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CONVERGENCE RATES OF SOLUTIONS FOR A TWO-SPECIES CHEMOTAXIS-NAVIER-STOKES SYTSTEM WITH COMPETITIVE KINETICS

Hai-Yang Jin

Department of Mathematics, South China University of Technology
Guangzhou 510640, China

Tian Xiang*

Institute for Mathematical Sciences, Renmin University of China
Beijing 100872, China

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Abstract. We study the convergence rates of solutions to the two-species chemotaxis-Navier-Stokes system with Lotka-Volterra competitive kinetics:

$$\begin{cases} (n_1)_t + u\cdot\nabla n_1 = \Delta n_1 - \chi_1\nabla\cdot(n_1\nabla c) + \mu_1 n_1(1-n_1-a_1n_2), \\ \qquad\qquad x\in\Omega,\ t>0, \\ (n_2)_t + u\cdot\nabla n_2 = \Delta n_2 - \chi_2\nabla\cdot(n_2\nabla c) + \mu_2 n_2(1-a_2n_1-n_2), \\ \qquad\qquad x\in\Omega,\ t>0, \\ c_t + u\cdot\nabla c = \Delta c - (\alpha n_1+\beta n_2)c, x\in\Omega,\ t>0, \\ u_t + \kappa(u\cdot\nabla)u = \Delta u + \nabla P + (\gamma n_1+\delta n_2)\nabla\phi,\quad \nabla\cdot u = 0, \\ \qquad\qquad x\in\Omega,\ t>0 \end{cases}$$

under homogeneous Neumann boundary conditions for n_1, n_2, c and no-slip boundary condition for u in a bounded domain $\Omega\subset\mathbb{R}^d (d\in\{2,3\})$ with smooth boundary. The global existence, boundedness and stabilization of solutions have been obtained in 2-D [8] and 3-D for $\kappa = 0$ and $\frac{\max\{\chi_1,\chi_2\}}{\min\{\mu_1,\mu_2\}}\|c_0\|_{L^\infty(\Omega)}$ being sufficiently small [4]. Here, we examine further convergence and derive the *explicit rates* of convergence for any supposedly given global bounded classical solution (n_1, n_2, c, u); more specifically, in L^∞-topology, we show that

$$(n_1(\cdot,t), n_2(\cdot,t), u(\cdot,t)) \xrightarrow{t\to\infty} \begin{cases} (\frac{1-a_1}{1-a_1a_2}, \frac{1-a_2}{1-a_1a_2}, 0) \text{ exponentially}, \\ \qquad\qquad \text{if } a_1, a_2\in(0,1); \\ (0,1,0) \text{ exponentially, if } a_1>1>a_2; \\ (0,1,0) \text{ algebraically, if } a_1=1>a_2; \\ (1,0,0) \text{ exponentially, if } a_2>1>a_1; \\ (1,0,0) \text{ algebraically, if } a_2=1>a_1. \end{cases}$$

In either cases, the c-solution component converges exponentially to 0.

Moreover, it is shown that only the rate of convergence for u is expressed

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* Corresponding author.

in terms of the model parameters and the first eigenvalue of in under homogeneous Dirichlet boundary conditions, and all other rates of convergence are explicitly expressed only in terms of the model parameters $a_i;\ _i;$ and and the space dimension d.

1. Introduction.

We consider the following two-species chemotaxis-fluid system with competitive terms:

$$\begin{cases} (n_1)_t + u\cdot\nabla n_1 = \ n_1 - \ _1\nabla\cdot(n_1\nabla c) + \ _1 n_1(1-n_1-a_1n_2); & x\in\ ;\ t>0;\\ (n_2)_t + u\cdot\nabla n_2 = \ n_2 - \ _2\nabla\cdot(n_2\nabla c) + \ _2 n_2(1-a_2n_1-n_2); & x\in\ ;\ t>0;\\ \quad c_t + u\cdot\nabla c = \ c - (\ n_1 + \ n_2)c; & x\in\ ;\ t>0;\\ u_t + \ (u\cdot\nabla)u = \ u + \nabla P + (\ n_1 + \ n_2)\nabla\ ;\quad \nabla\cdot u = 0; & x\in\ ;\ t>0;\\ \partial\ n_1 = \partial\ n_2 = \partial\ c = 0;\quad u = 0; & x\in\partial\ ;\ t>0;\\ n_i(x;0) = n_{i;0}(x);\ c(x;0) = c_0(x);\ u(x;0) = u_0(x); & x\in\ ;\ i=1;2; \end{cases} \tag{1.1}$$

where $\ \subset\mathbb{R}^d$ (throughout the whole paper, $d\in\{2;3\}$) is a bounded domain with smooth boundary ∂ and ∂ denotes differentiation with respect to the outward normal of $\partial\ ;\ \ \in\mathbb{R},\ \ _1;\ _2; a_1; a_2\geq 0$ and $\ _1;\ _2;\ ;\ ;\ ;\ >0$ are constants; $n_{1;0}; n_{2;0}; c_0; u_0;$ are known functions satisfying

$$0 < n_{1;0}; n_{2;0}\in C(\bar{\ });\quad 0<c_0\in W^{1;q}(\ );\quad u_0\in D(A^{\#}); \tag{1.2}$$

$$\in C^{1+}(\bar{\ }) \tag{1.3}$$

for some $q>d$, $\#\in\left(\frac{3}{4};1\right)$, $\ >0$ and A is the Stokes operator.

The system (1.1), an extension of the chemotaxis-fluid system introduced by Tuval et al. [28], depicts the evolution of two competing species which react on a single chemoattractant in a liquid surrounding environment. Here, n_1 and n_2 denote densities of species, c means the chemical concentration, and finally, u and P represent the fluid velocity field and its associated pressure. So, it is the mixed combination of the complex interaction between chemotaxis, the Lotka-Volterra kinetics and fluid.

Even in the absence of chemotaxis and fluid in (1.1), the reduced Lotka-Volterra competition system has been extensively studied. It is now well-known that its positive equilibrium $(N_1;N_2)$ with

$$N_1 := \frac{1-a_1}{1-a_1a_2};\qquad N_2 := \frac{1-a_2}{1-a_1a_2} \tag{1.4}$$

is globally asymptotically stable in weak competition case $a_1 < 1 < \frac{1}{a_2}$ [5, 6]. while, in the strong competition case $a_1 > 1 > \frac{1}{a_2}$, the positive equilibrium $(N_1;N_2)$ is unstable and the system admits non-constant steady states when the system parameters and the domain geometry are properly balanced [11, 15, 16]. In the one component chemotaxis-only context ($n_2\equiv 0\equiv u$), the global existence, boundedness and stabilization of solutions to (1.1) have also been studied widely, cf. [23, 24, 10, 14]; see also [30, 32, 34, 36] for classical Keller-Segel models with signal production instead of consumption, i.e., $-\ n_1c$ replaced with $-c+\ n_1$ in the third equation.

In one-species ($n_2\equiv 0$), global existence of weak (and/or eventual smoothness of weak solutions) and classical solutions and asymptotic behavior have been investigated, e.g., in [31, 33, 35] without logistic source ($\ _1 = 0$) and also the convergence rate has been explored [37] and in [13, 25, 27] with logistic source.

In two-species context, related studies rst begin with uid-free systems with signal production (in which the asymptotic stability usually depends on some smallness condition on the chemo-sensitivities) to understand the in uence of chemotaxis and the Lotka-Volterra kinetics [1, 2, 18, 17, 20, 19, 22]. For the two-species chemotaxis- uid system with competitive terms (1.1), the global existence, boundedness of classical solutions and stabilization to equilibria were very recently studied by Hirata et al. [8] in the 2-D setting and by Cao et al. [4] in the 3-D setting for $= 0$. For $= 1$, the global existence of weak solutions to the above system, and their eventual smoothness and stabilization were studied in [9]. We now state the bounded and classical solutions established in [8, 4] as follows:

(B2) **(Boundedness in 2-D [8])** In the case that $\subset \mathbb{R}^2$ is a bounded domain with smooth boundary, let $_1; \ _2; a_1; a_2 \geq 0$, $_1; \ _2; \ ; \ ; \ ; \ > 0$ and let (1.2) and (1.3) hold. The the IBVP (1.1) with $= 1$ possesses a unique classical solution $(n_1; n_2; c; u; P)$, up to addition of spatially constant functions to P, such that

$$\begin{aligned} &n_1; n_2 \in C(\bar{\ } \times [0;\infty)) \cap C^{2;1}(\bar{\ } \times (0;\infty));\\ &c \in C(\bar{\ } \times [0;\infty)) \cap C^{2;1}(\bar{\ } \times (0;\infty)) \cap L^\infty_{\mathrm{loc}}([0;\infty); W^{1;q}(\ ));\\ &u \in C(\bar{\ } \times [0;\infty)) \cap C^{2;1}(\bar{\ } \times (0;\infty)) \cap L^\infty_{\mathrm{loc}}([0;\infty); D(A^{\#}));\\ &P \in C^{1;0}(\bar{\ } \times (0;\infty)). \end{aligned}$$

Moreover, there exists a constant $C > 0$ such that for all $t > 0$

$$\|n_1(\cdot;t)\|_{L^1(\ )} + \|n_2(\cdot;t)\|_{L^1(\ )} + \|c(\cdot;t)\|_{W^{1;q}(\ )} + \|u(\cdot;t)\|_{L^1(\ )} \leq C. \qquad (1.5)$$

(B3) **(Boundedness in 3-D [4])** In the case that $\subset \mathbb{R}^3$ is a bounded domain with smooth boundary and that $= 0$, there exists a constant $_0 > 0$ such that whenever $\max\{\ _1; \ _2\}\|c_0\|_{L^1(\ )} < \ _0 \min\{\ _1; \ _2\}$, the statements of boundedness in (B2) hold.

(UC) **(Uniform Convergence [8, 4])** Let $(n_1; n_2; c; u; P)$ be the solution of (1.1) obtained from (B2) or (B3). Then it ful lls the following convergence properties:

(i) Assume that $a_1; a_2 \in (0;1)$. Then, with $(N_1; N_2)$ de ned in (1.4),

$$n_1(\cdot;t) \to N_1; \quad n_2(\cdot;t) \to N_2; \quad c(\cdot;t) \to 0; \quad u(\cdot;t) \to 0 \text{ in } L^\infty(\ ) \text{ as } t \to \infty;$$

(ii) Assume that $a_1 \geq 1 > a_2$. Then

$$n_1(\cdot;t) \to 0; \quad n_2(\cdot;t) \to 1; \quad c(\cdot;t) \to 0; \quad u(\cdot;t) \to 0 \text{ in } L^\infty(\ ) \text{ as } t \to \infty.$$

In this paper, we study dynamical properties for any supposedly global-in-time and bounded classical solution to (1.1), with particular focus on the model of convergence as well as their explicit rates of convergence. Before proceeding to our main results, let us observe that, the n_1- and n_2-equations in (1.1) are symmetric. Thus, we should naturally have a result about stabilization to $(0;1;0;0)$. This was not mentioned in related works, cf. [1, 8, 17]. Now, let $_P$ denote the Poincare constant, cf. (4.19), and, nally, let

$$= \frac{1}{2}\min\left\{\frac{(1-a_1a_2)\ _1\min\left\{\frac{1}{2}; \frac{a_1}{(1+a_1a_2)a_2}\right\}}{2\max\left\{\frac{1}{N_1}; \frac{a_1\ _1}{a_2\ _2N_2}\right\}}; \ (\ N_1 + \ N_2)\right\}; \qquad (1.6)$$

$$= \frac{1}{4}\min\Big\{(1-a_2)\ \ _2\min\Big\{\frac{1}{2};\frac{a_2}{1+a_2}\Big\};\ \ 2\ \ ;\ \ (a_1-1)\ \ _1\Big\}; \tag{1.7}$$

$$= \frac{1}{4}\min\{(1-a_1)\ \ _1\min\Big\{\frac{1}{2};\frac{a_1}{1+a_1}\Big\};\ \ 2\ \ ;\ \ (a_2-1)\ \ _2\}: \tag{1.8}$$

Then we are at the position to state our main results on exponential and algebraic convergence of global-in-time bounded solutions to model (1.1).

Theorem 1.1. *Let* $\subset \mathbb{R}^d (d \in \{2;3\})$ *be a bounded and smooth domain,* $= 0$ *if* $d = 3$, *and let* (1.2) *and* (1.3) *be in force, nally, let* $(n_1;n_2;c;u;P)$ *be a given global- and bounded-in time classical solution of* (1.1). *Then it enjoys the following decay properties.*

(I) When $a_1;a_2 \in (0;1)$, $(n_1;n_2;u)$ *converges exponentially to* $(N_1;N_2;0)$:

$$\begin{cases} \|n_1(\cdot;t)-N_1\|_{L^{1}(\ )} \le m_1 e^{-\frac{\ }{d+2}t}; & \forall t \ge 0;\\ \|n_2(\cdot;t)-N_2\|_{L^{1}(\ )} \le m_2 e^{-\frac{\ }{d+2}t}; & \forall t \ge 0;\\ \|u(\cdot;t)\|_{L^{1}(\ )} \le m_3 e^{-\frac{\ }{2(d+2)}\min\{\ _P;\ \}t}; & \forall t \ge 0: \end{cases} \tag{1.9}$$

(II) When $a_1 = 1 > a_2$, $(n_1;n_2;u)$ *converge algebraically to* $(0;1;0)$:

$$\begin{cases} \|n_1(\cdot;t)\|_{L^{1}(\ )} \le m_4(t+1)^{-\frac{1}{d+1}}; & \forall t \ge 0;\\ \|n_2(\cdot;t)-1\|_{L^{1}(\ )} \le m_5(t+1)^{-\frac{1}{d+2}}; & \forall t \ge 0;\\ \|u(\cdot;t)\|_{L^{1}(\ )} \le m_6(t+1)^{-\frac{\ }{d+2}}; & \forall t \ge 0: \end{cases} \tag{1.10}$$

(III) When $a_1 > 1 > a_2$, $(n_1;n_2;u)$ *converges exponentially to* $(0;1;0)$:

$$\begin{cases} \|n_1(\cdot;t)\|_{L^{1}(\ )} \le m_7 e^{-\frac{(a_1\ \ 1)\ \ _1}{2(d+1)}t}; & \forall t \ge 0;\\ \|n_2(\cdot;t)-1\|_{L^{1}(\ )} \le m_8 e^{-\frac{\ }{d+2}t}; & \forall t \ge 0;\\ \|u(\cdot;t)\|_{L^{1}(\ )} \le m_9 e^{-\frac{\ }{2(d+2)}\min\{\ _P;\ \}}; & \forall t \ge 0: \end{cases} \tag{1.11}$$

(II′) When $a_2 = 1 > a_1$, $(n_1;n_2;u)$ *converge algebraically to* $(1;0;0)$:

$$\begin{cases} \|n_1(\cdot;t)-1\|_{L^{1}(\ )} \le m_{10}(t+1)^{-\frac{1}{d+2}}; & \forall t \ge 0;\\ \|n_2(\cdot;t)\|_{L^{1}(\ )} \le m_{11}(t+1)^{-\frac{1}{d+1}}; & \forall t \ge 0;\\ \|u(\cdot;t)\|_{L^{1}(\ )} \le m_{12}(t+1)^{-\frac{\ }{d+2}}; & \forall t \ge 0: \end{cases} \tag{1.12}$$

(III′) When $a_2 > 1 > a_1$, $(n_1;n_2;u)$ *converge exponentially to* $(1;0;0)$:

$$\begin{cases} \|n_1(\cdot;t)-1\|_{L^{1}(\ )} \le m_{13} e^{-\frac{\ }{d+2}t}; & \forall t \ge 0;\\ \|n_2(\cdot;t)\|_{L^{1}(\ )} \le m_{14} e^{-\frac{(a_2\ \ 1)\ \ _2}{2(d+1)}t}; & \forall t \ge 0;\\ \|u(\cdot;t)\|_{L^{1}(\ )} \le m_{15} e^{-\frac{\ }{2(d+2)}\min\{\ _P;\ \}}; & \forall t \ge 0: \end{cases} \tag{1.13}$$

(IV) In either cases, the c*-solution component converges exponentially to* 0:

$$\|c(\cdot;t)\|_{L^{1}(\ )} \le m_{16} e^{-\frac{(\ \hat{N}_1+\ \hat{N}_2)}{2}t}; \quad \forall t \ge 0; \tag{1.14}$$

where $(\hat{N}_1;\hat{N}_2) = (N_1;N_2)$ *in Case (I),* $(\hat{N}_1;\hat{N}_2) = (0;1)$ *in cases (II) and (II′), and* $(\hat{N}_1;\hat{N}_2) = (1;0)$ *in cases (III) and (III′).*

Here, $\varepsilon \in (0,1)$ is arbitrarily given, only m_3, m_6, m_9 and m_{15} depend on ε; all $m_i (i = 1, 2, 3, \cdots, 16)$ are suitably large constants depending on the initial data $n_{1,0}, n_{2,0}, c_0, u_0, \varepsilon$, the model parameters and Sobolev embedding constants but not on time t, see Section 4. Moreover, we have the following lower estimates:

$$m_i = O(1), i = 1, 2, 7, 14; \qquad m_3 \geq O(1)(1 + (1 - a_1 a_2)^{-\frac{}{d+2}});$$

$$m_4 \geq O(1)\Big(1 + (1 - a_2)^{-\frac{1}{d+1}}\Big); \; m_5 \geq O(1)\Big(1 + (1 - a_2)^{-\frac{1}{d+2}}\Big);$$

$$m_6 \geq O(1)\Big(1 + (1 - a_2)^{-\frac{}{d+2}}\Big); \; m_8 \geq O(1)(1 + (a_1 - 1)^{-\frac{1}{d+2}} + (1 - a_2)^{-\frac{1}{d+2}});$$

$$m_9 \geq O(1)(1 + (a_1 - 1)^{-\frac{2}{d+2}} + (1 - a_2)^{-\frac{2}{d+2}}); \; m_{10} \geq O(1)\Big(1 + (1 - a_1)^{-\frac{1}{d+2}}\Big);$$

and

$$m_{11} \geq O(1)\Big(1 + (1 - a_1)^{-\frac{1}{d+1}}\Big); \; m_{13} \geq O(1)(1 + (1 - a_1)^{-\frac{1}{d+2}} + (a_2 - 1)^{-\frac{1}{d+2}});$$

as well as

$$m_{12} \geq O(1)\Big(1 + (1 - a_1)^{-\frac{}{d+2}}\Big); \quad m_{15} \geq O(1)(1 + (a_1 - 1)^{-\frac{2}{d+2}} + (1 - a_2)^{-\frac{2}{d+2}}).$$

From these estimates, we see the conditions that $1 - a_1 a_2 > 0$, $1 - a_2 > 0$, $1 - a_1 > 0$ and $a_1 > 1 > a_2$ etc are very crucial in their respective cases.

With certain regularity and dissipation properties of global bounded solutions, the idea for the proof of convergence is quite known and developed, cf. [1, 4, 8, 17, 25, 26, 27, 33] for example. The strategy for obtaining the explicit rates of convergence as stated in Theorem 1.1 consists mainly of four steps. In the first step, we present stronger regularity properties, e.g., $W^{1,\infty}$-regularity for n_i, $W^{1,p}$-regularity for u with any finite p, and $W^{2,\infty}$-regularity for c, for any bounded solution of (1.1) than those shown in [8, 4]; these are done in Section 2. In the crucial second step done in Section 3, we use refined computations to make those widely known Lyapunov functionals (cf. eg. [1, 4, 8, 17]) explicit, which will enable us to derive the explicit rates of convergence. Armed with the information provided by Step two, we then move on to calculate precisely the rates of convergence in L^1-and L^2-norm for the considered bounded solution, and related necessary estimates are also studied in great details. These constitute our Step three and are conducted in Section 4. Finally, thanks to the improved regularities, we apply the well-known Gagliardo-Nirenberg interpolation inequality to pass the obtained L^1- and L^2-convergence to the L^∞-convergence; these are our Step four and are also conducted in Section 4.

2. Regularity properties of bounded solutions.

Let (n_1, n_2, c, u, P) be a supposedly given global-in-time and bounded classical solution to (1.1) in the sense of (1.5). In this section, we provide more strong regularity properties for any such bounded solution than those shown in [8, 4], which are needed to achieve our desired rates of convergence in L^∞-norm. We start with the regularity of u and c.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^d$ be a bounded and smooth domain and $\kappa = 0$ if $d = 3$. For $d < p < \infty$, there exists a constant $C > 0$ such that*

$$\|u(\cdot, t)\|_{W^{1,p}} \leq C, \quad \forall t > 1 \tag{2.1}$$

and

$$\|c(\cdot, t)\|_{W^{1,\infty}} + \|\nabla c(\cdot, t)\|_{L^\infty} \leq C, \quad \forall t > 1. \tag{2.2}$$

Proof. Using the essentially same argument as in [33, Lemma 6.3], we obtain (2.1) for $d = 2$. When $d = 3$, we can obtain (2.1) for $k = 0$ by using the Stokes semigroup and noting the L^p-boundedness of $n_1 + n_2\nabla\phi$ for $p > 3$. The $W^{1,\infty}$-boundedness of c can be seen in [8, Lemma 3.9] and [27, Lemma 3.12]. With these and the L^∞-boundedness of n_1, n_2 and u, an direct application of the standard parabolic schauder theory (cf. [12, 21]) to the third equation in (1.1) yields (2.2). □

With the regularity properties in Lemma 2.1 at hand, we now utilize the quite commonly used arguments (cf. [26, 27]) to show the following $W^{1,\infty}$-regularity of n_1 and n_2. To this end, we first establish the following L^2-boundedness of the gradients of n_1 and n_2.

Lemma 2.2. *Let the conditions be the same as Lemma 2.1. Then there exists a constant $C > 0$ such that*

$$\|\nabla n_1(\cdot,t)\|_{L^2} + \|\nabla n_2(\cdot,t)\|_{L^2} \le C, \quad \forall t > 1. \tag{2.3}$$

Proof. Multiplying the first equation of system (1.1) by $-\Delta n_1$, integrating by parts and using the boundedness of n_1, n_2, u and (2.2), we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int |\nabla n_1|^2 + \int |\Delta n_1|^2 \\
&= \int u\cdot\nabla n_1 \Delta n_1 + \chi_1\int \nabla\cdot(n_1\nabla c)\Delta n_1 - \mu_1\int n_1(1-n_1-a_1n_2)\Delta n_1 \\
&\le \frac{1}{2}\int |\Delta n_1|^2 + c_1\int |\nabla n_1|^2 + c_2,
\end{aligned}$$

which yields

$$\frac{d}{dt}\int |\nabla n_1|^2 + \int |\Delta n_1|^2 \le 2c_1\int |\nabla n_1|^2 + 2c_2. \tag{2.4}$$

Here and below, unless otherwise stated, c_i (numbered within lemmas) or C denote some generic positive constants which may vary from line to line.

On the other hand, thanks to the boundedness of $\|n_1\|_{L^2}$ and the H^2-elliptic estimate, we apply the Gagliardo-Nirenberg inequality in $\Omega \subset \mathbb{R}^d$ to deduce

$$\begin{aligned}
2c_1\|\nabla n_1\|_{L^2}^2 &\le c_3\|D^2 n_1\|_{L^2}\|n_1\|_{L^2} + c_3\|n_1\|_{L^2}^2 \\
&\le c_4(\|\Delta n_1\|_{L^2} + \|n_1\|_{L^2})\|n_1\|_{L^2} + c_3\|n_1\|_{L^2}^2 \le \frac{2}{3}\|\Delta n_1\|_{L^2}^2 + c_5.
\end{aligned} \tag{2.5}$$

Substituting (2.5) into (2.4), one has

$$\frac{d}{dt}\|\nabla n_1\|_{L^2}^2 + c_1\|\nabla n_1\|_{L^2}^2 \le 2c_2 + \frac{3}{2}c_5,$$

which trivially gives

$$\|\nabla n_1(\cdot,t)\|_{L^2}^2 \le \|\nabla n_1(\cdot,1)\|_{L^2}^2 e^{-c_1(t-1)} + \frac{2c_2 + \frac{3}{2}c_5}{c_1} \le c_6, \qquad \forall t > 1.$$

Applying the similar arguments to the n_2-equation in (1.1), one can readily obtain that $\|\nabla n_2(\cdot,t)\|_{L^2}^2 \le c_6$ for all $t > 1$. We thus have shown the L^2-estimate (2.3). □

Lemma 2.3. *There exists a constant $C > 0$ such that*

$$\|n_1(\cdot,t)\|_{W^{1,\infty}} + \|n_2(\cdot,t)\|_{W^{1,\infty}} \le C, \quad \forall t > 1. \tag{2.6}$$

Proof. First, we show there exists a constant $c_1 > 0$ such that

$$\|n_1(\cdot,t)\|_{W^{1,1}} \le c_1, \quad \forall t > 1. \tag{2.7}$$

To this end, for any $T > 2$, we let

$$M(T) := \sup_{t\in(2,T)} \|\nabla n_1(\cdot,t)\|_{L^1}.$$

Since clearly ∇n_1 is continuous on $\overline{\Omega}\times[0,T]$, it follows that $M(T)$ is finite. Moreover, since by our universal assumption n_1 is bounded in $L^\infty(\Omega\times(0,\infty))$, to prove (2.7), it is sufficient to derive the existence of $c_2 > 0$ satisfying

$$M(T) \le c_2, \quad \forall T > 2. \tag{2.8}$$

To achieve (2.8), for any given $t \in (2,T)$, using the variation-of-constants formula to the first equation in (1.1), we get

$$\begin{aligned} n_1(\cdot,t) = e^{\Delta} n_1(\cdot,t-1) &- \chi_1\int_{t-1}^t e^{(t-s)\Delta}\,\nabla\cdot(n_1(\cdot,s)\nabla c(\cdot,s))ds \\ &- \int_{t-1}^t e^{(t-s)\Delta}\,u(\cdot,s)\cdot\nabla n_1(\cdot,s)ds \\ &+ \mu_1\int_{t-1}^t e^{(t-s)\Delta}\,n_1(\cdot,s)(1-n_1(\cdot,s)-a_1n_2(\cdot,s))ds, \end{aligned}$$

which implies

$$\begin{aligned} \|\nabla n_1(\cdot,t)\|_{L^1} \le{}& \|\nabla e^{\Delta} n_1(\cdot,t-1)\|_{L^1} + \chi_1\int_{t-1}^t \|\nabla e^{(t-s)\Delta}\,\nabla\cdot(n_1(\cdot,s)\nabla c(\cdot,s))\|_{L^1}ds \\ &+ \int_{t-1}^t \|\nabla e^{(t-s)\Delta}\,u(\cdot,s)\cdot\nabla n_1(\cdot,s)\|_{L^1}ds \\ &+ \mu_1\int_{t-1}^t \|\nabla e^{(t-s)\Delta}\,n_1(\cdot,s)(1-n_1(\cdot,s)-a_1n_2(\cdot,s))\|_{L^1}ds \\ ={}& I_1+I_2+I_3+I_4. \end{aligned} \tag{2.9}$$

Next, we shall employ the widely known smoothing L^p-L^q type estimates of the Neumann heat semigroup $\{e^{t\Delta}\}_{t\ge0}$ in Ω (see [29, 3, 7] for instance) to estimate $I_i, i = 1,2,3,4$.

Thanks to the boundedness of n_1,n_2,u in $\Omega\times(1,\infty)$, (2.1) and (2.2), we employ those smoothing Neumann heat semigroup estimates to obtain that

$$I_1 = \|\nabla e^{\Delta} n_1(\cdot,t-1)\|_{L^1} \le c_3\|n_1(\cdot,t-1)\|_{L^1} \le c_4 \tag{2.10}$$

and that

$$\begin{aligned} I_2 ={}& \chi_1\int_{t-1}^t \|\nabla e^{(t-s)\Delta}\,\nabla\cdot(n_1(\cdot,s)\nabla c(\cdot,s))\|_{L^1}ds \\ \le{}& c_5\int_{t-1}^t \left[1+(t-s)^{-\frac12-\frac{d}{2p}}\right]e^{-\lambda_1(t-s)}\|\nabla\cdot(n_1(\cdot,s)\nabla c(\cdot,s))\|_{L^p}ds \\ \le{}& c_5\int_{t-1}^t \left[1+(t-s)^{-\frac12-\frac{d}{2p}}\right]e^{-\lambda_1(t-s)}\|\nabla n_1(\cdot,s)\cdot\nabla c(\cdot,s)\|_{L^p}ds \\ &+ c_5\int_{t-1}^t \left[1+(t-s)^{-\frac12-\frac{d}{2p}}\right]e^{-\lambda_1(t-s)}\|n_1(\cdot,s)\Delta c(\cdot,s)\|_{L^p}ds \end{aligned}$$

$$\leq c_6 \int_{t-1}^{t} \Big[1 + (t-s)^{-\frac{1}{2}-\frac{d}{2p}}\Big] e^{-\lambda_1 (t-s)} \|\nabla n_1(\cdot, s)\|_{L^p} ds + c_7, \tag{2.11}$$

where $\lambda_1(>0)$ is the first nonzero eigenvalue of $-\Delta$ under homogeneous boundary condition and we have used the choice of $p > d$ to ensure the finiteness of the Gamma integral. Similarly, we can estimate I_3 as follows:

$$\begin{aligned} I_3 &= \int_{t-1}^{t} \|\nabla e^{(t-s)\Delta} u(\cdot, s) \cdot \nabla n_1(\cdot, s)\|_{L^\infty} ds \\ &\leq c_8 \int_{t-1}^{t} \Big[1 + (t-s)^{-\frac{1}{2}-\frac{d}{2p}}\Big] e^{-\lambda_1 (t-s)} \|\nabla n_1(\cdot, s)\|_{L^p} ds + c_9. \end{aligned} \tag{2.12}$$

At last, using the boundedness of n_1 and n_2 again, one has

$$\begin{aligned} I_4 &= \mu_1 \int_{t-1}^{t} \|\nabla e^{(t-s)\Delta} n_1(\cdot, s)(1 - n_1(\cdot, s) - a_1 n_2(\cdot, s))\|_{L^\infty} ds \\ &\leq c_{10} \int_{t-1}^{t} \Big[1 + (t-s)^{-\frac{1}{2}}\Big] e^{-\lambda_1 (t-s)} ds \\ &\leq c_{10} \int_{0}^{1} (1 + \sigma^{-\frac{1}{2}}) e^{-\lambda_1 \sigma} d\sigma \leq (\frac{1}{\lambda_1} + 2) c_{10}. \end{aligned} \tag{2.13}$$

Substituting (2.10), (2.11), (2.12) and (2.13) into (2.9), we infer that

$$\|\nabla n_1(\cdot, t)\|_{L^\infty} \leq c_{11} \int_{t-1}^{t} \Big[1 + (t-s)^{-\frac{1}{2}-\frac{d}{2p}}\Big] e^{-\lambda_1 (t-s)} \|\nabla n_1(\cdot, s)\|_{L^p} ds + c_{12}. \tag{2.14}$$

Then, by the L^2-boundedness of $\nabla n_1(\cdot, t)$ and $\nabla n_2(\cdot, t)$ in (2.3), the smoothness and hence boundedness of ∇n_1 on $\bar{\Omega} \times [1, 2]$ and the definition of $M(T)$, we estimate

$$\begin{aligned} \|\nabla n_1(\cdot, s)\|_{L^p} &\leq \|\nabla n_1(\cdot, s)\|_{L^\infty}^{\theta} \|\nabla n_1(\cdot, s)\|_{L^2}^{1-\theta} \\ &\leq c_{13}\big(\|\nabla n_1\|_{L^\infty(\bar{\Omega}\times[1,2])}^{\theta} + M^{\theta}(T)\big) \\ &\leq c_{14}(1 + M^{\theta}(T)), \qquad \forall s \in (1, T), \end{aligned} \tag{2.15}$$

where $\theta = \frac{p-2}{p} \in (0, 1)$ due to $p > 2$.

Finally, since $\frac{1}{2} + \frac{d}{2p} < 1$, then a substitution of (2.15) into (2.14) entails

$$M(T) \leq c_{15} M^{\theta}(T) + c_{16}, \quad \forall T > 2,$$

which upon a use of elementary inequality gives

$$M(T) \leq \max\{2c_{16}, (2c_{15})^{\frac{1}{1-\theta}}\} = \max\{2c_{16}, (2c_{15})^{\frac{p}{2}}\}, \quad \forall T > 2,$$

and hence (2.7) follows.

The argument done for n_1 can also be similarly applied to n_2 to find that

$$\|n_2(\cdot, t)\|_{W^{1,\infty}} \leq c_{17}, \quad \forall t > 1.$$

This along with (2.7) yields simply (2.6), finishing the proof of the lemma. □

3. Existence of explicit Lyapunov functionals. From boundedness to convergence, besides enough information on regularity, we still need some decaying estimates of bounded solutions under investigation. For the latter, the availability of a Lyapunov functional is crucial, see [1, 4, 8, 17] for instance. In this section, for our purpose, we particularize those known Lyapunov functionals used in those papers to obtain the explicit rates of convergence as stated in Theorem 1.1. Let us start with the case of $a_1; a_2 \in (0; 1)$. In this case, the explicit Lyapunov functional that we obtain for the chemotaxis- uid system (1.1) reads as follows:

Lemma 3.1. *De ne*

$$E_1 := \int \left(n_1 - N_1 - N_1 \log \frac{n_1}{N_1}\right) + \frac{a_1 \ _1}{a_2 \ _2} \int \left(n_2 - N_2 - N_2 \log \frac{n_2}{N_2}\right) + \frac{1}{2}\left(\frac{N_1 \ _1^2}{4} + \frac{a_1 \ _1 N_2 \ _2^2}{4a_2 \ _2} + 1\right) \int c^2 \tag{3.1}$$

and

$$F_1 := \int (n_1 - N_1)^2 + \int (n_2 - N_2)^2:$$

Then, in the case of $a_1; a_2 \in (0; 1)$, the nonnegative functions E_1 and F_1 satisfy

$$\frac{d}{dt} E_1(t) \le -(1 - a_1 a_2) \ _1 \min\left\{\frac{1}{2}; \frac{a_1}{(1 + a_1 a_2) a_2}\right\} F_1(t) =: - \ F_1(t); \quad \forall t > 0: \tag{3.2}$$

Proof. By honest di erentiation of E_1 in (3.1) and using elementary Cauchy-Schwarz inequality, one can easily derive the dissipation estimate (3.2); or alternatively, in the proof of [8, Lemma 4.1], by taking

$$k_1 = \frac{a_1 \ _1}{a_2 \ _2}; \quad l_1 = (\frac{N_1 \ _1^2}{4} + \frac{a_1 \ _1 N_2 \ _2^2}{4a_2 \ _2} + 1); \qquad = \frac{1}{2}(1 - a_1 a_2);$$

upon honest calculations, one can easily arrive at (3.2). □

For the purpose of deriving our explicit Lyapunov functionals in Cases (II)-(IV), we wish to perform honest computations here. We illustrate it for the case that $a_1 \ge 1 > a_2$. In this case, from (1.1), the fact that $a_1 \ge 1$ and the positivity of $n_1; n_2$, we calculate that

$$\begin{aligned} \frac{d}{dt} \int n_1 &= \ _1 \int (1 - n_1 - a_1 n_2) n_1 \le - \ _1 \int n_1^2 - \ _1 \int n_1 (n_2 - 1); \\ \frac{d}{dt} \int (n_2 - 1 - \log n_2) &= - \int \frac{|\nabla n_2|^2}{n_2^2} + \ _2 \int \frac{\nabla n_2}{n_2} \cdot \nabla c \\ &\quad - \ _2 \int (n_2 - 1)^2 - a_2 \ _2 \int n_1 (n_2 - 1) \end{aligned} \tag{3.3}$$

as well as

$$\frac{1}{2}\frac{d}{dt} \int c^2 = - \int |\nabla c|^2 - \int (\ n_1 + \ n_2) c^2:$$

Therefore, for any positive constants $_1; \ _2$ and $\in (0; 1)$, in view of the positivity of $n_i;$ and , a clear linear combination of the three estimates above shows

$$\begin{aligned} &- \frac{d}{dt} \int \left[n_1 + \ _1 (n_2 - 1 - \log n_2) + \frac{\ _2}{2} c^2\right] \\ &\ge \int \left[\ _1 n_1^2 + (\ _1 + a_2 \ _2 \ _1) n_1 (n_2 - 1) + \ _2 \ _1 (n_2 - 1)^2\right] \\ &\quad + \int \left[\ _1 \frac{|\nabla n_2|^2}{n_2^2} - \ _2 \ _1 \frac{\nabla n_2}{n_2} \cdot \nabla c + \ _2 |\nabla c|^2\right] \\ &= \int \left\{\left[\sqrt{\ _2 \ _1}(n_2 - 1) + \frac{(\ _1 + a_2 \ _2 \ _1)}{2\sqrt{\ _2 \ _1}} n_1\right]^2 + \left[\ _1 - \frac{(\ _1 + a_2 \ _2 \ _1)^2}{4 \ _2 \ _1}\right] n_1^2\right\} \end{aligned}$$

$$+\ {}_2\ {}_1(1-\ )\int(\ $$ [redacted] $$\Big[\Big(\sqrt{\ {}_1}\,\frac{\nabla n_2}{n_2}-\frac{{}_2\sqrt{\ {}_1}}{2}\nabla c\Big)^2$$

$$+\Big(\ {}_2-\frac{{}_2^2\ {}_1}{4}\Big)|\nabla c$$ [redacted]

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2 [redacted] $4\ \int$

With these calculations above, we obtain the next explicit decay property, which is a speci cation of [8, Lemma 4.3], see also [4, Section 4.2].

Lemma 3.2. *De ne*

$$E_2 := \int n_1 + \frac{1}{a_2\ 2}\int (n_2-1-\log n_2) + \frac{1\ \ {}_2^2}{8a_2\ \ 2}\int c^2$$

$2)n)(\ 2)2\alpha a a\ \ n$

Then, if a the *and*

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$$F_2 := \int n_1^2 + \int (n_2-1)^2.$$

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$r\ {}_1\int\ |\nabla c|^2.$

Now, thanks to $a_1 < 1$, we set

$$l_1 = \frac{a_1 \quad 1}{2}; \quad l_2 = \frac{\frac{2}{1}}{4}; \qquad " = \frac{1 + a_1}{2} \in (0;1);$$

and then we easily conclude (3.6) upon trivial computations from (3.7). □

4. **Convergence rates.** Aided by those dissipation estimates as provided in Lemmas 3.1, 3.2 and 3.3, even weaker regularity properties than those in Section 2, using the quite known arguments, cf. [1, 4, 8, 25, 26, 27, 33] for example, we know that any global-in-time and bounded classical solution of (1.1) satis es the convergence properties as follows:

$$\left\| (n_1(\cdot;t); n_2(\cdot;t); c(\cdot;t); u(\cdot;t)) - (\hat{N}_1; \hat{N}_2; 0; 0) \right\|_{L^1(\ )} \to 0 \text{ as } t \to \infty: \tag{4.1}$$

Here, $(\hat{N}_1;\hat{N}_2) = (N_1;N_2)$ when $a_1;a_2 \in (0;1)$, $(\hat{N}_1;\hat{N}_2) = (0;1)$ when $a_1 \geq 1 > a_2$, and $(\hat{N}_1;\hat{N}_2) = (1;0)$ when $a_2 \geq 1 > a_1$, where N_1 and N_2 are de ned by (1.4). In this section, we derive the explicit rates of convergence as described in Theorem 1.1 for any supposedly bounded and global-in-time classical solution to (1.1).

4.1. **Exponential convergence rate of c.** We rst take up the convergence rate of c, which is based on a parabolic comparison argument.

Lemma 4.1. *The c-solution component of any bounded solution of* (1.1) *stabilizes to zero exponentially, namely, there exists $t_0 > 1$ such that*

$$\|c(\cdot;t)\|_{L^1} \leq \|c_0\|_{L^1} e^{-\frac{\hat{N}_1 + \ \hat{N}_2}{2}(t-t_0)}; \quad \forall t \geq t_0:$$

Proof. We show the proof only for the case that $a_1;a_2 \in (0;1)$. Firstly, it follows from the convergence of (4.1), as $t \to \infty$, that $n_1(\cdot;t) \to \hat{N}_1$ and $n_2(\cdot;t) \to \hat{N}_2$ uniformly in . Henceforth, we can x a $t_0 > 1$ such that

$$\frac{\hat{N}_1}{2} \leq n_1 \leq \frac{3\hat{N}_1}{2} \quad \text{and} \quad \frac{\hat{N}_2}{2} \leq n_2 \leq \frac{3\hat{N}_2}{2} \quad \text{on } \bar{\ } \times [t_0;\infty): \tag{4.2}$$

This simply shows

$$n_1 + \ n_2 \geq \frac{N_1 + \ N_2}{2} \quad \text{on } \bar{\ } \times [t_0;\infty):$$

This along with the third equation in (1.1) and the positivity of c gives

$$c_t \leq \ c - u\cdot\nabla c - \frac{N_1 + \ N_2}{2}c \quad \text{on } \bar{\ } \times [t_0;\infty): \tag{4.3}$$

Let $z(t)$ be the solution of the following associated ODE problem:

$$\begin{cases} z'(t) + \frac{N_1 + \ N_2}{2} z(t) = 0; & t \geq t_0; \\ z(t_0) = \|c(\cdot;t_0)\|_{L^1}: \end{cases}$$

It is clear that $z(t)$ satis es (4.3) together with @ $z = 0$, and hence an application of the comparison principle and Hopf boundary point lemma immediately yields

$$c(x;t) \leq z(t) = \|c(\cdot;t_0)\|_{L^1} e^{-\frac{N_1 + \ N_2}{2}(t-t_0)} \quad \text{for all } x \in \bar{\ }; t \geq t_0:$$

Using the basic fact that $t \to \|c(\cdot;t)\|_{L^1}$ is non-increasing again by comparison principle, cf. [33, Lemma 2.1], one has

$$c(x;t) \leq \|c_0\|_{L^1} e^{-\frac{N_1 + \ N_2}{2}(t-t_0)} \quad \text{for all } x \in \bar{\ }; t \geq t_0:$$

This completes the proof of Lemma 4.1 by noting the positivity of c. □

4.2. **Exponential convergence rates in Case I:** $a_1; a_2 \in (0; 1)$. In this case, we will show that the solution components $(n_1; n_2; u)$ converge exponentially to $(N_1; N_2; 0)$.

4.2.1. *Convergence rates of n_1 and n_2 in Case I.* In this subsection, we shall establish the convergence rate of n_1 and n_2 on the basis of the convergence rate of c in Lemma 4.1 and the regularity of n_1 and n_2 provided by Lemma 2.3. We rst use Lemmas 4.1 and 3.1 to obtain the exponential convergence rates of $\|n_1 - N_1\|_{L^2}$ and $\|n_2 - N_2\|_{L^2}$.

Lemma 4.2. *The n_1- and n_2- solution components of bounded solution of (1.1) verify*

$$\|n_1(\cdot; t) - N_1\|_{L^2}^2 + \|n_2(\cdot; t) - N_2\|_{L^2}^2 \le K_1 e^{- \ (t-t_0)}; \quad \forall t \ge t_0; \tag{4.4}$$

where $K_1 = K_1(t_0) = O(1) > 0$ is de ned by

$$K_1 = \frac{9\left[E_1(t_0) + \frac{2(1-a_1a_2) \ _1| \ \ |||c_0\|_{L^1}^2 \min\left\{\frac{1}{2}; \frac{a_1}{(1+a_1a_2)a_2}\right\}\left(\frac{N_1 \ _1^2}{4} + \frac{a_2 \ _2 N_2 \ _2^2}{4a_1 \ _1} + 1\right)}{\min\left\{2(\ N_1 + \ N_2)\max\{\frac{1}{N_1}; \frac{a_1 \ _1}{a_2 \ _2 N_2}\}; \ (1-a_1a_2) \ _1 \min\{\frac{1}{2}; \frac{a_1}{(1+a_1a_2)a_2}\}\right\}e}\right]}{2\min\left\{\frac{1}{N_1}; \frac{a_1 \ _1}{a_2 \ _2 N_2}\right\}} \tag{4.5}$$

and the exponential decay rate is de ned by (1.6).

Proof. Applying Taylor's formula to the function $(z) = z - N_1 \log z$ at $z = N_1$, we obtain

$$n_1 - N_1 - N_1 \log\frac{n_1}{N_1} = (n_1) - (N_1) = \frac{''(\ )}{2}(n_1 - N_1)^2 = \frac{N_1}{2 \ ^2}(n_1 - N_1)^2 \tag{4.6}$$

for some > 0 between n_1 and N_1. Then a combination of (4.2) and (4.6) gives

$$\frac{2}{9N_1}(n_1 - N_1)^2 \le n_1 - N_1 - N_1 \log\frac{n_1}{N_1} \le \frac{2}{N_1}(n_1 - N_1)^2; \quad \forall t \ge t_0;$$

and hence

$$\frac{2}{9N_1}\int (n_1 - N_1)^2 \le \int \left(n_1 - N_1 - N_1 \log\frac{n_1}{N_1}\right) \le \frac{2}{N_1}\int (n_1 - N_1)^2; \quad \forall t \ge t_0: \tag{4.7}$$

Similarly, one gets for all $t \ge t_0$ that

$$\frac{2}{9N_2}\int (n_1 - N_2)^2 \le \int \left(n_2 - N_2 - N_2 \log\frac{n_2}{N_2}\right) \le \frac{2}{N_2}\int (n_2 - N_2)^2: \tag{4.8}$$

On the other hand, Lemma 4.1 quickly gives rise to

$$\int c^2 \le | \ \ |||c(\cdot; t)\|_{L^1}^2 \le | \ \ |||c_0\|_{L^1}^2 e^{-(\ N_1 + \ N_2)(t-t_0)}; \quad \forall t \ge t_0: \tag{4.9}$$

A substitution of (4.7), (4.8) and (4.9) into the de nition of E_1 in (3.1) gives

$$\begin{aligned} E_1(t) &\le \frac{2}{N_1}\int (n_1 - N_1)^2 + \frac{2a_1 \ _1}{a_2 \ _2 N_2}\int (n_2 - N_2)^2 + m_c e^{-(\ N_1 + \ N_2)(t-t_0)} \\ &\le F_1 + m_c e^{-(\ N_1 + \ N_2)(t-t_0)}; \quad \forall t \ge t_0; \end{aligned} \tag{4.10}$$

where

$$= 2\max\{\frac{1}{N_1};\frac{a_1 \ _1}{a_2 \ _2 N_2}\}; \quad m_c = | \quad |||c_0\|^2_{L^1}(\frac{N_1 \ ^2_1}{4}+\frac{a_2 \ _2 N_2 \ ^2_2}{4a_1 \ _1}+1): \tag{4.11}$$

Hence, from (4.10) and the dissipation estimate (3.2), we derive that

$$\frac{d}{dt}E_1 + -E_1 \le \frac{m_c}{\quad}e^{-(\ N_1+ \ N_2)(t-t_0)}; \quad \forall t \ge t_0;$$

and then, solving this Gronwall di erential inequality, we readily get

$$\begin{aligned} E_1(t) &\le E_1(t_0)e^{--(t-t_0)} + \frac{m_c}{\quad}e^{(\ N_1+ \ N_2)t_0}e^{--t}\int_{t_0}^t e^{[--(\ N_1+ \ N_2)]s}ds \\ &\le \Big[E_1(t_0) + \frac{2 \ m_c}{\min\{ \ N_1 + \ N_2; -\}e}\Big]e^{-\frac{\min f \ N_1+ \ N_2;-g}{2}(t-t_0)}; \quad \forall t \ge t_0; \end{aligned} \tag{4.12}$$

where we have used the following algebraic calculations:

$$\begin{aligned} &e^{(\ N_1+ \ N_2)t_0}e^{--t}\int_{t_0}^t e^{[--(\ N_1+ \ N_2)]s}ds \\ &= \begin{cases} (t-t_0)e^{--(t-t_0)}; & \text{if } - = (\ N_1 + \ N_2) \\ \frac{1}{--(\ N_1+ \ N_2)}\Big[e^{-(\ N_1+ \ N_2)(t-t_0)} - e^{--(t-t_0)}\Big]; & \text{if } - \ne (\ N_1 + \ N_2) \end{cases} \\ &\le (t-t_0)e^{-\min\{ \ N_1+ \ N_2;-\}(t-t_0)} \\ &\le \frac{2}{\min\{ \ N_1 + \ N_2; -\}e}e^{-\frac{\min f \ N_1+ \ N_2;-g}{2}(t-t_0)}. \end{aligned} \tag{4.13}$$

By the de nition of E_1 in (3.1) and the estimates (4.7), (4.8), we see that

$$E_1(t) \ge \frac{2}{9}\min\Big\{\frac{1}{N_1};\frac{a_1 \ _1}{a_2 \ _2N_2}\Big\}\Big[\int (n_1-N_1)^2 + \int (n_2-N_2)^2\Big]: \tag{4.14}$$

Joining (4.14) and (4.12) and substituting the de nitions of , and m_c in (3.2) and (4.11), we nally arrive at (4.4) with K_1 and given by (4.5) and (1.6), respectively. □

Thanks to the regularity provided by Lemma 2.3, we employ the well-known Gagliardo-Nirenberg interpolation inequality to pass the L^2-convergence of n_1 and n_2 in (4.4) to their L^∞-convergence.

Lemma 4.3. *Let $\subset \mathbb{R}^d$ be a bounded and smooth domain. Then the n_1- and n_2- solution components of any bounded solution of* (1.1) *decay exponentially to* $(N_1; N_2)$:

$$\|n_1(\cdot;t) - N_1\|_{L^1} + \|n_2(\cdot;t) - N_2\|_{L^1} \le Ce^{-\frac{\quad}{d+2}(t-t_0)}; \quad \forall t \ge t_0 \tag{4.15}$$

for some $C > 0$ independent of t. Here, the exponential decay rate is de ned by (1.6).

Proof. Due to the L^2-convergence of $n_1; n_2$ in (4.4) and the uniform $W^{1;\infty}$-boundedness of $n_1; n_2$ in (2.6), the Gagliardo-Nirenberg inequality enables us to conclude that

$$\begin{aligned}
&\|n_1(\cdot;t) - N_1\|_{L^\infty} + \|n_2(\cdot;t) - N_2\|_{L^\infty}\\
&\le c_1\Big(\|n_1(\cdot;t)\|_{W^{1;\infty}}^{\frac{d}{d+2}}\|n_1(\cdot;t) - N_1\|_{L^2}^{\frac{2}{d+2}} + \|n_2(\cdot;t)\|_{W^{1;\infty}}^{\frac{d}{d+2}}\|n_2(\cdot;t) - N_2\|_{L^2}^{\frac{2}{d+2}}\Big)\\
&\le c_2\Big(\|n_1(\cdot;t) - N_1\|_{L^2}^{\frac{2}{d+2}} + \|n_2(\cdot;t) - N_2\|_{L^2}^{\frac{2}{d+2}}\Big) \le c_3 e^{-\frac{\ }{d+2}(t-t_0)}; \quad \forall t \ge t_0:
\end{aligned} \tag{4.16}$$

This is nothing but the exponential decaying estimate (4.15). ☐

4.2.2. *Convergence rate of u in Case I.* With the information gained from Lemmas 4.2 and 2.1, we are now able to derive the convergence rate of u in L^∞-norm. To accomplish this goal, we again begin with its convergence rate in L^2-norm.

Lemma 4.4. *The u-solution component of* (1.1) *fulfills*

$$\|u(\cdot;t)\|_{L^2}^2 \le \Big(\|u(\cdot;t_0)\|_{L^2}^2 + \frac{2K_1\tilde{K}_1}{\min\{\ _P;\ \}e}\Big)e^{-\frac{\min\{\ _P;\ \}}{2}(t-t_0)}; \quad \forall t \ge t_0; \tag{4.17}$$

where $K_1; \tilde{K}_1$, and $_P$ are respectively defined by (4.5), (4.20), (1.6) *and* (4.19).

Proof. Recalling that $\nabla \cdot u = 0$ in and $u|_{@} = 0$, we multiply the fourth equation in (1.1) by u and integrate it over to obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int |u|^2 + \int |\nabla u|^2 &= \int (\ n_1 + \ n_2)\nabla \ \cdot u + \int |u|^2\nabla \cdot u\\
&= \int (n_1 - N_1)\nabla \ \cdot u + \int (n_2 - N_2)\nabla \ \cdot u;
\end{aligned} \tag{4.18}$$

where we also used the fact $\int \nabla \ \cdot u = 0$. Therefore, we apply the Poincare inequality:

$$_P \int |u|^2 \le \int |\nabla u|^2 \tag{4.19}$$

for the Poincare constant $_P$ to (4.18) deduce that

$$\begin{aligned}
&\frac{d}{dt}\int |u|^2 + 2\ _P\int |u|^2\\
&\le 2\ \int |n_1 - N_1||\nabla \ \cdot u| + 2\ \int |n_2 - N_2||\nabla \ \cdot u|\\
&\le \ _P\int |u|^2 + \frac{2\ ^2\|\nabla \ \|_{L^\infty}^2}{_P}\int |n_1 - N_1|^2 + \frac{2\ ^2\|\nabla \ \|_{L^\infty}^2}{_P}\int |n_2 - N_2|^2:
\end{aligned}$$

As a result, for

$$\tilde{K}_1 = 2\max\{\frac{\ ^2}{_P}; \frac{\ ^2}{_P}\}\|\nabla \ \|_{L^\infty}^2; \tag{4.20}$$

it follows that

$$\frac{d}{dt}\int |u|^2 + \ _P\int |u|^2 \le \tilde{K}_1\Big(\int |n_1 - N_1|^2 + \int |n_2 - N_2|^2\Big): \tag{4.21}$$

Substituting (4.4) into (4.21), we derive that

$$\frac{d}{dt}\int |u|^2 + \ _P\int |u|^2 \le K_1\tilde{K}_1 e^{-\ (t-t_0)}; \quad \forall t \ge t_0:$$

Solving this ODI and performing similar computations to (4.13), we readily obtain

$$\|u(\cdot;t)\|_{L^2}^2 \le \|u(\cdot;t_0)\|_{L^2}^2 e^{-\ _P(t-t_0)} + K_1 \bar{K}_1 e^{\ t_0} e^{-\ _P t}\int_{t_0}^{t} e^{(\ _P-\ )s}ds$$

$$\le\Big(\|u(\cdot;t_0)\|_{L^2}^2 + \frac{2K_1\bar{K}_1}{\min\{\ _P;\ \}e}\Big)e^{-\frac{\min\{\ _P;\ \}}{2}(t-t_0)};\quad \forall t\ge t_0;$$

which is precisely the desired exponential decay estimate (4.17). □

Lemma 4.5. *Let $\ \subset \mathbb{R}^d$ be a bounded and smooth domain. Then, for any $\ \in (0;1)$, there exists a constant*

$$\hat{K}_1 \ge O(1)(1 + (1-a_1a_2)^{-\frac{\ }{d+2}}) \tag{4.22}$$

such that

$$\|u(\cdot;t)\|_{L^1} \le \hat{K}_1 e^{-\frac{\ }{2(d+2)}\min\{\ _P;\ \}(t-t_0)};\quad \forall t\ge t_0; \tag{4.23}$$

where the exponent rate is de ned by (1.6).

Proof. Thanks to the L^2-convergence of u in (4.17) and the uniform-in-time $W^{1;p}$-boundedness of u in (2.1), the Gagliardo-Nirenberg inequality allows us to infer

$$\|u(\cdot;t)\|_{L^1} \le c_1\|u(\cdot;t)\|_{W^{1;p}}^{\frac{dp}{dp+2p-2d}}\|u(\cdot;t)\|_{L^2}^{\frac{2p-2d}{dp+2p-2d}} \le c_2 e^{-\frac{(p-d)}{dp+2p-2d}\min\{\frac{\ _P}{2};\frac{\ }{2}\}(t-t_0)};$$

$$\forall t\ge t_0;$$

which implies (4.23) upon choosing $p=[d+2(1-\ )]d/[(d+2)(1-\ )](>d)$. Noticing that $\ =O(1)(1-a_1a_2)$ by (1.6), then the lower bound for $\hat{K}_1$ in (4.22) follows from (4.17). □

4.3. **Algebraic convergence rates in Case II:** $a_1=1>a_2$. Here, we will show that the solution components $(n_1;n_2;u)$ converge at least algebraically to $(0;1;0)$.

4.3.1. *Convergence rates of n_1 and n_2 in Case II.* Again, we start with the derivation of the L^1- and L^2-convergence rates of n_1 and n_2.

Lemma 4.6. *There exists $t_1\ge\max\{1;t_0\}$ such that*

$$\|n_1(\cdot;t)\|_{L^1} + \|n_2(\cdot;t)-1\|_{L^2}^2 \le \frac{K_2}{t+t_1};\qquad \forall t\ge t_1; \tag{4.24}$$

where

$$K_2 = \frac{\max\Big\{2t_1E_2(t_1);\ \frac{k_1\left(k_1+\sqrt{k_1^2+2a}\right)}{\ }\Big\}}{\min\{1;\frac{2\ _1}{9a_2\ _2}\}} \ge O(1)(1+\frac{1}{\ }) = O(1)(1+(1-a_2)^{-1}) \tag{4.25}$$

with , $k_1;k_2$, k_3 and a de ned in (3.5), (4.29), (4.34) and (4.31), respectively.

Proof. Our proof makes use of the explicit Lyapunov functional provided by Lemma 3.2. To proceed, we rst apply the Hölder inequality to nd

$$\int n_1 \le |\ |^{\frac12}\Big(\int n_1^2\Big)^{\frac12}: \tag{4.26}$$

Next, since $\|n_2(\cdot;t)-1\|_{L^1}\to 0$ as $t\to\infty$ and

$$\lim_{z\to1}\frac{z-1-\log z}{z-1}=0;$$

we can take $\hat{t}_0 > 0$ such that $|n_2(x;t) - 1 - \log n_2(x;t)| \le |n_2(x;t) - 1|$ for all $x \in$ and $t \ge \hat{t}_0$. Accordingly, we have

$$\int (n_2 - 1 - \log n_2) \le \int |n_2 - 1| \le | \quad |^{\frac{1}{2}} \left(\int (n_2 - 1)^2 \right)^{\frac{1}{2}}; \quad \forall t \ge \hat{t}_0. \tag{4.27}$$

In this case, $(\hat{N}_1; \hat{N}_2) = (0;1)$, so the exponential decay of c in Lemma 4.1 warrants that

$$\int c^2 \le | \quad ||||c||^2_{L^1} \le | \quad ||||c_0||^2_{L^1} e^{- \quad (t-t_0)}; \quad \forall t \ge t_0. \tag{4.28}$$

From the de nitions of E_2 and F_2 in Lemma 3.2, upon a combination of (4.26), (4.27) and (4.28) and the fact that $\sqrt{A} + \sqrt{B} \le \sqrt{2(A+B)}$ for*2*

Then since $t_1 \geq t_0$, we infer from (4.8) and the definition of $E_2(t)$ in Lemma 3.2 that

$$\min\{1; \frac{2\ {}_{1}}{9a_2\ {}_{2}}\}\Big(\|n_1(\cdot;t)\|_{L^1} + \|n_2(\cdot;t) - 1\|_{L^2}^2\Big) \leq \frac{k_3}{t+t_1}; \qquad \forall t \geq t_1.$$

This, upon a substitution of the respective definitions of $k_1; k_2$, k_3 and a in (4.29), (4.34) and (4.31), proves our desired algebraic decay estimate (4.24). $\square$

Lemma 4.7. *There exist two constants K_3 and K_4 independent of t fulfilling*

$$K_3 \geq O(1)\Big(1 + (1-a_2)^{-\frac{1}{d+1}}\Big); \qquad K_4 \geq O(1)\Big(1 + (1-a_2)^{-\frac{1}{d+2}}\Big)$$

such that

$$\|n_1(\cdot;t)\|_{L^\infty} \leq \frac{K_3}{(t+t_1)^{\frac{1}{d+1}}}; \quad \forall t \geq t_1 \tag{4.35}$$

as well as

$$\|n_2(\cdot;t) - 1\|_{L^\infty} \leq \frac{K_4}{(t+t_1)^{\frac{1}{d+2}}}; \quad \forall t \geq t_1. \tag{4.36}$$

Proof. Equipped with the uniform $W^{1;\infty}$-bounds of $n_1; n_2$ in Lemma 2.3, as before, by means of the Gagliardo-Nirenberg inequality, we readily infer, for all $t \geq t_1$,

$$\|n_1(\cdot;t)\|_{L^\infty} \leq c_1\|n_1(\cdot;t)\|_{W^{1;\infty}}^{\frac{d}{d+1}}\|n_1(\cdot;t)\|_{L^1}^{\frac{1}{d+1}} \leq c_2\|n_1(\cdot;t)\|_{L^1}^{\frac{1}{d+1}} \tag{4.37}$$

and

$$\|n_2(\cdot;t) - 1\|_{L^\infty} \leq c_3\|n_2(\cdot;t)\|_{W^{1;\infty}}^{\frac{d}{d+2}}\|n_2(\cdot;t) - 1\|_{L^2}^{\frac{2}{d+2}}$$

Solving this ODI, we end up with

$$
\begin{aligned}
\|u(\cdot;t)\|_{L^2}^2 &\leq \|u(\cdot;t_1)\|_{L^2}^2 e^{-\ {}_p(t-t_1)} + K_2 \bar{K}_5 e^{-\ {}_p t} \int_{t_1}^{t} \frac{e^{\ {}_p s}}{s+t_1} ds \\
&\leq \|u(\cdot;t_1)\|_{L^2}^2 e^{-\ {}_p(t-t_1)} + \frac{K_2 \bar{K}_5 \hat{K}_5}{t+t_1} \\
&\leq \Big(\|u(\cdot;t_1)\|_{L^2}^2 e^{-\ {}_p t_1} K_5 + K_2 \bar{K}_5 \hat{K}_5 \Big)(t+t_1)^{-1}; \quad \forall t \geq t_1;
\end{aligned}
$$

from which (4.38) follows. Here, we have used the following facts

$$K_5 = \max\{(t+t_1)e^{-\ {}_p t} : \ t \geq t_1\} < \infty$$

and

$$\hat{K}_5 = \max\Big\{(t+t_1)e^{-\ {}_p t} \int_{t_1}^{t} \frac{e^{\ {}_p s}}{s+t_1} ds : \ t \geq t_1 \Big\} < \infty:$$

The latter is due to

$$\lim_{t\to\infty} \Big[(t+t_1)e^{-\ {}_p t} \int_{t_1}^{t} \frac{e^{\ {}_p s}}{s+t_1} ds \Big] = \lim_{t\to\infty} \frac{\int_{t_1}^{t} \frac{e^{\ {}_p s}}{s+t_1} ds}{(t+t_1)^{-1} e^{\ {}_p t}} = \frac{1}{p} < \infty:$$

□

With the L^2-convergence of u in Lemma 4.8 at hand, the same argument as done to Lemma 4.5 yields the following L^∞-convergence of u.

Lemma 4.9. *For any* $\ \in (0;1)$*, there exists a positive constant*

$$K_6 \geq O(1)\Big(1 + (1-a_2)^{-\frac{\ }{d+2}}\Big)$$

such that

$$\|u(\cdot;t)\|_{L^1} \leq \frac{K_6}{(t+t_1)^{\frac{\ }{d+2}}}; \qquad \forall t \geq t_1:$$

4.4. **Algebraic convergence rates in Case (II′):** $a_2 = 1 > a_1$. In this case, we shall show that $(n_1;n_2;u)$ converges at least algebraically to $(1;0;0)$. Comparing Lemmas 3.2 and 3.3 and the n_1-and n_2-equations in (1.1), we see that this subsection is fully parallel to Section 4.3, and so we simply write down their respective nal outcomes.

Lemma 4.10. *There exist* $t_2 \geq \max\{1;t_0\}$ *and positive constants* K_7 *and* K_8 *ful lling*

$$K_7 \geq O(1)\Big(1 + (1-a_1)^{-\frac{1}{d+2}}\Big); \qquad K_8 \geq O(1)\Big(1 + (1-a_1)^{-\frac{1}{d+1}}\Big)$$

such that

$$\|n_1(\cdot;t) - 1\|_{L^1} \leq \frac{K_7}{(t+t_2)^{\frac{1}{d+2}}}; \quad \forall t \geq t_2$$

and

$$\|n_2(\cdot;t)\|_{L^1} \leq \frac{K_8}{(t+t_2)^{\frac{1}{d+1}}}; \quad \forall t \geq t_2:$$

Lemma 4.11. *For any* $\ \in (0;1)$*, there exists* $C \ \geq O(1)(1 + (1-a_1)^{-\frac{\ }{d+2}})$ *such that*

$$\|u(\cdot;t)\|_{L^1} \leq \frac{C}{(t+t_2)^{\frac{\ }{d+2}}}; \qquad \forall t \geq t_2:$$

4.5. **Exponential convergence rates in Case III**: $a_1 > 1 > a_2$. In this section, we use the crucial fact that $a_1 > 1$ to show that the bounded solution components $(n_1; n_2; u)$ converge not only algebraically (as shown in Subsection 4.3) but also exponentially to $(0; 1; 0)$ and we shall compute out explicitly their respective rates of convergence. Armed with the knowledge from previous subsections, this section can be short and thus we include their stabilization results in a single lemma.

Lemma 4.12. *There exist $t_3 \geq \max\{1; t_0\}$ and $C = O(1) > 0$ such that the n_1-solution component of* (1.1) *fulfills*

$$\|n_1(\cdot; t)\|_{L^{1}} \leq Ce^{-\frac{(a_1\ 1)\ 1}{2(d+1)}(t-t_3)}; \qquad \forall t \geq t_3; \tag{4.40}$$

for some positive constant

$$K_9 \geq O(1)(1 + (a_1 - 1)^{-\frac{1}{d+2}} + (1 - a_2)^{-\frac{1}{d+2}}); \tag{4.41}$$

the n_2-solution component satisfies

$$\|n_2(\cdot; t) - 1\|_{L^{1}} \leq K_9 e^{-\frac{1}{4(d+2)}\min\{\frac{a_2\ 2}{1};\ 2\ ;\ (a_1-1)\ 1\}(t-t_3)}; \quad \forall t \geq t_3; \tag{4.42}$$

finally, for any $\in (0; 1)$, there exists

$$K_{10} \geq O(1)(1 + (a_1 - 1)^{-\frac{2}{d+2}} + (1 - a_2)^{-\frac{2}{d+2}}) \tag{4.43}$$

such that the u-solution component satisfies

$$\|u(\cdot; t)\|_{L^{1}} \leq K_{10} e^{-\frac{}{8(d+2)}\min\{\frac{a_2\ 2}{1};\ 2\ ;\ (a_1-1)\ 1;\ \ 4\ P\}(t-t_3)}; \quad \forall t \geq t_3. \tag{4.44}$$

Here, is defined by (3.5) *in Lemma* 3.2.

Proof. A simple integration of the first equation in (1.1) shows

$$\frac{d}{dt}\int n_1 = -\frac{(a_1-1)\ 1}{2}\int n_1 - \ 1\int\left[a_1 n_2 - 1 + n_1 - \frac{(a_1-1)}{2}\right]n_1. \tag{4.45}$$

The uniform convergence $n_1 \to 0$ and $n_2 \to 1$ (c.f. Subsection 4.3 or (4.1)) along with the fact $a_1 > 1$ allows us to find $t_3 \geq \max\{1; t_0\}$ such that

$$a_1 n_2 - 1 + n_1 \geq \frac{(a_1-1)}{2} \quad \text{on } \bar{\ } \times [t_3; \infty). \tag{4.46}$$

Thus, when $t > t_3$, based on (4.46) and (4.45), we derive a Gronwall inequality for $\|n_1\|_{L^1}$:

$$\frac{d}{dt}\int n_1 + \frac{(a_1-1)\ 1}{2}\int n_1 \leq 0; \qquad \forall t \geq t_3;$$

which trivially yields the exponential decay of $\|n_1\|_{L^1}$:

$$\|n_1(\cdot; t)\|_{L^1} \leq \|n_1(\cdot; t_3)\|_{L^1} e^{-\frac{(a_1\ 1)\ 1}{2}(t-t_3)}; \qquad \forall t \geq t_3. \tag{4.47}$$

Then, by the uniform $W^{1,\infty}$-bounds of n_1 in Lemma 2.3 and the Gagliardo-Nirenberg inequality, similar to (4.37), we readily obtain the exponential decay estimate (4.40).

Since $(\hat{N}_1; \hat{N}_2) = (0; 1)$, the exponential convergence of c in Lemma 4.1 entails

$$\int c^2 \leq |\ \ | \|c(\cdot; t)\|^2_{L^{1}} \leq |\ \ | \|c_0\|^2_{L^{1}} e^{-\ (t-t_0)}; \quad \forall t \geq t_3. \tag{4.48}$$

Using the essentially same argument leading to (4.8), we get

$$\frac{2}{9}\int (n_2-1)^2 \leq \int (n_2 - 1 - \log n_2) \leq 2\int (n_2-1)^2; \quad \forall t \geq t_3. \tag{4.49}$$

By (4.47), (4.48), (4.49) and the de nitions of E_2 and F_2 in Lemma 3.2, we bound

$$
\begin{aligned}
E_2 &= \int n_1 + \frac{1}{a_2 \ \ 2}\int (n_2 - 1 - \log n_2) + \frac{1 \ \ ^2_2}{8a_2 \ \ 2}\int c^2 \\
&\le \frac{2 \ \ 1}{a_2 \ \ 2}\int (n_2-1)^2 + \|n_1(\cdot, t_3)\|_{L^1} e^{-\frac{(a_1 \ \ 1) \ \ 1}{2}(t-t_3)} + \frac{1 \ \ ^2_2}{8a_2 \ \ 2}| \ \ |||c_0\|^2_{L^1} e^{- \ \ (t-t_0)} \\
&\le \frac{2 \ \ 1}{a_2 \ \ 2}F_2 + \Big[\|n_1(\cdot, t_3)\|_{L^1} + \frac{1 \ \ ^2_2}{8a_2 \ \ 2}| \ \ |||c_0\|^2_{L^1}\Big] e^{-\frac{1}{2}\min\{2 \ \ , (a_1-1) \ \ 1\}(t-t_3)} \\
&=: \frac{2 \ \ 1}{a_2 \ \ 2}F_2 + \hat{K}_9 e^{-\frac{1}{2}\min\{2 \ \ , (a_1-1) \ \ 1\}(t-t_3)}, \quad \forall t \ge t_3.
\end{aligned}
\tag{4.50}
$$

This along with the dissipation estimate (3.5) enables us to deduce that

$$
\frac{d}{dt}E_2(t) + \frac{a_2 \ \ 2}{2 \ \ 1}E_2(t) \le \frac{a_2 \ \ 2}{2 \ \ 1}\hat{K}_9 e^{-\frac{1}{2}\min\{2 \ \ , (a_1-1) \ \ 1\}(t-t_3)}, \quad \forall t \ge t_3,
$$

and then, solving this Gronwall di erential inequality, we infer, for $t \ge t_3$, that

$$
\begin{aligned}
E_2(t) &\le E_2(t_3) e^{-\frac{a_2 \ \ 2}{2 \ \ 1}(t-t_3)} \\
&\quad + \frac{a_2 \ \ 2}{2 \ \ 1}\hat{K}_9 e^{\frac{1}{2}\min\{2 \ \ , (a_1-1) \ \ 1\}t_3} e^{-\frac{a_2 \ \ 2}{2 \ \ 1}t}\int_{t_3}^t e^{[\frac{a_2 \ \ 2}{2 \ \ 1} - \frac{1}{2}\min\{2 \ \ , (a_1-1) \ \ 1\}]s}ds \\
&\le \Big[E_2(t_3) + \frac{2\hat{K}_9 \frac{a_2 \ \ 2}{1}}{\min\{\frac{a_2 \ \ 2}{1}, 2 \ \ , (a_1-1) \ \ 1\}e}\Big] e^{-\frac{1}{4}\min\{\frac{a_2 \ \ 2}{1}, 2 \ \ , (a_1-1) \ \ 1\}(t-t_3)},
\end{aligned}
\tag{4.51}
$$

where we have used the following algebraic computations similar to (4.13):

$$
\begin{aligned}
& e^{\frac{1}{2}\min\{2 \ \ , (a_1-1) \ \ 1\}t_3} e^{-\frac{a_2 \ \ 2}{2 \ \ 1}t}\int_{t_3}^t e^{[\frac{a_2 \ \ 2}{2 \ \ 1} - \frac{1}{2}\min\{2 \ \ , (a_1-1) \ \ 1\}]s}ds \\
&= \begin{cases} (t-t_3)e^{-\frac{a_2 \ \ 2}{2 \ \ 1}(t-t_3)}, & \text{if } \frac{a_2 \ \ 2}{2 \ \ 1} = \frac{1}{2}\min\{2 \ \ , (a_1-1) \ \ 1\} \\ \frac{\Big[e^{\ \ \frac{1}{2}\min f2 \ \ , (a_1 \ \ 1) \ \ 1 g(t \ \ t_3)} - e^{\ \ \frac{a_2 \ \ 2}{2 \ \ 1}(t \ \ t_3)}\Big]}{\frac{a_2 \ \ 2}{2 \ \ 1} - \frac{1}{2}\min\{2 \ \ , (a_1-1) \ \ 1\}}, & \text{if } \frac{a_2 \ \ 2}{2 \ \ 1} \ne \frac{1}{2}\min\{2 \ \ , (a_1-1) \ \ 1\} \end{cases} \\
&\le (t-t_3)e^{-\frac{1}{2}\min\{\frac{a_2 \ \ 2}{1}, 2 \ \ , (a_1-1) \ \ 1\}(t-t_3)} \\
&\le \frac{4}{\min\{\frac{a_2 \ \ 2}{1}, 2 \ \ , (a_1-1) \ \ 1\}e} e^{-\frac{1}{4}\min\{\frac{a_2 \ \ 2}{1}, 2 \ \ , (a_1-1) \ \ 1\}(t-t_3)}.
\end{aligned}
$$

From the de nition of E_2 in Lemma 3.2 or alternatively (4.50) and the estimates (4.49) and (4.51), we conclude nally that

$$
\|n_2(\cdot, t) - 1\|^2_{L^2} \le \breve{K}_9 e^{-\frac{1}{4}\min\{\frac{a_2 \ \ 2}{1}, 2 \ \ , (a_1-1) \ \ 1\}(t-t_3)}, \quad \forall t \ge t_3, \tag{4.52}
$$

where

$$
\breve{K}_9 = \frac{9a_2 \ \ 2}{2 \ \ 1}\Big[E_2(t_3) + \frac{4\hat{K}_9}{\min\{\frac{a_2 \ \ 2}{1}, 2 \ \ , (a_1-1) \ \ 1\}e}\Big]. \tag{4.53}
$$

Then, as we did in (4.16) of Lemma 4.3, we use the uniform $W^{1,\infty}$-boundedness of n_2 in (2.6) and the Gagliardo-Nirenberg inequality to improve the L^2-convergence of n_2 in (4.52) to the L^∞-convergence of n_2 in (4.42). The lower bound for K_9 in (4.41) follows from (4.53) and the fact that $\ \ = O(1)(1-a_2)$ by (3.5).

With the convergence rates of n_1 and n_2, we use the sprit of Lemma 4.4 to show rst the L^2-convergence of u. To start o , we utilize the L^1- and L^2-convergence of n_1 and n_2 in (4.47) and (4.52) to bound the counterpart of (4.21) as follows:

$$\begin{aligned}
&\frac{d}{dt}\int |u|^2 + \ _P\int |u|^2\\
&\le K_1\Big(\int n_1^2 + \int (n_2-1)^2\Big)\\
&\le K_1\|n_1\|_{L^1(\ \times(0;\infty))}\int n_1 + K_1\int (n_2-1)^2 \qquad (4.54)\\
&\le K_{10}e^{-\frac{(a_1\ \ 1)\ \ 1}{2}(t-t_3)} + K_{10}e^{-\frac{1}{4}\min\{\frac{a_2\ \ 2}{1};\ 2\ ;\ (a_1-1)\ \ 1\}(t-t_3)}\\
&\le 2K_{10}e^{-\frac{1}{4}\min\{\frac{a_2\ \ 2}{1};\ 2\ ;\ (a_1-1)\ \ 1\}(t-t_3)} =: 2K_{10}e^{-\ \ (t-t_3)};
\end{aligned}$$

where, upon notice of the fact from (3.5) that $\ = O(1)\min\{(a_1-1);(1-a_2)\}$ and the use of (4.20) and (4.53),

$$\begin{aligned}
K_{10} &= \max\Big\{K_1\|n_1\|_{L^1(\ \times(0;\infty))};\ \ K_1K_9\Big\}\\
&= O(1)\Big(1 + (\min\{(a_1-1);(1-a_2)\})^{-1}\Big):
\end{aligned} \qquad (4.55)$$

Solving the ordinary di erential inequality (4.54) and performing similar computations to the ones right after (4.51), we conclude from the fact that $\ = O(1)\min\{(a_1-1);(1-a_2)\}$ by (3.5) and (4.55) that for all $t\ge t_3$

$$\begin{aligned}
\|u(\cdot;t)\|_{L^2}^2 &\le \|u(\cdot;t_3)\|_{L^2}^2 e^{-\ \ _P(t-t_3)} + 2K_{10}e^{\ \ t_3}e^{-\ \ _Pt}\int_{t_3}^t e^{(\ _P-\ \ )s}ds\\
&\le\Big(\|u(\cdot;t_3)\|_{L^2}^2 + \frac{4K_{10}}{\min\{\ _P;\ \ \}e}\Big)e^{-\frac{\min\{\ _P;\ \}}{2}(t-t_3)} \qquad (4.56)\\
&\le O(1)\Big(1 + (a_1-1)^{-2} + (1-a_2)^{-2}\Big)e^{-\frac{\min\{\ _P;\ \}}{2}(t-t_3)}:
\end{aligned}$$

With this L^2-convergence and the uniform-in-time $W^{1;p}$-boundedness of u in (2.1), the Gagliardo-Nirenberg interpolation inequality allows us to deduce, as in Lemma 4.5, the L^∞-convergence (4.44). The lower bound for K_{10} in (4.43) can be seen from (4.56). □

4.6. **Exponential convergence rates in Case (III′):** $a_2 > 1 > a_1$. Using the Lyapunov functional provided by Lemma 3.3 and the arguments parallel to Subsection 4.5, we nd that any bounded solution $(n_1;n_2;u)$ will converge exponentially to $(1;0;0)$. We here shall omit the details and just write down their respective nal convergence results.

Lemma 4.13. *There exist $t_4\ge\max\{1;t_0\}$ and a positive constant*

$$K_{11}\ge O(1)(1 + (1-a_1)^{-\frac{1}{d+2}} + (a_2-1)^{-\frac{1}{d+2}});$$

such that the n_1-solution component of (1.1) ful lls

$$\|n_1(\cdot;t)-1\|_{L^1}\le K_{11}e^{-\frac{1}{4(d+2)}\min\{\ \ ;\ 2\ ;\ (a_2-1)\ \ 2\}(t-t_4)};\qquad \forall t\ge t_4;$$

the n_2-solution component satis es, for some $C = O(1) > 0$,

$$\|n_2(\cdot;t)\|_{L^1}\le Ce^{-\frac{(a_2\ \ 1)\ \ 2}{2(d+1)}(t-t_4)};\qquad \forall t\ge t_4;$$

nally, for any $\in (0;1)$*, there exists*

$$K_{12} \geq O(1)(1 + (a_1 - 1)^{-\frac{2}{d+2}} + (1 - a_2)^{-\frac{2}{d+2}})$$

such that the u-solution component satis es

$$\|u(\cdot;t)\|_{L^1} \leq K_{10} e^{-\frac{}{8(d+2)} \min\{\ ;\ 2\ ;\ (a_2-1)\ _2;\ \ 4\ _P\}(t-t_4)}; \quad \forall t \geq t_4.$$

Here, *is de ned by* (3.6) *in Lemma* 3.3*.*

Proof of Theorem 1.1*.* Notice that $t_i \geq \max\{t_0;1\} > 1 (i = 1;2;3;4)$; the respective decay estimates and their decay rates asserted in Theorem 1.1 follow from some lemmas in this section with perhaps some large constants m_i. More speci cally, the exponential decay estimate (1.9) follows from Lemmas 4.3 and 4.5; the algebraical decay estimate (1.10) follows from Lemmas 4.7 and 4.9; the exponential decay estimate (1.11) follows from Lemma 4.12 and the decay rate in (1.7) follows from a substitution of the de nition of in (3.5); the algebraical decay estimate (1.12) follows from Lemmas 4.10 and 4.11; the exponential decay estimate (1.13) follows from Lemma 4.13 and the decay rate in (1.8) follows from a substitution of the de nition of in (3.6); and, nally, the exponential decay estimate (1.14) follows from Lemma 4.1. □

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E-mail address: mahyjin@scut.edu.cn
E-mail address: txiang@ruc.edu.cn