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Boundedness of the attraction–repulsion Keller–Segel system



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ABSTRACT

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denoted by $u(x, t)$, $v(x, t)$ and $w(x, t)$ in the model (1.1) respectively. The positive parameters χ and ξ are called the chemosensitivity coefficients, and $\chi, \beta, \gamma, \delta > 0$ are chemical production and depredation rates. It is noted that the chemotaxis model with attractive and repulsive chemicals was also proposed in the paper [22] to interpret the quorum sensing effect in the chemotactic movement.

Proposed first by Keller and Segel [13], the classical (attractive) chemotaxis model was a system of two partial differential equations (i.e. the first two equations of (1.1) with $\xi = 0$) which possesses an apparent Lyapunov functional. This particular structure motivated a vast amount of mathematical studies in the past (see review articles [7,9,27]) and recent studies [3,8,26,28,29], where most of works were focused at whether the solution blows up or not (see some early works in [10,19,20] in this area). On the other hand, for the repulsive Keller–Segel model (i.e. the coupling of first and third equations of (1.1) with $\chi = 0$), a Lyapunov function (which was different from that of the attractive Keller–Segel model) was found in [4] which leads to the global existence of classical solutions in two dimensions and weak solutions in three and four dimensions. Compared to the classical Keller–Segel model, the three-component system of attraction–repulsion Keller–Segel (ARKS) model (1.1) is much harder to analyze due to the lack of an apparent Lyapunov functional. Since after a preliminary result on the linear stability analysis in one-dimensional space in the work [18], no progress has been made until a recent work by Tao and Wang [25] where the main contribution has three folds: (1) $\tau = 0$, the parameter regime of global boundedness and blowup of solutions were successfully identified by the Moser iteration method, which reveals the competing effect of attraction and repulsion plays a central role in determining the dynamics of solutions; (2) when $\tau = 1$ and $\beta = \delta$, numerous clean transformations were introduced to reduce the ARKS model (1.1) to the classical chemotaxis model so that the existing mathematical techniques (like Lyapunov functional) and results could be employed to derive various behaviors of solutions; (3) when $\tau = 1$ and $\beta \neq \delta$, an entropy inequality was provided to establish the time dependent global boundedness of solutions when the initial mass $\int_{\Omega} u_0 dx$ is small and repulsion prevails (i.e. $\xi\gamma - \chi\alpha > 0$).

The study of [25] leaves two evident gaps in the case of $\tau = 1$ and $\beta \neq \delta$: (a) existence of global solutions with uniform-in-time boundedness or with large data of initial value u_0 if the repulsion dominates; (b) behavior of solutions if the attraction prevails. All the past and current methods (e.g. see [10,19,20,28,29]) of proving the blowup of solutions of the attractive Keller–Segel model essentially depend on the existence of a Lyapunov functional. It appears to be hopeless at present due to the failure of finding a Lyapunov functional to establish the blowup of solutions for the case where the attraction prevails (i.e. $\xi\gamma - \chi\alpha < 0$). This paper is devoted to explore the questions left in the first gap. We specifically obtain the following main results in the present paper.

Theorem 1.1. *Assume that $0 \leq (u_0, v_0, w_0) \in [W^{1,\infty}(\Omega)]^3$ with $u_0 \not\equiv 0$ and $\xi\gamma > \chi\alpha$. Then the ARKS model (1.1) with $\tau = 1$ has a unique nonnegative classical solution*

such that

$$\nabla u, \nabla v, u \nabla v, u \nabla w \in L^1((0, T) \times \Omega)$$

and

$$\begin{aligned} \int_{\Omega} (u(t) - u_0) \phi dx + \int_0^t \int_{\Omega} \nabla u \cdot \nabla \phi dx d\tau + \int_0^t \int_{\Omega} u \nabla \phi \cdot (\xi \nabla w - \chi \nabla v) dx d\tau &= 0, \\ \int_{\Omega} (v(t) - v_0) \phi dx + \int_0^t \int_{\Omega} \nabla v \cdot \nabla \phi dx d\tau + \int_0^t \int_{\Omega} (\beta v - \alpha u) \phi dx d\tau &= 0, \\ \int_{\Omega} (w(t) - w_0) \phi dx + \int_0^t \int_{\Omega} \nabla w \cdot \nabla \phi dx d\tau + \int_0^t \int_{\Omega} (\delta w - \gamma u) \phi dx d\tau &= 0, \end{aligned} \quad (1.2)$$

for each $t \geq 0$ and $\phi \in W^{1,\infty}(\Omega)$.

The second result of this paper is concerned with the global existence of weak solutions to (1.1) in three dimensions.

Theorem 1.2. *Let Ω be a bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$. Assume $0 \leq (u_0, v_0, w_0) \in [W^{1,\infty}(\Omega)]^3$ and $\xi\gamma - \chi\alpha > 0$. Then there exists a global weak solution*

The following properties on mass can be easily derived by integrating each equation of (1.1) over Ω .

Lemma 2.2. *The solution (u, v, w) of (1.1) satisfies the following properties*

$$\begin{aligned} \|u(\cdot, t)\|_{L^1} &= \|u_0\|_{L^1}, \\ \|v(\cdot, t)\|_{L^1} &= \frac{\alpha}{\beta} \|u_0\|_{L^1} - \left(\frac{\alpha}{\beta} \|u_0\|_{L^1} - \|v_0\|_{L^1} \right) e^{-\beta t} \end{aligned} \tag{2.2}$$

Proof. Multiplying the first equation of (3.1) by $\ln u$ and integrating with respect to x over Ω yields that

$$\frac{d}{dt} \int_{\Omega} u \ln u dx + \int_{\Omega} \frac{|\nabla u|^2}{u} dx = - \int_{\Omega} \nabla u \cdot \nabla s dx. \quad (3.3)$$

Multiplying the second equation of (3.1) by $-\frac{1}{\xi\gamma - \chi\alpha} \Delta s$ and integrating over Ω , we have

$$\begin{aligned} & \frac{1}{2(\xi\gamma - \chi\alpha)} \frac{d}{dt} \int_{\Omega} |\nabla s|^2 dx + \frac{1}{\xi\gamma - \chi\alpha} \int_{\Omega} |\Delta s|^2 dx + \frac{\delta}{\xi\gamma - \chi\alpha} \int_{\Omega} |\nabla s|^2 dx \\ &= \int_{\Omega} \nabla u \cdot \nabla s dx - \frac{\chi(\beta - \delta)}{\xi\gamma - \chi\alpha} \int_{\Omega} v \Delta s dx \\ &\leq \int_{\Omega} \nabla u \cdot \nabla s dx + \frac{1}{2(\xi\gamma - \chi\alpha)} \int_{\Omega} |\Delta s|^2 dx + \frac{\chi^2(\beta - \delta)^2}{2(\xi\gamma - \chi\alpha)} \int_{\Omega} v^2 dx. \end{aligned} \quad (3.4)$$

From Lemma 2.3 and (2.2), we can find a constant $c_1 > 0$ such that $\|v\|_{W^{1,1}} \leq c_1$. Hence using the Sobolev inequality (cf. [6, p. 265]), we have

$$\|v\|_{L^2} \leq c_2 \|v\|_{W^{1,1}} \leq c_1 c_2. \quad (3.5)$$

The combination of (3.3), (3.4) and (3.5) entails

$$\begin{aligned} & \frac{d}{dt} \left\{ \int_{\Omega} u \ln u dx + \frac{1}{2(\xi\gamma - \chi\alpha)} \int_{\Omega} |\nabla s|^2 dx \right\} + \int_{\Omega} \frac{|\nabla u|^2}{u} dx \\ &+ \frac{1}{2(\xi\gamma - \chi\alpha)} \int_{\Omega} |\Delta s|^2 dx + \frac{\delta}{\xi\gamma - \chi\alpha} \int_{\Omega} |\nabla s|^2 dx \leq \frac{c_1^2 c_2^2 \chi^2 (\beta - \delta)^2}{2(\xi\gamma - \chi\alpha)}. \end{aligned} \quad (3.6)$$

Using the Gagliardo–Nirenberg inequality and the fact that $(X + Y)^d \leq 2^d(X^d + Y^d)$ for all $X \geq 0$ and $Y \geq 0$, we have

$$\|u^{\frac{1}{2}}\|_{L^3}^3 \leq c_3 (\|\nabla u^{\frac{1}{2}}\|_{L^2} \|u^{\frac{1}{2}}\|_{L^2}^2 + \|u^{\frac{1}{2}}\|_{L^2}^3) \leq 4 \|\nabla u^{\frac{1}{2}}\|_{L^2}^2 + \frac{c_3^2}{16} \|u_0\|_{L^1}^2 + c_3 \|u_0\|_{L^1}^{\frac{3}{2}}, \quad (3.7)$$

where we have used the fact that $\|u^{\frac{1}{2}}\|_{L^2} = \|u\|_{L^1}^{\frac{1}{2}} = \|u_0\|_{L^1}^{\frac{1}{2}}$. From (3.7) we can derive that

$$\begin{aligned} \int_{\Omega} u \ln u dx &\leq \int_{\Omega} u^{\frac{3}{2}} dx + c_4 = \|u^{\frac{1}{2}}\|_{L^3}^3 + c_4 \\ &\leq 4 \|\nabla u^{\frac{1}{2}}\|_{L^2}^2 + \frac{c_3^2}{16} \|u_0\|_{L^1}^2 + c_3 \|u_0\|_{L^1}^{\frac{3}{2}} + c_4 \\ &= \int_{\Omega} \frac{|\nabla u|^2}{u} dx + c_5, \end{aligned} \quad (3.8)$$

where $c_5 = \frac{c_3^2}{16} \|u_0\|_{L^1}^2 + c_3 \|u_0\|_{L^1}^{\frac{3}{2}} + c_4$. Substituting (3.8) into (3.6) and letting $c_6 = c_5 + \frac{c_1^2 c_2^2 \chi^2 (\beta - \delta)^2}{2(\xi\gamma - \chi\alpha)}$, we have

$$\frac{d}{dt} \left\{ \int_{\Omega} u \ln u dx + \frac{1}{2(\xi\gamma - \chi\alpha)} \int_{\Omega} |\nabla s|^2 dx \right\} + \int_{\Omega} u \ln u dx + \frac{\delta}{\xi\gamma - \chi\alpha} \int_{\Omega} |\nabla s|^2 dx \leq c_6. \quad (3.9)$$

Then applying the Gronwall's inequality to (3.9), we obtain (3.2). $\square$

Lemma 3.2. *Assume the conditions in Lemma 3.1 are satisfied. Then there exists a constant $C > 0$ such that*

$$\|u\|_{L^2} \leq C. \quad (3.10)$$

Proof. Multiplying the first equation of (3.1) by u , integrating the result with respect to x over Ω , we have

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} u^2 dx + \int_{\Omega} |\nabla u|^2 dx = \int_{\Omega} u^2 \Delta s dx \leq \|u\|_{L^3}^2 \|\Delta s\|_{L^3}. \quad (3.11)$$

Using the Gagliardo–Nirenberg inequality and (3.2), one has

$$\|\Delta s\|_{L^3} \leq c_2$$

where we have used

$$\int_{\Omega} u^2 dx \leq c_9 (\|\nabla u\|_{L^2} \|u\|_{L^1} + \|u\|_{L^1}^2) \leq \varepsilon \|\nabla u\|_{L^2}^2 + c_{10}. \quad (3.18)$$

Multiplying (3.15) and (3.18) by $2(\chi\alpha - \xi\gamma)^2$, (3.17) by $\chi^2(\delta - \beta)^2$, adding them to (3.16) and taking ε sufficiently small, we have

$$\frac{d}{dt} \left(2(\chi\alpha - \xi\gamma)^2 \int_{\Omega} u^2 dx + \right.$$

$$\frac{d}{dt} \int_{\Omega} u^3 dx + \int_{\Omega} |\nabla u^{\frac{3}{2}}|^2 dx \leq 3c_4. \quad (3.23)$$

Using the Gagliardo–Nirenberg inequality and the fact that $(X + Y)^d \leq 2^d(X^d + Y^d)$ for all $X \geq 0$ and $Y \geq 0$, we can derive that

$$\begin{aligned} \|u^{\frac{3}{2}}\|_{L^2}^6 &\leq [c_5(\|\nabla u^{\frac{3}{2}}\|_{L^2}^{\frac{1}{3}} \|u^{\frac{3}{2}}\|_{L^{\frac{4}{3}}}^{\frac{2}{3}} + \|u^{\frac{3}{2}}\|_{L^{\frac{4}{3}}})]^6 \\ &\leq 2^6 c_5^6 \|u^{\frac{3}{2}}\|_{L^{\frac{4}{3}}}^4 \|\nabla u^{\frac{3}{2}}\|_{L^2}^2 + 2^6 c_5^6 \|u^{\frac{3}{2}}\|_{L^{\frac{4}{3}}}^6 \\ &\leq 2^6 c_3^4 c_5^6 \|\nabla u^{\frac{3}{2}}\|_{L^2}^2 + 2^6 c_3^6 c_5^6 \\ &= \frac{1}{c_6} \|\nabla u^{\frac{3}{2}}\|_{L^2}^2 + c_7. \end{aligned} \quad (3.24)$$

Inserting (3.24) into (3.23) and using $\|u^{\frac{3}{2}}\|_{L^2}^6 = \|u\|_{L^3}^3$, we have

$$\frac{d}{dt} \|u\|_{L^3}^3 + c_6 \|u\|_{L^3}^3 \leq 3c_4 + c_6 c_7 = c_8,$$

which implies

$$\|u\|_{L^3}^3 \leq e^{-c_6 t} \|u_0\|_{L^3}^3 + \frac{c_8}{c_6} = c_9. \quad (3.25)$$

Then using Lemma 2.3 and (3.25), we obtain $\|(v, w)\|_{W^{1,\infty}} \leq c_{10}$ which implies (3.20). Then the

$$\frac{d}{dt} \int_{\Omega} u^p dx + p(p-1) \int_{\Omega} u^p dx \leq -\frac{2(p-1)}{p} \int_{\Omega} |\nabla u^{\frac{p}{2}}|^2 dx + c_2 p(p-1) \int_{\Omega} u^p dx \quad (3.27)$$

for all $t \in (0, T_{max})$ and for all $p \geq 2$, where $c_2 = 1 + \frac{c_1^2(\chi+\xi)^2}{2}$. Then it follows from (3.27) and the well-known Moser–Alikakos iteration procedure (cf. [1], [25, Lemma 4.1]) that there exists a constant $c_3 > 0$ such that

$$\|u(\cdot, t)\|_{L^\infty} \leq c_3 \quad \text{for all } t \in (0, T). \quad (3.28)$$

Theorem 1.1 is an immediate consequence of (3.28) and Lemma 2.1. $\square$

4. Proof of Theorem 1.2

In this section, we are devoted to proving Theorem 1.2 by constructing a global weak solution as the limit of global classical solution of appropriately ‘regularized’ problems. By using a similar argument as in [4], we consider the following approximate system of (3.1) for each $\varepsilon \geq 0$

$$\begin{cases} u_t^\varepsilon = \Delta u^\varepsilon + \nabla \cdot (u^\varepsilon (1 - \varepsilon u^\varepsilon) \nabla s^\varepsilon), & x \in \Omega, \ t > 0, \\ s_t^\varepsilon = \Delta s^\varepsilon - \delta s^\varepsilon + (\xi \gamma - \chi \alpha) u^\varepsilon + \chi(\beta - \delta) v^\varepsilon, & x \in \Omega, \ t > 0, \\ v_t^\varepsilon = \Delta v^\varepsilon + \alpha u^\varepsilon - \beta v^\varepsilon, & x \in \Omega, \ t > 0, \\ \frac{\partial u^\varepsilon}{\partial \nu} = \frac{\partial s^\varepsilon}{\partial \nu} = \frac{\partial v^\varepsilon}{\partial \nu} = 0, & x \in \partial\Omega, \ t > 0, \\ u^\varepsilon(x, 0) = u_0(x), \quad s^\varepsilon(x, 0) = \xi w_0(x) - \chi v_0(x) =: s_0(x), \quad v^\varepsilon(x, 0) = v_0(x), & x \in \Omega. \end{cases} \quad (4.1)$$

Observe that system (3.1) is obtained by taking $\varepsilon = 0$ in the approximate system (4.1). We have the following proposition for the approximate system (4.1).

Proposition 4.1. *Assume $0 \leq (u_0, v_0, w_0) \in [W^{1,\infty}(\Omega)]^3$. Then the system (4.1) has a local unique classical solution*

$$(u^\varepsilon, s^\varepsilon, v^\varepsilon) \in C(\bar{\Omega} \times [0, T_{max}^\varepsilon]; \mathbb{R}^3) \cap C^{2,1}(\bar{\Omega} \times (0, T_{max}^\varepsilon); \mathbb{R}^3)$$

and $u^\varepsilon(x, t), v^\varepsilon(x, t) \geq 0$ for each $(x, t) \in \bar{\Omega} \times [0, T_{max}^\varepsilon)$, T_{max}^ε denoting the maximal existence time. Furthermore, if there is a function $\omega : (0, \infty) \rightarrow (0, \infty)$ such that for each $T > 0$

$$\|(u^\varepsilon, s^\varepsilon, v^\varepsilon)(t)\|_{L^\infty} \leq \omega(T), \quad 0 < t < \min\{T, T_{max}^\varepsilon\},$$

then $T_{max}^\varepsilon = \infty$. In particular, if $\varepsilon \in (0, \varepsilon_0]$ with $\frac{1}{\varepsilon_0} = \max\{\|u_0\|_{L^\infty}, \frac{\beta}{\alpha}\|v_0\|_{L^\infty}\}$ then $0 \leq u^\varepsilon \leq \frac{1}{\varepsilon}$, $0 \leq v^\varepsilon \leq \frac{\alpha}{\beta\varepsilon}$ and $\|s^\varepsilon\|_{L^\infty} \leq \frac{C}{\varepsilon}$, hence $T_{max}^\varepsilon = +\infty$.

Remark 4.1. We should emphasize that the initial data $s_0(x) = \xi w_0(x) - \chi v_0(x)$ might be negative for our present setting.

Proof. First, we can apply [2, Theorem 14.6] to conclude the system (4.1) has a local unique classical solution for $\varepsilon \geq 0$. The non-negativity of u^ε and v^ε by using [2, Theorem 15.1] and the standard maximum principle for parabolic equations. The global existence criterion can be deduced from [2, Theorem 15.5]. At last, if $\varepsilon \in (0, \varepsilon_0]$ then $\frac{1}{\varepsilon} \geq \frac{1}{\varepsilon_0}$ hence $p^\varepsilon(x, 0) = -u_0 + \frac{1}{\varepsilon} \geq 0$. Substituting $p^\varepsilon = -u^\varepsilon + \frac{1}{\varepsilon}$ into the first equation of system (4.1), one has

$$p_t^\varepsilon = \Delta p^\varepsilon + \nabla \cdot (p^\varepsilon (1 - \varepsilon p^\varepsilon) \nabla s^\varepsilon).$$

Moreover, we use [2, Theorem 15.1] to obtain $p^\varepsilon \geq 0$ which yields $u^\varepsilon \leq \frac{1}{\varepsilon}$. Similarly, substituting $h^\varepsilon = -v^\varepsilon + \frac{\alpha}{\beta\varepsilon}$ into the third equation of system (4.1) yields that

$$h_t^\varepsilon = \Delta h^\varepsilon - \beta h^\varepsilon + \alpha p^\varepsilon.$$

Using the classical maximum principle and $p^\varepsilon \geq 0$ entails that $v^\varepsilon \leq \frac{\alpha}{\beta\varepsilon}$. Letting $q^\varepsilon = (\xi\gamma - \chi\alpha)u^\varepsilon + \chi(\beta - \delta)v^\varepsilon$, we have $\|q^\varepsilon\|_{L^\infty} \leq \frac{C}{\varepsilon}$. Noting that s^ε solves

$$\begin{cases} s_t^\varepsilon - \Delta s^\varepsilon + \delta s^\varepsilon = q^\varepsilon, & x \in \Omega, \ t > 0, \\ \frac{\partial s^\varepsilon}{\partial \nu} = 0, & x \in \partial\Omega, \ t > 0, \\ s^\varepsilon(x, 0) = \xi w_0(x) - \chi v_0(x) =: s_0(x), & x \in \Omega. \end{cases}$$

By a simple energy estimate we have $\|s^\varepsilon\|_{L^\infty} \leq \frac{C}{\varepsilon}$. Then we complete the proof of Proposition 4.1. $\square$

Next, we will derive some ε -independent estimates for these solutions obtained in Proposition 4.1. First, we derive the following useful inequality which is the core of the argument concerning the global existence of weak solutions.

Lemma 4.2. *Let $n = 3$ and $\xi\gamma > \chi\alpha$. For each $T_0 > 0$, there exists $C(T) > 0$, which may depend on $T := \min\{T_0, T_{max}^\varepsilon\}$ such that for any $\varepsilon \in (0, \varepsilon_0]$ the solution of (4.1) satisfies*

$$\int_0^t \int_\Omega \frac{|\nabla u^\varepsilon|^2}{u^\varepsilon} dx d\tau \leq C(T) \quad \text{for all } t \in (0, T) \quad (4.2)$$

and

$$\int_\Omega |\nabla s^\varepsilon|^2 dx + \int_0^t \int_\Omega |\Delta s^\varepsilon|^2 dx d\tau + 2\delta \int_0^t \int_\Omega |\nabla s^\varepsilon|^2 dx d\tau \leq C(T) \quad \text{for all } t \in (0, T). \quad (4.3)$$

Proof. Multiplying the first equation of (4.1) by $\ln u^\varepsilon - \ln(1 - \varepsilon u^\varepsilon)$ and the second by $-\frac{1}{\xi\gamma - \chi\alpha} \Delta s^\varepsilon$, and adding them, we end up with the following equation after integrating the result with respect to x over Ω

$$\begin{aligned} & \frac{d}{dt} \left\{ \int_\Omega \left(u^\varepsilon \ln u^\varepsilon + \frac{1}{\varepsilon} (1 - \varepsilon u^\varepsilon) \ln(1 - \varepsilon u^\varepsilon) \right) dx + \frac{1}{2(\xi\gamma - \chi\alpha)} \int_\Omega |\nabla s^\varepsilon|^2 dx \right\} \\ & + \int_\Omega \frac{|\nabla u^\varepsilon|^2}{u^\varepsilon(1 - \varepsilon u^\varepsilon)} dx + \frac{1}{(\xi\gamma - \chi\alpha)} \int_\Omega |\Delta s^\varepsilon|^2 dx + \frac{\delta}{(\xi\gamma - \chi\alpha)} \int_\Omega |\nabla s^\varepsilon|^2 dx \\ & = -\frac{\chi(\beta - \delta)}{\xi\gamma - \chi\alpha} \int_\Omega v^\varepsilon \Delta s^\varepsilon dx \leq \frac{1}{2(\xi\gamma - \chi\alpha)} \int_\Omega |\Delta s^\varepsilon|^2 dx + \frac{\chi^2(\beta - \delta)^2}{\xi\gamma - \chi\alpha} \int_\Omega (v^\varepsilon)^2 dx. \end{aligned} \quad (4.4)$$

Since $u^\varepsilon \in L^1(\Omega)$, then from Lemma 2.3, we can find a constant $c_1 > 0$ such that $\|v^\varepsilon\|_{W^{1, \frac{6}{5}}} \leq c_1$. Hence using the Sobolev inequality (cf. [6, p. 265]), we have

$$\|v^\varepsilon\|_{L^2} \leq c_2 \|v^\varepsilon\|_{W^{1, \frac{6}{5}}} \leq c_1 c_2. \quad (4.5)$$

Substituting (4.5) into (4.4), we can find a constant $c_3 > 0$ such that

$$\begin{aligned} & \frac{d}{dt} \left\{ \int_{\Omega} \left(u^{\varepsilon} \ln u^{\varepsilon} + \frac{1}{\varepsilon} (1 - \varepsilon u^{\varepsilon}) \ln(1 - \varepsilon u^{\varepsilon}) \right) dx + \frac{1}{2(\xi\gamma - \chi\alpha)} \int_{\Omega} |\nabla s^{\varepsilon}|^2 dx \right\} \\ & + \int_{\Omega} \frac{|\nabla u|^2}{u^{\varepsilon}(1 - \varepsilon u^{\varepsilon})} dx + \frac{1}{2(\xi\gamma - \chi\alpha)} \int_{\Omega} |\Delta s^{\varepsilon}|^2 dx + \frac{\delta}{(\xi\gamma - \chi\alpha)} \int_{\Omega} |\nabla s^{\varepsilon}|^2 dx \leq c_3. \end{aligned} \quad (4.6)$$

Since $r + \frac{1}{\varepsilon}(1 - \varepsilon r) \ln(1 - \varepsilon r) \geq 0$ for $r \in [0, \frac{1}{\varepsilon}]$ and $r \ln r \geq -\frac{1}{e}$ for $r \in [0, 1]$, we have

$$\begin{aligned} & \int_{\Omega} \left(u^{\varepsilon} \ln u^{\varepsilon} + \frac{1}{\varepsilon} (1 - \varepsilon u^{\varepsilon}) \ln(1 - \varepsilon u^{\varepsilon}) \right) dx \\ & = \int_{\Omega} \left(u^{\varepsilon} \ln u^{\varepsilon} - u^{\varepsilon} + u^{\varepsilon} + \frac{1}{\varepsilon} (1 - \varepsilon u^{\varepsilon}) \ln(1 - \varepsilon u^{\varepsilon}) \right) dx \\ & \geq \int_{\Omega} (u^{\varepsilon} \ln u^{\varepsilon} - u^{\varepsilon}) dx \geq \int_{\Omega} u^{\varepsilon} |\ln u^{\varepsilon}| dx - \left(\|u_0\|_{L^1} + \frac{|\Omega|}{e} \right). \end{aligned} \quad (4.7)$$

Furthermore, we have

$$\frac{|\nabla u^{\varepsilon}|^2}{u^{\varepsilon}(1 - \varepsilon u^{\varepsilon})} \geq \frac{|\nabla u^{\varepsilon}|^2}{u^{\varepsilon}}. \quad (4.8)$$

Integrating (4.4) with respect to t and using (4.7) and (4.8) to obtain

$$\begin{aligned} & \int_{\Omega} u^{\varepsilon} |\ln u^{\varepsilon}| dx + \frac{1}{2(\xi\gamma - \chi\alpha)} \int_{\Omega} |\nabla s^{\varepsilon}|^2 dx + \int_0^t \int_{\Omega} \frac{|\nabla u^{\varepsilon}|^2}{u^{\varepsilon}} dx d\tau \\ & + \frac{1}{2(\xi\gamma - \chi\alpha)} \int_0^t \int_{\Omega} |\Delta s^{\varepsilon}|^2 dx d\tau + \frac{\delta}{(\xi\gamma - \chi\alpha)} \int_0^t \int_{\Omega} |\nabla s^{\varepsilon}|^2 dx d\tau \leq C(T), \end{aligned} \quad (4.9)$$

which entails (4.2) and (4.3). $\square$

Next, we prove that at least a subsequence of the classical solutions $(u^{\varepsilon}, s^{\varepsilon}, v^{\varepsilon})$ to (4.1) converge in suitable topologies towards a (weak) solution to (1.1) as $\varepsilon \rightarrow 0$. First we deduce some bounds on $(u^{\varepsilon}, s^{\varepsilon}, v^{\varepsilon})$.

Lemma 4.3. *Let $n = 3$ and $\varepsilon \in (0, \varepsilon_0]$. For $T > 0$, the sequences $(u^{\varepsilon}, s^{\varepsilon}, v^{\varepsilon})_{\varepsilon}$ have the following properties:*

$$(u^{\varepsilon})_{\varepsilon} \text{ is bounded in } L^{\frac{5}{4}}((0, T); W^{1, \frac{5}{4}}(\Omega)), \quad (4.10)$$

$$(u_t^{\varepsilon})_{\varepsilon} \text{ is bounded in } L^1((0, T); C_0^1(\Omega)'), \quad (4.11)$$

$$(v^{\varepsilon})_{\varepsilon} \text{ is bounded in } L^{\frac{5}{3}}((0, T); W^{2, \frac{5}{3}}(\Omega)), \quad (4.12)$$

$$(v_t^{\varepsilon})_{\varepsilon} \text{ is bounded in } L^{\frac{5}{3}}(\Omega \times (0, T)), \quad (4.13)$$

$$(s^{\varepsilon})_{\varepsilon} \text{ is bounded in } L^{\infty}((0, T); W^{1,2}(\Omega)) \cap L^2((0, T); W^{2,2}(\Omega)), \quad (4.14)$$

$$(s_t^{\varepsilon})_{\varepsilon} \text{ is bounded in } L^{\frac{5}{3}}(\Omega \times (0, T)), \quad (4.15)$$

$$(u^{\varepsilon} \nabla s^{\varepsilon})_{\varepsilon} \text{ is bounded in } L^{\frac{10}{9}}(\Omega \times (0, T)). \quad (4.16)$$

Proof. First, we can obtain (4.14) from (4.3) directly. Next, we will show (4.10) and (4.11) hold. First, integrating the first equation of (4.1) and using the boundary conditions, one has

$$\|u^\varepsilon(t)\|_{L^1} \leq C \quad \text{for } t \in (0, T). \quad (4.17)$$

Using (4.2) and (4.17), we have

$$\int_0^T \|\sqrt{u^\varepsilon(t)}\|_{W^{1,2}}^2 dt = \int_0^T \|u^\varepsilon(t)\|_{L^1} dt + \frac{1}{4} \int_0^T \int_\Omega \frac{|\nabla u^\varepsilon|^2}{u^\varepsilon} dx dt \leq C(T), \quad (4.18)$$

which implies

$$\int_0^T \|u^\varepsilon(t)\|_{L^3} dt \leq C(T), \quad (4.19)$$

by using the continuous embedding of $W^{1,2}(\Omega)$ in $L^6(\Omega)$. Applying the interpolating inequality, using (4.17) and (4.19), we have

$$\int_0^T \|u^\varepsilon(t)\|_{L^{\frac{5}{3}}}^{\frac{5}{3}} dt \leq \int_0^T \|u^\varepsilon(t)\|_{L^1}^{\frac{2}{3}} \|u^\varepsilon(t)\|_{L^3} dt \leq C(T). \quad (4.20)$$

Using (4.2) and (4.20), one has

$$\begin{aligned} \int_0^T \int_\Omega |\nabla u^\varepsilon|^{\frac{5}{4}} dx dt &= \int_0^T \int_\Omega \left(\frac{|\nabla u^\varepsilon|}{(u^\varepsilon)^{1/2}} \right)^{\frac{5}{4}} (u^\varepsilon)^{\frac{5}{8}} dx dt \\ &\leq \int_0^T \left(\int_\Omega \frac{|\nabla u^\varepsilon|^2}{u^\varepsilon} dx \right)^{\frac{5}{8}} \left(\int_\Omega (u^\varepsilon)^{\frac{5}{3}} dx \right)^{\frac{3}{8}} dt \leq C(T). \end{aligned} \quad (4.21)$$

Then the combination of (4.20) and (4.21) entails (4.10). Furthermore, for any $\phi \in C_0^1(\Omega)$, we have

$$\begin{aligned} \left| \int_\Omega u_t^\varepsilon \phi dx \right| &\leq \int_\Omega |\nabla u^\varepsilon| |\nabla \phi| dx + \int_\Omega u^\varepsilon (1 - \varepsilon u^\varepsilon) |\nabla s^\varepsilon| |\nabla \phi| dx \\ &\leq \|\nabla \phi\|_{L^\infty} \left(\int_\Omega \frac{|\nabla u^\varepsilon|}{\sqrt{u^\varepsilon}} \sqrt{u^\varepsilon} dx + \|u^\varepsilon\|_{L^2} \|\nabla s^\varepsilon\|_{L^2} \right) \\ &\leq C(T) \left(\int_\Omega \frac{|\nabla u^\varepsilon|^2}{u^\varepsilon} dx + \|u^\varepsilon\|_{L^2} \right) \|\nabla \phi\|_{L^\infty}, \end{aligned}$$

where we have used $\|\nabla s^\varepsilon\|_{L^2} \leq C(T)$ (see (4.3)) and (4.17). Therefore we obtain

$$\|u_t^\varepsilon\|_{C_0^1(\Omega)'} \leq C(T) \left(\int_\Omega \frac{|\nabla u^\varepsilon|^2}{u^\varepsilon} dx + \|u^\varepsilon\|_{L^2} \right) \leq C(T) \left(\int_\Omega \frac{|\nabla u^\varepsilon|^2}{u^\varepsilon} dx + \|u^\varepsilon\|_{L^3} \right)$$

and the right side of the above inequality is bounded in $L^1(0, T)$ by (4.2) and (4.19). Hence, we complete the proof of (4.11).

We apply the classical parabolic regularity theory to the third equation of (4.1) and use (4.20) to obtain (4.12) and (4.13). Similarly, applying the classical parabolic regularity theory to the second equation of (4.1) and using (4.20) and (4.12), one has (4.15).

At last, we will show the key estimate (4.16) holds. Using the Sobolev embedding theorem and (4.14), we can obtain that $(\nabla s^\varepsilon)_\varepsilon$ is bounded in $L^2((0, T); L^6(\Omega))$ and $L^\infty((0, T); L^2(\Omega))$, which yields

$$\int_0^T \|\nabla s^\varepsilon\|_{L^{\frac{10}{3}}}^{\frac{10}{3}} dt \leq \int_0^T \|\nabla s^\varepsilon\|_{L^2}^{\frac{4}{3}} \|\nabla s^\varepsilon\|_{L^6}^2 dt \leq C(T), \quad (4.22)$$

where we have used the interpolating inequality

$$\|\nabla s^\varepsilon\|_{L^{\frac{10}{3}}} \leq \|\nabla s^\varepsilon\|_{L^2}^{\frac{2}{5}} \|\nabla s^\varepsilon\|_{L^6}^{\frac{3}{5}}.$$

The combination (4.20) and (4.22) with the Hölder inequality gives us that

$$\begin{aligned} \int_0^T \int_\Omega |\nabla s^\varepsilon u^\varepsilon|^{\frac{10}{9}} dx dt &\leq \int_0^T \left(\int_\Omega |\nabla s^\varepsilon|^{\frac{10}{3}} dx \right)^{\frac{1}{3}} \left(\int_\Omega (u^\varepsilon)^{\frac{5}{3}} dx \right)^{\frac{2}{3}} dt \\ &\leq \left(\int_0^T \int_\Omega |\nabla s^\varepsilon|^{\frac{10}{3}} dx dt \right)^{\frac{1}{3}} \left(\int_0^T \int_\Omega (u^\varepsilon)^{\frac{5}{3}} dx dt \right)^{\frac{2}{3}} \\ &\leq C(T), \end{aligned}$$

which yields (4.16). Then the proof of Lemma 4.3 is completed. $\square$

Now, we consider the relative compactness of the sequences $(u^\varepsilon, s^\varepsilon, v^\varepsilon)_\varepsilon$.

Lemma 4.4. *There are functions (u, s, v)*

$$\begin{aligned} u &\in L^{\frac{5}{4}}((0, T); W^{1, \frac{5}{4}}(\Omega)) \cap C((0, T); C_0^1(\Omega)'), \\ s &\in L^\infty((0, T); W^{1, 2}(\Omega)) \cap C((0, T); L^2(\Omega)) \cap L^2((0, T); W^{2, 2}(\Omega)), \\ v &\in L^{\frac{5}{3}}((0, T); W^{2, \frac{5}{3}}(\Omega)), \end{aligned}$$

and a sequence of $(u^\varepsilon)_\varepsilon$, $(s^\varepsilon)_\varepsilon$ and $(v^\varepsilon)_\varepsilon$ such that

$$\begin{aligned} u^\varepsilon &\rightarrow u \text{ in } L^{\frac{5}{4}}(\Omega \times (0, T)) \cap C((0, T); C_0^1(\Omega)'), \\ s^\varepsilon &\rightarrow s \text{ in } L^2((0, T); W^{1, 2}(\Omega)) \cap C((0, T); L^2(\Omega)), \\ v^\varepsilon &\rightarrow v \text{ in } L^{\frac{5}{3}}((0, T); W^{1, \frac{5}{3}}(\Omega)), \end{aligned} \quad (4.23)$$

and

$$\begin{aligned} \int_\Omega (s(t) - s_0) \phi dx + \int_0^t \int_\Omega (\nabla s \cdot \nabla \phi + [\delta s - (\xi \gamma - \chi \alpha) u - \chi(\beta - \delta) v] \phi) dx d\tau &= 0, \\ \int_\Omega (v(t) - v_0) \phi dx + \int_0^t \int_\Omega [\nabla v \cdot \nabla \phi + (\beta v - \alpha u) \phi] dx d\tau &= 0 \end{aligned} \quad (4.24)$$

for each $t \in [0, T]$ and $\phi \in W^{1, \infty}(\Omega)$.

Proof. The proof of this lemma closely follows an argument in [4, Lemma 4.2]. Noting that (4.11), (4.17) and Ascoli theorem entail that

$$(u^\varepsilon)_\varepsilon \text{ is relatively compact in } C((0, T); C_0^1(\Omega)') . \quad (4.25)$$

Combing (4.10) and (4.11), and using Aubin–Lions Lemma [15, Théorème 5.1] gives

$$(u^\varepsilon)_\varepsilon \text{ is relatively compact in } L^{\frac{5}{4}}(\Omega \times (0, T)) . \quad (4.26)$$

Similarly, using (4.12), (4.13) and Aubin–Lions Lemma, we have

$$(v^\varepsilon)_\varepsilon \text{ is relatively compact in } L^{\frac{5}{3}}((0, T); W^{1, \frac{5}{3}}(\Omega)) . \quad (4.27)$$

Furthermore, it follows from (4.14), (4.15) and [24, Corollary 4] that

$$(s^\varepsilon)_\varepsilon \text{ is relatively compact in } L^2((0, T); W^{1,2}(\Omega)) \cap C((0, T); L^2(\Omega)) . \quad (4.28)$$

Then one can obtain (4.23) from (4.25)–(4.28). Furthermore, for all $\phi \in W^{1,\infty}(\Omega)$, we have the following identities from the second equation and the third equation of (4.1)

$$\begin{aligned} \int_{\Omega} (s^\varepsilon(t) - s_0^\varepsilon) \phi dx + \int_0^t \int_{\Omega} (\nabla s^\varepsilon \cdot \nabla \phi + [\delta s^\varepsilon - (\xi \gamma - \chi \alpha) u^\varepsilon - \chi(\beta - \delta) v^\varepsilon] \phi) dx d\tau &= 0, \\ \int_{\Omega} (v^\varepsilon(t) - v_0^\varepsilon) \phi dx + \int_0^t \int_{\Omega} [\nabla v^\varepsilon \cdot \nabla \phi + (\beta v^\varepsilon - \alpha u^\varepsilon) \phi] dx d\tau &= 0, \end{aligned}$$

which entail (4.24) as $\varepsilon \rightarrow 0$. The proof of Lemma 4.4 is completed. $\square$

We are now in a position to prove Theorem 1.2.

Proof of Theorem 1.2. We multiply the first equation of (4.1) by $\phi \in W^{1,\infty}(\Omega)$, and integrate by part, we end up with

$$-\int_{\Omega} u^\varepsilon \phi dx + \int_{\Omega} u_0^\varepsilon \phi dx = \int_0^t \int_{\Omega} \nabla u^\varepsilon \cdot \nabla \phi dx d\tau + \int_0^t \int_{\Omega} u^\varepsilon (1 - \varepsilon u^\varepsilon) \nabla s^\varepsilon \cdot \nabla \phi dx d\tau . \quad (4.29)$$

Noting that (4.16) entails that

$$\int_0^t \int_{\Omega} |u^\varepsilon (1 - \varepsilon u^\varepsilon) \nabla s^\varepsilon|^{\frac{10}{9}} dx d\tau \leq \int_0^t \int_{\Omega} |u^\varepsilon \nabla s^\varepsilon|^{\frac{10}{9}} dx d\tau \leq C(T) .$$

Hence, the sequence $(u^\varepsilon (1 - \varepsilon u^\varepsilon) \nabla s^\varepsilon)_\varepsilon$ is bounded in $L^{\frac{10}{9}}(\Omega \times (0, T))$ and thus weakly compact in $L^1(\Omega \times (0, T))$. From Lemma 4.4 we can derive that $u^\varepsilon (1 - \varepsilon u^\varepsilon) \nabla s^\varepsilon \rightarrow u \nabla s$ a.e. in $\Omega \times (0, T)$ as $\varepsilon \rightarrow 0$. By Vitali convergence theorem, we obtain

$$u^\varepsilon (1 - \varepsilon u^\varepsilon) \nabla s^\varepsilon \rightarrow u \nabla s \quad \text{as } \varepsilon \rightarrow 0 \text{ in } L^1(\Omega \times (0, T)) . \quad (4.30)$$

Letting $\varepsilon \rightarrow 0$ and substituting (4.30) into (4.29), we derive that

$$\int_{\Omega} (u(t) - u_0) \phi dx + \int_0^t \int_{\Omega} \nabla u \cdot \nabla \phi dx d\tau + \int_0^t \int_{\Omega} u \nabla \phi \cdot \nabla s dx d\tau = 0. \quad (4.31)$$

Combining (4.24) and (4.31), and using the transformation $s = \xi w - \chi v$, one has (1.2). Then the proof of Theorem 1.2 is completed. $\square$

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- 27 Z.A. , *Discrete Contin. Dyn. Syst. B* 18 (3) (2013) 601–641.
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