

Negligibility of haptotaxis effect in a chemotaxis–haptotaxis model

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In this work, we rigorously study chemotaxis effect versus haptotaxis effect on boundedness, blow-up and asymptotical behavior of solutions for a chemotaxis-haptotaxis model in 2D settings. It is well-known that the corresponding Keller–Segel chemotaxis-only model possesses a striking feature of critical mass blowup phenomenon, namely, subcritical mass ensures boundedness, whereas, supercritical mass induces the existence of blow-ups. Herein, we show that this critical mass blow-up phenomenon stays almost the same in the full chemotaxis-haptotaxis model and that any global-in-time haptotaxis solution component vanishes exponentially and the other two solution components converge exponentially to that of chemotaxis-only model in a global sense for suitably large chemo-sensitivity and in the usual sense for suitably small chemo-sensitivity. Therefore, haptotaxis is neither good nor bad than chemotaxis, showing negligibility of haptotaxis effect in the underlying chemotaxis-haptotaxis model.

Keywords: Chemotaxis–haptotaxis; negligibility of haptotaxis; global existence; boundedness; blow-up.

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1. Introduction

Chemotaxis, the oriented movement of cells (or organisms) toward higher concentrations of diffusible chemical substances secreted by cells themselves, has received great attentions both in biological and mathematical communities. In 1970s, Keller and Segel introduced a celebrated minimal mathematical partial differential system to describe the collective behavior of cells under the influence of chemotaxis,¹⁸ which reads as

$$\begin{aligned} u_t &= \Delta u - \chi \cdot \nabla(u \cdot v), & x &\in \Omega, t > 0, \\ \tau v_t &= \Delta v - v + u, & x &\in \Omega, t > 0, \\ \frac{\partial u}{\partial \nu} &= \frac{\partial v}{\partial \nu} = 0, & x &\in \partial\Omega, t > 0, \\ u(x, 0) &= u_0(x), \tau v(x, 0) = \tau v_0(x), & x &\in \Omega, \end{aligned} \quad (1.1)$$

where $\chi > 0, \tau \in [0, 1]$, u and v are, respectively, the cell density and the chemical concentration, $\Omega \subset \mathbb{R}^n (n \geq 1)$ is a bounded domain with the smooth boundary $\partial\Omega$ and $\frac{\partial}{\partial \nu}$ means the outward normal derivative on $\partial\Omega$. The seminal Keller–Segel (KS) minimal model (1.1) and its numerous variants have been widely investigated since 1970. The striking feature of KS type models is the possibility of blow-up of solutions in a finite/infinite time, which strongly depends on the space dimension. A finite/infinite time blow-up never occurs in 1D,³⁰ a critical mass blow-up occurs in 2D: when the initial mass $\|u_0\|_{L^1} < \frac{4\pi}{\chi}$, solutions exist globally and are uniformly bounded, whereas, when $\|u_0\|_{L^1} > \frac{4\pi}{\chi}$, there exist solutions blowing up in finite or infinite time, cf. Refs. 11, 14, 27, 28 and 32, and even small initial mass can result in blow-ups in ≥ 3 D.^{47,49} See Refs. 1, 13, 47 and 49 for more surveys on the classical KS model and its variants.

It is now well known that such chemotactic aggregation will be prevented by suitable introduction of logistic source of the form $au - bu^2 (a \in \mathbb{R}, b > 0)$ into the u -equation in (1.1):

$$\begin{aligned} u_t &= \Delta u - \chi \cdot \nabla(u \cdot v) + au - bu^2, & x &\in \Omega, t > 0, \\ \tau v_t &= \Delta v - v + u, & x &\in \Omega, t > 0. \end{aligned} \quad (1.2)$$

Indeed, for $n \leq 2$, any $b > 0$ will be sufficient to rule out any blow-up.^{29,30,44,51} A recent subtle study from⁵³ further shows that the chemotactic aggregation can be even prevented by a sub-logistic source like $au - \frac{bu^2}{\ln^\gamma(u+1)}$ or $au - \frac{bu^2}{\ln(\ln(u+e))}$ for some $a \in \mathbb{R}, b > 0, \gamma \in (0, 1)$. These results convey to us, for $n \leq 2$, that blow-up is fully precluded as long as a logistic or sub-logistic source presents, and, in this case, the blow-up phenomenon possessed by (1.1) completely disappears.

For $n \geq 3$, the blow-up prevention in (1.2) by logistic source becomes increasingly intricate, and it has been explored qualitatively and quantitatively in a series of works.^{48,52,54} In summary, it is only known thus far that properly strong logistic damping in (1.2) can prevent blow-up driven by the chemotactic cross-diffusion

in (1.1). More precisely, in the parabolic-elliptic case $\tau = 0$, the logistic damping outweighs chemotactic aggregation when $b \geq \frac{(n-2)}{n}\chi$.^{44,55} In the fully parabolic case $\tau = 1$, the issue becomes even more delicate: for $n \geq 4$, sufficiently strong logistic damping can prevent blow-up,⁴⁸ and, in the case of $n = 3$ or in smooth convex domains, explicit smallness of $\frac{\chi}{\mu}$ on boundedness and convergence is available.^{48,52} We would add that, in 3D bounded, smooth and convex domains, even through logistic damping guarantees global existence of weak solutions,²⁰ weak damping sources may fail to suppress blow-up for (1.1). Indeed, for $n \geq 3$, radially symmetrical blow-up has been observed in a parabolic-elliptic simplification of (1.1) under a proper sub-quadratic damping source.⁵⁰ For more dynamical properties like mass persistence and long time behavior, etc., one can consult^{41,52} for instance.

Besides chemotaxis influence, cells are observed to direct their movement also towards higher concentration of certain non-diffusible substance, known widely as haptotaxis. Such an important extension of chemotaxis to a more complex cell migration mechanism has been introduced by Chaplain and Lolas^{3,4} to describe processes of cancer invasion into surrounding healthy tissue. In that process, cancer invasion is associated with the degradation of the extracellular matrix (ECM) with density w , which is degraded by matrix degrading enzymes (MDEs) with density v secreted by tumor cells with density u . Besides random motion, the migration of invasive cells is oriented both by a chemotaxis mechanism and by a haptotaxis mechanism (cellular locomotion directed in response to a concentration gradient of the non-diffusible adhesive molecules within ECM). In this way, the evolution of (u, v, w) satisfies the following combined chemotaxis–haptotaxis model with logistic source:

$$\begin{aligned} u_t &= \Delta u - \chi \cdot (u - v) - \xi \cdot (u - w) + \mu u(1 - u - w), & x \in \Omega, t > 0, \\ \tau v_t &= \Delta v - v + u, & x \in \Omega, t > 0, \\ w_t &= -vw, & x \in \Omega, t > 0, \end{aligned} \quad (1.3)$$

where the newly introduced parameters $\xi, \mu > 0$. In the past decades, the global solvability, boundedness and asymptotic behavior for the corresponding no-flux or homogeneous Neumann boundary and initial value problem (1.3) and its numerous variants have been widely investigated for certain smooth initial data. To get a clear picture for comparison between (1.3) and (1.2), we here try our best to collect the most relevant available results about (1.3). First, for haptotaxis-only models, i.e. $\chi = 0$ in $\leq 3D$: the local existence and uniqueness of classical solutions for $\mu = 0$ is shown in Ref. 26, boundedness of weak solutions for a similar model (where $+u$ in the second equation in (1.3) is replaced with $+uw$) is proved in Ref. 25 and asymptotic behavior of solutions is examined in Ref. 23, global existence of classical solutions is studied in Refs. 43 and 45, and, boundedness of solution is studied in Ref. 33 with $\mu > \eta\xi$. Next, for combined chemotaxis–haptotaxis model: for $\tau = 0$, the global existence and boundedness of classical solutions to (1.3) is

established in Ref. 36 for any $\mu > 0$ in 2D, and for large enough $\mu > 0$ in 3D; later on, global boundedness is further subsequently studied under the condition $\mu > \chi^{40}$ and $\mu > \frac{(n-2)^+}{n}\chi$ (cf. non-borderline boundedness for the chemotaxis-only system (1.2)),^{39,42} and also the exponential decay of w under smallness of w_0 and largeness of $\frac{\mu}{\chi^2}$ is also discussed; for $\tau = 1$, global existence and boundedness of classical solutions are established for any $\chi > 0$ in 1D in Refs. 35 and 12 with asymptomatic behavior, and first for large $\frac{\mu}{\chi}$ ³⁵ and then for any $\mu > 0$ in 2D;³⁴ very recently, instead of requiring logistic damping in (1.3), implicitly small initial mass of u_0 or weaker damping sub-logistic source like $u(1 - w - \frac{u}{\ln^\gamma(u+1)})$ with $\gamma \in (0, 1)$ or $u(1 - w - \frac{u}{\ln(\ln(u+e))})$ is demonstrated to guarantee boundedness for (1.3) in 2D.

where and below, $\chi, \xi, > 0, \tau \in (0, 1)$ and $\eta \geq 0$, as for the initial data $(u_0, \tau v_0, w_0)$, for convenience, we assume, for some $\theta \in (0, 1)$, that

$$(u_0, \tau v_0, w_0) \in C(\bar{\Omega}) \times W^{1,\infty}(\Omega) \times C^{2+\theta}(\bar{\Omega}) \quad \text{with } u_0 \geq 0, \tau v_0 \geq 0, w_0 \geq 0, \quad \frac{\partial w_0}{\partial \nu} \Big|_{\partial \Omega} = 0. \quad (1.5)$$

Although haptotaxis may have some influence on the properties of the underlying system on short or intermediate time scales,⁴ our next main findings manifest, for small η , that haptotaxis effect is almost negligible in terms of global existence, boundedness, blow-up and long time behavior.

Theorem 1.1. (Negligibility of haptotaxis in the chemotaxis–haptotaxis model (1.4)) *Let $\Omega \subset \mathbb{R}^2$ be a bounded, smooth and convex-plus domain in the sense of Definition 2.1 below, the initial data $(u_0, \tau v_0, w_0)$ satisfy (1.5) and the parameters $\chi, \xi > 0, \tau \in (0, 1)$ and $\eta \geq 0$.*

(B1) (Negligibility of haptotaxis on global existence for arbitrary η) *Assume*

$$m := \|u_0\|_{L^1} < \frac{4\pi}{\chi}. \quad (1.6)$$

Then the corresponding chemotaxis–haptotaxis model (1.4) possesses a unique global-in-time, positive and classical solution which is locally bounded in time.

(B2) (Negligibility of haptotaxis on boundedness for small η) *Besides the subcritical mass condition (1.6), assume further that*

$$\eta < v_\infty^m := m \int_0^\infty \frac{1}{4\pi s} e^{-\left(s + \frac{(\text{diam}(\Omega))^2}{4s}\right)} ds. \quad (1.7)$$

Then the global solution of (1.4) obtained in (B1) is uniformly bounded in time in the sense there exists $C_1 = C_1(u_0, \tau v_0, w_0, \Omega) > 0$ such that

$$\|u(t)\|_{L^\infty} + \|v(t)\|_{W^{1,\infty}} + \|w(t)\|_{W^{1,\infty}} \leq C_1, \quad t \geq 0. \quad (1.8)$$

(B3) (Almost negligibility of haptotaxis on blow-up for small η) *For $\epsilon > 0, m > 0$ and $x_0 \in \partial\Omega$, we define $(U_\epsilon, V_\epsilon) \in [C(\bar{\Omega}) \times W^{1,\infty}(\Omega)]^2$ as follows:*

$$V_\epsilon(x) = \frac{1}{\chi} \ln \frac{\epsilon^2}{(\epsilon^2 + \pi |x - x_0|^2)^2} - \frac{1}{\Omega} \int_\Omega \ln \frac{\epsilon^2}{(\epsilon^2 + \pi |x - x_0|^2)^2}$$

and

$$U_\epsilon(x) = \frac{me^{\chi V_\epsilon(x)}}{\int_\Omega e^{\chi V_\epsilon(x)}}.$$

Then, if m is supercritical and η is small in the sense that

$$m > \frac{4\pi}{\chi}, \quad m \leq \frac{4\pi l}{\chi} : l \in \mathbb{N}^+ \quad \text{and} \quad \eta < v_\infty^m, \quad (1.9)$$

the corresponding solution $(u^\epsilon, v^\epsilon, w^\epsilon)$ of (1.4) with $(u_0, \tau v_0, w_0) = (U_\epsilon, \tau V_\epsilon, \tau \inf_\Omega V_\epsilon, w_0)$ fulfills either (A) or (B); here (A) means, for some $\epsilon_0 > 0$, the

corresponding solution $(u^{\epsilon_0}, v^{\epsilon_0}, w^{\epsilon_0})$ blows up in finite or infinite time, and (B) means ‘almost’ (approximate) blow-up in the sense, for all $\epsilon > 0$, that the resulting solutions $(u^\epsilon, v^\epsilon, w^\epsilon)$ exist globally and are uniformly bounded in time but

$$\liminf_{\epsilon \rightarrow 0+} \frac{u^\epsilon v^\epsilon \quad L^\infty((0, \infty); L^1(\Omega))}{\ln \epsilon} \geq \frac{4(m\chi \quad 4\pi)(v_\infty^m \quad \eta)}{K\chi\xi[2 + (v_\infty^m \quad \eta)\delta]} \quad (1.10)$$

and

$$\begin{aligned} \liminf_{\epsilon \rightarrow 0+} \frac{\min \quad u^\epsilon \quad L^\infty(\Omega \times (0, \infty)), \quad v^\epsilon \quad L^\infty(\Omega \times (0, \infty))}{\ln \epsilon} \\ \geq \frac{4(m\chi \quad 4\pi)(v_\infty^m \quad \eta)}{mK\chi\xi[2 + (v_\infty^m \quad \eta)\delta]}, \end{aligned} \quad (1.11)$$

where $K = \max \quad 1, \quad w_0 \quad L^\infty$ and (due to (1.7) and (1.9)) δ is uniquely determined by

$$m \int_0^\delta \frac{1}{4\pi s} e^{-\left(s + \frac{(\text{diam}(\Omega))^2}{4s}\right)} ds = \frac{\eta + v_\infty^m}{2}. \quad (1.12)$$

- (B4) (Convergence of chemotaxis–haptotaxis model (1.4) to chemotaxis-only model (1.1) in a global sense) Under (1.7), any global-in-time w vanishes exponentially in the sense for any $\lambda \in (0, v_\infty^m \quad \eta)$, there exists $C_2 = C_2(u_0, \tau v_0, w_0, \lambda, \Omega) > 0$ such that

$$w(t) \quad L^\infty \leq C_2 e^{-\lambda t}, \quad t \geq 0. \quad (1.13)$$

Under (1.7), any global bounded solution (u, v, w) satisfying (1.8) of (1.4) converges exponentially to that of chemotaxis-only model (1.1) in a global sense: for any $\rho \in (0, \min \quad \lambda_1, v_\infty^m \quad \eta)$, there exists $C_3 = C_3(u_0, \tau v_0, w_0, \rho, \Omega) > 0$ such that

$$u(t) \quad \phi(t; u, v) \quad L^\infty \leq C_3 \xi e^{-\rho t}, \quad t \geq 0; \quad (1.14)$$

the v solution component satisfies $v(t) = \psi(t; u, v)$ for all $t \geq 0$ and, finally, for any $\kappa \in (0, v_\infty^m \quad \eta)$, there exists $C_4 = C_4(u_0, \tau v_0, w_0, \kappa, \Omega) > 0$ such that

$$w(t) \quad W^{1,\infty} \leq C_4 e^{-\kappa t}, \quad t \geq 0. \quad (1.15)$$

- (B5) (Convergence of chemotaxis–haptotaxis model (1.4) to chemotaxis-only model (1.1) in the usual sense) Under (1.7), there exists a small $\chi_0 \in (0, \frac{4\pi}{\|u_0\|_{L^1}})$ such that, whenever $\chi \leq \chi_0$, the w solution component vanishes exponentially as in (1.15) and the solution component (u, v) of the chemotaxis–haptotaxis model (1.4) converges exponentially to the solution (u^0, v^0) of chemotaxis-only model (1.1) in the usual sense: for any $\lambda \in (0, \min \quad \lambda_1, v_\infty^m \quad \eta)$, there exists $C_5 = C_5(u_0, v_0, w_0, \lambda, \Omega) > 0$ such that

$$u(t) \quad u^0(t) \quad L^\infty + \quad v(t) \quad v^0(t) \quad L^\infty \leq C_5 e^{-\lambda t}, \quad t \geq 0. \quad (1.16)$$

Here and below, $\lambda_1(>0)$ is the first nonzero eigenvalue of $-\Delta$ under homogeneous Neumann boundary condition. The symbols ϕ and ψ are solution operator for (1.1) via variation-of-constants formula:

$$\phi(t; u, v) = e^{t\Delta}u_0 - \chi \int_0^t e^{(t-s)\Delta} \cdot ((u - v)(s)) ds$$

and $\psi(t; u, v) = (-\Delta + 1)^{-1}u(t)$ if $\tau = 0$, and, if $\tau = 1$,

$$\psi(t; u, v) = e^{t(\Delta-1)}v_0 + \int_0^t e^{(t-s)(\Delta-1)}u(s) ds.$$

We adopt commonly abbreviated notations: for instance, for a function f

$$\begin{aligned} f(t)_{L^p} &= f(\cdot, t)_{L^p(\Omega)} = \left(\int_{\Omega} f(x, t)^p dx \right)^{\frac{1}{p}}, \\ f_{L^q(0, T; L^p(\Omega))} &= \left(\int_0^T f(t)_{L^p}^q dt \right)^{\frac{1}{q}}. \end{aligned}$$

Remark 1.1. Comments on negligibility of haptotaxis versus chemotaxis in (1.4)

- (R1) We mention that convex-plus domains contain convex domains, cf. the text right after Definition 2.1; also, we mention that Theorem 1.1 (cf. Remark 2.1), at least, global existence in (B1) and in Lemma 3.10 and assertions for the limiting case $\xi = 0$ hold for bounded smooth but not necessarily convex-plus domains.
- (R2) In light of (B1) and (B2), cf. details in Sec. 3, in the limiting case of $\chi = 0$, any solution to the resulting haptotaxis-only system (1.4) exists globally for all η , and, moreover, when $\eta < v_\infty^m$ or $\eta = v_\infty^m$, $\tau = 0$, then the solution is uniformly bounded as in (1.8) and w decays exponentially or algebraically. This again shows the negligibility of haptotaxis on global existence, boundedness, blow-up and long time behavior.
- (R3) In view of (1.10) and (1.11), in the limiting case of $\xi = 0$, the alternative (A) must happen. This together with (B1), (B2) and (R1) recover exactly the well-known 2D critical mass blow-up phenomenon in the chemotaxis-only setting.^{13,14,28}
- (R4) By (1.10) and Remark 3.1, a control of $\|u(t)v(t)\|_{L^\infty(0, T_m; L^1(\Omega))}$ in time or initial data is crucial to derive boundedness versus blow-up. This gives a different (perhaps equivalent) criterion than the widely known $L^{\frac{n}{2}+}$ -criterion in the chemotaxis-only systems.^{1,51}
- (R5) We comment that (B4) (more general, every maximal solution of (1.4) is comparable to that of (1.1) in this sense, cf. Lemma 5.1) merely says solutions of (1.4) converge exponentially to that of (1.1) in the solution operator (global) sense.¹² But, under a further smallness of χ , this convergence can be lifted to the usual sense that solutions of (1.4) converge exponentially to that of (1.1) in accordance with (1.15) and (1.16).

Theorem 1.1 indicates rigorously the negligibility of haptotaxis versus chemotaxis in (1.4) on global existence, boundedness, blow-up and long time behavior within (1.7). This opens up new directions for us to explore haptotaxis effect in more complex chemotaxis–haptotaxis settings.

In the rest of this section, we outline the plan of this work, which comprises five main sections.

In this section, we observe from existing literature empirically that haptotaxis plays little role in chemotaxis models, which motivate us to study rigorously the haptotaxis effect in our chosen chemotaxis–haptotaxis model (1.4). Finally, we state the negligibility of haptotaxis versus chemotaxis in Theorem 1.1 on global existence, boundedness, blow-up and long time behavior.

In Sec. 2, we first state the local existence and a convenient extensibility of smooth solutions to the IBVP (1.4). Afterwards, we obtain a standard $W^{1,q}$ -estimate for an inhomogeneous heat/elliptic equation, cf. Lemma 2.3, and then, we develop important *a priori* estimates on v and w in Lemmas 2.4–2.6, in particular, the explicit lower bound of v and the exponential decay of w for suitably small η . Finally, for convenience of reference, we state the widely-used 2D Gagliardo–Nirenberg (GN) inequality⁶ and a consequence of the Trudinger–Moser inequality,²⁸ cf. Lemmas 2.7 and 2.8.

To make our presentation more smooth, we divide Sec. 3 into three subsections to show the negligibility of haptotaxis in (1.4) on global existence (cf. Sec. 3.1) and boundedness (cf. Sec. 3.2). As an added result, we also exhibit in Sec. 3.3 the negligibility of pure haptotaxis effect by showing that the system (1.4) with $\chi = 0$ always has a global-in-time classical solution for any mass $u_0 \in L^1$, which becomes uniformly bounded for $\eta < v_\infty^m$ or $\eta = v_\infty^m$, $\tau = 0$. Our analysis starts with an important identity associated with (1.4), cf. Lemma 3.1, which along with smallness of $u_0 \in L^1$ or smallness of η enables us to apply the Trudinger–Moser inequality in Lemma 2.8 to derive two integral-type Gronwall inequalities. As a result, we obtain the key improved L^1 -bound of $u \ln u$ rather than L^1 -bound of u . Then, using quite known testing procedure and semi-group estimates,^{37,38,51,53,56} we conclude the desired global existence and global boundedness in respective cases, cf. Lemmas 3.5, 3.8 and 3.10.

In Sec. 4, we shall illustrate the almost negligibility of haptotaxis in (1.4) on blow-up as detailed in (B3). We observe, if $\eta < v_\infty$, the stationary problem of (1.4) is the same as that of the minimal chemotaxis-only model. Making use of this observation and based on the existing knowledge on the minimal chemotaxis-only system,^{11,14} we essentially build our almost blow-up argument on the use of the energy identity provided in Lemma 3.1 to an equivalent system (4.16) with initial data $(U_\epsilon, \tau V_\epsilon - \tau \inf_\Omega V_\epsilon, w_0)$ of (1.4). Under the conditions in (1.9) and assuming that the resulting solution exists globally and is uniformly bounded, we can show that the functional acted on our stationary problem both has a finite lower bound and an explicit upper bound involving $L^\infty(0, \infty; L^1(\Omega))$ -norm of $U^\epsilon(V^\epsilon)^+$,

cf. Lemma 4.3 and its proof. Finally, upon simple translations via the link (4.15), we readily recover our almost blow-up assertion for our original system as in (B3).

In Sec. 5, we shall show the negligibility of haptotaxis in (1.4) on long time behavior as detailed in (B4) and (B5), which indeed are direct consequences of our more general results provided in Lemmas 5.1 and 5.2. The proofs are shown conveniently via Neumann semigroup type arguments and the key ingredient relies on the fact that v has a positive lower bound and that solutions to the haptotaxis-only system (1.4) with $\chi = 0$ are uniformly-in-time bounded (cf. Lemma 3.10) under the smallness on η in (1.7).

2. Preliminary Knowledge and *a Priori* Estimates

For convenience and completeness, we begin with the local well-posedness and a convenient extendibility of classical solutions to the chemotaxis–haptotaxis system (1.4), which are well-established via a proper fixed-point framework and parabolic regularity theory.

Lemma 2.1. *Let $\chi, \xi, \tau, \eta \geq 0$, $\Omega \subset \mathbb{R}^n$ ($n \geq 1$) be a bounded and smooth domain and let the initial data $(u_0, \tau v_0, w_0)$ satisfy (1.5). Then there exist a maximal existence time $T_m \in (0, \infty]$ and a unique triple (u, v, w) of functions from $[C^0(\bar{\Omega} \times [0, T_m)) \times C^{2,1}(\bar{\Omega} \times (0, T_m))]^3$ solving the IBVP (1.4) classically on $\Omega \times (0, T_m)$ and such that*

$$0 < u, \quad 0 < v \quad \text{and} \quad 0 \leq w \leq \max\{1, w_0\}_{L^\infty(\Omega)} := K \quad \text{on } \bar{\Omega} \times (0, T_m). \quad (2.1)$$

Moreover, the following convenient extensibility criterion holds:

$$\text{either } T_m = +\infty \quad \text{or} \quad \limsup_{t \rightarrow T_m -} (\|u(t)\|_{L^\infty} + \|v(t)\|_{W^{1,\infty}}) = +\infty. \quad (2.2)$$

Proof. By well-developed fixed point arguments based on the Banach contraction principle and the standard parabolic regularity theory^{16,21,26,37,38,40,44,48} for detailed discussions, one can readily derive the local existence and uniqueness of classical solutions as well as the following extensibility criterion:

$$\text{either } T_m = +\infty \quad \text{or} \quad \limsup_{t \rightarrow T_m -} (\|u(t)\|_{L^\infty} + \|v(t)\|_{W^{1,\infty}} + \|w(t)\|_{L^4}) = +\infty. \quad (2.3)$$

Due to the nonnegativity of initial data, especially, $u_0 \geq 0$ in (1.5), the positivity of solution components (u, v, w) in (2.1) follows from the (strong) maximum principle along with the Hopf boundary point lemma. Next, we show that the extensibility criterion (2.2) is equivalent to (2.3). To this end, for any $p \geq 2$, we compute from the w -equation in (1.4) and use (2.1) and Young's inequality to deduce that

$$\begin{aligned} \frac{1}{p} \frac{d}{dt} \int_{\Omega} w^p &= \int_{\Omega} w^{p-2} v \cdot w + \int_{\Omega} (\eta - v - 2\eta w) w^p \\ &\leq (1 + \eta) \int_{\Omega} w^p + \frac{K^p}{p} \int_{\Omega} v^p, \quad t \in (0, T_m). \end{aligned} \quad (2.4)$$

Solving this Gronwall inequality directly, we find, for $t \in (0, T_m)$, that

$$w(t) \leq [w_0 + K^p \sup_{s \in (0, t)} v(s)] e^{(1+\eta)pt}. \quad (2.5)$$

Based on this, it follows easily that the criterion (2.2) is equivalent to (2.3). $\square$

Henceforth, we shall assume that the basic conditions in Lemma 2.1 are satisfied. C, C_i (numbering within lemmas or theorems) and C_ϵ , etc. will denote some generic constants which may vary line-by-line.

Thanks to the no-flux boundary condition, the next L^1 -information follows easily.

Lemma 2.2. *The local-in-time classical solution (u, v, w) of system (1.4) satisfies*

$$u(t) \leq u_0 := m, \quad t \in (0, T_m) \quad (2.6)$$

and

$$v(t) \leq u_0 + \begin{cases} 0 & \text{if } \tau = 0, \\ (v_0 - u_0)e^{-t} & \text{if } \tau = 1, \end{cases} \quad t \in (0, T_m). \quad (2.7)$$

Proof. A direct integration of the u -equation in (1.4) and a use of the no-flux boundary condition yield (2.6). Then, an integration of the v -equation entails

$$\tau \frac{d}{dt} v + v = u = u_0, \quad (2.8)$$

which directly gives rise to (2.7). $\square$

The following widely used reciprocal bounds, turning information on u into control on v , are derived by using the elliptic regularity if $\tau = 0$ or the variation-of-constants formula for v and L^p - L^q -estimates for the heat semigroup $e^{t\Delta}$ in Ω if $\tau > 0$.

Lemma 2.3. *Let $\Omega \subset \mathbb{R}^2$ be a bounded and smooth domain and let*

$$q = \begin{cases} 1, \frac{2p}{p-2} & \text{if } 1 \leq p \leq 2, \\ [1, \infty] & \text{if } p > 2. \end{cases} \quad (2.9)$$

Then there exists $C_1 = C_1(p, q, \tau v_0, \Omega) > 0$ such that the unique local-in-time classical solution (u, v, w) of the IBVP (1.4) verifies

$$v(t) \leq C_1 \left(1 + \sup_{s \in (0, t)} u(s) \right), \quad t \in (0, T_m). \quad (2.10)$$

In particular, for any $q \in [1, 2)$, there exists $C_2 = C_2(q, \tau v_0, \Omega) > 0$ such that

$$v(t) \leq C_2 + \sup_{s \in (0, t)} u(s), \quad t \in (0, T_m). \quad (2.11)$$

Proof. In the case of $\tau = 1$, by the variation-of-constants formula, it follows that

$$v(t) = e^{-t} e^{t\Delta} v_0 + \int_0^t e^{-(t-s)} e^{(t-s)\Delta} u(s) ds. \quad (2.12)$$

Then using the widely known smoothing L^p - L^q estimates of the Neumann heat semigroup $e^{t\Delta}$ $t \geq 0$ in Ω , see, e.g. Refs. 15 and 47 and applying those estimates to (2.12), one can easily derive (2.10), cf. Refs. 15 and 51. In the case of $\tau = 0$, the standard well-known $W^{2,p}$ - or $W^{1,q}$ -elliptic theory readily entails (2.10). Due to the mass conservations of u in (2.6), we first take $p = 1$ in (2.9), and then from (2.10) and the Sobolev embedding $W^{1,q}(\Omega) \hookrightarrow L^{\frac{2q}{2-q}}(\Omega)$ for $q < 2$, we readily obtain the desired estimate (2.11). $\square$

By the ODE satisfied by w in (1.4), we get more detailed information about w in terms of v .

Lemma 2.4. *Let (u, v, w) be the solution of system (1.4) obtained in Lemma 2.1. Then for any $t \in (0, T_m)$, the unique solution component w of (1.4) is given by*

$$w(x, t) = \frac{w_0(x) e^{-\int_0^t [v(x, r) - \eta] dr}}{1 + \eta w_0(x) \int_0^t e^{-\int_0^s [v(x, r) - \eta] dr} ds}, \quad (2.13)$$

which satisfies $0 \leq w(x, t) \leq \max\{1, w_0\}_{L^\infty}$ and

$$\frac{w_0(x)}{1 + w_0(x)(e^{\eta t} - 1)} e^{-\int_0^t [v(x, r) - \eta] dr} \leq w(x, t) \leq w_0(x) e^{-\int_0^t [v(x, r) - \eta] dr}. \quad (2.14)$$

Proof. When $w = 0$, upon dividing the w -equation in (1.4) by w^2 and then multiplying an integrating factor and rearranging, we rewrite the w -equation as

$$\frac{d}{ds} \frac{1}{w(x, s)} e^{-\int_0^s [v(x, r) - \eta] dr} = \eta e^{-\int_0^s [v(x, r) - \eta] dr},$$

which, upon being integrated from $s = 0$ to t and being rearranged, gives (2.13). Next, by (2.13), we can readily derive the lower bound for w in (2.14). We further use the nonnegativity of v to obtain

$$\begin{aligned} w(x, t) &= \frac{w_0(x) e^{\eta t} e^{-\int_0^t v(x, r) dr}}{1 + \eta w_0(x) \int_0^t e^{\eta s} e^{-\int_0^s v(x, r) dr} ds} \leq \frac{w_0(x) e^{\eta t} e^{-\int_0^t v(x, r) dr}}{1 + w_0(x) (e^{\eta t} - 1) e^{-\int_0^t v(x, r) dr}} \\ &\leq \max\{1, w_0(x)\}, \end{aligned}$$

which gives the upper bound of w and hence completes the proof of this lemma. $\square$

From the expression of w in (2.13), one can compute its gradient (cf. (5.7) with $\tau\alpha$ replaced by 0) and then find, if $\frac{\partial w_0}{\partial \nu} = 0$ on $\partial\Omega$, the no flux boundary conditions in (1.4) are equivalent to the homogeneous Neumann boundary conditions:

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0 \quad \text{on } \partial\Omega \times (0, \infty).$$

Indeed, only in a couple of convenient places involving Neumann heat semigroup for instance, we shall use these facts tacitly, cf. (3.32), (5.10) and (5.14).

Nowadays, a vast of literature uses the point-wise lower bound (without even mentioning any source) for the heat Neumann semigroup $(e^{t\Delta})_{t>0}$ in Ω :

$$e^{t\Delta}z(x) \geq \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{(\text{diam}(\Omega))^2}{4t}} z(x) \quad \Omega, t > 0 \quad \text{for all nonnegative } z \in C(\bar{\Omega}). \quad (2.15)$$

As we traced, the first rigorous proof for (2.15) with $n = 1$ appears to be Lemma 3.1 in Ref. 12 for Ω being an interval in $\mathbb{R}^1$. Later, the authors in Lemma 2.2 in Ref. 7 and Lemma 2.1 in Ref. 10 simply say that “In the same way as in the proof of Lemma 3.1 in Ref. 12, we can obtain the point-wise estimate (2.15)”. To clarify those gaps and give a simple justification for a class of domains including convex domains, we make the following definition.

Definition 2.1. Let $\Omega \subset \mathbb{R}^n (n \geq 1)$ be a nonempty measurable set. We say that Ω is convex-plus if the Neumann semigroup in Ω outweighs the Cauchy semigroup in the sense that

$$e^{t\Delta}z(x) \geq \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\Omega} e^{-\frac{|x-y|^2}{4t}} z(y) dy \quad \Omega, t > 0 \quad \text{for all nonnegative } z \in C(\bar{\Omega}). \quad (2.16)$$

Obviously, the inequality (2.16) becomes equality for $\Omega = \mathbb{R}^n$. We also comment that the point-wise lower estimate (2.16) holds for smooth convex domains (hence, the name of convex-plus). Indeed, for such a domain and for a continuous function z with compact support contained in Ω , one can readily check that the lower bound function in (2.16) forms a sub-solution to the corresponding Neumann initial-boundary value problem:

$$f_t = \Delta f \quad \text{in } \Omega, \quad \frac{\partial f}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \quad f|_{t=0} = z.$$

Therefore, by comparison principle, the point-wise lower estimate (2.16) holds for all $z \in C_c(\Omega)$ (space of continuous functions having compact supported in Ω). Then, by the density of $C_c(\Omega)$ in $C(\bar{\Omega})$, the point-wise lower bound (2.16) follows immediately.

Of course, the point-wise lower bound estimate (2.15) follows trivially from (2.16) upon noticing $\text{diam}(\Omega) = \sup_{x, y \in \Omega} |x - y|$. With such clarifications, the mass conservation of u in (2.6) and the second equation in (1.4) give rise to a uniform positive point-wise lower bound for v .^{7,9,10,12} This is crucial in our forthcoming analysis.

Lemma 2.5. Let $\Omega \subset \mathbb{R}^2$ be a bounded, smooth and convex-plus domain. Then, for any given $\sigma \in (0, T_m)$, the local solution component v of (1.4) fulfills

$$v(x, t) \geq v_{\sigma}^m := \begin{cases} m\zeta(\infty), & (x, t) \in \Omega \times (0, T_m) \quad \text{if } \tau = 0, \\ m\zeta(\sigma), & (x, t) \in \Omega \times (\sigma, T_m) \quad \text{if } \tau = 1, \end{cases} \quad (2.17)$$

where the function ζ is defined by

$$\zeta(t) = \int_0^t \frac{1}{4\pi s} e^{-\left(s + \frac{(\text{diam}(\Omega))^2}{4s}\right)} ds. \quad (2.18)$$

Therefore, the local solution component w of (1.4) satisfies

$$\begin{aligned} & \frac{(v_\sigma^m - \eta)w(x, \sigma)}{v_\sigma^m - \eta + \eta w(x, \sigma) \left[1 - e^{-(v_\sigma^m - \eta)(t - \sigma)}\right]} e^{-\int_\sigma^t (v(x, r) - \eta) dr} \\ & \leq w(x, t) \leq K e^{-(v_\sigma^m - \eta)(t - \sigma)}, \quad (x, t) \in \Omega \times (\sigma, T_m). \end{aligned} \quad (2.19)$$

Moreover, in the case of $\tau = 0$ and $\eta = v_\infty^m$, the local solution w of (1.4) satisfies

$$w(x, t) \leq \frac{K}{1 + \eta t}, \quad (x, t) \in \Omega \times (0, T_m). \quad (2.20)$$

Remark 2.1. We particularly thank the referee for a comment on this lemma, which motivates us to formulate the notion of convex-plus in Definition 2.1. This is the only place where we need this additional requirement on Ω for the purpose of deriving explicit bounds. Without the convex-plus assumption on Ω , implicit positive lower bound involving m and Ω for v can also be found Lemma 3.1 of Ref. 8. Thus, with modifications, Theorem 1.1 holds for bounded smooth domains.

Proof. With the clarifications above, the key idea is to employ the point-wise lower bound in (2.15) with $n = 2$ and the representation of the v -equation in (1.4). In the case of $\tau = 0$, see details in (Lemma 2.1 with $n = 2$).¹⁰ When $\tau = 1$, one simply utilizes the point-wise lower bound in (2.15) to (2.12) and then uses the order property of $(e^{t\Delta})_{t \geq 0}$ by the maximum principle and (2.6) to obtain readily (2.17), see similar arguments.^{7,12}

Then applying the lower bound of v in (2.17) to (2.13) or alternatively solving the differential inequality $w_t \leq$ (

Lemma 2.6. *Given $\sigma \in (0, T_m)$ and given $p > 1$, for any $\epsilon > 0$, there exists $C = C(p, \epsilon, \sigma, \tau v_0) > 0$ such that the local solution of (1.4) verifies, for $t \in (\sigma, T_m)$,*

$$\begin{aligned} w(t)^p \leq & e^{-p(v_\sigma^m - \eta - \epsilon)(t - \sigma)} w(\sigma)^p \\ & + K^p \epsilon^{-(p-1)} \int_\sigma^t e^{-p(v_\sigma^m - \eta - \epsilon)(t-s)} v(s)^p ds. \end{aligned} \quad (2.22)$$

Proof. For any $\epsilon > 0$ and $t \in (\sigma, T_m)$, we use the lower bound of v provided by Lemma 2.5 and Young's inequality with epsilon to refine (2.4) as

$$\begin{aligned} \frac{1}{p} \frac{d}{dt} \int_\Omega w^p &= \int_\Omega w^{p-2} v \cdot w + \int_\Omega (\eta - v - 2\eta w) w^p \\ &\leq (v_\sigma^m - \eta - \epsilon) \int_\Omega w^p + \frac{K^p}{p\epsilon^{p-1}} \int_\Omega v^p. \end{aligned} \quad (2.23)$$

Solving this differential inequality via integrating factor method, we quickly infer (2.22). $\square$

The properties of w provided in Lemmas 2.5 and 2.6 will be very important in deriving global boundedness of solutions in Sec. 3.2. Next, for convenience of reference, we collect the widely known 2D GN interpolation inequality for direct use in the sequel.

Lemma 2.7. (2D GN inequality Refs. 6 and 21) *Let $\Omega \subset \mathbb{R}^2$ be a bounded smooth domain and let $p \geq 1$, $q \in (0, p)$ and $r > 0$. Then there exists a positive constant $C_{GN} = C(p, q, r, \Omega)$ such that*

$$\|w\|_{L^p} \leq C_{GN} \left(\|w\|_{L^2}^{1-\frac{q}{p}} \|w\|_{L^q}^{\frac{q}{p}} + \|w\|_{L^r} \right), \quad w \in H^1(\Omega) \cap L^q(\Omega).$$

Finally, we present a consequence of a frequently used Trudinger–Moser inequality from Ref. 28, which will be employed in our subsequent boundedness analysis.

Lemma 2.8. *Let $\Omega \subset \mathbb{R}^2$ be a bounded and smooth domain. Then, for any $\epsilon > 0$, there exists a positive constant $C_\epsilon = C(\epsilon, \Omega) > 0$ such that*

$$\begin{aligned} \int_\Omega fg &\leq \int_\Omega f \ln f + \frac{1}{8\pi} + \epsilon \|a\|_{L^1}^2 \|f\|_{L^2}^2 + \frac{2}{\Omega} \|a\|_{L^1} \|f\|_{L^1} \|g\|_{L^1} \\ &\quad + \|f\|_{L^1} \ln \frac{C_\epsilon}{\|f\|_{L^1}}, \quad a \in \mathbb{R}, \quad 0 < f \in L^1(\Omega), \quad g \in H^1(\Omega). \end{aligned} \quad (2.24)$$

Proof. Note that $\ln z$ is a concave function in z and $\int_\Omega \frac{f}{\|f\|_{L^1}} = 1$; then a simple use of Jensen's inequality implies that

$$\ln \frac{1}{\|f\|_{L^1}} \int_\Omega e^{ag} = \ln \int_\Omega \frac{e^{ag}}{f} \frac{f}{\|f\|_{L^1}} \geq \frac{1}{\|f\|_{L^1}} \int_\Omega f \ln \frac{e^{ag}}{f}.$$

Now, for any $\epsilon > 0$, we apply the Trudinger–Moser inequality²⁸ to find a positive constant $C_\epsilon = C(\epsilon, \Omega) > 0$ such that

$$\begin{aligned} \int_{\Omega} f \ln \frac{e^{ag}}{f} &\leq \int_{\Omega} f \ln \frac{1}{f} + \int_{\Omega} e^{ag} \\ &\leq \int_{\Omega} f \ln \frac{C_\epsilon}{f} + \frac{1}{8\pi} + \epsilon \int_{\Omega} a^2 + \frac{2}{\Omega} \int_{\Omega} a, \end{aligned}$$

which easily yields the desired estimate (2.24). $\square$

3. Negligibility of Haptotaxis on Global Existence and Boundedness

In this section, using the subcritical mass condition $\|u_0\|_{L^1} < \frac{4\pi}{\chi}$ for model (1.1), we shall demonstrate (B1) and (B2) by showing that global existence of classical solutions to (1.4) for any $\eta \geq 0$ and uniform-in-time boundedness for small η , indicating that the haptotaxis effect on global existence and boundedness in 2D is negligible. Our purposes are based on the following evolution identity, which along with subcritical mass $\|u_0\|_{L^1}$ enables us to derive an integral-type Gronwall inequality. As a result, we can improve the L^1 -bound information on u to the L^1 -bound of $u \ln u$, which is the key step to establish the global existence of solutions to the system (1.4).

Lemma 3.1. *The local-in-time classical solution (u, v, w) of (1.4) fulfills*

$$\begin{aligned} & \left(\int_{\Omega} u (\ln u - \chi v - \xi w)^2 \right)'(t) + \tau \chi \int_{\Omega} v_t^2 + \int_{\Omega} u (\ln u - \chi v - \xi w)^2 \\ &= \xi \int_{\Omega} uvw + \eta \xi \int_{\Omega} uw(w-1), \quad t \in (0, T_m), \end{aligned} \quad (3.1)$$

where $\mathcal{E}(t)$ is defined by

$$\mathcal{E}(u, v, w)(t) = \int_{\Omega} u \ln u - \chi \int_{\Omega} uv - \xi \int_{\Omega} uw + \frac{\chi}{2} \int_{\Omega} (v^2 + w^2), \quad t \in (0, T_m). \quad (3.2)$$

Proof. Multiplying the first equation in (1.4) by $\ln u - \chi v - \xi w$, integrating by parts over Ω via the no-flux boundary condition, we derive upon noticing $\int_{\Omega} u_t = 0$ that

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} u (\ln u - \chi v - \xi w)^2 &= \int_{\Omega} u_t (\ln u - \chi v - \xi w)^2 \\ &= \frac{d}{dt} \int_{\Omega} (u \ln u - \chi uv - \xi uw) + \chi \int_{\Omega} uv_t + \xi \int_{\Omega} uw_t. \end{aligned} \quad (3.3)$$

Next, from the facts that $u = \tau v_t - \Delta v + v$ and $w_t = -vw + \eta w(1 - w)$ because of (1.4), we infer from integration by parts that

$$\int_{\Omega} uv_t = \int_{\Omega} (\tau v_t - \Delta v + v)v_t = \tau \int_{\Omega} v_t^2 + \frac{1}{2} \frac{d}{dt} \int_{\Omega} (v^2 + |v|^2), \quad (3.4)$$

and

$$\int_{\Omega} uw_t = \int_{\Omega} uvw + \eta \int_{\Omega} uw(1 - w). \quad (3.5)$$

Finally, substituting (3.4) and (3.5) into (3.3) and then applying simple manipulations, we readily conclude (3.1) with $\phi(t)$ given by (3.2). $\square$

3.1. Global existence without restriction on η

With the help of Lemma 3.1, we now use the subcritical mass condition $\|u_0\|_{L^1} < \frac{4\pi}{\chi}$ to derive an integral-type Gronwall inequality and then to obtain a time-dependent bound for $\|u \ln u\|_{L^1}$, which is indeed sufficient for us to perform the bootstrap argument to conclude global existence.

Lemma 3.2. *Under the subcritical mass condition $m := \|u_0\|_{L^1} < \frac{4\pi}{\chi}$ in (1.6), there exists $C = C(u_0, \tau v_0, w_0, \Omega) > 0$ such that*

$$\|u \ln u\|(t)_{L^1} + \|v(t)\|_{H^1}^2 \leq C e^{\frac{\xi K}{\gamma} t}, \quad t \in (0, T_m), \quad (3.6)$$

where K and γ are defined by (2.1) and (3.9), respectively.

Proof. First, we apply the estimates in (2.1) and (2.6) to (3.1) to discover that

$$\begin{aligned} & \phi'(t) + \tau \chi \int_{\Omega} v_t^2 + \int_{\Omega} u (\ln u - \chi v - \xi w)^2 \leq \xi K \int_{\Omega} uv \\ & + \eta m \xi K(K - 1), \quad t \in (0, T_m), \end{aligned} \quad (3.7)$$

which, upon being integrated from 0 to t , yields simply that

$$\phi(t) \leq \phi(0) + \xi K \int_0^t \int_{\Omega} uv + m \eta \xi K(K - 1)t, \quad t \in (0, T_m). \quad (3.8)$$

For our later purpose, since $m\chi < 4\pi$, we first select positive constants ϵ and γ as follows:

$$\begin{aligned} \epsilon &= \frac{4\pi - m\chi}{16\pi m\chi} > 0, \\ \gamma &= \frac{\frac{2\pi}{4\pi + m\chi}}{\frac{1}{2} - \chi} = \frac{(4\pi - m\chi)\chi}{2[4\pi + m\chi + 2\frac{2\pi}{4\pi + m\chi}]} > 0. \end{aligned} \quad (3.9)$$

Then by straightforward computations, we see that

$$\begin{aligned} A &:= \frac{\chi}{2} - m \frac{1}{8\pi} + \epsilon (\chi + \gamma)^2 \\ &= \frac{(4\pi + m\chi)\chi}{16\pi} - \frac{2\pi}{4\pi + m\chi} - \frac{1}{2} + 3 \frac{2\pi}{4\pi + m\chi} + \frac{1}{2} > 0. \end{aligned} \quad (3.10)$$

By the definition of $\varphi(t)$ in (3.2), we use (2.1) and (2.6) to deduce that

$$\begin{aligned} \varphi(t) &= \int_{\Omega} u \ln u - (\chi + \gamma) \int_{\Omega} uv - \xi \int_{\Omega} uv + \gamma \int_{\Omega} uv + \frac{\chi}{2} \int_{\Omega} (v^2 + |v|^2) \\ &\geq \int_{\Omega} u \ln u - (\chi + \gamma) \int_{\Omega} uv - m\xi K + \gamma \int_{\Omega} uv + \frac{\chi}{2} \int_{\Omega} (v^2 + |v|^2). \end{aligned} \quad (3.11)$$

Next, for ϵ, γ as specified in (3.9), we apply the consequence of Trudinger–Moser inequality (2.24) with $(a, f, g) = (\chi + \gamma, u, v)$ along with the L^1 -boundedness of v in (2.7) to find a positive constant $C_1 = C_1(\Omega) > 0$ such that

$$\int_{\Omega} u \ln u - (\chi + \gamma) \int_{\Omega} uv \geq -m \frac{1}{8\pi} + \epsilon \int_{\Omega} (\chi + \gamma)^2 |v|^2 - C_2, \quad (3.12)$$

where C_2 is a finite number and is defined by

$$C_2 = m \ln \frac{C_1}{m} + \frac{2(\chi + \gamma)}{\Omega} \max_{u_0 \in L^1, \tau \in v_0 \in L^1} \dots$$

Plugging (3.12) into (3.11) and employing (3.10), we infer that

$$\varphi(t) + m\xi K + C_2 \geq \gamma \int_{\Omega} uv + \frac{\chi}{2} \int_{\Omega} v^2 + A \int_{\Omega} |v|^2, \quad t \in (0, T_m). \quad (3.13)$$

Combining (3.8) and (3.13) and observing $A > 0$ due to (3.10), we conclude an integral-type Gronwall inequality as follows:

$$\gamma \int_{\Omega} uv \leq \xi K \int_0^t \int_{\Omega} uv + m\eta\xi K(K-1)t + C_3, \quad t \in (0, T_m). \quad (3.14)$$

where $C_3 = \varphi(0) + m\xi K + C_2$ is a finite number. Here and below, we have assumed for convenience that $u_0 \ln u_0$ belongs to $L^1(\Omega)$ for convenience. Otherwise, we replace the initial time $t = 0$ by $t = \sigma \in (0, T_m)$ so that $u(\sigma) \ln u(\sigma)$ belongs to $L^1(\Omega)$ since $u(\sigma) \in C(\bar{\Omega})$ and $u(\sigma) > 0$ in $\bar{\Omega}$.

Solving the integral-type Gronwall inequality (3.14) via integrating factor method, we infer that

$$\int_{\Omega} uv + \int_0^t \int_{\Omega} uv \leq C_4 e^{\frac{\xi K}{\gamma} t}, \quad t \in (0, T_m). \quad (3.15)$$

Then by (3.8), one can simply deduce that $\varphi(t)$ grows no great than exponentially as well:

$$\varphi(t) \leq C_5 e^{\frac{\xi K}{\gamma} t}, \quad t \in (0, T_m). \quad (3.16)$$

Similarly, this along with (3.13) shows that $\int_{\Omega} |v|^2$ grows no great than exponentially:

$$\int_{\Omega} v^2 + \int_{\Omega} |v|^2 \leq C_6 e^{\frac{\xi K}{\gamma} t}, \quad t \in (0, T_m). \quad (3.17)$$

Finally, in view of (3.2), (3.15) and (3.16) and the fact $s \ln s \leq e^{-1}$ for $s > 0$, we conclude that

$$\begin{aligned} \int_{\Omega} u \ln u &= \int_{\Omega} u \ln u - 2 \int_{\{u \leq 1\}} u \ln u \\ &\leq \chi(t) + \chi \int_{\Omega} uv + \xi \int_{\Omega} uw - 2 \int_{\{u \leq 1\}} u \ln u \\ &\leq \chi(t) + \chi \int_{\Omega} uv + m\xi K + 2e^{-1} |\Omega| \\ &\leq C_7 e^{\frac{\xi K}{\nu} t}, \quad t \in (0, T_m), \end{aligned}$$

which together with (3.17) yields precisely our desired estimate (3.6). $\square$

Next, we wish to raise the regularity of u based on our obtained local L^1 -boundedness of $u \ln u$.

Lemma 3.3. *Under the condition $m < \frac{4\pi}{\chi}$ in (1.6), for any $T \in (0, \infty)$, there exists $C(T) = C(u_0, \tau v_0, w_0, T) > 0$ such that the local solution (u, v, w) of the IBVP (1.4) verifies that*

$$\|u(t)\|_{L^2} + \tau \|v(t)\|_{L^4} + \|w(t)\|_{L^6} \leq C(T), \quad t \in (0, \min\{T, T_m\}). \quad (3.18)$$

Moreover, for any $q \in (1, \infty)$, there exists $C_q(T) = C(q, u_0, \tau v_0, w_0, T) > 0$ such that

$$\|v(t)\|_{W^{1,q}} + \|w(t)\|_{W^{1,q}} \leq C_q(T), \quad t \in (0, \min\{T, T_m\}). \quad (3.19)$$

Proof. Testing the u -equation by u and then integrating over Ω by parts, we find that

$$\frac{d}{dt} \int_{\Omega} u^2 + 2 \int_{\Omega} u^2 = 2\chi \int_{\Omega} u \cdot u \cdot v + 2\xi \int_{\Omega} u \cdot u \cdot w. \quad (3.20)$$

For the v -equation, we first take gradient of the v -equation and then multiply it by $v - v^2$ and, finally integrate by parts and note the fact $2 \int_{\Omega} v \cdot \Delta v = \Delta \int_{\Omega} v^2 - 2 \int_{\Omega} D^2 v^2$ to see that

$$\begin{aligned} &\tau \frac{d}{dt} \int_{\Omega} v^4 + 2 \int_{\Omega} v^2 + 4 \int_{\Omega} v^2 D^2 v^2 + 4 \int_{\Omega} v^4 \\ &= 4 \int_{\Omega} u \Delta v - v^2 - 4 \int_{\Omega} u \cdot v \cdot v^2 + 2 \int_{\partial\Omega} v^2 \frac{\partial}{\partial \nu} v^2. \end{aligned} \quad (3.21)$$

By the ODE for w , we set $p = 6$ in (2.4) to discover

$$\frac{d}{dt} \int_{\Omega} w^6 + 6 \int_{\Omega} (v - \eta + 2\eta w) w^6 = 6 \int_{\Omega} w \cdot w^4 \cdot v \cdot w. \quad (3.22)$$

Then the combinations of (3.20)–(3.22) gives

$$\begin{aligned}
& \frac{d}{dt} \int_{\Omega} (u^2 + \tau v^4 + w^6) + 2 \int_{\Omega} u^2 + 2 \int_{\Omega} v^2 \\
& + 4 \int_{\Omega} v^2 D^2 v^2 + 4 \int_{\Omega} v^4 + 6 \int_{\Omega} (v - \eta + 2\eta w) w^6 \\
& = 2\chi \int_{\Omega} u \cdot u \cdot v + 2\xi \int_{\Omega} u \cdot u \cdot w + 4 \int_{\Omega} u \Delta v v^2 + 4 \int_{\Omega} u \cdot v \cdot v^2 \\
& + 2 \int_{\partial\Omega} v^2 \frac{\partial v^2}{\partial \nu} - 6 \int_{\Omega} w w^4 v \cdot w, \quad t \in (0, \min\{T, T_m\}).
\end{aligned} \tag{3.23}$$

Next, we use similar ideas^{53,56} to bound the terms on the right-hand side of (3.23) in terms of the dissipation terms on its left-hand side. First, using Young's inequality with epsilon and the facts that $|\Delta v| \leq \sqrt{2} D^2 v$ and $0 \leq w \leq K$ thanks to (2.1), we estimate, for any $\epsilon_1 > 0$, that

$$\begin{aligned}
& 2\chi \int_{\Omega} u \cdot u \cdot v + 2\xi \int_{\Omega} u \cdot u \cdot w + 4 \int_{\Omega} u \Delta v v^2 \\
& + 4 \int_{\Omega} u \cdot v \cdot v^2 - 6 \int_{\Omega} w w^4 v \cdot w \\
& \leq \int_{\Omega} u^2 + 2(3 + \chi^2) \int_{\Omega} u^2 v^2 + 2\xi^2 \int_{\Omega} u^2 w^2 \\
& + 4 \int_{\Omega} v^2 D^2 v^2 + \int_{\Omega} v^2 + 6K\epsilon_1 \int_{\Omega} v^6 + \frac{5K}{(6\epsilon_1)^{\frac{1}{5}}} \int_{\Omega} w^6 \\
& \leq \int_{\Omega} u^2 + 4 \int_{\Omega} v^2 D^2 v^2 + \int_{\Omega} v^2 + \frac{4(3 + \chi^2 + \xi^2)}{3(3\epsilon_1)^{\frac{1}{2}}} \int_{\Omega} u^3 \\
& + 2(3 + \chi^2 + 3K)\epsilon_1 \int_{\Omega} v^6 + \frac{5K}{(6\epsilon_1)^{\frac{1}{5}}} + 2\xi^2 \epsilon_1 \int_{\Omega} w^6.
\end{aligned} \tag{3.24}$$

As for the boundary integral in (3.23), one can use^{52–54} the boundary trace embedding to bound it in terms of the boundedness of $\|v\|_{L^2}^2$ in (3.17) to conclude, for any $\epsilon_2 > 0$, that

$$\begin{aligned}
& \int_{\partial\Omega} v^2 \frac{\partial v^2}{\partial \nu} v^2 \leq \epsilon_2 \int_{\Omega} v^2 + C_{\epsilon_2} \int_{\Omega} v^2 \\
& \leq \epsilon_2 \int_{\Omega} v^2 + C_{\epsilon_2} e^{\frac{2\xi K}{\gamma} T}, \quad t \in (0, \min\{T, T_m\}).
\end{aligned} \tag{3.25}$$

Proof. Multiplying both sides of the u equation in (1.4) by $3u^2$, integrating over Ω by parts and applying the boundedness information in Lemma 3.3, Young’s inequality with epsilon and the 2D GN inequality, we estimate, for $t \in (0, \min\{T, T_m\})$, that

$$\begin{aligned}
\frac{d}{dt} \int_{\Omega} u^3 + 6 \int_{\Omega} u |u|^2 &= 6 \chi \int_{\Omega} u^2 |u| \cdot |v| + 6 \xi \int_{\Omega} u^2 |u| \cdot |w| \\
&\leq 2 \int_{\Omega} u |u|^2 + 9 \chi^2 \int_{\Omega} u^3 |v|^2 + 9 \xi^2 \int_{\Omega} u^3 |w|^2 \\
&\leq 2 \int_{\Omega} u |u|^2 + 9 \chi^2 \int_{\Omega} u^4 + \frac{3^5 \chi^2}{4^4} \int_{\Omega} |v|^8 \\
&\quad + 9 \xi^2 \int_{\Omega} u^4 + \frac{3^5 \xi^2}{4^4} \int_{\Omega} |w|^8 \\
&\leq 2 \int_{\Omega} u |u|^2 + 9(\chi^2 + \xi^2) \int_{\Omega} u^{\frac{3}{2}} |v|^{\frac{8}{3}} + C_1(T) \\
&\leq 2 \int_{\Omega} u |u|^2 + 9(\chi^2 + \xi^2) C_2 \\
&\quad \times \int_{\Omega} u^{\frac{3}{2}} |v|^{\frac{4}{3}} |u|^{\frac{3}{2}} |v|^{\frac{4}{3}} + \int_{\Omega} u^{\frac{3}{2}} |v|^{\frac{8}{3}} + C_1(T) \\
&\leq 2 \int_{\Omega} u |u|^2 + 9(\chi^2 + \xi^2) C_3(T) \int_{\Omega} u^{\frac{3}{2}} |v|^{\frac{4}{3}} + C_4(T) \\
&\leq 3 \int_{\Omega} u |u|^2 + C_5(T),
\end{aligned}$$

which, upon being integrated from 0 to t , shows the local boundedness of $u|u|^3$. Then an application of Lemma 2.3 with $p = 3$ yields that

$$u(t)|u|^3 + |v|(t)$$

either Moser iteration or Neumann semigroup method, shown widely in Refs. 21, 37, 38, 46, 51, 53 and 56.

Lemma 3.5. *Under the condition $m < \frac{4\pi}{\chi}$ in (1.6), for any $T \in (0, \infty)$, there exists $C(T) = C(u_0, \tau v_0, w_0, T) > 0$ such that the local solution (u, v, w) of the IBVP (1.4) fulfills that*

$$u(t)_{L^\infty} + v(t)_{W^{1,\infty}} + w(t)_{W^{1,\infty}} \leq C(T), \quad t \in (0, \min\{T, T_m\}). \quad (3.31)$$

Thus, $T_m = \infty$ and (u, v, w) of (1.4) exists globally in time and is locally bounded.

Proof. By the variation-of-constants formula for the u -equation in (1.4) and the well-known smoothing L^p - L^q -estimates for the Neumann heat semigroup $e^{t\Delta}_{t \geq 0}$,^{15,47} we utilize the local boundedness provided in (3.29) to infer, for $t \in (0, \min\{T, T_m\})$, that

$$\begin{aligned} u(t)_{L^\infty} &\leq e^{t\Delta} u_0_{L^\infty} + \chi \int_0^t e^{(t-s)\Delta} \cdot ((u-v)(s))_{L^\infty} ds \\ &\quad + \xi \int_0^t e^{(t-s)\Delta} \cdot ((u-w)(s))_{L^\infty} ds \\ &\leq u_0_{L^\infty} + C_1 \chi \int_0^t (1 + (t-s)^{-\frac{1}{2}-\frac{1}{3}}) e^{-\lambda_1(t-s)} (u-v)(s)_{L^3} ds \\ &\quad + C_2 \xi \int_0^t (1 + (t-s)^{-\frac{1}{2}-\frac{1}{3}}) e^{-\lambda_1(t-s)} (u-w)(s)_{L^3} ds \\ &\leq u_0_{L^\infty} + C_1 \chi \int_0^t (1 + (t-s)^{-\frac{1}{2}-\frac{1}{3}}) e^{-\lambda_1(t-s)} u_{L^3} + v_{L^\infty} ds \\ &\quad + C_2 \xi \int_0^t (1 + (t-s)^{-\frac{1}{2}-\frac{1}{3}}) e^{-\lambda_1(t-s)} u_{L^3} + w_{L^\infty} ds \\ &\leq u_0_{L^\infty} + C_3(T)(\chi + \xi) \int_0^1 (1 + z^{-\frac{5}{6}}) dz + 2 \int_1^\infty e^{-\lambda_1 z} dz \\ &\leq u_0_{L^\infty} + C_3(T)(\chi + \xi) \left(7 + \frac{2}{\lambda_1} \right). \end{aligned} \quad (3.32)$$

Here, λ_1 is the first positive eigenvalue of $-\Delta$ under homogeneous Neumann boundary condition. Then the desired local boundedness (3.31) follows directly from (3.29) and (3.32). Hence, by the extensibility criterion (2.2) in Lemma 2.1, we must have that $T_m = \infty$, that is, the classical solution (u, v, w) of (1.4) exists globally in time and is locally bounded as in (3.31). $\square$

3.2. Uniform boundedness under a smallness on η

With the aid of the energy identity in Lemma 3.1, in the sequel, besides the condition $u_0_{L^1} < \frac{4\pi}{\chi}$, we impose the smallness condition $\eta < v_\infty^m$ in (1.7) to

derive a weighted Gronwall inequality of integral form and then to obtain a time-independent bound for $u \ln u$ in L^1 , which is indeed sufficient for us to perform the bootstrap argument to conclude uniform boundedness.

Lemma 3.6. *Assume that both the condition $m < \frac{4\pi}{\chi}$ in (1.6) and the condition $\eta < v_\infty^m$ in (1.7) hold. Then there exists $C = C(u_0, \tau v_0, w_0, \Omega) > 0$ such that*

$$(u \ln u)(t) \leq C \left(1 + \frac{1}{(4\pi - m\chi)\chi} \left(1 + \frac{\xi K}{v_\infty^m \eta} e^{\frac{C\xi K}{(4\pi - m\chi)(v_\infty^m - \eta)\chi}} \right) \right), \quad t \geq \delta, \quad (3.33)$$

where $\delta > 0$ is determined by (1.12).

Proof. Recalling we have shown in Sec. 3.1 that $T_m = \infty$, and thus, we need only to show uniform-in-time boundedness. First, by $\eta < v_\infty^m$ in (1.7) and the definition of δ in (1.12), we deduce from Lemma 2.5 that

$$\eta < \frac{\eta + v_\infty^m}{2} = m \int_0^\delta \frac{1}{4\pi s} e^{-(s + \frac{(\text{diam}(\Omega))^2}{4s})} ds := v_\delta^m \leq v \quad \text{on } \Omega \times [\delta, \infty), \quad (3.34)$$

which entails the exponential decay of w in (2.19) of Lemma 2.5 as

$$w \leq K e^{-\frac{(v_\infty^m - \eta)}{2}(t - \delta)} \quad \text{on } \Omega \times [\delta, \infty). \quad (3.35)$$

Now, substituting the estimates (2.1), (2.6) and (3.35) into (3.1), we have

$$\begin{aligned} & (u \ln u)_t + \chi \int_\Omega u (\ln u - \chi v - \xi w)^2 \\ & \leq \xi K e^{-\frac{(v_\infty^m - \eta)}{2}(t - \delta)} \int_\Omega uv + \eta m \xi K (K - 1) e^{-\frac{(v_\infty^m - \eta)}{2}(t - \delta)}, \quad t \geq \delta, \end{aligned} \quad (3.36)$$

which, upon being integrated from δ to t , shows trivially that

$$(u \ln u)(t) \leq (u \ln u)(\delta) + \xi K \int_\delta^t e^{-\frac{(v_\infty^m - \eta)}{2}(s - \delta)} \int_\Omega uv + \frac{2m\eta\xi K(K - 1)}{v_\infty^m \eta}, \quad t \geq \delta. \quad (3.37)$$

Under the subcritical mass condition $m < \frac{4\pi}{\chi}$, we still have exactly the same lower bound of $(u \ln u)(t)$ as in (3.13) and hence

$$\gamma \int_\Omega uv \leq (u \ln u)(t) + m\xi K + C_2 \quad t \in (0, T_m),$$

which, combined with (3.37) and a weighted integral form of Gronwall inequality, gives

$$\gamma \int_\Omega uv \leq \xi K \int_\delta^t e^{-\frac{(v_\infty^m - \eta)}{2}(s - \delta)} \int_\Omega uv + B, \quad t \geq \delta, \quad (3.38)$$

where since $\eta < v_\infty^m$ in (1.7) and (3.13), B is a positive finite number and is given by

$$B = (u \ln u)(\delta) + m\xi K + C_2 + \frac{2m\eta\xi K(K - 1)}{v_\infty^m \eta} = O(1) \left(1 + \frac{\xi K}{v_\infty^m \eta} \right). \quad (3.39)$$

In the case of $\xi > 0$, setting

$$y(t) = \int_{\delta}^t e^{-\frac{(v_{\infty}^m - \eta)}{2}(s - \delta)} uv, \quad (3.40)$$

then rewriting (3.38) and multiplying an integrating factor, we find, for $t \geq \delta$, that

$$\frac{\gamma}{B} y(t) e^{-\frac{2\xi K}{\gamma(v_{\infty}^m - \eta)} \left[1 - e^{-\frac{(v_{\infty}^m - \eta)}{2}(t - \delta)} \right]} \leq e^{-\frac{(v_{\infty}^m - \eta)}{2}(t - \delta) - \frac{2\xi K}{\gamma(v_{\infty}^m - \eta)} \left[1 - e^{-\frac{(v_{\infty}^m - \eta)}{2}(t - \delta)} \right]},$$

which, upon integration from δ to t and rearrangement, shows that

$$\frac{\xi K}{B} y(t) \leq e^{\frac{2\xi K}{\gamma(v_{\infty}^m - \eta)} [1 - e^{-\frac{(v_{\infty}^m - \eta)}{2}(t - \delta)}]} \quad 1 \leq e^{\frac{2\xi K}{\gamma(v_{\infty}^m - \eta)}} \quad 1, \quad t \geq \delta. \quad (3.41)$$

Using (3.40) and substituting (3.41) into (3.38) and (3.37), we, respectively, conclude that

$$\int_{\Omega} uv \leq \frac{B}{\gamma} e^{\frac{2\xi K}{\gamma(v_{\infty}^m - \eta)}}, \quad t \geq \delta \quad (3.42)$$

and

$$(t) \leq B e^{\frac{2\xi K}{\gamma(v_{\infty}^m - \eta)}}, \quad t \geq \delta. \quad (3.43)$$

The latter along with the lower bound of χ in (3.13) shows that

$$\int_{\Omega} v^2 + \int_{\Omega} v^2 \leq \frac{B}{\min\{\frac{\chi}{2}, A\}} e^{\frac{2\xi K}{\gamma(v_{\infty}^m - \eta)}}, \quad t \geq \delta. \quad (3.44)$$

Finally, in view of (3.2), (3.42) and (3.43), we finally conclude that

$$\begin{aligned} \int_{\Omega} u \ln u &= \int_{\Omega} u \ln u - 2 \int_{\{u \leq 1\}} u \ln u \\ &\leq (t) + \chi \int_{\Omega} uv + \xi \int_{\Omega} uw - 2 \int_{\{u \leq 1\}} u \ln u \\ &\leq (t) + \chi \int_{\Omega} uv + m\xi K + 2e^{-1} \int_{\Omega} \\ &\leq B e^{\frac{2\xi K}{\gamma(v_{\infty}^m - \eta)}} \left(1 + \frac{\chi}{\gamma} \right) + m\xi K + 2e^{-1} \int_{\Omega}, \quad t \geq \delta. \end{aligned}$$

which together with (3.44) and the bounds for B, γ and A , respectively, in (3.39), (3.9) and (3.10) yields our desired estimate (3.33). $\square$

Next, we again use the exponential decay property of w in (2.19) to refine the arguments in (3.3) to obtain time-independent bound for the terms on the left-hand side of (3.18).

Lemma 3.7. *If $m < \frac{4\pi}{\chi}$ and $\eta < v_{\infty}^m$, then there exist $k \geq 1$ and $C(k) = C(u_0, \tau v_0, w_0, k) > 0$ such that the global classical and positive solution (u, v, w) of (1.4) verifies that*

$$\|u(t)\|_{L^2} + \tau \|v(t)\|_{L^4} + \|w(t)\|_{L^6} \leq C(k), \quad t \geq k\delta; \quad (3.45)$$

and, for any $q \in (1, \infty)$, there exists $C_q(k) = C(q, u_0, \tau v_0, w_0, k) > 0$ such that

$$v(t)_{W^{1,q}} + w(t)_{W^{1,q}} \leq C_q(k), \quad t \geq k\delta, \quad (3.46)$$

where $\delta > 0$ is determined by (1.12).

Proof. Observing, besides $0 \leq w \leq K$, that w decays exponentially as in (3.35) on $\Omega \times (\delta, \infty)$, we then can easily refine (3.24) as follows: for any $\epsilon_1 > 0$ and $t \geq \delta$,

$$\begin{aligned} & 2\chi_{\Omega} u \cdot u \cdot v + 2\xi_{\Omega} u \cdot u \cdot w - 4_{\Omega} u \Delta v - v^2 \\ & - 4_{\Omega} u \cdot v \cdot v^2 - 6_{\Omega} w \cdot w^4 \cdot v \cdot w \\ & \leq \int_{\Omega} u^2 + 4 \int_{\Omega} v^2 D^2 v^2 + \int_{\Omega} v^2 + \frac{4(3 + \chi^2 + \xi^2)}{3(3\epsilon_1)^{\frac{1}{2}}} \int_{\Omega} u^3 \\ & + 2(3 + \chi^2 + 3K)\epsilon_1 \int_{\Omega} v^6 + \frac{5K}{(6\epsilon_1)^{\frac{1}{5}}} e^{-\frac{(v_{\infty}^m - \eta)}{2}(t - \delta)} + 2\xi^2 \epsilon_1 \int_{\Omega} w^6. \end{aligned} \quad (3.47)$$

Since we have shown the uniform boundedness of $\|u \ln u\|_{L^1} + \|v\|_{L^2}$ in (3.33), applying the 2D GN inequality and using the same arguments used to show (3.25)–(3.27), we can readily improve them in the following manners, for $t \geq \delta$:

$$\begin{aligned} & 2 \int_{\partial\Omega} v^2 \frac{\partial}{\partial \nu} v^2 \leq \int_{\Omega} v^2 + C_1, \\ & \int_{\Omega} v^6 \leq C \end{aligned}$$

Finally, substituting the estimate above into (3.49), we end up with

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} (u^2 + \tau |v|^4 + |w|^6) \\ & + \min\{4, 4, (v_{\infty}^m - \eta)\} \int_{\Omega} (u^2 + \tau |v|^4 + |w|^6) \leq C_4, \quad t \geq k\delta, \end{aligned} \quad (3.50)$$

which quickly entails the uniform boundedness information:

$$\begin{aligned} & \int_{\Omega} (u^2 + \tau |v|^4 + |w|^6)(t) \\ & \leq \int_{\Omega} (u^2 + \tau |v|^4 + |w|^6)(k\delta) + \frac{C_4}{\min\{4, 4, (v_{\infty}^m - \eta)\}}, \quad t \geq k\delta, \end{aligned} \quad (3.51)$$

from which (3.45) follows easily. On the other hand, taking $\sigma = \delta$, noting $v_{\delta}^m = (v_{\infty}^m + \eta)/2 > \eta$ in (3.34) and choosing $\epsilon = (v_{\infty}^m - \eta)/4$ in Lemma 2.6, we derive from (2.22), for any $p > 1$, that

$$w(t)_{L^p} \leq w(\delta)_{L^p} + \frac{4K}{v_{\infty}^m - \eta} \sup_{s \in (\delta, t)} v(s)_{L^p}, \quad t \geq \delta. \quad (3.52)$$

Finally, based on (3.51), we first take $p = 2$ in Lemma 2.3 to obtain the uniform $W^{1,q}$ -boundedness of v for any $q \in (1, \infty)$, which combined with (3.52) gives our desired estimate (3.46). $\square$

Armed with the uniform boundedness in Lemma 3.7, one can easily adapt the arguments done in Lemmas 3.4 and 3.5 to obtain first uniform $(L^3, W^{1,\infty}, W^{1,\infty})$ -global boundedness of (u, v, w) thanks to Lemma 2.3 and (3.52), and then uniform L^{∞} -boundedness of u . Finally, we obtain the following desired uniform boundedness.

Lemma 3.8. *Under the subcritical condition $m < \frac{4\pi}{\chi}$ in (1.6) and the condition $\eta < v_{\infty}^m$ in (1.7), the global solution (u, v, w) of the IBVP (1.4) is uniformly bounded according to (1.8).*

3.3. Haptotaxis-only ($\chi = 0$) is unable to induce finite time blow-up in (1.4)

From the crucial starting boundedness provided in Lemmas 3.2 and 3.6, one can easily see that the obtained bounds become unbounded when $\chi = 0$. Will haptotaxis induce finite blow-up in (1.4) with $\chi = 0$? In the sequel, we indeed shall give a negative answer by showing that all classical solutions to the IBVP (1.4) with $\chi = 0$ exists globally-in-time and is uniformly bounded if $\eta \geq 0$ is small.

Lemma 3.9. *There exists $C = C(u_0, \tau v_0, w_0, \Omega) > 0$ such that the local-in-time classical solution (u, v, w) of the IBVP (1.4) with $\chi = 0$ fulfills*

$$(u \ln u)(t)_{L^1} + \|v(t)\|_{H^1}^2 \leq C, \quad t \in (0, T_m). \quad (3.53)$$

Proof. It follows from (3.1) with $\chi = 0$, for $t \in (0, T_m)$, that

$$\frac{d}{dt} \int_{\Omega} u(\ln u - \xi w) + \int_{\Omega} u (\ln u - \xi w)^2 = \xi \int_{\Omega} uvw + \eta \xi \int_{\Omega} uw(w-1). \quad (3.54)$$

Now, setting $\rho = \bar{u}e^{-\frac{\xi w}{2}}$, by the facts $0 \leq w \leq K$ and $\int_{\Omega} u = m$ from (2.1) and (2.6), we see that $\rho \frac{2}{L^2} \leq \int_{\Omega} u = m$. Then using the L^3 -boundedness of v ensured by (2.11) with $q = \frac{6}{5}$, Young's inequality and the 2D GN interpolation inequality, from (3.54) one has

$$\begin{aligned} & 2 \frac{d}{dt} \int_{\Omega} e^{\xi w} \rho^2 \ln \rho + 4 \int_{\Omega} \rho^2 + 2 \int_{\Omega} e^{\xi w} \rho^2 \ln \rho \\ & \leq K \xi e^{\xi K} \int_{\Omega} v \rho^2 + 2 e^{\xi K} \int_{\Omega} \rho^2 \ln \rho + \eta \xi m K (K-1) \\ & \leq \frac{1}{3} K^3 \xi^3 e^{\xi K} \int_{\Omega} v^3 + \frac{2}{3} e^{\xi K} \int_{\Omega} \rho^3 + 2 e^{\xi K} \int_{\Omega} \rho^3 + \eta \xi m K (K-1) \\ & \leq C_1 + \frac{8}{3} e^{\xi K} (C_2 \int_{\Omega} \rho \frac{2}{L^2} + C_2 \int_{\Omega} \rho \frac{3}{L^2}) \\ & \leq \int_{\Omega} \rho \frac{2}{L^2} + C_3, \end{aligned} \quad (3.55)$$

which gives

$$\frac{d}{dt} \int_{\Omega} e^{\xi w} \rho^2 \ln \rho + \int_{\Omega} e^{\xi w} \rho^2 \ln \rho \leq \frac{C_3}{2}. \quad (3.56)$$

Solving the Gronwall differential inequality (3.56) and recalling the facts that $0 \leq w \leq K$, $\int_{\Omega} u = m$ and the algebraic fact $-2s^2 \ln s \leq e^{-1}$ for $s > 0$, we readily infer that

$$\begin{aligned} \int_{\Omega} \rho^2 \ln \rho & \leq \int_{\Omega} e^{\xi w} \rho^2 \ln \rho = \int_{\Omega} e^{\xi w} \rho^2 \ln \rho \Big|_{\{0 < \rho < 1\}} e^{\xi w} \rho^2 \ln \rho \\ & \leq \int_{\Omega} e^{\xi w} \rho^2 \ln \rho + e^{K\xi-1} \int_{\Omega} \rho^2 \leq C_4, \quad t \in (0, T_m) \end{aligned} \quad (3.57)$$

and, with the elementary fact $-s \ln s \leq e^{-1}$ for $s > 0$, we get that

$$\begin{aligned} \int_{\Omega} u \ln u & = \int_{\Omega} u \ln u \Big|_{\{0 < u < 1\}} u \ln u \\ & = 2 \int_{\Omega} e^{\xi w} \rho^2 \ln \rho + \xi \int_{\Omega} uvw \Big|_{\{0 < u < 1\}} u \ln u \\ & \leq 2C_4 + K \xi m + 2e^{-1} \int_{\Omega} \rho^2, \quad t \in (0, T_m). \end{aligned} \quad (3.58)$$

In the case of $\tau = 0$, we integrate the v -equation in (1.4) by part to see

$$\int_{\Omega} v^2 + \int_{\Omega} v^2 = \int_{\Omega} uv. \quad (3.59)$$

To bound the term on the right, we employ (2.24) with $(a, \epsilon, f, g) = (1, \frac{1}{8\pi}, \frac{mu}{2\pi}, \frac{2\pi v}{m})$ and use the L^1 -boundedness of v in (2.7) and (3.58) to deduce that

$$\begin{aligned} uv &\leq \frac{mu}{2\pi} \ln \frac{mu}{2\pi} + \frac{1}{2} v^2_{L^2} + \frac{2m}{\Omega} v_{L^1} + \left\| \frac{mu}{2\pi} \right\|_{L^1} \ln \frac{C_5}{\frac{mu}{2\pi}_{L^1}} \\ &\leq C_6 + \frac{1}{2} v^2_{L^2}. \end{aligned}$$

Inserting this into (3.59), we quickly get the H^1 -boundedness of v :

$$v(t)_{H^1}^2 \leq C_7, \quad t \in (0, T_m). \quad (3.60)$$

In the case of $\tau > 0$, upon integration by part from the v -equation in (1.4) and a use of Young's inequality, we find that

$$\tau \frac{d}{dt} \int_{\Omega} v^2 + 2 \int_{\Omega} v^2 + \int_{\Omega} \Delta v^2 \leq \int_{\Omega} u^2 \leq e^{2K\xi} \int_{\Omega} \rho^4. \quad (3.61)$$

Due to the uniform L^1 -boundedness of $\rho^2 \ln \rho$ in (3.57), the 2D GN inequality involving logarithmic functions in Lemma A.5 in Ref. 38 or Lemma 3.4 in Ref. 56 entails that

$$e^{2K\xi} \int_{\Omega} \rho^4 \leq \int_{\Omega} \rho^2 + C_8.$$

Substituting this into (3.61) and combining (3.55), we obtain an ODI as follows:

$$\frac{d}{dt} \int_{\Omega} (2e^{\xi w} \rho^2 \ln \rho + \tau v^2) + \min \left\{ 1, \frac{2}{\tau} \right\} \int_{\Omega} (2e^{\xi w} \rho^2 \ln \rho + \tau v^2) \leq C_9.$$

Solving this ODI and noting (3.57), we get the L^2 -boundedness of v and then by (2.11) we obtain the H^1 -boundedness of v . This along with (3.60) and (3.58) gives rise to (3.53). $\square$

Lemma 3.10. *The unique classical solution (u, v, w) of the IBVP (1.4) with $\chi = 0$ exists globally in time. Moreover, when $\eta < v_{\infty}^m$ if $\tau > 0$ and $\eta \leq v_{\infty}^m$ if $\tau = 0$ with v_{∞}^m defined in (1.7), then the solution is uniformly bounded as in (1.8).*

Proof. In light of the key boundedness in Lemma 3.9, global existence follows easily from Lemmas 3.3–3.5. When $\eta < v_{\infty}^m$, we first see from (2.19) of Lemma 2.5 that w decays exponentially on $\Omega \times (\delta, \infty)$, and then we easily follow Lemmas 3.7 and 3.8 to derive the desired global boundedness. When $\eta = v_{\infty}^m$ and $\tau = 0$, we get from (2.20) of Lemma 2.5 that w decays algebraically on $\Omega \times (0, \infty)$, and then (3.47) correspondingly becomes: for any $\epsilon_1 > 0$ and $t \geq 0$

$$\begin{aligned} &2\xi \int_{\Omega} u \cdot u \cdot w + 4 \int_{\Omega} u \Delta v + v^2 + 4 \int_{\Omega} u \cdot v \cdot v^2 + 6 \int_{\Omega} w \cdot w^4 + v \cdot w \\ &\leq \int_{\Omega} u^2 + 4 \int_{\Omega} v^2 + D^2 v^2 + \int_{\Omega} v^2 + \frac{4(3+\xi^2)}{3(3\epsilon_1)^{\frac{1}{2}}} \int_{\Omega} u^3 \\ &\quad + 2(3+3K)\epsilon_1 \int_{\Omega} v^6 + \frac{5K}{(6\epsilon_1)^{\frac{1}{5}}(1+\eta t)} + 2\xi^2 \epsilon_1 \int_{\Omega} w^6. \end{aligned}$$

With these preparations at hand, one can easily adapt the arguments in Lemmas 3.7 and 3.8 to derive the desired global boundedness. $\square$

Remark 3.1. Rechecking our arguments, we can easily find a critical global existence criterion for (1.4), namely, if

$$u(t)$$

reduced to

$$\begin{aligned}
 & \left\{ \begin{array}{ll} \Delta v + v = u, & x \in \Omega, \\ u = \frac{me^{\chi v}}{\Omega e^{\chi v}}, & x \in \Omega, \\ \frac{\partial u}{\partial \nu} = 0 = \frac{\partial v}{\partial \nu}, & x \in \partial\Omega, \\ u = m = v. & \end{array} \right. \quad (U, V) \stackrel{=}{\Longleftrightarrow} (u, v - \bar{v}) \quad \left\{ \begin{array}{ll} \Delta V + V = U - \frac{m}{\Omega}, & x \in \Omega, \\ U = \frac{me^{\chi V}}{\Omega e^{\chi V}}, & x \in \Omega, \\ \frac{\partial U}{\partial \nu} = 0 = \frac{\partial V}{\partial \nu}, & x \in \partial\Omega, \\ U = m, \quad V = 0. & \end{array} \right. \quad (4.2)
 \end{aligned}$$

We point out, even though, the steady state problem (4.2) is the same as that of the chemotaxis-only model (1.1), while, the functional (not a Lyapunov functional) associated with our chemotaxis-haptotaxis model (1.4) is more complex; indeed, by (3.2), the functional reads as

$$\begin{aligned}
 (u, v, w) := & \int_{\Omega} k_s(u, v) \, dx - \int_{\Omega} uw, \quad k_s(u, v) = \int_{\Omega} u \ln u \, dx - \int_{\Omega} uv \\
 & + \frac{\chi}{2} \int_{\Omega} (v^2 + v^2); \quad (4.3)
 \end{aligned}$$

the latter is a Lyapunov functional of the chemotaxis-only system (4.2). First, by the arguments in (2.1) and Lemma 3.5 of Ref. 14, we obtain a lower bound for $\int_{\Omega} k_s$ when $\int_{\Omega} u_0 = \frac{4\pi l}{\chi}$ for any $l \in \mathbb{N}^+$.

Lemma 4.1. Suppose $m = \int_{\Omega} u_0 = \frac{4\pi l}{\chi}$ for all $l \in \mathbb{N}^+$. Then, with k_s defined by (4.3), it follows that

$$\begin{aligned}
 M = & \inf_{k_s(U, V) : (U, V) \text{ is a solution of the stationary system (4.2)}} \\
 & (0, \infty). \quad (4.4)
 \end{aligned}$$

Next, for convenience, for $\epsilon > 0$, $m > 0$ and $x_0 \in \partial\Omega$, we redefine (U_ϵ, V_ϵ) as follows:

$$V_\epsilon(x) = \frac{1}{\chi} \ln \frac{\epsilon^2}{(\epsilon^2 + \pi |x - x_0|^2)^2} - \frac{1}{\Omega} \int_{\Omega} \ln \frac{\epsilon^2}{(\epsilon^2 + \pi |x - x_0|^2)^2} \, dx \quad (4.5)$$

and

$$U_\epsilon(x) = \frac{me^{\chi V_\epsilon(x)}}{\int_{\Omega} e^{\chi V_\epsilon(x)} \, dx}. \quad (4.6)$$

Lemma 4.2. Let $(U_\epsilon, V_\epsilon)_{\epsilon>0}$ be defined by (4.5) and (4.6). Then $(U_\epsilon, V_\epsilon) \in [C(\bar{\Omega}) \times W^{1,\infty}(\Omega)]^2$, $\int_{\Omega} V_\epsilon = 0$, $\int_{\Omega} U_\epsilon = m$ and the infimum of V_ϵ is uniformly bounded in ϵ

and it is given by

$$\inf_{x \in \Omega} V_\epsilon(x) = \frac{2}{\chi} \ln[\epsilon^2 + \pi(\text{diam } \Omega)^2] - \frac{1}{\Omega} \ln(\epsilon^2 + \pi |x - x_0|^2). \quad (4.7)$$

In addition, we also have

$$k_s(U_\epsilon, V_\epsilon) \leq 4m \frac{4\pi}{\chi} \ln \frac{1}{\epsilon} + R_\epsilon, \quad (4.8)$$

where R_ϵ is defined in (4.14) below and R_ϵ is uniformly bounded in ϵ .

Proof. By direct computations, the first three assertions follow. Simple calculations along with the integrability of $\ln |x - x_0|$ on Ω yield the uniform boundedness for $\inf_\Omega V_\epsilon$ and (4.7). In the sequel, we shall show (4.8) by assuming $x_0 = 0$ for convenience. By the definition of U_ϵ in (4.6), we compute

$$\begin{aligned} U_\epsilon \ln U_\epsilon - \chi \int_\Omega U_\epsilon V_\epsilon &= \frac{m}{\Omega} \int_\Omega \frac{e^{\chi V_\epsilon}}{e^{\chi V_\epsilon}} \ln m + \chi V_\epsilon - \ln \int_\Omega e^{\chi V_\epsilon} \\ &\quad - \frac{\chi m}{\Omega} \int_\Omega \frac{e^{\chi V_\epsilon}}{e^{\chi V_\epsilon}} V_\epsilon \\ &= m \ln m - m \ln \int_\Omega e^{\chi V_\epsilon}, \end{aligned}$$

which, together with the definition of k_s in (4.3), allows us to deduce that

$$k_s(U_\epsilon, V_\epsilon) = m \ln m - m \ln \int_\Omega e^{\chi V_\epsilon} + \frac{\chi}{2} \int_\Omega |V_\epsilon|^2 + \frac{\chi}{2} \int_\Omega V_\epsilon^2. \quad (4.9)$$

Next, we estimate the terms on the right. Using the definition of V_ϵ in (4.5), we compute that

$$\begin{aligned} m \ln \int_\Omega e^{\chi V_\epsilon} &= m \ln \int_\Omega \frac{\epsilon^2}{(\epsilon^2 + \pi |x|^2)^2} = \frac{1}{\Omega} \ln \int_\Omega \frac{\epsilon^2}{(\epsilon^2 + \pi |x|^2)^2} \\ &= 2m \ln \epsilon - \frac{m}{\Omega} \int_\Omega \ln(\epsilon^2 + \pi |x|^2)^2 - m \ln \int_\Omega \frac{\epsilon^2}{(\epsilon^2 + \pi |x|^2)^2}. \end{aligned} \quad (4.10)$$

Noting $\Omega \subset B(0, R)$ with R being the maximum distance between $x_0 = 0$ and $\partial\Omega$, we further get

$$\frac{\Omega \epsilon^2}{(\epsilon^2 + \pi R^2)^2} \leq \frac{\epsilon^2}{\Omega (\epsilon^2 + \pi |x|^2)^2} \leq 2\pi \epsilon^2 \int_0^R \frac{r}{(\epsilon^2 + \pi r^2)^2} dr = 1 - \frac{\epsilon^2}{\epsilon^2 + \pi R^2}$$

as well as

$$\begin{aligned} \frac{\chi}{2} \int_{\Omega} V_{\epsilon}^2 &= \frac{8\pi^2}{\chi} \int_{\Omega} \frac{x^2}{(\epsilon^2 + \pi x^2)^2} \leq \frac{16\pi^3}{\chi} \int_0^R \frac{r^3}{(\epsilon^2 + \pi r^2)^2} dr \\ &= \frac{8\pi}{\chi} \left[\ln(\epsilon^2 + \pi R^2) - 2 \ln \epsilon + \frac{\epsilon^2}{\epsilon^2 + \pi R^2} \right]. \end{aligned} \quad (4.11)$$

Moreover, after some direct calculations, we obtain

$$\begin{aligned} \frac{\chi^2}{4} V_{\epsilon}^2 &= \int_{\Omega} \ln(\epsilon^2 + \pi x^2) \left[\frac{1}{\Omega} \int_{\Omega} \ln(\epsilon^2 + \pi x^2) \right]^2 \\ &= \ln^2(\epsilon^2 + \pi x^2) \int_{\Omega} \frac{2}{\Omega} \ln(\epsilon^2 + \pi x^2) \int_{\Omega} \ln(\epsilon^2 + \pi x^2) \\ &\quad + \frac{1}{\Omega^2} \int_{\Omega} \ln(\epsilon^2 + \pi x^2)^2, \end{aligned}$$

which enables us to infer that

$$\frac{\chi}{2} \int_{\Omega} V_{\epsilon}^2 = \frac{1}{2\chi} \int_{\Omega} \ln^2(\epsilon^2 + \pi x^2)^2 - \frac{1}{2\chi\Omega} \int_{\Omega} \ln(\epsilon^2 + \pi x^2)^2. \quad (4.12)$$

Finally, we substitute (4.10)–(4.12) into (4.9) to conclude that

$$k_s(U_{\epsilon}, V_{\epsilon}) \leq 4 - m \left[\frac{4\pi}{\chi} \ln \frac{1}{\epsilon} + R_{\epsilon} \right], \quad (4.13)$$

where

$$\begin{aligned} R_{\epsilon} &= m \ln m - \frac{m}{\Omega} \int_{\Omega} \ln(\epsilon^2 + \pi x^2)^2 - m \ln \frac{\Omega}{(\epsilon^2 + \pi R^2)^2} \\ &\quad + \frac{8\pi}{\chi} \left[\ln(\epsilon^2 + \pi R^2) + \frac{\epsilon^2}{\epsilon^2 + \pi R^2} \right] - 1 \\ &\quad + \frac{1}{2\chi} \int_{\Omega} \ln^2(\epsilon^2 + \pi x^2)^2 - \frac{1}{2\chi\Omega} \int_{\Omega} \ln(\epsilon^2 + \pi x^2)^2. \end{aligned} \quad (4.14)$$

By the integrability of $\ln x$ on Ω , it follows that R_{ϵ} is uniformly bounded in $\epsilon \rightarrow 0$. Therefore, the desired estimate (4.8) follows from (4.13) and (4.14). $\square$

Thanks to the properties of $(U_{\epsilon}, V_{\epsilon})$ provided in Lemma 4.2, we can set $(u_0, v_0, w_0) = (U_{\epsilon}, V_{\epsilon} - \inf_{\Omega} V_{\epsilon}, w_0)$ in our original system (1.4). In the sequel, we study the unboundedness of such emanating solutions, denoted by, $(u^{\epsilon}, v^{\epsilon}, w^{\epsilon})$. To that purpose, we use the following change of variables

$$U = u, \quad V = v - \bar{v} = v - \inf_{\Omega} V_{\epsilon} + \frac{m}{\Omega} e^{-t} + \frac{m}{\Omega}, \quad (4.15)$$

$$W = w \quad \text{on } \bar{\Omega} \times [0, T_m^{\epsilon})$$

to transform our original chemotaxis–haptotaxis system (1.4) equivalently as

$$\begin{aligned}
 U_t &= \Delta U - \chi \cdot (U - V) - \xi \cdot (U - W), & x \in \Omega, t > 0, \\
 V_t &= \Delta V - V + U - \frac{m}{\Omega}, & x \in \Omega, t > 0, \\
 W_t &= -V - \inf_{\Omega} V_{\epsilon} + \frac{m}{\Omega} e^{-t} + \frac{m}{\Omega} W + \eta W(1 - W), & x \in \Omega, t > 0, \\
 \frac{\partial U}{\partial \nu} - \chi U \frac{\partial V}{\partial \nu} - \xi U \frac{\partial W}{\partial \nu} &= \frac{\partial V}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\
 U(x, 0) &= U_{\epsilon}(x), \quad V(x, 0) = V_{\epsilon}(x), \quad W(x, 0) = w_0(x), & x \in \Omega.
 \end{aligned} \tag{4.16}$$

Let us denote the resulting solution of (4.16) by $(U^{\epsilon}, V^{\epsilon}, W^{\epsilon})$. Then it follows from (4.15) that

$$U^{\epsilon} = u^{\epsilon}, \quad V^{\epsilon} = v^{\epsilon} - \inf_{\Omega} V_{\epsilon} + \frac{m}{\Omega} e^{-t} + \frac{m}{\Omega}, \quad W^{\epsilon} = w^{\epsilon} \quad \text{on } \bar{\Omega} \times [0, T_m^{\epsilon}). \tag{4.17}$$

Because of this relation, we only need to focus on the existence of blowup solutions to (4.16) under supercritical mass condition $m > \frac{4\pi}{\chi}$.

To start off, using the functional $\Phi(t)$ defined in (4.3), performing the same computations as in Lemma 3.1 to the transformed system (4.16), we find the resulting differential equality that

$$\begin{aligned}
 & \Phi'(U^{\epsilon}, V^{\epsilon}, W^{\epsilon})(t) + \chi \int_{\Omega} (V_t^{\epsilon})^2 + \int_{\Omega} U^{\epsilon} (\ln U^{\epsilon} - \chi V^{\epsilon} - \xi W^{\epsilon})^2 \\
 &= \xi \int_{\Omega} U^{\epsilon} V^{\epsilon} W^{\epsilon} + \eta \xi \int_{\Omega} U^{\epsilon} (W^{\epsilon})^2 + \xi \int_{\Omega} \left(\inf_{\Omega} V_{\epsilon} + \frac{m}{\Omega} e^{-t} + \frac{m}{\Omega} \right) \eta \\
 & \quad \times \int_{\Omega} U^{\epsilon} W^{\epsilon}.
 \end{aligned} \tag{4.18}$$

Lemma 4.3. *Let $\eta < v_{\infty}^m$ with v_{∞}^m defined in (1.7). For given $m > 0$ and for any $\epsilon > 0$, suppose that $(U^{\epsilon}, V^{\epsilon}, W^{\epsilon})$ is a global and uniformly bounded-in-time solution of (4.16). Then there exists a subsequence of times $t_k^{\epsilon} \rightarrow \infty$ such that $(U^{\epsilon}, V^{\epsilon}, W^{\epsilon})(t_k) \rightarrow (U_{\infty}^{\epsilon}, V_{\infty}^{\epsilon}, 0)$ in $[C^2(\bar{\Omega})]^2$ for some functions $(U_{\infty}^{\epsilon}, V_{\infty}^{\epsilon})$ $[C^2(\bar{\Omega})]^2$. Furthermore, $(U_{\infty}^{\epsilon}, V_{\infty}^{\epsilon})$ is a solution of (4.2) and*

$$\begin{aligned}
 (U_{\infty}^{\epsilon}, V_{\infty}^{\epsilon}, 0) &\leq (U_{\epsilon}, V_{\epsilon}, w_0) + mK\xi - K\eta + \max \left\{ \frac{m}{\Omega}, \inf_{\Omega} V_{\epsilon} \right\} \delta + \frac{2}{v_{\infty}^m \eta} \\
 &+ K\xi \left(\delta + \frac{2}{v_{\infty}^m \eta} \right) \|U^{\epsilon}(V^{\epsilon})^+\|_{L^{\infty}(0, \infty; L^1(\Omega))},
 \end{aligned} \tag{4.19}$$

where $K = \max_{\Omega} 1, w_0, L^\infty$, δ is a positive and finite number defined by (1.12) and $\inf_{\Omega} V_\epsilon$ is defined in (4.7) of Lemma 4.2 and it is uniformly bounded in ϵ .

Proof. For $\epsilon > 0$, notice from (4.17) that $(u^\epsilon, v^\epsilon, w^\epsilon)$ is a global and bounded classical solution to the system (1.4) with $(u_0, v_0, w_0) = (U_\epsilon, V_\epsilon - \inf_{\Omega} V_\epsilon, w_0)$. Then we first use the standard bootstrap arguments involving interior and boundary parabolic (Schauder) regularity theory¹⁹ to the second equation in (1.4) to infer the $C^{2+\theta, 1+\theta/2}$ -estimate for v^ϵ . Then we use the formula for w in (2.13) to infer the same type estimate for w^ϵ . Finally, we derive the same type estimate for u^ϵ from the first equation in (1.4). Turning back to $(U^\epsilon, V^\epsilon, W^\epsilon)$ via (4.17), we altogether have, for some $\theta \in (0, 1)$, that

$$(U^\epsilon, V^\epsilon, W^\epsilon)_{C^{2+\theta, 1+\frac{\theta}{2}}(\bar{\Omega} \times [t, t+1])} \leq C_1(\epsilon), \quad t \geq 1. \quad (4.20)$$

This along with the Arzelà–Ascoli compactness theorem shows that $(U^\epsilon, V^\epsilon, W^\epsilon)(t)_{t \geq 1}$ is relatively compact in $[C^2(\bar{\Omega})]^3$, and then it follows that $(U^\epsilon, V^\epsilon, W^\epsilon)(t)$ defined in (4.3) is bounded for $t \geq 1$. Hence, by the exponential decay of W^ϵ in (2.19) of Lemma 2.18, there exists a subsequence $t_k^\epsilon \rightarrow \infty$ such that $(U^\epsilon, V^\epsilon, W^\epsilon)(t_k^\epsilon) \rightarrow (U_\infty^\epsilon, V_\infty^\epsilon, 0)$ in $(C^2(\bar{\Omega}))^3$ for some functions $U_\infty^\epsilon, V_\infty^\epsilon \in C^2(\bar{\Omega})$. This immediately shows that $(U_\infty^\epsilon, V_\infty^\epsilon)$ verifies the last two lines in (4.2). Moreover, it further follows from (4.3) that

$$(U^\epsilon, V^\epsilon, W^\epsilon)(t_k^\epsilon) \rightarrow (U_\infty^\epsilon, V_\infty^\epsilon, 0) \quad \text{as } t_k^\epsilon \rightarrow \infty, \quad (4.21)$$

By $\eta < v_\infty^m$ in (1.7) and the definition of δ in (1.12), we use (2.17) of Lemma 2.5 to conclude $v^\epsilon \geq \frac{\eta + v_\infty^m}{2}$ on $\bar{\Omega} \times [\delta, \infty)$. Finally, we use (2.19) of Lemma 2.5 (cf. (3.34) and (3.35)) to see that

$$W^\epsilon = w^\epsilon \leq \max_{\Omega} 1, w_0, L^\infty e^{-\frac{(v_\infty^m - \eta)}{2}(t - \delta)} := K e^{-\frac{(v_\infty^m - \eta)}{2}(t - \delta)} \quad \text{on } \bar{\Omega} \times [\delta, \infty). \quad (4.22)$$

Now, integrating (4.18) from 0 to t , using the nonnegativity of U^ϵ, W^ϵ and the fact $\int_{\Omega} U^\epsilon = m$ as well as the bound $0 \leq W^\epsilon \leq K$, we conclude, for $t > \delta$, that

$$\begin{aligned} & (U^\epsilon, V^\epsilon, W^\epsilon)(t) + \chi \int_0^t \int_{\Omega} (V_t^\epsilon)^2 + \int_0^t \int_{\Omega} U^\epsilon (\ln U^\epsilon - \chi V^\epsilon - \xi W^\epsilon)^2 \\ &= (U_\epsilon, V_\epsilon, w_0) + \xi \int_0^t \int_{\Omega} U^\epsilon V^\epsilon W^\epsilon + \eta \xi \int_0^t \int_{\Omega} U^\epsilon (W^\epsilon)^2 \\ &+ \xi \int_0^t \int_{\Omega} \inf_{\Omega} V_\epsilon + \frac{m}{\Omega} e^{-s} + \frac{m}{\Omega} \int_0^t \int_{\Omega} U^\epsilon W^\epsilon \\ &\leq (U_\epsilon, V_\epsilon, w_0) + \xi \int_0^\delta \int_{\Omega} U^\epsilon V^\epsilon W^\epsilon + \eta \xi \int_0^\delta \int_{\Omega} U^\epsilon (W^\epsilon)^2 \\ &+ K \xi \int_0^t \int_{\Omega} U^\epsilon (V^\epsilon)^+ e^{-\frac{(v_\infty^m - \eta)}{2}(s - \delta)} ds \end{aligned}$$

$$\begin{aligned}
& + mK^2 \eta \xi \int_{\delta}^t e^{-(v_{\infty}^m - \eta)(s - \delta)} ds \\
& + \xi \max \left\{ \frac{m}{\Omega}, \inf_{\Omega} V_{\epsilon} \right\} \int_0^{\delta} \int_{\Omega} U^{\epsilon} W^{\epsilon} + mK \int_{\delta}^t e^{-\frac{(v_{\infty}^m - \eta)}{2}(s - \delta)} ds \\
& \leq (U_{\epsilon}, V_{\epsilon}, w_0) + K \xi \int_{\delta}^t \int_{\Omega} U^{\epsilon} (V^{\epsilon})^{+} \,_{L^{\infty}((0, \infty); L^1(\Omega))} \\
& + mK^2 \xi \eta \int_{\delta}^t \int_{\Omega} U^{\epsilon} + \frac{1}{v_{\infty}^m} \int_{\Omega} U^{\epsilon} (\ln U^{\epsilon} - \chi V^{\epsilon} - \xi W^{\epsilon})^2 \leq C_2(\epsilon),
\end{aligned}$$

which, upon an obvious use of (4.21), trivially implies (4.19), and

$$\chi \int_1^{\infty} \int_{\Omega} (V_t^{\epsilon})^2 + \int_1^{\infty} \int_{\Omega} U^{\epsilon} (\ln U^{\epsilon} - \chi V^{\epsilon} - \xi W^{\epsilon})^2 \leq C_2(\epsilon). \quad (4.23)$$

Furthermore, we employ (4.20), (4.22) and (4.23) to extract a further subsequence, still denoted $(t_k^{\epsilon})_{k \geq 1}$ for convenience, such that

$$\int_{\Omega} (V_t^{\epsilon})^2(t_k^{\epsilon}) \rightarrow 0 \quad \text{as } t_k^{\epsilon} \rightarrow \infty \quad (4.24)$$

and

$$\int_{\Omega} U^{\epsilon}(t_k^{\epsilon}) (\ln U^{\epsilon}(t_k^{\epsilon}) - \chi V^{\epsilon}(t_k^{\epsilon}))^2 \rightarrow 0 \quad \text{as } t_k^{\epsilon} \rightarrow \infty. \quad (4.25)$$

Then using (4.24), we evaluate the second equation in (4.16) at $t = t_k^{\epsilon}$ and send $k \rightarrow \infty$ to infer

$$\Delta V_{\infty}^{\epsilon} + V_{\infty}^{\epsilon} = U_{\infty}^{\epsilon} - \frac{m}{\Omega}. \quad (4.26)$$

By a connectedness argument, one gets $U_{\infty}^{\epsilon} > 0$ (cf. Lemma 3.1 of Ref. 47). Then we send $k \rightarrow \infty$ in (4.25) to obtain $(\ln U_{\infty}^{\epsilon} - \chi V_{\infty}^{\epsilon})^2 = 0$ in $\bar{\Omega}$, which gives rise to

$$U_{\infty}^{\epsilon} = \frac{m e^{\chi V_{\infty}^{\epsilon}}}{\Omega e^{\chi V_{\infty}^{\epsilon}}}. \quad (4.27)$$

Finally, collecting (4.26) and (4.27), we know that $(U_{\infty}^{\epsilon}, V_{\infty}^{\epsilon})$ is a solution of (4.2). $\square$

Lemma 4.4. *Let $\eta < v_{\infty}^m$ with v_{∞}^m defined in (1.7) and let $m > \frac{4\pi}{\chi}$ and $m \frac{4\pi l}{\chi} : l \in \mathbb{N}^+$. Then either (I): for some $\epsilon_0 > 0$, the corresponding solution $(U^{\epsilon_0}, V^{\epsilon_0}, W^{\epsilon_0})$ of (4.16) blows up in finite or infinite time, or (II): for all $\epsilon > 0$, the resulting solutions $(U^{\epsilon}, V^{\epsilon}, W^{\epsilon})$ of (4.16) exist globally and are uniformly bounded in time but*

$$\liminf_{\epsilon \rightarrow 0^+} \frac{U^{\epsilon} (V^{\epsilon})^{+} \,_{L^{\infty}((0, \infty); L^1(\Omega))}}{\ln \epsilon} \geq \frac{4(m\chi - 4\pi)(v_{\infty}^m - \eta)}{K\chi\xi[2 + (v_{\infty}^m - \eta)\delta]} \quad (4.28)$$

and

$$\liminf_{\epsilon \rightarrow 0+} \frac{\min_{U^\epsilon \in L^\infty(\Omega \times (0, \infty)), (V^\epsilon)^+ \in L^\infty(\Omega \times (0, \infty))} U^\epsilon}{\ln \epsilon} \geq 4(m\chi - 4\pi)(v_\infty^m)$$

Hence, the lower bound estimates (1.10) and (1.11) follow simply from (4.28) and (4.29). $\square$

5. Negligibility of Haptotaxis on Long Time Behavior

In this section, we first show that any local-in-time classical solution of (1.4) is comparable to that of (1.1) in the solution operator sense, from which (B4) follows. Moreover, we show that any solution of chemotaxis–haptotaxis model (1.4) converges exponentially to that of chemotaxis-only model (1.1) in the sense of (B5) for small χ .

Lemma 5.1. *Let (u, v, w) denote the maximal classical solution of the IBVP (1.4) defined on $(0, T_m)$. Assume that*

$$\eta < \eta_m := u_0|_{L^1} + \int_0^{\frac{T_m}{\tau}} \frac{1}{4\pi s} e^{-\left(s + \frac{(\text{diam}(\Omega))^2}{4s}\right)} ds, \quad (5.1)$$

where $\frac{T_m}{\tau}$ is understood as ∞ if $\tau = 0$ or $T_m = \infty$. Then, for any $\lambda \in (0, \eta_m - \eta)$, there exist positive constants $K_i = K_i(u_0, \tau v_0, w_0, \lambda, \Omega) > 0$ such that, for any $t \in [0, T_m)$,

$$\begin{aligned} w &\leq K_1 e^{-\lambda t}, \\ w(t)|_{L^\infty} &\leq K_2 [1 + t \sup_{s \in [0, t]} v(s)|_{L^\infty}] \left[1 + \frac{\eta}{\lambda} e^{-\lambda t}\right]. \end{aligned} \quad (5.2)$$

Moreover, with ψ given in (B5) of Theorem 1.1, it follows $v(t) = \psi(t; u, v)$ for all $t \in (0, T_m)$ and, for any $\mu \in (0, \min\{\lambda_1, \eta_m - \eta\})$, there exists $K_3 = K_3(u_0, \tau v_0, w_0, \mu, \Omega) > 0$ such that, for any $t \in [0, T_m)$,

$$\begin{aligned} u(t) &= e^{t\Delta} u_0 + \chi \int_0^t e^{(t-s)\Delta} \cdot (u(s) - v(s)) ds|_{L^\infty} \\ &\leq K_3 \xi \sup_{s \in [0, t]} u(s)|_{L^\infty} [1 + t \sup_{s \in [0, t]} v(s)|_{L^\infty}] \left[1 + \frac{\eta}{\mu} e^{-\mu t}\right]. \end{aligned} \quad (5.3)$$

Proof. By (5.1) and the fact $\lambda \in (0, \eta_m - \eta)$, we first fix a unique $\alpha \in (0, T_m)$ according to

$$u_0|_{L^1} + \int_0^\alpha \frac{1}{4\pi s} e^{-\left(s + \frac{(\text{diam}(\Omega))^2}{4s}\right)} ds = \eta + \lambda < \eta_m. \quad (5.4)$$

It then follows from Lemma 2.5 that

$$v \geq \eta + \lambda \quad \text{on } \Omega \times [\tau\alpha, T_m) \quad \text{and} \quad w \leq K e^{-\lambda(t-\tau\alpha)} \quad \text{on } \Omega \times [\tau\alpha, T_m). \quad (5.5)$$

Recalling from the expression of w in (2.21), we have, for $t \in [\tau\alpha, T_m)$, that

$$w(t) = \frac{w(\tau\alpha) e^{-\int_{\tau\alpha}^t [v(r) - \eta] dr}}{1 + \eta w(\tau\alpha) \int_{\tau\alpha}^t e^{-\int_{\tau\alpha}^s [v(r) - \eta] dr} ds}. \quad (5.6)$$

Thus, for $t \in [\tau\alpha, T_m)$, we compute from (5.6) that

$$\begin{aligned} & (w)e^{\int_{\tau\alpha}^t [v(r)-\eta]dr} \\ &= \frac{w(\tau\alpha) - \frac{w(\tau\alpha)}{\tau\alpha} \int_{\tau\alpha}^t v(r)dr}{1 + \eta w(\tau\alpha) \int_{\tau\alpha}^t e^{-\int_{\tau\alpha}^s [v(r)-\eta]dr} ds} \\ &= \frac{\eta w(\tau\alpha) \int_{\tau\alpha}^t e^{-\int_{\tau\alpha}^s [v(r)-\eta]dr} \left[\frac{w(\tau\alpha)}{\tau\alpha} \int_{\tau\alpha}^s v(r)dr \right] ds}{[1 + \eta w(\tau\alpha) \int_{\tau\alpha}^t e^{-\int_{\tau\alpha}^s [v(r)-\eta]dr} ds]^2}. \end{aligned} \quad (5.7)$$

Now, for $t \in [\tau\alpha, T_m)$, we estimate from (5.5)–(5.7) that

$$\begin{aligned} & w(t) \leq \left[w(\tau\alpha) + K(t - \tau\alpha) \sup_{r \in [\tau\alpha, t]} v(r) \right] e^{-\lambda(t - \tau\alpha)}. \end{aligned} \quad (5.8)$$

The exponential decay estimate of w in (5.2) then follows from (5.5) and (5.8) upon taking suitably large positive constants K_i .

Since the equations for v in the chemotaxis–haptotaxis model (1.4) and in the chemotaxis-only model (1.1) are identical, they have the same solution operator ψ , and so $v(t) = \psi(t; u, v)$.

Next, we utilize the variation-of-constants formula to the u -equation in (1.4) to get

$$u(t) = e^{t\Delta} u_0 - \chi \int_0^t e^{(t-s)\Delta} \cdot (u(s) - v(s)) ds - \xi \int_0^t e^{(t-s)\Delta} \cdot (u(s) - w(s)) ds. \quad (5.9)$$

Therefore, we use the smoothing L^p – L^q -estimates for $e^{t\Delta}$, $t \geq 0$ ^{15,47} to bound, for any $\mu \in (0, \min\{\lambda_1, \eta_m - \eta\})$, that

$$\begin{aligned} & \left\| u(t) - e^{t\Delta} u_0 - \chi \int_0^t e^{(t-s)\Delta} \cdot (u(s) - v(s)) ds \right\|_{L^\infty} \\ & \leq \xi \int_0^t e^{(t-s)\Delta} \cdot (u(s) - w(s)) \, ds \\ & \leq C_1 \xi \int_0^t (1 + (t-s)^{-\frac{1}{2}}) e^{-\lambda_1(t-s)} \|u(s) - w(s)\|_{L^\infty} ds \\ & \leq C_2 \xi \int_0^t (1 + (t-s)^{-\frac{1}{2}}) e^{-\lambda_1(t-s)} e^{-\mu s} ds \\ & = C_2 \xi e^{-\mu t} \int_0^t (1 + z^{-\frac{1}{2}}) e^{-(\lambda_1 - \mu)z} dz \\ & \leq C_3 \xi e^{-\mu t}, \quad t \in [0, T_m), \end{aligned} \quad (5.10)$$

where we have applied (5.2) with $\lambda = \mu$, the fact $\mu < \min \lambda_1, \eta_m - \eta$ and C_2 is given by

$$C_2 = C_1 K_2 \sup_{s \in [0, t]} \|u(s)\|_{L^\infty} [1 + t \sup_{s \in [0, t]} \|v(s)\|_{L^\infty}] + 1 + \frac{\eta}{\mu}. \quad (5.11)$$

The desired estimate (5.3) follows trivially from (5.10) and (5.11). $\square$

Lemma 5.2. *Let $\eta < v_\infty^m$ with v_∞^m defined in (1.7) (or (5.1) with T_m/τ replace by ∞). Then there exists $\chi_0 \in (0, \frac{4\pi}{\|u_0\|_{L^1}})$ such that, whenever $\chi \leq \chi_0$, the global solution component (u, v) of the chemotaxis–haptotaxis model (1.4) converges exponentially to the solution (u^0, v^0) of chemotaxis-only model (1.1) in the sense, for any $\lambda \in (0, \min \lambda_1, \eta_m - \eta)$, there exists a positive constant $K_4 = K_4(u_0, \tau v_0, w_0, \lambda, \Omega) > 0$ such that*

$$\|u(t) - u^0(t)\|_{L^\infty} + \|v(t) - v^0(t)\|_{L^\infty} \leq K_4 e^{-\lambda t}, \quad t \geq 0. \quad (5.12)$$

Proof. From the chemotaxis–haptotaxis model (1.4) and chemotaxis-only model (1.1), we first observe that the differences $\rho := u - u^0$ and $c := v - v^0$ solve the following system:

$$\begin{aligned} \rho_t &= \Delta \rho - \chi \cdot (\rho - v^0) - \chi \cdot (u - c) - \xi \cdot (u - w), & x \in \Omega, t > 0, \\ \tau c_t &= \Delta c - c + \rho, & x \in \Omega, t > 0, \\ \frac{\partial \rho}{\partial \nu} - \xi u \frac{\partial w}{\partial \nu} &= \frac{\partial v^0}{\partial \nu} = \frac{\partial c}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ \rho(x, 0) &= 0, \quad \tau c(x, 0) = 0, \quad w(x, 0) = w_0(x), & x \in \Omega. \end{aligned} \quad (5.13)$$

Henceforth, we shall assume $\chi \in [0, \frac{4\pi}{\|u_0\|_{L^1}})$; by Sec. 3 on global existence and boundedness, we see that u, v, w, ρ, c exist globally-in-time and are uniformly bounded in the sense of (1.8).

To proceed further, we apply the variation-of-constants formula to the first equation in (5.13) to estimate ρ as

$$\begin{aligned} \|\rho(t)\|_{L^\infty} &\leq \chi \int_0^t e^{(t-s)\Delta} \cdot (\rho(s) - v^0(s)) \|_{L^\infty} ds \\ &\quad + \chi \int_0^t e^{(t-s)\Delta} \cdot (u(s) - c(s)) \|_{L^\infty} ds \\ &\quad + \xi \int_0^t e^{(t-s)\Delta} \cdot (u(s) - w(s)) \|_{L^\infty} ds \\ &:= I_1 + I_2 + I_3. \end{aligned} \quad (5.14)$$

By the choice of $\lambda \in (0, \min \lambda_1, \eta_m - \eta)$, we see

$$\mu = \frac{\lambda + \min \lambda_1, \eta_m - \eta}{2} \implies \mu \in (\lambda, \min \lambda_1, \eta_m - \eta).$$

Recalling that, for such μ , we have indeed estimated I_3 in (5.10) and (5.11) as

$$\begin{aligned} I_3 &\leq C_1 \|u\|_{L^\infty(\Omega_\infty)} (1 + t \|v\|_{L^\infty(\Omega_\infty)}) \left(1 + \frac{\eta}{\mu}\right) \xi e^{-\mu t} \\ &\leq M e^{-\lambda t}, \quad t > 0, \end{aligned} \quad (5.15)$$

where $\Omega_\infty = \Omega \times (0, \infty)$, the fact $\mu > \lambda$ is used, M is finite and it is given by

$$M = C_1 \sup_{t>0} \|u\|_{L^\infty(\Omega_\infty)} (1 + t \|v\|_{L^\infty(\Omega_\infty)}) \left(1 + \frac{\eta}{\mu}\right) \xi e^{-\frac{(\min\{\lambda_1, \eta m - \eta\} - \lambda)t}{2}} < \infty. \quad (5.16)$$

We note that M depends only on $u_0, \tau v_0, w_0, \lambda, \Omega$; for $M > M$ to be determined below as in (5.21), let us define

$$T = \sup \{T > 0 : \|\rho(t)\|_{L^\infty} \leq M e^{-\lambda t}, \quad t \in (0, T)\}. \quad (5.17)$$

By continuity of ρ and the fact $\rho(0) = 0$, it follows that T is well-defined and $T > 0$. In the sequel, we shall (via connectedness argument) show that $T = \infty$. To this purpose, in the case of $\tau = 1$, we use the variation-of-constants formula for c in (5.13)

$$c(t) = \int_0^t e^{(t-s)(\Delta-1)} \rho(s) ds$$

and use the smoothing L^p - L^q -estimates for $e^{t\Delta}$ $t \geq 0$ (c.f. ^{15,47}) to estimate

$$\begin{aligned} \|c(t)\|_{L^\infty} &\leq \int_0^t \|e^{(t-s)(\Delta-1)} \rho(s)\|_{L^\infty} ds \\ &\leq C_2 \int_0^t (1 + (t-s)^{-\frac{1}{2}}) e^{-(\lambda_1+1)(t-s)} \|\rho(s)\|_{L^\infty} ds \\ &\leq C_2 M \int_0^t (1 + (t-s)^{-\frac{1}{2}}) e^{-(\lambda_1+1)(t-s)} e^{-\lambda s} ds \\ &= C_2 M e^{-\lambda t} \int_0^t (1 + z^{-\frac{1}{2}}) e^{-(\lambda_1+1-\lambda)z} dz \leq C_3 M e^{-\lambda t}, \quad t \in [0, T], \end{aligned} \quad (5.18)$$

where we have applied (5.17) and the fact $\lambda < \lambda_1 < \lambda_1 + 1$ in the last line. In the case of $\tau = 0$, the estimate (5.18) follows readily by $W^{2,p}$ -elliptic estimate and Sobolev embedding.

Now, we employ the semi-group properties, (5.17) and (5.18) to bound $I_1 + I_2$ in (5.14) as

$$\begin{aligned} I_1 + I_2 &\leq C_4 \chi \|v^0\|_{L^\infty(\Omega_\infty)} \int_0^t (1 + (t-s)^{-\frac{1}{2}}) e^{-\lambda_1(t-s)} \|\rho(s)\|_{L^\infty} ds \\ &\quad + C_4 \chi \|u\|_{L^\infty(\Omega_\infty)} \int_0^t (1 + (t-s)^{-\frac{1}{2}}) e^{-\lambda_1(t-s)} \|c(s)\|_{L^\infty} ds \end{aligned}$$

$$\begin{aligned}
&\leq C_4 M \chi \left(v^0 \right)_{L^\infty(\Omega_\infty)} \int_0^t (1 + (t-s)^{-\frac{1}{2}}) e^{-\lambda_1(t-s)} e^{-\lambda s} ds \\
&\quad + C_3 C_4 M \chi \left(u \right)_{L^\infty(\Omega_\infty)} \int_0^t (1 + (t-s)^{-\frac{1}{2}}) e^{-\lambda_1(t-s)} e^{-\lambda s} ds \\
&\leq C_5 M \chi \left(\left(v^0 \right)_{L^\infty(\Omega_\infty)} + \left(u \right)_{L^\infty(\Omega_\infty)} \right) e^{-\lambda t}, \quad t \in [0, T]. \quad (5.19)
\end{aligned}$$

Recalling from Sec. 3.3 that the solution of (1.4) is uniformly bounded in χ $[0, \frac{4\pi}{\|u_0\|_{L^1}})$. Therefore, by continuity, we first choose a (perhaps small) χ_0 $(0, \frac{4\pi}{\|u_0\|_{L^1}})$ fulfilling

$$C_5 \chi_0 \left(\left(v^0 \right)_{L^\infty(\Omega_\infty)} + \left(u \right)_{L^\infty(\Omega_\infty)} \right) < 1, \quad (5.20)$$

and then we fix $M > M$ in (5.17) according to

$$M = 2M[1 - C_5 \chi_0 \left(\left(v^0 \right)_{L^\infty(\Omega_\infty)} + \left(u \right)_{L^\infty(\Omega_\infty)} \right)]^{-1}. \quad (5.21)$$

Finally, substituting (5.15) and (5.19) into (5.14) and using (5.16), for any $\chi \leq \chi_0$, we infer that

$$\begin{aligned}
\rho(t) \left(\right)_{L^\infty} &\leq C_5 M \chi_0 C_5 \chi_0 \left(\left(v^0 \right)_{L^\infty(\Omega_\infty)} + \left(u \right)_{L^\infty(\Omega_\infty)} \right) e^{-\lambda t} + M e^{-\lambda t} \\
&= \frac{1}{2} [1 + C_5 \chi_0 \left(\left(v^0 \right)_{L^\infty(\Omega_\infty)} + \left(u \right)_{L^\infty(\Omega_\infty)} \right)] M e^{-\lambda t}, \quad t \in [0, T].
\end{aligned} \quad (5.22)$$

Given the fact in (5.20), comparing (5.22) and (5.17), one can easily conclude from the maximality of T (or the nonempty set $(0, T)$ is both open and closed) that $T = \infty$. Therefore,

$$u(t) \left(\right)_{L^\infty} = \rho(t) \left(\right)_{L^\infty} \leq M e^{-\lambda t}, \quad t \geq 0. \quad (5.23)$$

Then the maximum principle applied to the second equation in (5.13) yields easily

$$v(t) \left(\right)_{L^\infty} = c(t) \left(\right)_{L^\infty} \leq \rho(t) \left(\right)_{L^\infty} \leq M e^{-\lambda t}, \quad t \geq 0. \quad (5.24)$$

The desired convergence estimate (5.12) follows directly from (5.23) and (5.24). $\square$

Proof. Proof of negligibility of haptotaxis on long time behavior in (B4) and (B5). In the case of global existence, i.e. when $T_m = \infty$, upon identifying $\eta_m = v_\infty^m$, the exponential decay in (1.13) is merely the first estimate in (5.2). The exponential decay in (1.16) is simply (5.12). To see (1.15), for any $\kappa \in (0, v_\infty^m - \eta)$, taking

$$\lambda = \frac{(\kappa + v_\infty^m - \eta)}{2} = \kappa + \frac{(v_\infty^m - \eta - \kappa)}{2}$$

and noticing, since (u, v, w) is assumed to be bounded according to (1.8), that

$$\sup_{t>0} K_2 [1 + t \sup_{s \in [0, t]} v(s) \left(\right)_{L^\infty}] \leq 1 + \frac{\eta}{\lambda} e^{-\frac{(v_\infty^m - \eta - \kappa)}{2} t} < \infty,$$

we readily conclude the $W^{1,\infty}$ -exponential decay of w in (1.15) from (5.2). Similarly, for any $\rho \in (0, \min\{\lambda_1, \eta_m - \eta\})$, taking

$$\mu = \frac{(\rho + \min\{\lambda_1, \eta_m - \eta\})}{2} = \rho + \frac{(\min\{\lambda_1, \eta_m - \eta\} - \rho)}{2}$$

and observing by the boundedness of (u, v, w) in (1.8) that

$$\sup_{t>0} K_3 \sup_{s \in [0, t]} u(s)_{L^\infty} [1 + t \sup_{s \in [0, t]} v(s)_{L^\infty}] \left(1 + \frac{\eta}{\lambda}\right) e^{-\frac{(\min\{\lambda_1, \eta_m - \eta\} - \rho)}{2} t} < \infty,$$

we quickly derive the desired exponential decay in (1.14) from (5.3). $\square$

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