

CRITICAL MASS ON THE KELLER-SEGEL SYSTEM WITH SIGNAL-DEPENDENT MOTILITY

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ABSTRACT. This paper is concerned with the global boundedness and blow-up of solutions to the Keller-Segel system with density-dependent motility in a two-dimensional bounded smooth domain with Neumann boundary conditions. We show that if the motility function decays exponentially, then a critical mass phenomenon similar to the minimal Keller-Segel model will arise. That is, there is a number $m_* > 0$, such that the solution will globally exist with uniform-in-time bound if the initial cell mass (i.e., L^1 -norm of the initial value of cell density) is less than m_* , while the solution may blow up if the initial cell mass is greater than m_* .

1. INTRODUCTION

To show how individual cell paths can result in an average cell flux proportional to the macroscopic chemical gradient, Keller and Segel derived the following chemotaxis system based on a Brownian motion framework in their seminal work [26]:

$$(1.1) \quad \begin{aligned} & \square \\ u_t &= \nabla \cdot (\gamma(v) \nabla u - u \phi(v) \nabla v), \\ v_t &= \Delta v + u - v, \end{aligned}$$

where u denotes the cell density and v stands for the concentration of the chemical signal emitted by cells. $\gamma(v) > 0$ is the diffusion coefficient and $\chi(v)$ is called the chemotactic coefficient; both of them depend on the chemical signal concentration and satisfy the following proportionality relation:

$$(1.2) \quad \phi(v) = (\alpha - 1)\gamma'(v),$$

where α denotes the ratio of effective body length (i.e., maximal distance between receptors) to the cell step size. We refer the detailed derivation of (1.1)-(1.2) to [26]. The prominent feature of the Keller-Segel system (1.1) is that two motility coefficients $\gamma(v)$ and $\phi(v)$ depend on the chemical signal concentration and have a prescribed relationship to each other. Recently this proportionality relation with $\alpha = 0$ and $\gamma'(v) < 0$ has been advocated as “density-suppressed motility

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mechanism" to interpret the stripe pattern formation of engineered *Escherichia Coli* in [11, 29], which will be elaborated later. Such a signal-dependent motility mechanism has also been used in preytaxis to describe the spatially inhomogeneous distribution of coexistence in the predator-prey system (see [24, 25]). There are some other chemotaxis models where the diffusive and chemotactic coefficients depend on the chemical concentration gradient (cf. [10]) or cell density (cf. [43]), which clearly have different modeling views from the system (1.1)-(1.2).

The study of the Keller-Segel system (1.1) was started with simplified cases. If $\gamma(v) = 1$ and $\phi(v) = \chi > 0$ (χ is a constant), the system (1.1) is simplified to the so-called minimal Keller-Segel (abbreviated as KS) model:

$$(1.3) \quad \begin{aligned} u_t &= \Delta u - \chi \nabla \cdot (u \nabla v), & x \in \Omega, t > 0, \\ v_t &= \Delta v + u - v, & x \in \Omega, t > 0, \end{aligned}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary. Under homogeneous Neumann boundary conditions, the dynamics of (1.3) such as boundedness, blow-up, and pattern formation have been extensively studied; see the review papers [7, 16–18] for more details. The most prominent phenomenon is the existence of critical mass depending on the space dimensions. Precisely, the global bounded solutions exist in one dimension [31]. In a space of two dimensions ($n = 2$), there exists a critical mass $m_* = \frac{4\pi}{\chi}$ such that the solution converges to its unique constant equilibrium if $\int_{\Omega} u_0 dx < m_*$ [30, 38] and blows up if $\int_{\Omega} u_0 dx > m_*$ [19], where $\int_{\Omega} u_0 dx$ denotes the initial cell mass. In the higher dimensions ($n \geq 3$), for any $\int_{\Omega} u_0 dx > 0$, the solution may blow up in finite time [42]. The mathematical analysis for the KS model (1.3) on the boundedness vs. blow-up was essentially based on the following Lyapunov functional:

$$(1.4) \quad F(u, v) = \int_{\Omega} u \ln u dx + \frac{\chi}{2} \int_{\Omega} (v^2 + |\nabla v|^2) dx - \chi \int_{\Omega} uv dx.$$

If $\gamma(v) = 1$ and $\phi(v) = \frac{\chi}{v}$, the system (1.1) becomes the so-called singular Keller-Segel system:

$$(1.5) \quad \begin{aligned} u_t &= \Delta u - \chi \nabla \cdot \left(\frac{u}{v} \nabla v \right), & x \in \Omega, t > 0, \\ v_t &= \Delta v + u - v, & x \in \Omega, t > 0, \end{aligned}$$

and there are various results in the literature indicating the nonexistence of blow-up of solutions. With homogeneous Neumann boundary conditions, the existence of globally bounded solutions of (1.5) was established if $n = 2$ and $\chi \leq \chi_0$ for some $\chi_0 > 1$ in [28] or $n \geq 3$ and $\chi < \frac{2}{n}$ in [12, 40]. Moreover, when $\chi < \frac{2}{n}$ and $n \geq 2$, the asymptotic stability of constant steady states was obtained in [41]. More results on the radially symmetric case or weak solutions can be found in [8, 13–15, 32] and we refer to [7] for more details.

Turning to the full KS system (1.1) where $\gamma(v)$ and $\phi(v)$ are nonconstant functions satisfying (1.2), to our knowledge, the known results are only limited to the special case $\phi(v) = -\gamma'(v)$ (i.e., $\alpha = 0$ in (1.2)) which simplifies the KS system (1.1) into

$$(1.6) \quad \begin{aligned} u_t &= \Delta(\gamma(v)u), & x \in \Omega, t > 0, \\ v_t &= \Delta v + u - v, & x \in \Omega, t > 0. \end{aligned}$$

Here the parameter $\alpha = 0$ in (1.2) means that “the distance between receptors is zero and the chemotaxis occurs because of an undirected effect on activity due to the presence of a chemical sensed by a single receptor” as stated in [26, p. 228]. Recently, to describe the stripe pattern formation observed in the experiment of [29], a so-called density-suppressed motility model was proposed in [11] as follows:

$$(1.7) \quad \begin{aligned} u_t &= \Delta(\gamma(v)u) + \sigma u(1 - u), & x \in \Omega, \ t > 0, \\ v_t &= \Delta v + u - v, & x \in \Omega, \ t > 0 \end{aligned}$$

with $\gamma'(v) < 0$ and $\sigma \geq 0$ denotes the intrinsic cell growth rate. Clearly the density-suppressed motility model (1.7) with $\sigma = 0$ coincides with the simplified KS model (1.6).

When the homogeneous Neumann boundary conditions are imposed, there are some results available to (1.6) and (1.7). First for the system (1.6), it was shown [35] that globally bounded solutions exist in two-dimensional spaces by assuming that the motility function $\gamma(v) \in C^3([0, \infty) \cap W^{1,\infty}(0, \infty))$ has both positive lower and upper bounds. It turns out that the uniformly positive assumption on $\gamma(v)$ (i.e., $\gamma(v)$ has a positive lower bound) is not necessary to ensure the global boundedness of solutions. For example, if $\gamma(v) = \frac{\chi}{v^k}$ (i.e., $\gamma(v)$ decays algebraically), it has been proved that global bounded solutions exist in all dimensions provided $\chi > 0$ is small enough [44] or in two-dimensional spaces for parabolic-elliptic simplification of the system (1.6) (see [3]). For the system (1.7) with $\gamma'(v) < 0$, it was shown that global bounded solutions exist in two dimensions for any $\sigma > 0$ [22] and in higher dimensions ($n \geq 3$) for large $\sigma > 0$ [36]. The results of [22] essentially rely on the assumption $\sigma > 0$. Therefore a natural question is whether the solution of (1.7) with $\sigma = 0$ (i.e., KS system (1.6) with $\gamma'(v) < 0$) is globally bounded. This question has been partially confirmed in [3, 44] for algebraically decreasing function $\gamma(v)$ with various conditions as mentioned above. The purpose of this paper is to investigate the same question for exponentially decreasing motility function $\gamma(v) = e^{-\chi v}$ with $\chi > 0$. That is we consider the following problem:

$$(1.8) \quad \begin{aligned} u_t &= \Delta(e^{-\chi v}u), & x \in \Omega, \ t > 0, \\ v_t &= \Delta v + u - v, & x \in \Omega, \ t > 0, \\ \frac{\partial u}{\partial \nu} &= \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, \ t > 0, \\ u(x, 0) &= u_0(x), \ v(x, 0) = v_0(x), & x \in \Omega, \end{aligned}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary and ν stands for the outward unit normal vector on $\partial\Omega$. Surprisingly, we find that uniform-in-time boundedness of solutions of (1.8) is no longer true and the solution may blow up in two dimensions, which is quite different from the results of [3, 44] for algebraically decreasing function $\gamma(v)$. Our result indicates that the solution behavior of the system (1.6) may essentially depend on the decay rate of the motility function $\gamma(v)$. The main results of this paper are the following.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary. Assume that $0 \leq (u_0, v_0) \in [W^{1,\infty}(\Omega)]^2$. Then the following results hold true:*

(i) *If $\int_{\Omega} u_0 dx < \frac{4\pi}{\chi}$, then the system (1.8) admits a unique classical solution $(u, v) \in [C^0(\bar{\Omega} \times [0, \infty)) \cap C^{2,1}(\bar{\Omega} \times (0, \infty))]^2$ satisfying*

$$\|u(\cdot, t)\|_{L^\infty} + \|v(\cdot, t)\|_{W^{1,\infty}} \leq C,$$

where C is a constant independent of t .

(ii) For any $M > \frac{4\pi}{\chi}$ and $M \notin \{\frac{4\pi m}{\chi} : m \in \mathbb{N}^+\}$ where $\mathbb{N}^+$ denotes the set of positive integers, there exist initial data (u_0, v_0) satisfying $\int_{\Omega} u_0 dx = M$ such that the corresponding solution blows up in finite/infinite time.

We remark that the blow-up result in Theorem 1.1(ii) does not assert the finiteness or infiniteness of blow-up time, which leaves out an interesting question for future study. Moreover, as we know the nonlinear diffusion may play an important role in blow-up dynamics such as blow-up rate (see [9, 21] and the references therein). Hence it would be of interest to study qualitative properties of blow-up solutions to the system (1.8) in the future.

The new contribution of this paper lies in the finding of the critical mass phenomenon for the system (1.6) with exponentially decreasing motility function $\gamma(v)$. This new finding along with the existing results in [3, 44] for (1.6) with algebraically decreasing function $\gamma(v)$ shows that the dynamics of (1.6) is very rich and complex where the decay rate of the motility function $\gamma(v)$ will play a key role. This provides us a heuristic direction to further explore the dynamics of the full Keller-Segel system (1.1) whose dynamics have been only partially understood so far for the special case $\alpha = 0$ in (1.2), namely for (1.6). Technically to overcome the possible degeneracy, we develop the weighted energy estimates by treating the degenerate term as a weight function to achieve the results in Theorem 1.1. This technique may become a common (if not necessary) tool to study chemotaxis systems with the signal-dependent degenerate diffusion.

One can check that the system (1.8) has the same Lyapunov functional (1.4) as for the minimal KS model (1.3), which can be used to construct some initial data with large negative energy such that the solution of (1.8) blows up for supercritical mass (i.e., $\int_{\Omega} u_0 dx > \frac{4\pi}{\chi}$). Moreover, under the subcritical mass (i.e., $\int_{\Omega} u_0 dx < \frac{4\pi}{\chi}$), using the same Lyapunov functional and Trudinger-Moser inequality, we can find a constant $c_1 > 0$ such that

$$(1.9) \quad \|u \ln u\|_{L^1} + \|\nabla v\|_{L^2} + \int_0^t \|v_t\|_{L^2}^2 ds \leq c_1$$

which has been a key to proving the boundedness of solutions of the minimal KS system (1.3). However, there are some significant differences between systems (1.3) and (1.8). For the minimal KS model (1.3), the estimate (1.9) is enough to establish the existence of global classical solutions (see [30]). However for the system (1.8), the motility coefficient $e^{-\chi v}$ may touch down to zero (degenerate) as $v \rightarrow \infty$, and hence the method for the constant diffusion as in [30] no longer works and new ideas are demanded. In this paper, we shall develop the weighted energy estimates by taking $e^{-\chi v}$ as the weight function based on the Lyapunov functional to establish our results.

2. LOCAL EXISTENCE AND BASIC INEQUALITIES

Using Amann's theorem [5, 6] (cf. also [37, Lemma 2.6]) or the well-established fixed point argument together with the parabolic regularity theory [22, 34], we can show the existence and uniqueness of local solutions of (1.8). We omit the details of the proof for brevity.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary. Assume that $0 \leq (u_0, v_0) \in [W^{1,\infty}(\Omega)]^2$. Then there exists $T_{\max} \in (0, \infty]$ such that the*

problem (1.8) has a unique classical solution $(u, v) \in [C(\bar{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\max}))]^2$. Moreover, $u, v > 0$ in $\Omega \times (0, T_{\max})$ and

if $T_{\max} < \infty$, then $\|u(\cdot, t)\|_{L^\infty} + \|v(\cdot, t)\|_{W^{1,\infty}} \rightarrow \infty$ as $t \nearrow T_{\max}$.

Lemma 2.2. If (u, v) is a solution of (1.8) in $\Omega \times (0, T)$ for some $T > 0$, then

$$(2.1) \quad \|u(\cdot, t)\|_{L^1} = \|u_0\|_{L^1} \equiv M_0 \text{ for all } t \in (0, T)$$

and

$$(2.2) \quad \|v(\cdot, t)\|_{L^1} \leq \|u_0\|_{L^1} + \|v_0\|_{L^1} \text{ for all } t \in (0, T).$$

Proof. Integrating the first equation of (1.8) and using the Neumann boundary conditions, we obtain (2.1) directly. On the other hand, integrating the second equation of (1.8) with respect to x over Ω , one has

$$\frac{d}{dt} \int_{\Omega} v dx + \int_{\Omega} v dx = \int_{\Omega} u dx = \int_{\Omega} u_0 dx,$$

which immediately gives (2.2). $\square$

Lemma 2.3. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary. Assume $\mathcal{A}$ is a self-adjoint realization of $-\Delta$ defined on $D(\mathcal{A}) := \{\psi \in W^{2,2}(\Omega) \cap L^2(\Omega) \mid \int_{\Omega} \psi = 0 \text{ and } \frac{\partial \psi}{\partial \nu} = 0 \text{ on } \partial\Omega\}$. Then for any $L > 0$ and a nonnegative function f satisfying

$$(2.3) \quad \int_{\Omega} f \ln f dx \leq L,$$

it holds that

$$(2.4) \quad \int_{\Omega} |\mathcal{A}^{-\frac{1}{2}}(f - \bar{f})|^2 dx \leq C(L),$$

where $\bar{f} = \frac{1}{|\Omega|} \int_{\Omega} f dx$.

Proof. Using (2.3) and noting the fact $z \ln z \geq -\frac{1}{e}$ for all $z > 0$, we have

$$\begin{aligned} \|f\|_{L^1} &= \int_{f \geq e} f dx + \int_{f < e} f dx \\ &\leq \int_{f \geq e} f \ln f dx + \int_{f < e} f dx = \int_{\Omega} f \ln f - \int_{f < e} f \ln f dx + \int_{f < e} f dx \\ &\leq L + \frac{|\Omega|}{e} + e|\Omega|, \end{aligned}$$

and hence

$$(2.5) \quad \|f - \bar{f}\|_{L^1} \leq 2\|f\|_{L^1} \leq 2L + \frac{2|\Omega|}{e} + 2e|\Omega|.$$

Next, we consider the following system:

$$(2.6) \quad \begin{aligned} -\Delta \phi &= f - \bar{f}, & x \in \Omega, \\ \frac{\partial \phi}{\partial \nu} &= 0, & x \in \partial\Omega. \end{aligned}$$

Let G denote the Green's function of $-\Delta$ in Ω with the homogeneous Neumann boundary condition. From (2.6), one has

$$(2.7) \quad \phi(x) = \int_{\Omega} G(x-y)(f(y) - \bar{f}) dy.$$

Then using the similar argument as in [33, Lemma A.3] along with (2.5), from (2.7) one can find a constant $\kappa > 0$ such that

$$(2.8) \quad \int_{\Omega} e^{\kappa|\phi|} dx \leq c_1.$$

Recall a result (see [33, Lemma A.2]): for $\kappa > 0$, it holds

$$XY \leq \frac{1}{\kappa} X \ln X + \frac{1}{\kappa e} e^{\kappa Y} \text{ for all } X > 0 \text{ and } Y > 0.$$

Then multiplying the first equation of (2.6) by ϕ , and integrating it by parts, we end up with

$$(2.9) \quad \int_{\Omega} |\nabla \phi|^2 dx = \int_{\Omega} f\phi - \int_{\Omega} \bar{f}\phi dx \leq \int_{\Omega} f|\phi| + \int_{\Omega} \bar{f}|\phi| dx \\ \leq \frac{1}{\kappa} \int_{\Omega} f \ln f dx + \frac{2}{\kappa e} \int_{\Omega} e^{\kappa|\phi|} dx + \frac{|\Omega|}{\kappa} \bar{f} \ln \bar{f}.$$

Substituting (2.3) and (2.8) into (2.9), and using the boundedness of $\bar{f} \ln \bar{f}$, one has

$$(2.10) \quad \int_{\Omega} |\nabla \phi|^2 dx \leq c_2$$

where c_2 depends on L . The definition of $\mathcal{A}$ defines the self-adjoint fractional powers $\mathcal{A}^{-\delta}$ for any $\delta > 0$. Then from (2.6) we have $\phi = \mathcal{A}^{-1}(f - \bar{f})$ and hence

$$\int_{\Omega} |\mathcal{A}^{-\frac{1}{2}}(f - \bar{f})|^2 dx = \int_{\Omega} \mathcal{A}^{-1}(f - \bar{f})(f - \bar{f}) dx = \int_{\Omega} \phi(-\Delta \phi) dx = \int_{\Omega} |\nabla \phi|^2 dx \leq c_2,$$

which gives (2.4). $\square$

Lemma 2.4. *Let (u, v) be a solution of the system (1.8). Then there exists a constant $C > 0$ independent of t such that*

$$(2.11) \quad \|\Delta v\|_{L^2} \leq C(\|u\|_{L^2} + \|v_t\|_{L^2}).$$

Proof. Note that v satisfies the following system:

$$(2.12) \quad \begin{aligned} -\Delta v + v &= u - v_t, & x \in \Omega, \quad t > 0, \\ \frac{\partial v}{\partial \nu} &= 0, & x \in \Omega, \quad t > 0. \end{aligned}$$

Then applying the Agmon-Douglis-Nirenberg L^p estimates (see [1, 2]) to the system (2.12), we can find a constant $c_1 > 0$ such that

$$\|v\|_{W^{2,2}} \leq c_1\|(u - v_t)\|_{L^2} \leq 2c_1(\|u\|_{L^2} + \|v_t\|_{L^2}),$$

which gives (2.11). $\square$

3. PROOF OF THEOREM 1.1

In this section, we shall prove Theorem 1.1, which includes the global existence of classical solutions for subcritical mass and blow-up of solutions for supercritical mass.

Lemma 3.1. *Let $F(u, v)$ be defined in (1.4). Then the solutions of (1.8) satisfy*

$$(3.1) \quad \frac{d}{dt} F(u, v) + E(u, v) = 0,$$

where

$$E(u, v) = \chi \int_{\Omega} v_t^2 dx + \int_{\Omega} e^{-\chi v} u |\nabla(\ln u - \chi v)|^2 dx.$$

Proof. We multiply the first equation of (1.8) by $(\ln u - \chi v)$

Then the combination of (3.7) and (3.8) gives

$$(3.9) \quad F(u, v) \geq -M_0 \ln \frac{1}{M_0} \int_{\Omega} e^{(\chi+\eta)v} dx + \frac{\chi}{2} \int_{\Omega} (v^2 + |\nabla v|^2) dx + \eta \int_{\Omega} uv dx.$$

Noting the fact $\|v\|_{L^1} \leq c_1$, and using the Trudinger-Moser inequality in two-dimensional spaces [30], one has

$$(3.10) \quad \int_{\Omega} e^{(\chi+\eta)v} dx \leq c_2 e^{(\frac{1}{8\pi} + \varepsilon)(\chi+\eta)^2 \|\nabla v\|_{L^2}^2},$$

which substituted into (3.9) gives

$$(3.11) \quad F(u, v) \geq \frac{\chi}{2} - \frac{1}{8\pi} + \varepsilon (\chi + \eta)^2 M_0 \int_{\Omega} |\nabla v|^2 dx + \frac{\chi}{2} \int_{\Omega} v^2 dx + \eta \int_{\Omega} uv dx - c_3,$$

where $c_3 := M_0 \ln \frac{c_2}{M_0}$. Since $M_0 = \int_{\Omega} u_0 dx < \frac{4\pi}{\chi}$, it holds that

$$(3.12) \quad \frac{\chi}{2} - \frac{1}{8\pi} + \varepsilon (\chi + \eta)^2 M_0 > 0,$$

by choosing $\varepsilon > 0$ and $\eta > 0$ small enough. Substituting (3.12) into (3.11), one has

$$F(u, v) \geq \frac{\chi}{2} \int_{\Omega} v^2 dx + \eta \int_{\Omega} uv dx - c_3,$$

which gives $F(u, v) \geq -c_3$ and $\int_{\Omega} uv dx \leq \frac{F(u, v) + c_3}{\eta}$ by the fact $F(u, v) \leq F(u_0, v_0)$. Then using the definition of $F(u, v)$ in (1.4) and the fact $F(u, v) \leq F(u_0, v_0)$ again, we obtain

$$\int_{\Omega} u \ln u dx \leq F(u, v) + \chi \int_{\Omega} uv dx \leq 1 + \frac{\chi}{\eta} F(u_0, v_0) + \frac{\chi c_3}{\eta},$$

which gives (3.5). Moreover, we have the following estimate:

$$(3.13) \quad \begin{aligned} \frac{\chi}{2} \int_{\Omega} |\nabla v|^2 dx &\leq F(u, v) + \chi \int_{\Omega} uv dx - \int_{\Omega} u \ln u dx \\ &\leq F(u, v) + \chi \int_{\Omega} uv dx + \frac{|\Omega|}{e} \leq 1 + \frac{\chi}{\eta} F(u_0, v_0) + \frac{\chi c_3}{\eta} + \frac{|\Omega|}{e}. \end{aligned}$$

Integrating (3.1) and using the fact $F(u, v) \geq -c_3$, it follows that

$$\chi \int_0^t \int_{\Omega} v_t^2 dx dt + \int_0^t \int_{\Omega} e^{-\chi v} u |\nabla (\ln u - \chi v)|^2 dx dt \leq F(u_0, v_0) - F(u, v) \leq F(u_0, v_0) + c_3,$$

which yields

$$(3.14) \quad \int_0^t \int_{\Omega} v_t^2 dx dt \leq \frac{F(u_0, v_0) + c_3}{\chi}.$$

Thus the combination of (3.13)-(3.14) gives (3.6) and completes the proof. $\square$

Lemma 3.3. *Let (u, v) be a solution of (1.8). If $\int_{\Omega} u_0(x) dx < \frac{4\pi}{\chi}$, then there exists a constant $C > 0$ independent of t such that the following inequality holds:*

$$(3.15) \quad \int_t^{t+\tau} \int_{\Omega} e^{-\chi v} u^2 dx ds \leq C \quad \text{for all } t \in (0, \bar{T}_{\max}),$$

where

$$(3.16) \quad \tau := \min\{1, \frac{1}{2}T_{\max}\} \text{ and } T_{\max} = \begin{cases} T_{\max} - \tau & \text{if } T_{\max} < \infty, \\ \infty & \text{if } T_{\max} = \infty. \end{cases}$$

Proof. Using the definition of $\mathcal{A}$ in Lemma 2.3, we can rewrite the system (1.8) as follows:

$$(3.17) \quad \begin{aligned} (u - \bar{u})_t &= -\mathcal{A}(e^{-\chi v}u - \overline{e^{-\chi v}u}), \quad x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \nu} &= 0, \quad x \in \partial\Omega, t > 0. \end{aligned}$$

Then multiplying (3.17) by $\mathcal{A}^{-1}(u - \bar{u})$ and integrating the result by parts, we have

$$(3.18) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\Omega} |\mathcal{A}^{-\frac{1}{2}}(u - \bar{u})|^2 dx &= - \int_{\Omega} \mathcal{A}^{-1}(u - \bar{u}) \cdot \mathcal{A} \left(e^{-\chi v}u - \overline{e^{-\chi v}u} \right) dx \\ &= - \int_{\Omega} (u - \bar{u}) \cdot \left(e^{-\chi v}u - \overline{e^{-\chi v}u} \right) dx. \end{aligned}$$

On the other hand, with some direct calculations and noting that $\bar{u} = \frac{1}{|\Omega|} \int_{\Omega} u dx = \frac{M_0}{|\Omega|}$, it holds

$$(3.19) \quad \begin{aligned} &- \int_{\Omega} (u - \bar{u}) \cdot \left(e^{-\chi v}u - \overline{e^{-\chi v}u} \right) dx \\ &= - \int_{\Omega} (u - \bar{u}) \left(e^{-\chi v}(u - \bar{u}) + e^{-\chi v}\bar{u} - \overline{e^{-\chi v}u} \right) dx \\ &= - \int_{\Omega} e^{-\chi v}(u - \bar{u})^2 dx + \bar{u} \int_{\Omega} (\bar{u} - u) e^{-\chi v} dx \\ &\leq - \int_{\Omega} e^{-\chi v}(u - \bar{u})^2 dx + \frac{M_0^2}{|\Omega|}. \end{aligned}$$

Then we substitute (3.19) into (3.18) to get

$$(3.20) \quad \frac{d}{dt} \int_{\Omega} |\mathcal{A}^{-\frac{1}{2}}(u - \bar{u})|^2 dx + 2 \int_{\Omega} e^{-\chi v}(u - \bar{u})^2 dx \leq \frac{2M_0^2}{|\Omega|}.$$

Since $\int_{\Omega} u_0 dx < \frac{4\pi}{\chi}$, then from Lemmas 3.2 and 2.3, we can find a constant $c_1 > 0$ such that

$$(3.21) \quad \int_{\Omega} |\mathcal{A}^{-\frac{1}{2}}(u - \bar{u})|^2 dx \leq c_1.$$

Then integrating (3.20) over $(t, t + \tau)$ and using (3.21), one has

$$\int_t^{t+\tau} \int_{\Omega} e^{-\chi v}(u - \bar{u})^2 dx ds \leq \frac{M_0^2}{|\Omega|} \tau \leq \frac{M_0^2}{|\Omega|},$$

which gives

$$\begin{aligned} \int_t^{t+\tau} \int_{\Omega} e^{-\chi v} u^2 dx ds &= \int_t^{t+\tau} \int_{\Omega} e^{-\chi v} (u - \bar{u} + \bar{u})^2 dx ds \\ &\leq 2 \int_t^{t+\tau} \int_{\Omega} e^{-\chi v} (u - \bar{u})^2 dx ds + 2 \int_t^{t+\tau} \int_{\Omega} \bar{u}^2 dx ds \leq \frac{4M_0^2}{|\Omega|}, \end{aligned}$$

and hence (3.15) follows. Then we complete the proof. $\square$

Lemma 3.4. *Suppose the conditions in Lemma 3.3 hold. Then there exists a constant $C > 0$ independent of t such that*

$$(3.22) \quad \int_t^{t+\tau} \|v(\cdot, s)\|_{L^\infty} ds \leq C \quad \text{for all } t \in (0, \bar{T}_{\max}),$$

where τ is defined by (3.16).

Proof. Using the Sobolev embedding theorem and applying the Agmon-Douglis-Nirenberg L^p estimates (see [1, 2]) to the system (2.12), we have

$$(3.23) \quad \begin{aligned} \|v\|_{L^\infty} &\leq c_1 \|v\|_{W^{2, \frac{3}{2}}} \leq c_2 \|(u - v_t)\|_{L^{\frac{3}{2}}} \\ &\leq 2c_2 (\|u\|_{L^{\frac{3}{2}}} + \|v_t\|_{L^{\frac{3}{2}}}) \\ &\leq 2c_2 \left(\int_\Omega u^2 e^{-\chi v} dx + \int_\Omega e^{3\chi v} dx + 2c_2 \int_\Omega v_t^2 dx \right)^{\frac{1}{6}} |\Omega|^{\frac{1}{6}} \\ &\leq c_2^2 \int_\Omega u^2 e^{-\chi v} dx + \int_\Omega e^{3\chi v} dx + c_2^2 \|v_t\|_{L^2}^2 + |\Omega|^{\frac{1}{3}}. \end{aligned}$$

On the other hand, using the fact $\|v\|_{L^1} + \|\nabla v\|_{L^2} \leq c_3$ (see Lemmas 3.2 and 2.3) and applying the Trudinger-Moser inequality in two-dimensional spaces [30], one has $\int_\Omega e^{3\chi v} dx \leq c_4$, which, substituted into (3.23) and combined with (3.15) and (3.6), gives

$$\int_t^{t+\tau} \|v(\cdot, s)\|_{L^\infty} ds \leq c_2^2 \int_t^{t+\tau} \int_\Omega u^2 e^{-\chi v} dx + c_2^2 \int_t^{t+\tau} \|v_t(\cdot, s)\|_{L^2}^2 ds + c_4 \leq c_5,$$

which yields (3.22). $\square$

With the above results in hand, we shall show that there exists a constant $C > 0$ such that $\|u(\cdot, t)\|_{L^2} \leq C$ for any $t \in (0, T_{\max})$, which will be used to rule out the possibility of degeneracy. Precisely, we have the following results.

Lemma 3.5. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary and $\int_\Omega u_0 dx < \frac{4\pi}{\chi}$. If (u, v) is a solution of system (1.8) in $\Omega \times (0, T_{\max})$, then there exists a positive constant C independent of t such that*

$$(3.24) \quad \|u(\cdot, t)\|_{L^2} \leq C \quad \text{for all } t \in (0, T_{\max}).$$

Proof. We multiply the first equation of (1.8) by u and integrate the result by parts with respect to x . Then using the Hölder inequality and Young's inequality, we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_\Omega u^2 dx + \int_\Omega e^{-\chi v} |\nabla u|^2 dx &= \chi \int_\Omega e^{-\chi v} u \nabla u \cdot \nabla v dx \\ &\leq \chi \int_\Omega e^{-\chi v} |\nabla u|^2 dx + \frac{1}{2} \int_\Omega e^{-\chi v} u^2 |\nabla v|^2 dx \\ &\leq \frac{1}{2} \int_\Omega e^{-\chi v} |\nabla u|^2 dx + \frac{\chi^2}{2} \int_\Omega e^{-\chi v} u^2 |\nabla v|^2 dx, \end{aligned}$$

which yields

$$(3.25) \quad \frac{d}{dt} \int_\Omega u^2 dx + \int_\Omega e^{-\chi v} |\nabla u|^2 dx \leq \chi^2 \int_\Omega e^{-\chi v} u^2 |\nabla v|^2 dx.$$

On the other hand, using the fact $|X + Y|^2 \geq \frac{1}{2}X^2 - Y^2$ and $e^{-\frac{\chi}{2}v}\nabla u = \nabla(e^{-\frac{\chi}{2}v}u) + \frac{\chi}{2}e^{-\frac{\chi}{2}v}u\nabla v$, we have

$$e^{-\chi v}|\nabla u|^2 \geq \frac{1}{2}|\nabla(e^{-\frac{\chi}{2}v}u)|^2 - \frac{\chi^2}{4}e^{-\chi v}u^2|\nabla v|^2,$$

which substituted into (3.25) gives

$$(3.26) \quad \frac{d}{dt} \int_{\Omega} u^2 dx + \frac{1}{2} \int_{\Omega} |\nabla(e^{-\frac{\chi}{2}v}u)|^2 dx \leq \frac{5\chi^2}{4} \int_{\Omega} e^{-\chi v} u^2 |\nabla v|^2 dx \leq \frac{5\chi^2}{4} \|\nabla v\|_{L^4}^2 \|e^{-\frac{\chi}{2}v}u\|_{L^4}^2.$$

Moreover, the Gagliardo-Nirenberg inequality along with the facts $\|\nabla v\|_{L^2} \leq c_1$ and $\|\nabla v\|_{L^4} \leq c_2(\|\Delta v\|_{L^2}^{\frac{1}{2}}\|\nabla v\|)^{\frac{1}{2}}$

Lemma 3.6. *Let the conditions in Lemma 3.5 hold. Suppose (u, v) is a solution of (1.8) in $\Omega \times (0, T_{\max})$; then one has*

$$(3.30) \quad \|u(\cdot, t)\|_{L^\infty} \leq C \text{ for all } t \in (0, T_{\max}),$$

where $C > 0$ is a constant independent of t .

Proof. Noting (3.24) and applying the parabolic regularity estimates to the second equation of (1.8), one can find a positive constant c_1 such that

$$(3.31) \quad \|v\|_{L^\infty} + \|\nabla v\|_{L^4} \leq c_1,$$

which gives

$$(3.32) \quad e^{-\chi v} \geq e^{-\chi c_1} := d_1 > 0.$$

Then multiplying the first equation of (1.8) by u^{p-1} with $p \geq 3$ and integrating the result by parts, we end up with

$$\begin{aligned} & \frac{1}{p} \frac{d}{dt} \int_{\Omega} u^p dx + (p-1) \int_{\Omega} e^{-\chi v} u^{p-2} |\nabla u|^2 dx \\ &= -(p-1) \chi \int_{\Omega} e^{-\chi v} u^{p-1} \nabla u \cdot \nabla v dx \\ &\leq \frac{p-1}{2} \int_{\Omega} e^{-\chi v} u^{p-2} |\nabla u|^2 dx + \frac{(p-1)\chi^2}{2} \int_{\Omega} e^{-\chi v} u^p |\nabla v|^2 dx, \end{aligned}$$

which, combined with (3.32) and the fact $e^{-\chi v} \leq 1$, gives

$$(3.33) \quad \frac{d}{dt} \int_{\Omega} u^p dx + \frac{2(p-1)d_1}{p} \int_{\Omega} |\nabla u^{\frac{p}{2}}|^2 dx \leq \frac{p(p-1)\chi^2}{2} \int_{\Omega} u^p |\nabla v|^2 dx.$$

Then with (3.24) and (3.31), we can use the Hölder's inequality and Gagliardo-Nirenberg inequality to get

$$\begin{aligned} (3.34) \quad & \frac{p(p-1)\chi^2}{2} \int_{\Omega} u^p |\nabla v|^2 dx \leq \frac{p(p-1)\chi^2}{2} \int_{\Omega} u^{2p} dx^{\frac{1}{2}} \cdot \int_{\Omega} |\nabla v|^4 dx^{\frac{1}{2}} \\ &\leq \frac{p(p-1)\chi^2}{2} \|u^{\frac{p}{2}}\|_{L^4}^2 \|\nabla v\|_{L^4}^2 \\ &\leq \frac{p(p-1)\chi^2}{2} c_2 (\|\nabla u^{\frac{p}{2}}\|_{L^2}^{2(1-\frac{1}{p})} \|u^{\frac{p}{2}}\|_{L^{\frac{4}{p}}}^{\frac{2}{p}} + \|u^{\frac{p}{2}}\|_{L^{\frac{4}{p}}}^2) \\ &\leq \frac{(p-1)d_1}{p} \|\nabla u^{\frac{p}{2}}\|_{L^2}^2 + c_3. \end{aligned}$$

On the other hand, using the Gagliardo-Nirenberg inequality and (3.24) again, one has

$$\begin{aligned} \int_{\Omega} u^p dx &= \|u^{\frac{p}{2}}\|_{L^2}^2 \leq c_4 (\|\nabla u^{\frac{p}{2}}\|_{L^2}^{2(1-\frac{2}{p})} \|u^{\frac{p}{2}}\|_{L^{\frac{4}{p}}}^{\frac{4}{p}} + \|u^{\frac{p}{2}}\|_{L^{\frac{4}{p}}}^2) \\ &\leq (p-1)d_1 \end{aligned}$$

where $c_6 > 0$ is constant depending on p but independent of t . Applying the parabolic regularity theory to the second equation of (1.8), and choosing $p = 4$ in (3.36), one can find a positive constant d_2 independent of p such that $\|\nabla v(\cdot, t)\|_{L^\infty} \leq d_2$. Then by the well-known Moser iteration [4](or see [22]), we can show (3.30). $\square$

Lemma 3.7. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary. Assume that $0 \leq (u_0, v_0) \in [W^{1,\infty}(\Omega)]^2$ and then if $\int_{\Omega} u_0 dx < \frac{4\pi}{\chi}$, the system (1.8) admits a unique classical solution $(u, v) \in [C^0(\bar{\Omega} \times [0, \infty)) \cap C^{2,1}(\bar{\Omega} \times (0, \infty))]^2$ with uniform-in-time bound.*

Proof. Using Lemma 3.6, we have $\|u(\cdot, t)\|_{L^\infty} \leq C_1$. Then applying the parabolic regularity to the second equation of (1.8), one has $\|v(\cdot, t)\|_{W^{1,\infty}} \leq C_2$. Hence Lemma 3.7 follows directly by using Lemma 2.1. $\square$

3.2. Blow-up for supercritical mass. In this subsection, we shall construct some initial data with supercritical mass (i.e., $\int_{\Omega} u_0 dx > \frac{4\pi}{\chi}$) such that the corresponding solution of (1.8) blows up based on some ideas in [19, 23]. Noting $M_0 = \int_{\Omega} u_0 dx$, then the stationary solution of system (1.8) satisfies the following problem:

$$(3.37) \quad \begin{cases} -\Delta v + v = \frac{M_0 e^{\chi v}}{\int_{\Omega} e^{\chi v} dx}, & x \in \Omega, \\ u = \frac{M_0 e^{\chi v}}{\int_{\Omega} e^{\chi v} dx}, & x \in \Omega, \\ \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, \\ \int_{\Omega} v dx = \int_{\Omega} u dx = M_0. \end{cases}$$

For convenience, we introduce the following change of variable: $V = v - \frac{1}{|\Omega|} \int_{\Omega} v dx = v - \frac{M_0}{|\Omega|}$. Then the system (3.37) can be rewritten as

$$(3.38) \quad \begin{cases} -\Delta V + V = \frac{M_0 e^{\chi V}}{\int_{\Omega} e^{\chi V} dx} - \frac{M_0}{|\Omega|}, & x \in \Omega, \\ U = \frac{M_0 e^{\chi V}}{\int_{\Omega} e^{\chi V} dx}, & x \in \Omega, \\ \frac{\partial V}{\partial \nu} = 0, & x \in \partial\Omega, \\ \int_{\Omega} V dx = 0, \quad \int_{\Omega} U dx = M_0. \end{cases}$$

We point out that the steady state problem (3.38) and the Lyapunov function (1.4) for (1.8) are the same as those for the minimal Keller-Segel system (1.3) whose blow-up of solutions has been studied in [19, 20]. Hence we use the same arguments as in [19, Lemma 3.5] to establish the lower bound for the steady-state energy when $\int_{\Omega} u_0 dx \neq \frac{4\pi m}{\chi}$ for any $m \in \mathbb{N}^+$. For convenience, we present the results without proof.

Lemma 3.8. *Suppose $M_0 \neq \frac{4\pi m}{\chi}$ for all $m \in \mathbb{N}^+$. Then there exists a constant $K > 0$ such that*

$$(3.39) \quad F(U, V) \geq -K$$

holds for any solution (U, V) of the system (3.38).

Next, we show that there exist some initial data with supercritical mass (i.e., $M_0 > \frac{4\pi}{\chi}$) such that the energy is below any prescribed bound. To this end, we first prove that there is a sequence $(U_\varepsilon, V_\varepsilon)_{\varepsilon>0}$ satisfying $\int_{\Omega} V_\varepsilon(x) dx = 0$ and

$\int_{\Omega} U_{\varepsilon}(x) dx = M_0$ such that $\lim_{\varepsilon \rightarrow 0} F(U_{\varepsilon}, V_{\varepsilon}) = -\infty$ if $M_0 > \frac{4\pi}{\chi}$. Let $(U_{\varepsilon}, V_{\varepsilon})$ be defined as follows:

$$(3.40) \quad V_{\varepsilon}(x) = \frac{1}{\chi} \ln \frac{\varepsilon^2}{(\varepsilon^2 + \pi|x - x_0|^2)^2} - \frac{1}{|\Omega|} \ln \frac{\varepsilon^2}{(\varepsilon^2 + \pi|x - x_0|^2)^2} dx$$

and

$$(3.41) \quad U_{\varepsilon}(x) = \frac{M_0 e^{\chi V_{\varepsilon}(x)}}{\int_{\Omega} e^{\chi V_{\varepsilon}(x)} dx},$$

where x_0 is an arbitrary point on $\partial\Omega$. One can easily check that $\int_{\Omega} V_{\varepsilon}(x) dx = 0$ and $\int_{\Omega} U_{\varepsilon}(x) dx = M_0$. Next, we shall show that $\lim_{\varepsilon \rightarrow 0} F(U_{\varepsilon}, V_{\varepsilon}) = -\infty$ if $M_0 > \frac{4\pi}{\chi}$.

Lemma 3.9. *Let $(U_{\varepsilon}, V_{\varepsilon})_{\varepsilon > 0}$ be defined by (3.40)–(3.41) and $x_0 \in \partial\Omega$. If $M_0 > \frac{4\pi}{\chi}$, then it holds that*

$$(3.42) \quad F(U_{\varepsilon}, V_{\varepsilon}) \rightarrow -\infty \quad \text{as } \varepsilon \rightarrow 0.$$

Proof. Since x_0 is an arbitrary point on $\partial\Omega$, we assume $x_0 = 0$ without loss of generality. With the definition of $F(u, v)$ and (3.41), one has

$$(3.43) \quad \begin{aligned} F(U_{\varepsilon}, V_{\varepsilon}) &= \int_{\Omega} U_{\varepsilon} \ln U_{\varepsilon} dx - \chi \int_{\Omega} U_{\varepsilon} V_{\varepsilon} dx + \frac{\chi}{2} \int_{\Omega} |\nabla V_{\varepsilon}|^2 dx + \frac{\chi}{2} \int_{\Omega} V_{\varepsilon}^2 dx \\ &= M_0 \ln M_0 - M_0 \ln \int_{\Omega} e^{\chi V_{\varepsilon}} dx + \frac{\chi}{2} \int_{\Omega} |\nabla V_{\varepsilon}|^2 dx + \frac{\chi}{2} \int_{\Omega} V_{\varepsilon}^2 dx, \end{aligned}$$

where we have used the fact

$$\begin{aligned} &\int_{\Omega} U_{\varepsilon} \ln U_{\varepsilon} dx - \chi \int_{\Omega} U_{\varepsilon} V_{\varepsilon} dx \\ &= \frac{M_0}{\int_{\Omega} e^{\chi V_{\varepsilon}} dx} \int_{\Omega} e^{\chi V_{\varepsilon}} \ln M_0 + \chi V_{\varepsilon} - \ln \int_{\Omega} e^{\chi V_{\varepsilon}} dx \quad dx - \frac{\chi M_0}{\int_{\Omega} e^{\chi V_{\varepsilon}} dx} \int_{\Omega} e^{\chi V_{\varepsilon}} V_{\varepsilon} dx \\ &= M_0 \ln M_0 - M_0 \ln \int_{\Omega} e^{\chi V_{\varepsilon}} dx. \end{aligned}$$

On the other hand, we use (3.40) and the polar coordinates around origin $0 \in \partial\Omega$, with R denoting the maximum distance between the pole and boundary of Ω , to derive that

$$(3.44) \quad \begin{aligned} \frac{\chi}{2} \int_{\Omega} |\nabla V_{\varepsilon}|^2 dx &\leq \frac{8\pi^2}{\chi} \int_0^{\frac{R}{\varepsilon}} \int_0^{\pi} \frac{r^3}{(1 + \pi r^2)^2} dr d\theta \\ &\leq \frac{4\pi}{\chi} \ln \frac{1}{\varepsilon^2} + \ln(\varepsilon^2 + \pi R^2) - 1 + \frac{\varepsilon^2}{\varepsilon^2 + \pi R^2} \\ &\leq \frac{8\pi}{\chi} \ln \frac{1}{\varepsilon} + O_1(1), \end{aligned}$$

where $|O_1(1)| \leq C$ as $\varepsilon \rightarrow 0$. Moreover, direct calculations give

$$(3.45) \quad \frac{\chi}{2} \int_{\Omega} V_{\varepsilon}^2 dx = \frac{1}{2\chi} \int_{\Omega} (\ln(\varepsilon^2 + \pi|x|^2))^2 dx - \frac{1}{2\chi|\Omega|} \int_{\Omega} \ln(\varepsilon^2 + \pi|x|^2) dx = O_2(1),$$

where $|O_2(1)| \leq C$ as $\varepsilon \rightarrow 0$. Furthermore, it has that

$$(3.46) \quad \ln \int_{\Omega} e^{\chi V_{\varepsilon}} dx = \ln |\Omega| \int_{\Omega} \frac{\varepsilon^2}{(\varepsilon^2 + \pi|x|^2)^2} dx - \frac{1}{|\Omega|} \ln \int_{\Omega} \frac{\varepsilon^2}{(\varepsilon^2 + \pi|x|^2)^2} dx$$

and

$$1 - \frac{\varepsilon^2}{\pi R_1^2 + \varepsilon^2} \leq \int_{\Omega} \frac{\varepsilon^2}{(\varepsilon^2 + \pi|x|^2)^2} dx \leq 1 - \frac{\varepsilon^2}{\pi R_2^2 + \varepsilon^2},$$

where R_1 and R_2 denote the maximum and minimum distance between the pole and the boundary of Ω . Then from (3.46), one can show that

$$(3.47) \quad \begin{aligned} & -M_0 \ln \int_{\Omega} e^{\chi V_{\varepsilon}} dx \\ &= -M_0 \ln |\Omega| \int_{\Omega} \frac{\varepsilon^2}{(\varepsilon^2 + \pi|x|^2)^2} dx - \frac{1}{|\Omega|} \ln \int_{\Omega} \frac{\varepsilon^2}{(\varepsilon^2 + \pi|x|^2)^2} dx \\ &= \frac{M_0}{|\Omega|} \ln \varepsilon^2 dx + \frac{M_0}{|\Omega|} \ln(\varepsilon^2 + \pi|x|^2)^2 dx - M_0 \ln |\Omega| \int_{\Omega} \frac{\varepsilon^2}{(\varepsilon^2 + \pi|x|^2)^2} dx \\ &= 2M_0 \ln \varepsilon + O_3(1) \end{aligned}$$

with $|O_3(1)| \leq C$ as $\varepsilon \rightarrow 0$. Finally substituting (3.44), (3.45), and (3.47) into (3.43) gives

$$(3.48) \quad F(U_{\varepsilon}, V_{\varepsilon}) \leq 2 \frac{4\pi}{\chi} - M_0 \ln \frac{1}{\varepsilon} + O(1),$$

where $O(1) = O_1(1) + O_2(1) + O_3(1)$ and $|O(1)| \leq C$ as $\varepsilon \rightarrow 0$. Since $M_0 > \frac{4\pi}{\chi}$, (3.42) follows directly from (3.48). $\square$

Next, we shall establish the connection between the energy of steady states and the initial data. More precisely, we have the following results.

Lemma 3.10. *Let (u, v) be a global-in-time bounded solution of (1.8). Then there exist a sequence of times $t_k \rightarrow \infty$ and nonnegative function $(U_{\infty}, V_{\infty}) \in [C^2(\bar{\Omega})]^2$ such that $(u(\cdot, t_k), v(\cdot, t_k)) \rightarrow (U_{\infty}, V_{\infty})$ in $[C^2(\bar{\Omega})]^2$. Furthermore, (U_{∞}, V_{∞}) is a solution of (3.38) satisfying*

$$(3.49) \quad F(U_{\infty}, V_{\infty}) \leq F(u_0, v_0).$$

Proof. Since (u, v) is the global classical solution with uniform-in-time bound of the system (1.8), then we can use the standard bootstrap arguments involving interior parabolic regularity theory [27] to find a constant $c_1 > 0$ independent of t such that

$$(3.50) \quad \|u(\cdot, t)\|_{C^{2+\sigma, 1+\frac{\sigma}{2}}(\bar{\Omega} \times [1, \infty))} + \|v(\cdot, t)\|_{C^{2+\sigma, 1+\frac{\sigma}{2}}(\bar{\Omega} \times [1, \infty))} \leq c_1,$$

where $\sigma \in (0, 1)$. From (3.50), we know that $(u(\cdot, t), v(\cdot, t))_{t \geq 1}$ is relatively compact in $[C^2(\bar{\Omega})]^2$ and $F(u, v)$ is bounded for $t > 1$. Hence there exists a suitable time sequence $t_k \rightarrow \infty$ such that $(u(\cdot, t_k), v(\cdot, t_k)) \rightarrow (U_{\infty}, V_{\infty})$ in $[C^2(\bar{\Omega})]^2$ for some nonnegative $U_{\infty}, V_{\infty} \in C^2(\bar{\Omega})$. Then we have

$$F(u(\cdot, t_k), v(\cdot, t_k)) \rightarrow F(U_{\infty}, V_{\infty}), \text{ as } t_k \rightarrow \infty,$$

which gives (3.49) by the fact $F(u, v) \leq F(u_0, v_0)$ from (3.1). On the other hand, using the facts $0 < c_2 \leq e^{-\chi v}$ and $F(u, v)$ is bounded for $t > 1$, from Lemma 3.1, one has

bounded in time, then from Lemma 3.10, we have $F(U_\infty, V_\infty) \leq F(u_0, v_0) < -K$, which combined with (3.56) raises the following contradiction:

$$-K \leq F(U_\infty, V_\infty) \leq F(u_0, v_0) < -K.$$

Then the Lemma 3.11 is proved. $\square$

3.2.1. *Proof of Theorem 1.1.* Theorem 1.1 is a direct consequence of Lemmas 3.7 and 3.11.

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