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BOUNDEDNESS OF A CHEMOTAXIS-CONVECTION MODEL DESCRIBING TUMOR-INDUCED ANGIOGENESIS*

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Abstract This paper is concerned with the parabolic-parabolic-elliptic system

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(u\nabla v) + \xi_1\nabla\cdot(u^m\nabla w), & x\in\Omega, t>0,\\ v_t = \Delta v + \xi_2\nabla\cdot(v\nabla w) + u - v, & x\in\Omega, t>0,\\ 0 = \Delta w + u - \dfrac{1}{|\Omega|}\displaystyle\int_\Omega u, \int_\Omega w = 0, & x\in\Omega, t>0,\\ \dfrac{\partial u}{\partial\nu} = \dfrac{\partial v}{\partial\nu} = \dfrac{\partial w}{\partial\nu} = 0, & x\in\partial\Omega, t>0,\\ u(x,0) = u_0(x), v(x,0) = v_0(x), & x\in\Omega \end{cases}$$

in a bounded domain $\Omega\subset\mathbb{R}^n$ with a smooth boundary, where the parameters χ,ξ_1,ξ_2 are positive constants and $m\geq 1$. Based on the coupled energy estimates, the boundedness of the global classical solution is established in any dimensions ($n\geq 1$) provided that $m>1$.

Key words boundedness; convection; chemotaxis; tumor invasion

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1 Introduction

To study the convection effect on tumor-induced angiogenesis, Orme and Chaplain [22] proposed the system

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(u\nabla v) + \xi_1\nabla\cdot(u\nabla w),\\ v_t = \Delta v + \xi_2\nabla\cdot(v\nabla w) + \alpha u - \beta v,\\ w_t = \Delta w + \gamma u - \delta w, \end{cases}\tag{1.1}$$

where $u(x,t)$ is the density of endothelial cells (ECs), $v(x,t)$ refers to the density of adhesive sites, and $w(x,t)$ describes the density of the matrix; one can find more details about the background of this in [13, 15, 25].

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When $\xi_1 = \xi_2 = 0$, system (1.1) becomes the classical Keller-Segel (KS) chemotaxis model

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v), \\ v_t = \Delta v + \alpha u - \beta v, \end{cases} \tag{1.2}$$

whose solution behaviors, including boundedness, blow-up and pattern formation, have been studied for a long time [2, 6]. An interesting feature of the KS model (1.2) is the blow-up of the solution in higher dimensions ($n \geq 2$). More precisely, it has been proven that for any initial mass $m := \int_\Omega u_0 \mathrm{d}x > 0$, there exists some initial data such that the corresponding solution blows up in finite time [28] in the case where $n \geq 3$. In the two dimensional spaces ($n = 2$), there exists a critical mass $m_* = \frac{4\pi}{\chi\alpha}$ such that the solution exists globally with a uniform-in-time bound when $\int_\Omega u_0 \mathrm{d}x < m_*$ [21], while the solution blows up if $\int_\Omega u_0 \mathrm{d}x > m_*$ [7].

For when $\xi_1 > 0$ and $\xi_2 = 0$, system (1.1) has been referred to as the attraction-repulsion Keller-Segel (ARKS) chemotaxis system,

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + \xi_1 \nabla \cdot (u \nabla w), \\ v_t = \Delta v + \alpha u - \beta v, \\ w_t = \Delta w + \gamma u - \delta w; \end{cases} \tag{1.3}$$

this has been used to describe the aggregation of Microglia in Alzheimer's disease in [20]. Compared with the classical KS chemotaxis model (1.2), the ARKS model (1.3) has the mechanism to prevent the blow-up of the solution if repulsion dominates in the sense that $\xi_1\gamma \geq \chi\alpha$. More precisely, if $\xi_1\gamma \geq \chi\alpha$, the global bounded solution exists in two dimensions for the full ARKS model [10, 18] and in higher dimensions ($n \geq 3$) for the parabolic-elliptic-elliptic ARKS model [24]. However, if $\xi_1\gamma < \chi\alpha$ (i.e., attraction prevails over repulsion), based on the existence of a Lyapunov functional, a critical mass phenomenon was found in two dimensional spaces; refer to [5, 11, 14] for more details. Recently, the global stabilization of a constant steady state was studied in [12, 16, 17], in the cases where $\xi_1\gamma \geq \chi\alpha$.

For when $\xi_1 > 0$ and $\xi_2 > 0$, the research results on system (1.1) seem to be at a rather rudimentary stage, due to the coupling of chemotaxis and convection. For a one dimensional space, the boundedness and large time behavior of a solution were studied in [13, 15]. Recently, for a parabolic-parabolic-elliptic simplified model in a bounded domain $\Omega \subset \mathbb{R}^n (1 \leq n \leq 3)$, Tao & Winkler [25] proved the global boundedness of a solution for large ξ_1 by using a novel Moser-type iteration to derive a priori bounds of $\|v\|_{L^\infty}$.

To study the convection effect ξ_2 more deeply, in this paper, we shall study the existence of global classical solutions of the system

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + \xi_1 \nabla \cdot (u^m \nabla w), & x \in \Omega, t > 0, \\ v_t = \Delta v + \xi_2 \nabla \cdot (v \nabla w) + u - v, & x \in \Omega, t > 0, \\ 0 = \Delta w + u - \frac{1}{|\Omega|} \int_\Omega u, \int_\Omega w = 0, & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.4}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with a smooth boundary and the initial value satisfies the following conditions:

$$\begin{cases} 0 \le u_0 \in C^0(\bar{\Omega}) \text{ with } u_0 \not\equiv 0, \\ 0 \le v_0 \in W^{1,\infty}(\Omega). \end{cases} \tag{1.5}$$

The third equation of (1.4) arises by using a quasi-stationary approximation procedure as in [9], based on the assumption that the matrix diffuses much faster than adhesive sites and endothelial cells. For the system (1.4) with $m = 1$, it was proven in [4] that the global classical solution exists for all dimensions, provided that ξ_1 is large. In this paper, we shall show the boundedness of a solution for all dimensions with $\xi_1 > 0$ and $m > 1$. More precisely, we have the following result:

Theorem 1.1 (Boundedness of the solution) Let $m > 1$ and let Ω be a bounded domain in $\mathbb{R}^n (n \ge 1)$ with a smooth boundary. Then the system (1.4)–(1.5) admits a unique global classical solution (u, v, w) satisfying

$$\begin{cases} u \in C^0(\bar{\Omega} \times [0, \infty)) \cap C^{2,1}(\bar{\Omega} \times (0, \infty)), \\ v \in \bigcap\limits_{n<p<\infty} C^0([0, \infty); W^{1,p}(\Omega)) \cap C^{2,1}(\bar{\Omega} \times (0, \infty)) \text{ and} \\ w \in C^{2,0}(\bar{\Omega} \times (0, \infty)). \end{cases}$$

Moreover, there exists a constant $C_1 > 0$ independent of t such that

$$\|u(\cdot, t)\|_{L^\infty} + \|v(\cdot, t)\|_{W^{1,\infty}} + \|w(\cdot, t)\|_{W^{1,\infty}} \le C_1.$$

2 Local Existence and Preliminaries

Based on some ideas in [24, 25, 27], one can show the local existence of a solution for system (1.4)–(1.5). We omit the details of the proof for the solve of simplicity.

Lemma 2.1 Let $m > 1$ and let Ω be a bounded domain in $\mathbb{R}^n$ with a smooth boundary. Assume that (u_0, v_0) satisfies (1.5). Then there exists $T_{\max} \in (0, \infty]$ such that system (1.4) has a unique classical solution (u, v, w), where

$$\begin{cases} u \in C^0(\bar{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\max})), \\ v \in \bigcap\limits_{n<p<\infty} C^0([0, T_{\max}); W^{1,p}(\Omega)) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\max})) \text{ and} \\ w \in C^{2,0}(\bar{\Omega} \times (0, T_{\max})). \end{cases}$$

Moreover, the solutions u, v are positive in $\Omega \times (0, T_{\max})$ and

$$\text{if } T_{\max} < \infty, \text{ then } \limsup_{t \nearrow T_{\max}} \{\|u(\cdot, t)\|_{L^\infty} + \|v(\cdot, t)\|_{W^{1,p}}\} = \infty \text{ for all } p > n.$$

Lemma 2.2 Let (u, v, w) be the solution obtained in Lemma 2.1. Then, for all $t \in (0, T_{\max})$, it holds that

$$\|u(\cdot, t)\|_{L^1} = \|u_0\|_{L^1} := m_1 \tag{2.1}$$

and

$$\|v(\cdot, t)\|_{L^1} \le \|v_0\|_{L^1} + \|u_0\|_{L^1}, \tag{2.2}$$

w1 L()1(c)-2(2()34(2)-6(.)1(3)-6(2))]Te322()-6(r)-33l1(gl-6(o)18(wt)-22.1T111714.28Td[(a)-6(N)3((o)1(n)1(g

as well as

wuW- d (L)Tj 0T1_3 9.923 Tf 4.44 -4.08 d (L)Tj /T1_5 6.974 Tf 5.04 -1.8 Td (W)Tj /T1_6 4.971 Tf 5.52 1.92 21.8

$$\|D^2 w\|_{L^p} \leq \|\nabla w\|_{W^{1,p}} \leq C\|u\|_{L^p}, \text{ for some } p > 1. \tag{2.3}$$

W$W\psi T\psi\psi Tf\psi\psi\psi T\psi Tj\psi T\psi\psi Tf\psi\psi$.

Proof

is valid for all $f \in L^q(\Omega)$;

(ii) if $2 \le q \le p < \infty$, then

$$\|\nabla \mathrm{e}^{t\Delta} f\|_{L^p} \le c_2 \left(1 + t^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}\right) \mathrm{e}^{-\lambda_1 t} \|\nabla f\|_{L^q} \text{ for all } t > 0 \tag{2.8}$$

is true for all $f \in W^{1,p}(\Omega)$.

3 Proof of Theorem 1.1

In this section, we shall prove the global boundedness of the solution. To this end, we first show the boundedness of $\|v\|_{L^\infty}$ for all dimensions based on some ideas in [25].

Lemma 3.1 Assume that the conditions in Lemma 2.1 hold. Let (u, v, w) be the solution of (1.4). Then for all $n \ge 1$, it holds that

$$\|v(\cdot, t)\|_{L^\infty} \le M. \tag{3.1}$$

Proof Multiplying the second equation of (1.4) by $v^{p-1} (p \ge 2)$ and integrating it by parts, we obtain

$$\begin{aligned} \frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega v^p + (p-1)\int_\Omega v^{p-2}|\nabla v|^2 &= \xi_2 \int_\Omega v^{p-1}\nabla\cdot(v\nabla w) + \int_\Omega uv^{p-1} - \int_\Omega v^p \\ &= -\xi_2(p-1)\int_\Omega v^{p-1}\nabla v\cdot\nabla w + \int_\Omega uv^{p-1} - \int_\Omega v^p \\ &= \xi_2 \frac{p-1}{p}\int_\Omega v^p \Delta w + \int_\Omega uv^{p-1} - \int_\Omega v^p. \end{aligned} \tag{3.2}$$

Noting (2.1), one has that $\Delta w = \frac{1}{|\Omega|}\int_\Omega u - u = \frac{m_1}{|\Omega|} - u$. Then we can rewrite (3.2) as follows:

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega v^p + p(p-1)\int_\Omega v^{p-2}|\nabla v|^2 + p\int_\Omega v^p \\ =\ &\frac{\xi_2 m_1(p-1)}{|\Omega|}\int_\Omega v^p - \xi_2(p-1)\int_\Omega uv^p + p\int_\Omega uv^{p-1}. \end{aligned} \tag{3.3}$$

On the other hand, using Young's inequality, we can derive that

$$\begin{aligned} p\int_\Omega uv^{p-1} &\le p\int_\Omega u\left\{\frac{\xi_2(p-1)}{p}v^p + \left(\frac{\xi_2(p-1)}{p}\frac{p}{p-1}\right)^{-(p-1)}\frac{1}{p}\cdot 1\right\} \\ &= \xi_2(p-1)\int_\Omega uv^p + \xi_2^{-(p-1)}\int_\Omega u \\ &\le \xi_2(p-1)\int_\Omega uv^p + m_1\xi_2^{1-p}; \end{aligned}$$

that is,

$$p\int_\Omega uv^{p-1} - \xi_2(p-1)\int_\Omega uv^p \le m_1\xi_2^{1-p}. \tag{3.4}$$

Then, substituting (3.4) into (3.3) and using the fact that $\frac{4(p-1)}{p} \ge 2$, one obtains

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega v^p + 2\int_\Omega |\nabla v^{\frac{p}{2}}|^2 + p\int_\Omega v^p \le \frac{\xi_2 m_1(p-1)}{|\Omega|}\int_\Omega v^p + m_1\xi_2^{1-p}. \tag{3.5}$$

Finally, we can estimate the first term on the right hand of (3.5) as

$$\frac{\xi_2 m_1(p-1)}{|\Omega|}\int_\Omega v^p = \frac{\xi_2 m_1(p-1)}{|\Omega|}\|v^{\frac{p}{2}}\|_{L^2}^2 \le 2\int_\Omega |\nabla v^{\frac{p}{2}}|^2 + c_1 p(1 + p^{\frac{n}{2}})\left(\int_\Omega v^{\frac{p}{2}}\right)^2, \tag{3.6}$$

where we have used the estimate (see [24])

$$\|U\|_{L^2}^2 \le \varepsilon\|\nabla U\|_{L^2}^2 + C_{1N}(1+\varepsilon^{-\frac{n}{2}})\|U\|_{L^1}^2$$

by letting $U = v^{\frac{p}{2}}$, $\varepsilon := \frac{2|\Omega|}{\xi_2 m_1(p-1)+1}$ and $c_1 = C_{1N}\max\left\{\left(\frac{\xi_2 m_1+1}{2|\Omega|}\right)^{\frac{n}{2}}, 1\right\}$ depend on n and Ω but remain independent of p.

Then substituting (3.6) into (3.5) and noting the fact that $1+p^{\frac{n}{2}} \le 2(1+p)^{\frac{n}{2}}$, one obtains

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega v^p + p\int_\Omega v^p &\le c_1 p(1+p^{\frac{n}{2}})\left(\int_\Omega v^{\frac{p}{2}}\right)^2 + m_1\xi_2^{1-p}\\ &\le 2c_1 p(1+p)^{\frac{n}{2}}\left(\int_\Omega v^{\frac{p}{2}}\right)^2 + m_1\xi_2^{1-p},\end{aligned}$$

which entails that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{pt}\int_\Omega v^p\right) \le 2\mathrm{e}^{pt}c_1 p(1+p)^{\frac{n}{2}}\left(\int_\Omega v^{\frac{p}{2}}\right)^2 + \mathrm{e}^{pt}m_1\xi_2^{1-p}. \tag{3.7}$$

For any $T\in(0,T_{\max})$, integrating (3.7) over $[0,t]$ for $0\le t\le T$, we obtain

$$\int_\Omega v^p(x,t) \le \int_\Omega v_0^p + 2c_1(1+p)^{\frac{n}{2}}\sup_{0\le t\le T}\left(\int_\Omega v^{\frac{p}{2}}(x,t)\right)^2 + \frac{m_1\xi_2^{1-p}}{p}. \tag{3.8}$$

Letting

$$\mathcal{N}(p) := \max\left\{\sup_{0\le t\le T}\left(\int_\Omega v^p(x,t)\right)^{\frac{1}{p}},\quad \frac{1}{\xi_2},\quad \|v_0\|_{L^\infty}\right\},$$

from (3.8), one can find a positive constant $c_2 := 3\max\{2c_1, m_1\xi_2, |\Omega|, 1\}$ such that

$$\mathcal{N}(p) \le c_2^{\frac{1}{p}}(1+p)^{\frac{n}{2p}}\mathcal{N}(\frac{p}{2}), \quad \text{for all } \ p\ge 2.$$

Taking $p=2^j, j=1,2,\cdots$, one derives that

$$\begin{aligned}\mathcal{N}(2^j) &\le c_2^{2^{-j}}(1+2^j)^{2^{-j}\cdot\frac{n}{2}}\mathcal{N}(2^{j-1})\\ &\cdots\\ &\le c_2^{2^{-j}+\cdots+2^{-1}}[(1+2^j)^{2^{-j}}\cdots(1+2)^{2^{-1}}]^{\frac{n}{2}}\mathcal{N}(1)\\ &\le c_2\left([2^{j2^{-j}}(2^{-j}+1)^{2^{-j}}]\cdots[2^{2^{-1}}(2^{-1}+1)^{2^{-1}}]\right)^{\frac{n}{2}}\mathcal{N}(1)\\ &\le c_2 2^{\frac{n}{2}[j2^{-j}+(j-1)2^{-(j-1)}+\cdots+2^{-1}]}\cdot 2^{\frac{n}{2}(2^{-j}+2^{-(j-1)}+\cdots+2^{-1})}\mathcal{N}(1)\\ &\le c_2 2^{\frac{3n}{2}}\mathcal{N}(1).\end{aligned}$$

Letting $j\to\infty$ and noting the fact that $\|v(\cdot,t)\|_{L^1}\le m_1+\|v_0\|_{L^1}$, we have that

$$\|v(\cdot,t)\|_{L^\infty} \le c_2 2^{\frac{3n}{2}}\mathcal{N}(1) \le c_2 2^{\frac{3n}{2}}\max\{\|v_0\|_{L^\infty}, m_1+\|v_0\|_{L^1}, \frac{1}{\xi_2}\} \le c_3,$$

which gives (3.1). □

3.1 Coupled Energy Estimates

In this subsection, we shall show the coupled energy estimate of $\int_\Omega u^p + \int_\Omega |\nabla v|^{2p}$ for $p>1$.

Lemma 3.2 Let (u,v,w) be the solution of system (1.4). Then, for all $p>1$, it holds that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p + \frac{p(p-1)}{2}\int_\Omega u^{p-2}|\nabla u|^2 + \frac{p(p-1)\xi_1}{2(p-1+m)}\int_\Omega u^{p+m} \le \frac{p(p-1)\chi^2}{2}\int_\Omega u^p|\nabla v|^2 + C, \tag{3.9}$$

where $C>0$ is a constant independent of t.

Proof Multiplying the first equation of system (1.4) by pu^{p-1} and integrating it over Ω and noting the fact that $\Delta w=\frac{m_1}{|\Omega|}-u$, we have that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p =& p\int_\Omega u^{p-1}\Delta u-\chi p\int_\Omega u^{p-1}\nabla\cdot(u\nabla v)+\xi_1 p\int_\Omega u^{p-1}\nabla\cdot(u^m\nabla w)\\ =&-p(p-1)\int_\Omega u^{p-2}|\nabla u|^2+\chi p(p-1)\int_\Omega u^{p-1}\nabla u\cdot\nabla v\\ &-\xi_1 p(p-1)\int_\Omega u^{p-2+m}\nabla u\cdot\nabla w\\ =&-p(p-1)\int_\Omega u^{p-2}|\nabla u|^2+\chi p(p-1)\int_\Omega u^{p-1}\nabla u\cdot\nabla v\\ &+\frac{\xi_1 p(p-1)m_1}{(p+m-1)|\Omega|}\int_\Omega u^{p+m-1}-\frac{\xi_1 p(p-1)}{p+m-1}\int_\Omega u^{p+m}.\end{aligned}\tag{3.10}$$

Then, using Hölder's and Young's inequalities, one has that

$$\chi p(p-1)\int_\Omega u^{p-1}\nabla u\cdot\nabla v\le\frac{p(p-1)}{2}\int_\Omega u^{p-2}|\nabla u|^2+\frac{p(p-1)\chi^2}{2}\int_\Omega u^p|\nabla v|^2\tag{3.11}$$

and

$$\frac{\xi_1 p(p-1)m_1}{(p+m-1)|\Omega|}\int_\Omega u^{p+m-1}\le\frac{\xi_1 p(p-1)}{2(p+m-1)}\int_\Omega u^{p+m}+c_1.\tag{3.12}$$

Then substituting (3.11) and (3.12) into (3.10), one has that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p+\frac{p(p-1)}{2}\int_\Omega u^{p-2}|\nabla u|^2+\frac{p(p-1)\xi_1}{2(p-1+m)}\int_\Omega u^{p+m}\le\frac{p(p-1)\chi^2}{2}\int_\Omega u^p|\nabla v|^2+c_1,$$

which yields (3.9). The proof of Lemma 3.2 is complete. □

Lemma 3.3 Let (u,v,w) be the solution of (1.4). Then, for all $p\ge\max\{n,2\}$, there exists a positive constant C such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|\nabla v|^{2p}+2p\int_\Omega|\nabla v|^{2p}+p\int_\Omega|\nabla v|^{2(p-1)}|D^2v|^2\le C\int_\Omega u^{p+1}+C.\tag{3.13}$$

Proof Recalling the second equation of system (1.4) and using the identity $\Delta|\nabla v|^2=2\nabla v\cdot\nabla\Delta v+2|D^2v|^2$, we have that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|\nabla v|^{2p}=&2p\int_\Omega(|\nabla v|^2)^{p-1}\nabla v\cdot\nabla v_t\\ =&2p\int_\Omega(|\nabla v|^2)^{p-1}\nabla v\cdot\nabla\Delta v+2\xi_2 p\int_\Omega(|\nabla v|^2)^{p-1}\nabla v\cdot\nabla(\nabla\cdot(v\nabla w))\\ &+2p\int_\Omega(|\nabla v|^2)^{p-1}\nabla v\cdot\nabla u-2p\int_\Omega|\nabla v|^{2p}\\ =&p\int_\Omega|\nabla v|^{2(p-1)}\Delta|\nabla v|^2-2p\int_\Omega|\nabla v|^{2(p-1)}|D^2v|^2\\ &+2\xi_2 p\int_\Omega|\nabla v|^{2(p-1)}\nabla v\cdot\nabla(\nabla v\cdot\nabla w)+2\xi_2 p\int_\Omega|\nabla v|^{2(p-1)}\nabla v\cdot\nabla(v\Delta w)\\ &+2p\int_\Omega|\nabla v|^{2(p-1)}\nabla v\cdot\nabla u-2p\int_\Omega|\nabla v|^{2p},\end{aligned}$$

which gives that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|\nabla v|^{2p}+2p\int_\Omega|\nabla v|^{2(p-1)}|D^2v|^2+2p\int_\Omega|\nabla v|^{2p}+p(p-1)\int_\Omega|\nabla v|^{2(p-2)}|\nabla|\nabla v|^2|^2$$

$$
\begin{aligned}
&= p\int_{\partial\Omega}|\nabla v|^{2(p-1)}\frac{\partial|\nabla v|^2}{\partial\nu}\mathrm{d}S + 2\xi_2 p\int_\Omega|\nabla v|^{2(p-1)}\nabla v\cdot\nabla(\nabla v\cdot\nabla w)\\
&\quad + 2\xi_2 p\int_\Omega|\nabla v|^{2(p-1)}\nabla v\cdot\nabla(v\Delta w) - 2p\int_\Omega u\cdot\nabla(|\nabla v|^{2(p-1)}\nabla v)\\
&= \sum_{i=1}^4 I_i. \qquad (3.14)
\end{aligned}
$$

To control the term I_1, we first state the following well-known fact, which is due to homogeneous Neumann conditions [19, Lemma 4.2] (see also [8]):

$$\frac{\partial|\nabla v|^2}{\partial\nu}\le 2\kappa|\nabla v|^2 \text{ on } \partial\Omega. \qquad (3.15)$$

Here κ denotes the maximum curvature of $\partial\Omega$. Now, combining (3.15) and using the trace inequality $\|\psi\|_{L^2(\partial\Omega)}\le\varepsilon\|\nabla\psi\|_{L^2(\Omega)}+C_\varepsilon\|\psi\|_{L^2(\Omega)}$ for any $\varepsilon>0$ (cf. [23, Remark 52.9]), we have that

$$
\begin{aligned}
I_1 &= p\int_{\partial\Omega}|\nabla v|^{2(p-1)}\frac{\partial|\nabla v|^2}{\partial\nu}\mathrm{d}S\\
&\le 2\kappa p\||\nabla v|^p\|^2_{L^2(\partial\Omega)}\\
&\le \frac{p(p-1)}{2}\int_\Omega|\nabla v|^{2(p-2)}|\nabla|\nabla v|^2|^2 + c_1\int_\Omega|\nabla v|^{2p}. \qquad (3.16)
\end{aligned}
$$

Using Young's inequality, as well as (2.6) and the fact that $\|v(\cdot,t)\|_{L^\infty}\le M$, we infer that

$$c_1\int_\Omega|\nabla v|^{2p}\le\frac{p}{2\left(2p+\sqrt{n}\right)^2M^2}\int_\Omega|\nabla v|^{2(p+1)}+c_2\le\frac{p}{2}\int_\Omega|\nabla v|^{2(p-1)}|D^2v|^2+c_2,$$

which, substituted into (3.16), gives that

$$I_1\le\frac{p(p-1)}{2}\int_\Omega|\nabla v|^{2(p-2)}|\nabla|\nabla v|^2|^2+\frac{p}{2}\int_\Omega|\nabla v|^{2(p-1)}|D^2v|^2+c_2. \qquad (3.17)$$

On the other hand, we can use (2.4), (2.6) and (2.3) to derive that

$$
\begin{aligned}
I_2 &= 2\xi_2 p\int_\Omega|\nabla v|^{2(p-1)}\nabla v\cdot\nabla(\nabla v\cdot\nabla w)\\
&\le 2\xi_2 p\left(1+\frac{\sqrt{n}}{2p}\right)\|\nabla v\|^{2p}_{L^{2(p+1)}}\|D^2w\|_{L^{p+1}}\\
&\le 2\xi_2 p\left(1+\frac{\sqrt{n}}{2p}\right)(2p+\sqrt{n})^{\frac{2p}{p+1}}\|v\|^{\frac{2p}{p+1}}_{L^\infty}\cdot\left\{\int_\Omega|\nabla v|^{2(p-1)}|D^2v|^2\right\}^{\frac{p}{p+1}}\cdot\|D^2w\|_{L^{p+1}}\\
&\le 2c_3\xi_2 p\left(1+\frac{\sqrt{n}}{2p}\right)(2p+\sqrt{n})^{\frac{2p}{p+1}}M^{\frac{2p}{p+1}}\cdot\left\{\int_\Omega|\nabla v|^{2(p-1)}|D^2v|^2\right\}^{\frac{p}{p+1}}\cdot\|u\|_{L^{p+1}}. \qquad (3.18)
\end{aligned}
$$

Similarly, combining (2.5) with (2.6) and using (2.3), one has that

$$
\begin{aligned}
I_3 &= 2\xi_2 p\int_\Omega|\nabla v|^{2(p-1)}\nabla v\cdot\nabla(v\Delta w)\\
&= -2\xi_2 p\int_\Omega v\Delta w\nabla\cdot(|\nabla v|^{2(p-1)}\nabla v)\\
&\le 2\xi_2 p(2p-2+\sqrt{n})\|v\|_{L^\infty}\cdot\|\Delta w\|_{L^{p+1}}\cdot\|\nabla v\|^{p-1}_{L^{2(p+1)}}\cdot\left\{\int_\Omega|\nabla v|^{2(p-1)}\cdot|D^2v|^2\right\}^{\frac12}\\
&\le 2\xi_2 p(2p-2+\sqrt{n})(2p+\sqrt{n})^{\frac{p-1}{p+1}}\|v\|^{\frac{2p}{p+1}}_{L^\infty}\cdot\left\{\int_\Omega|\nabla v|^{2(p-1)}\cdot|D^2v|^2\right\}^{\frac{p}{p+1}}\cdot\|\Delta w\|_{L^{p+1}}
\end{aligned}
$$

$$\leq c_4 M^{\frac{2p}{p+1}} \cdot \left\{ \int_\Omega |\nabla v|^{2(p-1)} \cdot |D^2 v|^2 \right\}^{\frac{p}{p+1}} \cdot \|u\|_{L^{p+1}}, \tag{3.19}$$

with $c_4 := 2\sqrt{n} c_3 \xi_2 p (2p-2+\sqrt{n})(2p+\sqrt{n})^{\frac{p-1}{p+1}}$, where the fact that $|\Delta w| \leq \sqrt{n}|D^2 w|$ is used. Finally, we can use (2.6) to estimate the term I_4 as

$$\begin{aligned}
I_4 &= -2p \int_\Omega u \cdot \nabla(|\nabla v|^{2(p-1)} \nabla v) \\
&= -2p \int_\Omega u \cdot \{2(p-1)|\nabla v|^{2(p-2)} \cdot (D^2 v \cdot \nabla v)\nabla v + |\nabla v|^{2(p-1)} \Delta v\} \\
&\leq 2p(2p-2+\sqrt{n}) \int_\Omega u |\nabla v|^{2(p-1)} |D^2 v| \\
&\leq 2p(2p-2+\sqrt{n}) \cdot \left\{ \int_\Omega u^2 |\nabla v|^{2(p-1)} \right\}^{\frac{1}{2}} \cdot \left\{ \int_\Omega |\nabla v|^{2(p-1)} |D^2 v|^2 \right\}^{\frac{1}{2}} \\
&\leq 2p(2p-2+\sqrt{n}) \|u\|_{L^{p+1}} \|\nabla v\|^{p-1}_{L^{2(p+1)}} \cdot \left\{ \int_\Omega |\nabla v|^{2(p-1)} |D^2 v|^2 \right\}^{\frac{1}{2}} \\
&\leq 2p(2p-2+\sqrt{n})(2p+\sqrt{n})^{\frac{p-1}{p+1}} M^{\frac{p-1}{p+1}} \cdot \left\{ \int_\Omega |\nabla v|^{2(p-1)} |D^2 v|^2 \right\}^{\frac{p}{p+1}} \cdot \|u\|_{L^{p+1}}
\end{aligned} \tag{3.20}$$

for all $t \in (0, T_{\max})$. Then, substituting (3.17)–(3.20) into (3.14) and using Young's inequality, we end up with

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |\nabla v|^{2p} + \frac{3p}{2} \int_\Omega |\nabla v|^{2(p-1)} |D^2 v|^2 + 2p \int_\Omega |\nabla v|^{2p} \\
\leq\ & c_5 \left(M^{\frac{2p}{p+1}} + M^{\frac{p-1}{p+1}} \right) \cdot \left\{ \int_\Omega |\nabla v|^{2(p-1)} \cdot |D^2 v|^2 \right\}^{\frac{p}{p+1}} \cdot \|u\|_{L^{p+1}} \\
\leq\ & \frac{p}{2} \int_\Omega |\nabla v|^{2(p-1)} \cdot |D^2 v|^2 + c_6 \|u\|^{p+1}_{L^{p+1}} + c_7,
\end{aligned}$$

which gives (3.13). The proof of Lemma 3.3 is complete. □

Lemma 3.4 Let (u, v, w) be the solution of system (1.4) and $m > 1$. Then, for all $p > \max\{nm, 2\}$, one has that

$$\|u(\cdot, t)\|_{L^p} + \|\nabla v(\cdot, t)\|_{L^{2p}} \leq C, \text{ for all } t \in (0, T_{\max}), \tag{3.21}$$

where C is a constant independent of t.

Proof Using Young's inequality with ε and the fact that $m > 1$, from Lemma 3.3 we can derive that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |\nabla v|^{2p} + 2p \int_\Omega |\nabla v|^{2p} + p \int_\Omega |\nabla v|^{2(p-1)} |D^2 v|^2 \\
\leq\ & c_1 \int_\Omega u^{p+1} + c_1 \leq \frac{p(p-1)\xi_1}{4(p-1+m)} \int_\Omega u^{p+m} + c_2.
\end{aligned} \tag{3.22}$$

On the other hand, using Young's inequality with ε again, from Lemma 3.2, one can derive that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega u^p + p \int_\Omega u^p + \frac{p(p-1)}{2} \int_\Omega u^{p-2} |\nabla u|^2 + \frac{p(p-1)\xi_1}{2(p-1+m)} \int_\Omega u^{p+m} \\
\leq\ & \frac{p(p-1)\chi^2}{2} \int_\Omega u^p |\nabla v|^2 + p \int_\Omega u^p + c_3 \\
\leq\ & \frac{p(p-1)\xi_1}{4(p-1+m)} \int_\Omega u^{p+m} + c_4 \int_\Omega |\nabla v|^{\frac{2(p+m)}{m}} + c_5,
\end{aligned} \tag{3.23}$$

which yields that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p + p\int_\Omega u^p + \frac{p(p-1)\xi_1}{4(p-1+m)}\int_\Omega u^{p+m} \le c_4\int_\Omega |\nabla v|^{\frac{2(p+m)}{m}} + c_5. \tag{3.24}$$

Letting $y(t) := \int_\Omega u^p + \int_\Omega |\nabla v|^{2p}$, then combining (3.24) and (3.22), and using (2.6) and the fact that $m > 1$, we have that

$$\begin{aligned} y'(t) + py(t) + p\int_\Omega |\nabla v|^{2(p-1)}|D^2 v|^2 &\le c_4\int_\Omega |\nabla v|^{\frac{2(p+m)}{m}} + c_6 \\ &\le \frac{p}{(2p+\sqrt{n})^2M^2}\int_\Omega |\nabla v|^{2(p+1)} + c_7 \\ &\le p\int_\Omega |\nabla v|^{2(p-1)}|D^2 v|^2 + c_7, \end{aligned}$$

which entails that

$$y'(t) + py(t) \le c_7. \tag{3.25}$$

Solving (3.25), and noting the definition of $y(t)$, it follows that

$$\int_\Omega u^p + \int_\Omega |\nabla v|^{2p} = y(t) \le y(0) + \frac{c_7}{p},$$

which gives (3.21). □

3.2 L^∞-estimate of u

In this subsection, we shall show the boundedness of $\|u(\cdot,t)\|_{L^\infty}$, provided that $m > 1$, based on the semigroup estimates.

Lemma 3.5 Assume that the conditions in Lemma 3.4 hold. Then there exists a constant $C_1 > 0$ independent of t such that

$$\|u(\cdot,t)\|_{L^\infty} \le C_1 \qquad \text{for all } t \in (0, T_{\max}). \tag{3.26}$$

Proof From Lemma 3.4, we can derive that there exists a $p_0 > \max\{nm, 2\}$ such that

$$\|u\|_{L^{p_0}} + \|\nabla v\|_{L^{p_0}} \le c_1. \tag{3.27}$$

Then one can use (2.3) to obtain

$$\|\nabla w\|_{W^{1,p_0}} \le c_2\|u\|_{L^{p_0}} \le c_1c_2, \tag{3.28}$$

which, together with the Sobolev theorem and the fact that $p_0 > \max\{nm, 2\} > n$, gives that

$$\|\nabla w\|_{L^\infty} \le c_3. \tag{3.29}$$

Then we can use (3.27)–(3.29), $\|v\|_{L^\infty} \le M$ and the fact that $|\Delta w| \le \sqrt{n}|D^2 w|$ to derive that

$$\|\nabla v\cdot\nabla w\|_{L^{p_0}} \le \|\nabla w\|_{L^\infty}\|\nabla v\|_{L^{p_0}} \le c_1c_3 \tag{3.30}$$

and

$$\|v\Delta w\|_{L^{p_0}} \le \|v\|_{L^\infty}\|\Delta w\|_{L^{p_0}} \le c_1c_2M\sqrt{n}. \tag{3.31}$$

Applying the variation-of-constant method to the second equation of (1.4), one has that

$$\nabla v(\cdot,t) = \nabla \mathrm{e}^{(\Delta-1)t}v_0 + \xi_2\int_0^t \nabla\mathrm{e}^{(\Delta-1)(t-s)}\left(\nabla v\cdot\nabla w + v\Delta w\right)\mathrm{d}s + \int_0^t \nabla\mathrm{e}^{(\Delta-1)(t-s)}u\mathrm{d}s,$$

which gives that

$$\begin{aligned}\|\nabla v\|_{L^\infty} &\le \|\nabla \mathrm{e}^{(\Delta-1)t} v_0\|_{L^\infty} + \xi_2 \int_0^t \|\nabla \mathrm{e}^{(\Delta-1)(t-s)} \left(\nabla v \cdot \nabla w + v\Delta w\right) \|_{L^\infty} \mathrm{d}s \\ &\quad + \int_0^t \|\nabla \mathrm{e}^{(\Delta-1)(t-s)} \ u\|_{L^\infty} \mathrm{d}s \\ &= K_1 + K_2 + K_3. \end{aligned} \tag{3.32}$$

Using the semigroup estimate (2.8), one obtains

$$K_1 = \|\nabla \mathrm{e}^{(\Delta-1)t} v_0\|_{L^\infty} \le c_4 \|\nabla v_0\|_{L^\infty}. \tag{3.33}$$

We can use semigroup estimate (2.7) and (3.30), as well as (3.31), to derive that

$$\begin{aligned} K_2 &= \xi_2 \int_0^t \|\nabla \mathrm{e}^{(\Delta-1)(t-s)} \left(\nabla v \cdot \nabla w + v\Delta w\right) \|_{L^\infty} \mathrm{d}s \\ &\le c_5 \int_0^t \left(1 + (t-s)^{-\frac{1}{2} - \frac{n}{2p_0}}\right) \mathrm{e}^{-(\lambda_1+1)(t-s)} (\|\nabla v \cdot \nabla w\|_{L^{p_0}} + \|v\Delta w\|_{L^{p_0}}) \mathrm{d}s \\ &\le \left(c_1c_3c_5 + c_1c_2c_5 M\sqrt{n}\right) \int_0^t \left(1 + (t-s)^{-\frac{1}{2} - \frac{n}{2p_0}}\right) \mathrm{e}^{-(\lambda_1+1)(t-s)} \mathrm{d}s \\ &\le c_6, \end{aligned} \tag{3.34}$$

where we have used the fact that $\int_0^t \left(1 + (t-s)^{-\frac{1}{2} - \frac{n}{2p_0}}\right) \mathrm{e}^{-(\lambda_1+1)(t-s)} \mathrm{d}s < \infty$, due to $p_0 > n$.

Similarly, using semigroup estimate (2.7) again and noting (3.27), one has that

$$K_3 \le c_7 \int_0^t \left(1 + (t-s)^{-\frac{1}{2} - \frac{n}{2p_0}}\right) \mathrm{e}^{-(\lambda_1+1)(t-s)} \|u\|_{L^{p_0}} \mathrm{d}s \le c_8. \tag{3.35}$$

Then substituting (3.33)–(3.35) into (3.32), one obtains that

$$\|\nabla v\|_{L^\infty} \le c_9. \tag{3.36}$$

Similarly, we rewrite the first equation of (1.4) as follows:

$$u_t - \Delta u + u = -\chi \nabla \cdot (u \nabla v) + \xi_1 \nabla \cdot (u^m \nabla w) + u. \tag{3.37}$$

Then, applying the variation-of-constant method to (3.37), one obtains that

$$\begin{aligned} u(x,t) &= \mathrm{e}^{(\Delta-1)t} u_0 - \chi \int_0^t \mathrm{e}^{(\Delta-1)(t-s)} \nabla \cdot (u\nabla v) \mathrm{d}s \\ &\quad + \xi_1 \int_0^t \mathrm{e}^{(\Delta-1)(t-s)} \nabla \cdot (u^m \nabla w) \mathrm{d}s + \int_0^t \mathrm{e}^{(\Delta-1)(t-s)} u \mathrm{d}s, \end{aligned}$$

which, together with the semigroup estimates, gives that

$$\begin{aligned} \|u(\cdot,t)\|_{L^\infty} &\le \|\mathrm{e}^{t(\Delta-1)} u_0\|_{L^\infty} + \chi \int_0^t \|\mathrm{e}^{(t-s)(\Delta-1)} \nabla \cdot (u\nabla v)\|_{L^\infty} \mathrm{d}s \\ &\quad + \xi_1 \int_0^t \|\mathrm{e}^{(t-s)(\Delta-1)} \nabla \cdot (u^m \nabla w)\|_{L^\infty} \mathrm{d}s + \int_0^t \|\mathrm{e}^{(t-s)(\Delta-1)} u\|_{L^\infty} \mathrm{d}s \\ &\le \mathrm{e}^{-t} \|u_0\|_{L^\infty} + c_{10} \int_0^t \left(1 + (t-s)^{-\frac{1}{2} - \frac{n}{2p_0}}\right) \mathrm{e}^{-(t-s)} \|u\nabla v\|_{L^{p_0}} \mathrm{d}s \\ &\quad + c_{11} \int_0^t \left(1 + (t-s)^{-\frac{1}{2} - \frac{nm}{2p_0}}\right) \mathrm{e}^{-(t-s)} \|u^m \nabla w\|_{L^{\frac{p_0}{m}}} \mathrm{d}s \end{aligned}$$

$$+ c_{12}\int_0^t \left(1+(t-s)^{-\frac{n}{2p_0}}\right) \mathrm{e}^{-(t-s)}\|u\|_{L^{p_0}}\mathrm{d}s. \tag{3.38}$$

On the other hand, using (3.29) and (3.36), as well as $\|u\|_{L^{p_0}} \leq c_1$ in (3.27), one can derive that

$$\|u\nabla v\|_{L^{p_0}} \leq \|\nabla v\|_{L^\infty}\|u\|_{L^{p_0}} \leq c_1c_9 \tag{3.39}$$

and

$$\|u^m\nabla w\|_{L^{p_0}} \leq \|\nabla w\|_{L^\infty}\|u^m\|_{L^{\frac{p_0}{m}}} \leq c_1^m c_3. \tag{3.40}$$

Substituting (3.39) and (3.40) into (3.38), and noting the fact that $p_0 > nm > n$, one has that

$$\begin{aligned}\|u(\cdot,t)\|_{L^\infty} \leq{}& \|u_0\|_{L^\infty} + c_9c_{10}c_1\int_0^t \left(1+(t-s)^{-\frac{1}{2}-\frac{n}{2p_0}}\right)\mathrm{e}^{-(t-s)}\mathrm{d}s\\ &+ c_3c_{11}c_1^m\int_0^t \left(1+(t-s)^{-\frac{1}{2}-\frac{nm}{2p_0}}\right)\mathrm{e}^{-(t-s)}\mathrm{d}s\\ &+ c_1c_{12}\int_0^t \left(1+(t-s)^{-\frac{n}{2p_0}}\right)\mathrm{e}^{-(t-s)}\mathrm{d}s\\ \leq{}& c_{13},\end{aligned}$$

which gives (3.26). This completes the proof of the lemma. □

Proof of Theorem 1.1 From Lemma 3.5, we can find a constant $c_1 > 0$ independent of t such that

$$\|u(\cdot,t)\|_{L^\infty} + \|v(\cdot,t)\|_{W^{1,\infty}} \leq c_1$$

for all $t \in (0, T_{\max})$, which, combined with Lemma 2.1, completes the proof of Theorem 1.1. □

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