag{3.72}
$$

n

$$
\begin{aligned}
(3K_6+1) \int_\Omega w^3 &= (3K_6+1)\|w^{\frac{3}{2}}\|_{L^2}^2 \\
&\le c_3 \|\nabla w^{\frac{3}{2}}\|_{L^2}^{\frac{2}{3}} \|w^{\frac{3}{2}}\|_{L^{\frac{4}{3}}}^{\frac{4}{3}} + \|w^{\frac{3}{2}}\|_{L^{\frac{4}{3}}}^2 \\
&\le \int_\Omega w|\nabla w|^2 + c_4 \qquad t \in (0, T_{max}).
\end{aligned} \tag{3.73}
$$

n , , n (3.72) n (3.73) n (3.71), n $\frac{d}{dt}\int_\Omega w^3 + \int_\Omega w^3 \le c_3 + c_4$, (3.68) n ' n , . □

Lemma 3.10. *Let the assumptions in Lemma 2.1 hold and* (u, v, w) *be the solution of the system* (1.3)*. Then there exists a constant* $K_{11} > 0$ *independent of* t *such that*

$$
\|\nabla v(\cdot, t)\|_{L^\infty} \le K_{11}, \qquad t \in (0, T_{max}). \tag{3.74}
$$

Proof. n 2.1, (3.74) , $t \in (0, \tau]$ τ n (3.13). n n n (3.74) $t \in (\tau, T_{max})$. , n n , n (1.3) :

$$
v_t - d_2\Delta v + v = -\xi \nabla v \cdot \nabla u - \xi v \Delta u + uv - b_2 vw + (1-\theta_1)v. \quad /\_5 \quad 1 \quad .572\ 53.74)
$$

$$
\begin{aligned}
\|\nabla v(\cdot,t)\|_{L^\infty} \le & \|\nabla e^{(t-\tau)(d_2\Delta-1)} v(\cdot,\tau)\|_{L^\infty} + \xi \int_\tau^t \|\nabla e^{(t-s)(d_2\Delta-1)}(\nabla v\cdot\nabla u + v\Delta u)\|_{L^\infty} ds \\
& + \int_\tau^t \|\nabla e^{(t-s)(d_2\Delta-1)}[uv - b_2 vw + (1-\theta_1)v]\|_{L^\infty} ds \\
\le & c_1 + \xi\gamma_2 \int_\tau^t (1+(t-s)^{-\frac{3}{4}}) e^{-(d_2\lambda_1+1)(t-s)} \|\nabla v\cdot\nabla u\|_{L^4} ds \\
& + \xi\gamma_2 \int_\tau^t (1+(t-s)^{-\frac{2}{3}}) e^{-(d_2\lambda_1+1)(t-s)} \|v\Delta u\|_{L^6} ds \\
& + \gamma_2 \int_\tau^t (1+(t-s)^{-\frac{5}{6}}) e^{-(d_2\lambda_1+1)(t-s)} \|uv - b_2 vw + (1-\theta_1)v\|_{L^3} ds \\
:= & c_1 + I_1 + I_2 + I_3 \qquad t\in(\tau,T_{max}).
\end{aligned} \tag{3.76}
$$

n. $\|\nabla u(\cdot,t)\|_{L^\infty} + \|\nabla v(\cdot,t)\|_{L^4} \le c_2$ (3.39) n (3.55), n

$$
\|\nabla v\cdot\nabla u\|_{L^4} \le \|\nabla u\|_{L^\infty}\|\nabla v\|_{L^4} \le c_2^2, \qquad t\in(\tau,T_{max})
$$

n n

$$
\begin{aligned}
I_1 = & \xi\gamma_2 \int_\tau^t (1+(t-s)^{-\frac{3}{4}}) e^{-(d_2\lambda_1+1)(t-s)} \|\nabla v\cdot\nabla u)\|_{L^4} ds \\
\le & c_2^2 \xi\gamma_2 \int_\tau^t (1+(t-s)^{-\frac{3}{4}}) e^{-(d_2\lambda_1+1)(t-s)} ds \\
\le & c_3 \qquad t\in(\tau,T_{max}).
\end{aligned} \tag{3.77}
$$

n n , n. $\|v(\cdot,t)\|_{L^\infty} \le K_6$ n $\int_\tau^t e^{-6(t-s)}\|\Delta u\|_{L^6}^6 \le K_7$ (3.5 n 3.6),

$$
\begin{aligned}
I_2 = & \xi\gamma_2 \int_\tau^t (1+(t-s)^{-\frac{2}{3}}) e^{-(d_2\lambda_1+1)(t-s)} \|v\Delta u\|_{L^6} ds \\
\le & \xi\gamma_2 K_6 \left(\int_\tau^t (1+(t-s)^{-\frac{2}{3}})^{\frac{6}{5}} e^{-\frac{6d_2\lambda_1}{5}(t-s)} ds \right)^{\frac{5}{6}} \left(\int_\tau^t e^{-6(t-s)} \|\Delta u\|_{L^6}^6 ds \right)^{\frac{1}{6}}
\end{aligned} \tag{3.78}
$$

$$
\begin{aligned}
&\le K_7^{\frac{1}{6}}\xi\gamma_2 K_6\left(\int_\tau^t (1+(t-s)^{-\frac{2}{3}})^{\frac{6}{5}}e^{-\frac{6d_2\lambda_1}{5}(t-s)}ds\right)^{\frac{5}{6}}\\
&\le c_4 \qquad t\in(\tau,T_{max}).
\end{aligned}
$$

A , , n $\|u(\cdot,t)\|_{L^\infty}\le K$ (2.2), $\|v(\cdot,t)\|_{L^\infty}\le K_6$ (3.5) n $\|w(\cdot,t)\|_{L^3}\le K_{10}$ (3.9), n

$$
\|uv-b_2vw+(1-\theta_1)v\|_{L^3}\le KK_6|\Omega|^{\frac{1}{3}}+b_2K_6K_{10}+(1+\theta_1)K_6|\Omega|^{\frac{1}{3}}\le c_5 \qquad t\in(\tau,T_{max}),
$$

n n

$$
\begin{aligned}
I_3&=\gamma_2\int_\tau^t(1+(t-s)^{-\frac{5}{6}})e^{-(d_2\lambda_1+1)(t-s)}\|uv-b_2vw+(1-\theta_1)v\|_{L^3}ds\\
&\le c_5\gamma_2\int_\tau^t(1+(t-s)^{-\frac{5}{6}})e^{-(d_2\lambda_1+1)(t-s)}ds\\
&\le c_6 \qquad t\in(\tau,T_{max}).
\end{aligned}
\tag{3.79}
$$

n (3.77), (3.78) n (3.79) n (3.76) (3.74) n n . □

Lemma 3.11. *Let the assumptions in Lemma 2.1 hold and* (u,v,w) *be the solution of* (1.3). *Then it holds that*

$$
\|w(\cdot,t)\|_{L^\infty}\le K_{12} \qquad t\in(0,T_{max}),
\tag{3.80}
$$

where $K_{12}>0$ *is a constant independent of* t.

Proof. n n n n n (1.3),

$$
\begin{aligned}
w(\cdot,t)=&e^{t(\Delta-1)}w_0-\int_0^t e^{(t-s)(\Delta-1)}\nabla\cdot(\chi w\nabla v)ds\\
&+\int_0^t e^{(t-s)(\Delta-1)}w\,(1-\theta_2+v)\,ds \qquad t\in(0,T_{max}).
\end{aligned}
$$

n n n $\|v(\cdot,t)\|_{W^{1,\infty}}$ n $\|w(\cdot,t)\|_{L^3}$, n n n n c_1 n c_2

$$
\|\chi w\nabla v\|_{L^3}\le\chi\|\nabla v\|_{L^\infty}\|w\|_{L^3}\le c_1 \qquad t\in(0,T_{max}),
$$

n

$$\|w(1-\theta_2+v)\|_{L^3} \le (1+\theta_2+\|v\|_{L^\infty})\|w\|_{L^3} \le c_2 \qquad t \in (0, T_{max}).$$

2.8,

$$\begin{aligned}
\|w(\cdot,t)\|_{L^\infty} \le & \|e^{t(\Delta-1)}w_0\|_{L^\infty} + \int_0^t \|e^{(t-s)(\Delta-1)}\nabla\cdot(\chi w\nabla v)\|_{L^\infty} ds \\
& + \int_0^t \left\| e^{(t-s)(\Delta-1)} w(1-\theta_2+v) \right\|_{L^\infty} \\
\le & \|w_0\|_{L^\infty} + \int_0^t (1+(t-s)^{-\frac{5}{6}})e^{-(\lambda_1+1)(t-s)}\|\chi w\nabla v\|_{L^3} ds \\
& + \int_0^t (1+(t-s)^{-\frac{1}{3}})e^{-(t-s)}\|w(1-\theta_2+v)\|_{L^3} ds \\
\le & \|w_0\|_{L^\infty} + c_1\int_0^t (1+(t-s)^{-\frac{5}{6}})e^{-(\lambda_1+1)(t-s)} ds \\
& + c_2 \int_0^t (1+(t-s)^{-\frac{1}{3}})e^{-(t-s)} ds \\
\le & c_3 \qquad t\in(0,T_{max}),
\end{aligned}$$

(3.80). 3.11 . □

Proof of Theorem 1.1. 2.2 (3.39)

$$\|u(\cdot,t)\|_{W^{1,\infty}} \le c_1 \qquad t\in(0,T_{max}).$$

3.5 3.10, $c_2 > 0$ $\|v(\cdot,t)\|_{W^{1,\infty}} \le c_2$. A 3.11,

$$\|u(\cdot,t)\|_{W^{1,\infty}} + \|v(\cdot,t)\|_{W^{1,\infty}} + \|w(\cdot,t)\|_{L^\infty} \le c_3 \qquad t\in(0,T_{max}),$$

2.1 1.1. □

4. Global stability

(23, 3). (5,14),

9) - (. 1,17). , n
.

4.1. Case of prey-only

n , n, , - n ($\theta_1 \geq 1$). n

$$\frac{d}{dt}\mathcal{E}_1(t) \le 0,$$

= n n $(u, v, w) = (1, 0, 0)$.
n ' n n n (23, 3) (u, v, w)
n $(1, 0, 0)$ $t \to \infty$ n L^∞. □

4.2. Case of semi-coexistence

n n, n n $\theta_1 < 1$
n $\theta_1 + b_1\theta_2 \ge 1$. n , n n n n :

$$\mathcal{E}_2(t) := \mathcal{E}_2(u, v, w) = \int_\Omega \left(u - \theta_1 - \theta_1 \ln \frac{u}{\theta_1}\right) + b_1 \int_\Omega \left(v - v^* - v^* \ln \frac{v}{v^*}\right) + b_1 b_2 \int_\Omega w,$$

$v^* = \frac{1-\theta_1}{b_1} > 0$.

Lemma 4.2. *Let* (u, v, w) *be the solution of* (1.3) *obtained in Theorem 1.1. Then if* $\theta_1 < 1$, $\theta_1 + b_1\theta_2 > 1$ *and*

$$4d_1 d_2 \theta_1 > \xi^2 (1 - \theta_1)\|u\|^2_{L^\infty}, \tag{4.2}$$

we have

$$\lim_{t\to\infty} \left(\|u(\cdot, t) - \theta_1\|_{L^\infty} + \|v(\cdot, t) - \frac{1-\theta_1}{b_1}\|_{L^\infty} + \|w(\cdot, t)\|_{L^\infty} \right) = 0.$$

Proof. , n ' n n n (4.1) n η_2 n u n θ_1

$$u - \theta_1 - \theta_1 \ln \frac{u}{\theta_1} = \frac{\theta_1}{2\eta_2^2}(u - \theta_1)^2 \ge 0$$

n = n n $u = \theta_1$. , $v - v^* - v^* \ln \frac{v}{v^*} \ge 0$ n = $v = v^*$.
n $\mathcal{E}_2(\theta_1, \frac{1-\theta_1}{b_1}, 0) = 0$ n $\mathcal{E}_2(u, v, w) > 0$ $(u, v, w) \ne (\theta_1, \frac{1-\theta_1}{b_1}, 0)$.
n n , n n n $\mathcal{E}_2(t)$ n n (1.3),

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}_2(t) &= \int_\Omega \frac{u - \theta_1}{u} u_t + b_1 \int_\Omega \frac{v - \frac{1-\theta_1}{b_1}}{v} v_t + b_1 b_2 \int_\Omega w_t \\
&= \underbrace{-d_1\theta_1 \int_\Omega \frac{|\nabla u|^2}{u^2} - d_2(1-\theta_1)\int_\Omega \frac{|\nabla v|^2}{v^2} + \xi(1-\theta_1)\int_\Omega \frac{\nabla u \cdot \nabla v}{v}}_{I_1}
\end{aligned} \tag{4.3}$$

$$+\underbrace{\int_\Omega (u-\theta_1)(1-u-b_1v)+b_1\int_\Omega \left(v-\frac{1-\theta_1}{b_1}\right)(u-\theta_1)+b_2(1-\theta_1-b_1\theta_2)\int_\Omega w}_{I_2}.$$

n $X_1=(\frac{\nabla u}{u},\frac{\nabla v}{v})$, n n I_1

$$I_1=-\int_\Omega X_1A_1X_1^T, \qquad A_1=\begin{pmatrix} d_1\theta_1 & \frac{-\xi(1-\theta_1)u}{2} \\ \frac{-\xi(1-\theta_1)u}{2} & d_2(1-\theta_1)\end{pmatrix}.$$

A , n, n n (4.2) , A_1 n n n n n $\alpha_1>0$,

$$I_1=-\int_\Omega X_1A_1X_1^T\le -\alpha_1\int_\Omega\left(\frac{|\nabla u|^2}{u^2}+\frac{|\nabla v|^2}{v^2}\right). \tag{4.4}$$

n n, n I_2

$$\begin{aligned} I_2=&\int_\Omega (u-\theta_1)(\theta_1-u+1-\theta_1-b_1v)\\ &+b_1\int_\Omega\left(v-\frac{1-\theta_1}{b_1}\right)(u-\theta_1)+b_2(1-\theta_1-b_1\theta_2)\int_\Omega w\\ =&-\int_\Omega (u-\theta_1)^2+b_2(1-\theta_1-b_1\theta_2)\int_\Omega w.\end{aligned}\tag{4.5}$$

, , n (4.4) n (4.5) n (4.3), $\theta_1+b_1\theta_2>1$, n

$$\frac{d}{dt}\mathcal{E}_2(t)=-\int_\Omega X_1A_1X_1^T-\int_\Omega (u-\theta_1)^2+b_2(1-\theta_1-b_1\theta_2)\int_\Omega w\le 0,$$

n = n n $u=\theta_1, w=0$ n $\nabla v=0$. n $\nabla v=0$ $v=\bar v$ n n $\bar v>0$. , $(u,v,w)=(\theta_1,\bar v,0)$, n (1.3), n n

$$\theta_1(1-\theta_1-b_1\bar v)=0,$$

$\bar v=\frac{1-\theta_1}{b_1}$. n $\frac{d}{dt}\mathcal{E}_2(t)=0$ $(u,v,w)=(\theta_1,\frac{1-\theta_1}{b_1},0)$. n n , n n n (23, 3), , n (u,v,w) n $(\theta_1,\frac{1-\theta_1}{b_1},0)$ $t\to\infty$ n L^∞. □

4.3. Case of coexistence

n , n, n (u_*, v_*, w_*) n (1.4) n $\theta_1 < 1$ n $\theta_1 + b_2\theta_2 < 1.$ n , n n n n n :

$$\mathcal{E}_3(t) := \mathcal{E}_3(u, v, w) = \mathcal{J}_u(t) + b_1\mathcal{J}_v(t) + b_1b_2\mathcal{J}_w(t),$$

$$\mathcal{J}_\ell(t) = \int_\Omega \left(\ell - \ell_* - \ell_* \, \mathrm{n} \frac{\ell}{\ell_*} \right), \ \ell = u, v, w.$$

Lemma 4.3. *Let* (u, v, w) *be the solution of* (1.3) *obtained in Theorem 1.1. Then if the parameters satisfy* $\theta_1 < 1$ *and* $\theta_1 + b_1\theta_2 < 1$ *as well as*

$$4d_1d_2u_*v_* > \xi^2 b_1 v_*^2 \|u\|_{L^\infty}^2 + \chi^2 b_2 d_1 u_* w_* \|v\|_{L^\infty}^2. \tag{4.6}$$

Then it holds that

$$_{t\to\infty} (\|u - u_*\|_{L^\infty} + \|v - v_*\|_{L^\infty} + \|w - w_*\|_{L^\infty}) = 0.$$

Proof. B n n 4.1 n 4.2, n (4.1) $\mathcal{E}_3(u_*, v_*, w_*) = 0$ n $\mathcal{E}_3(u, v, w) > 0$ $(u, v, w) \neq (u_*, v_*, w_*)$. , n $1 - u_* - b_1v_* = 0,$ n (1.3)

$$\begin{aligned} \frac{d}{dt}\mathcal{J}_u(t) &= \int_\Omega \left(1 - \frac{u_*}{u}\right) u_t \\ &= -d_1u_* \int_\Omega \frac{|\nabla u|^2}{u^2} + \int_\Omega (u - u_*)(1 - u - b_1v) \\ &= -d_1u_* \int_\Omega \frac{|\nabla u|^2}{u^2} - \int_\Omega (u - u_*)^2 - b_1 \int_\Omega (u - u_*)(v - v_*). \end{aligned} \tag{4.7}$$

, n n n (1.3) n $\theta_1 = u_* - b_2w_*$ n

$$\begin{aligned} b_1\frac{d}{dt}\mathcal{J}_v(t) &= b_1 \int_\Omega \left(1 - \frac{v_*}{v}\right) v_t \\ &= -d_2b_1v_* \int_\Omega \frac{|\nabla v|^2}{v^2} + \xi v_* b_1 \int_\Omega \frac{\nabla u \cdot \nabla v}{v} + b_1 \int_\Omega (v - v_*)(u - b_2w - \theta_1) \\ &= -d_2b_1v_* \int_\Omega \frac{|\nabla v|^2}{v^2} + \xi v_* b_1 \int_\Omega \frac{\nabla u \cdot \nabla v}{v} + b_1 \int_\Omega (u - u_*)(v - v_*) \end{aligned} \tag{4.8}$$

$$-\,b_1b_2\int_\Omega (v-v_*)(w-w_*).$$

$v_*=\theta_2$ n , n (1.3),

$$\begin{aligned}
b_1b_2\frac{d}{dt}\mathcal{J}_w(t) &= b_1b_2\int_\Omega\left(1-\frac{w_*}{w}\right)w_t\\
&= -w_*b_1b_2\int_\Omega\frac{|\nabla w|^2}{w^2}+w_*b_1b_2\chi\int_\Omega\frac{\nabla v\cdot\nabla w}{w}+\int_\Omega(w-w_*)(v-\theta_2) \qquad (4.9)\\
&= -w_*b_1b_2\int_\Omega\frac{|\nabla w|^2}{w^2}+\chi w_*b_1b_2\int_\Omega\frac{\nabla v\cdot\nabla w}{w}+b_1b_2\int_\Omega(v-v_*)(w-w_*).
\end{aligned}$$

n n (4.7), (4.8) n (4.9),

$$\frac{d}{dt}\mathcal{E}_3(t)=-\int_\Omega X_2A_2X_2^T-\int_\Omega(u-u_*)^2, \tag{4.10}$$

$X_2=\left(\frac{\nabla u}{u},\frac{\nabla v}{v},\frac{\nabla w}{w}\right)$ n A_2 n

$$A_2:=\begin{pmatrix} d_1u_* & -\frac{\xi b_1v_*u}{2} & 0\\ -\frac{\xi b_1v_*u}{2} & d_2b_1v_* & \end{pmatrix}$$

n (4.11) n (4.10), n

$$\frac{d}{dt}\mathcal{E}_3(t) \le -\alpha_2 \int_\Omega \left(\frac{|\nabla u|^2}{u^2} + \frac{|\nabla v|^2}{v^2} + \frac{|\nabla w|^2}{v^2} \right) - \int_\Omega (u - u_*)^2,$$

$\frac{d}{dt}\mathcal{E}_3(t) \le 0$ u, v, w n $u = u_*, \nabla v = 0$ n $\nabla w = 0$ $\frac{d}{dt}\mathcal{E}_3(t) = 0.$
n $\nabla v = 0$ n $\nabla w = 0$

$$v = \tilde{v}_* \quad \text{n} \quad w = \tilde{w}_*,$$

$\tilde{v}$

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