

BOUNDEDNESS AND LARGE TIME BEHAVIOR OF AN ATTRACTION-REPULSION CHEMOTAXIS MODEL WITH LOGISTIC SOURCE

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Abstract. In this paper, we study an attraction-repulsion Keller-Segel chemotaxis model with logistic source

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v) + \nabla \cdot (u \nabla w) + f(u); & x \in \Omega; t > 0; \\ v_t = \Delta v + u - v; & x \in \Omega; t > 0; \\ w_t = \Delta w + u - w; & x \in \Omega; t > 0; \end{cases} \quad (*)$$

in a smooth bounded domain $\Omega \subset \mathbb{R}^n (n \geq 1)$, with homogeneous Neumann boundary conditions and nonnegative initial data $(u_0; v_0; w_0)$ satisfying suitable regularity, where $u \geq 0; v \geq 0; w \geq 0$ and f is a smooth growth source satisfying $f(0) \geq 0$ and

$$f(u) \leq a - bu; \quad u \geq 0; \quad \text{with some } a \geq 0; b > 0; \gamma \geq 1.$$

When $\gamma = 1$ (i.e. repulsion cancels attraction), the boundedness of classical solution of system (*) is established if the dampening parameter b and the space dimension n satisfy

$$\begin{cases} b > \max\{1; 3 - \frac{6}{n}\}; & \text{when } 1 \leq n \leq 5; \\ b \geq 2; & \text{when } 6 \leq n \leq 9; \\ b > 1 + \frac{2(n-4)}{n+2}; & \text{when } n \geq 10. \end{cases}$$

Furthermore, when $f(u) = u(1-u)$ and repulsion cancels attraction, by constructing appropriate Lyapunov functional, we show that if $b > \frac{2}{8}(\frac{n-2}{2})^2$, the solution $(u; v; w)$ exponentially stabilizes to the constant stationary solution $(1; -; -)$ in the case of $1 \leq n \leq 9$. Our results implies that when repulsion cancels attraction the logistic source play an important role on the solution behavior of the attraction-repulsion chemotaxis system.

1. Introduction and main results. We consider the initial-boundary value problem of the attraction-repulsion Keller-Segel model with logistic source

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$$\begin{cases} u_t = \mu \nabla \cdot (\nabla u) - \chi \nabla \cdot (u \nabla v) + \rho \nabla \cdot (u \nabla w) + f(u); & x \in \mathbb{R}^2; t > 0; \\ v_t = \kappa \nabla \cdot (\nabla v) + \alpha u - \beta v; & x \in \mathbb{R}^2; t > 0; \\ w_t = \eta \nabla \cdot (\nabla w) + \gamma u - \delta w; & x \in \mathbb{R}^2; t > 0; \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0; & x \in \partial \Omega; t > 0; \\ u(x; 0) = u_0(x); v(x; 0) = v_0(x); w(x; 0) = w_0(x); & x \in \mathbb{R}^2; \end{cases} \quad (1)$$

in a bounded domain $\Omega \subset \mathbb{R}^n (n \geq 1)$ with smooth boundary, where u denotes the cell density, v represents the concentration of chemoattractant and w accounts for the concentration of chemorepellent. The positive parameters $\mu, \chi, \rho, \kappa, \alpha, \beta, \eta, \gamma, \delta$ are called the chemotactic coefficients, and $f(u)$ is a logistic source. Here μ is a nonnegative scaling constant. The logistic source $f(u)$ describes the cell proliferation and death. The system (1) with two chemical signals (chemoattractant and chemorepellent) may interact to produce aggregation of microglia, which also was proposed in [34] to describe the quorum effect in the chemotactic process.

The attraction-repulsion Keller-Segel model (1) can be viewed as a generalization of the following Keller-Segel chemotaxis model

$$\begin{cases} u_t = \mu \nabla \cdot (\nabla u) - \chi \nabla \cdot (u \nabla v) + f(u); & x \in \mathbb{R}^2; t > 0; \\ v_t = \kappa \nabla \cdot (\nabla v) + \alpha u - \beta v; & x \in \mathbb{R}^2; t > 0; \end{cases} \quad (2)$$

whose solution behavior has been extensively studied in the past four decades in various perspectives (see the survey articles [3, 12] and references therein). When $f(u) = 0$, the system (2) was called minimal Keller-Segel chemotaxis model. A striking feature of the minimal model (2) is the blow-up of solutions in two or higher dimensions [7, 11, 14, 31, 46, 48], which limits the application of the model to explain the aggregation phenomena observed in experiment. To prevent blow-up of solutions, differential mechanisms have been proposed (see [8, 9, 10] and references therein). The chemotaxis model (2) with non-trivial logistic source has been studied [20, 33, 44, 45], and the result showed that this mechanism can enforce the boundedness of solutions so that blow-up is inhibited. More precisely, when $\mu = 1$ and $f(u) = u(1 - u)$ with $\chi > 0$, the system (2) has a uniform-in-time bounded classical solution in two dimensional bounded domain [33] and a weak solution in three dimensional convex domain [20]. In higher dimensions ($n \geq 3$), the existence of the classical solution with uniform-in-time bound was also established if the logistic source $f(u) = a - bu^2$ for some $a \geq 0$ and $b > 0$ with $\chi = 0$ in [41] and $\chi = 1$ in [45], respectively. Moreover, for more general logistic $f(u) = a - bu^\theta$ with $\theta > 2 - \frac{1}{n}(n - 2)$, Winkler [44] also constructed some global 'very weak' solutions to the system (2), which however whether or not there exist global classical solutions was left as an interesting and challenging open problem. We should point out that from the point of mathematical intuition, the logistic damping has a balance effect on the formation of possible singularity. However, the blow-up is possible in a slightly modified version of system (2) with logistic source in [47].

Although the attraction-repulsion Keller-Segel model (1) is a direct generalization of the Keller-Segel system (2), the mathematical analysis on the boundedness and blow-up of solutions confront great challenges due to the complicated interactions between three species u, v and w , and the difficulty of constructing a Lyapunov functional. For the attraction-repulsion chemotaxis system (1) without growth source

(i.e. $f(u) = 0$), the global existence of classical solutions, non-trivial stationary state, asymptotic behavior and pattern formation of the system (1) with Neumann boundary conditions were studied [16, 27, 28] with $\beta = 1$ in one dimension. By introducing a novel transformation $s = w - v$, Tao and Wang [40] studied the global solvability, boundedness, blow-up, existence of steady states in a bounded domain with homogeneous Neumann boundary conditions in higher dimensions ($n \geq 2$) and first found that the solution behavior of (1) essentially depends on the sign of parameter $\beta := \frac{\alpha}{\beta} - \frac{\gamma}{\delta}$, which interprets the competing effect between attraction and repulsion as follows (see also [18]):

- $\beta < 0$, repulsion dominates;
- $\beta = 0$, repulsion cancels attraction;
- $\beta > 0$, attraction dominates.

The recent progress on the solution behavior of the system (1) without logistic source can be found in [5, 15, 18, 21, 23, 24, 25, 26] for bounded domain and in [17, 36] for whole space.

However, to our knowledge, there are few results on the attraction-repulsion chemotaxis model (1) with non-trivial logistic source. When $\beta = 0$, based on the L^p energy estimates and Moser iteration, Zhang and Li [50] established the existence of global classical solutions of the system (1) with $f(u) = u(1 - u)$ if one of the following conditions holds: (1) $n = 1$; (2) $n = 2$; (3) $\frac{n-2}{n}(\frac{\alpha}{\beta} - \frac{\gamma}{\delta}) < 1$ and $n \geq 3$. Moreover, they also showed that the global classical solution will converge to the unique constant steady $(1; \frac{\alpha}{\beta}; \frac{\gamma}{\delta})$ if $\beta > 2$. The global existence of classical solution also was studied by Li and Xiang [22] for the logistic source $f(u) = a - bu^\theta$ with some $a \geq 0$ and $b > 0$. Specially, Li and Xiang [22] proved that the classical solution will exist for all $n \geq 2$ in the case $\beta > 2$. Moreover, when $\beta = 0$ (i.e. repulsion cancels attraction), they proved the classical solution with uniform-in-time bound exists if $\beta > \frac{1}{2}(\frac{\alpha}{\beta} - \frac{\gamma}{\delta} - \frac{n^2 + 4n}{n} - n + 2)$. However if $\beta = 1$, they only established the boundedness of solutions when $n = 1$ with $\beta = 1$ or $n = 2$ with $\beta = 2$ [22]. According to the above results [22, 50], the blow-up of solution is prevented when $\beta = 0$ and the power parameter θ is large in higher dimensions. Whileas for the full parabolic attraction-repulsion chemotaxis model (1) (i.e. $\beta = 1$) with logistic source, the global existence of classical solutions was only obtained when the space dimension $n \geq 2$, which was left as an open problem for the higher dimensions ($n \geq 3$). Moreover, as far as we know, there is not any result on the large time behavior of solutions for the full attraction-repulsion chemotaxis model (1) with logistic source.

The main purpose of this paper is to investigate the effect of the logistic source on the solution behavior of the following full attraction-repulsion chemotaxis model

$$\begin{cases} u_t = u - r(u - v) + r(u - w) + f(u); & x \in \mathbb{R}^n; t > 0; \\ v_t = v + u - v; & x \in \mathbb{R}^n; t > 0; \\ w_t = w + u - w; & x \in \mathbb{R}^n; t > 0; \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0; & x \in \partial \Omega; t > 0; \\ u(x; 0) = u_0(x); v(x; 0) = v_0(x); w(x; 0) = w_0(x); & x \in \mathbb{R}^n; \end{cases} \quad (3)$$

where the kinetic term f satisfies $f(0) = 0$ and

$$f(u) = a - bu^\theta; u \geq 0; \text{ with some } a \geq 0; b > 0; \theta \geq 1; \quad (4)$$

To study the dampening effect of the logistic source, we focus our study on the case of $\beta = 1$. We find the lower bound of the power parameter α depending on n to guarantee the existence of global bounded solutions. Moreover, by constructing Lyapunov functional, we also study the large time behavior of the solution for the system (3) with logistic source $f(u) = u(1 - u)$. Our first main results are stated as follows.

Theorem 1.1. Assume that $(u_0; v_0; w_0) \in [W^{1,\infty}(\Omega)]^3$ and $\Omega \subset \mathbb{R}^n$ ($n \geq 1$) is a bounded domain with smooth boundary. Suppose $\beta = 1$ and $f(u)$ satisfy (4). If the power parameter α and the space dimension n satisfy the following relations

$$\begin{cases} \alpha > \max\{1; 3 - \frac{6}{n}\}; & \text{when } 1 \leq n \leq 5; \\ \alpha > 2; & \text{when } 6 \leq n \leq 9; \\ \alpha > 1 + \frac{2(n-4)}{n+2}; & \text{when } n \geq 10; \end{cases}$$

then system (3) has a unique triple of non-negative solution $(u; v; w) \in C^0([0; \infty)) \cap C^{2,1}(\bar{\Omega} \times (0; \infty))$ satisfying

$$\|u(\cdot; t)\|_{L^1} + \|v(\cdot; t)\|_{W^{1,1}} + \|w(\cdot; t)\|_{W^{1,1}} \leq C; \quad (5)$$

where $C > 0$ is a constant independent of t .

We have several remarks concerning the boundedness results in Theorem 1.1.

Remarks.

In Theorem 1.1, we give the lower bound of α (which may be not optimal) to prevent the blow-up of solutions. For the prototype logistic source $f(u) = u(1 - u)$ (i.e. $\beta = 2$), when $1 \leq n \leq 9$ we can obtain the existence of the classical solution with uniform-in-time bound directly from Theorem 1.1 by noting $\alpha = 2 > \max\{1; 3 - \frac{6}{n}\}$ for $1 \leq n \leq 5$.

For the cubic growth source $f(u) = u(u - c)(d - u)$ with $c, d > 0$ as originally introduced by Mimura and Tsujikawa in [27], it satisfies the condition (4) with $\beta = 3$ and some $a, b > 0$. For any n , one can easily check that $1 + \frac{2(n-4)}{n+2} < 3$, hence the system (3) with cubic growth source has a unique global classical solution satisfying (5) for all biologically meaningful parameters.

Theorem 1.2. Assume that $(u_0; v_0; w_0) \in [W^{1,\infty}(\Omega)]^3$ and $\Omega \subset \mathbb{R}^n$ ($1 \leq n \leq 9$) is a bounded domain with smooth boundary, if $\beta = 1$ and $f(u) = u(1 - u)$, then for any $\alpha > 0$, system (3) has a non-negative solution $(u; v; w) \in C^0([0; \infty)) \cap C^{2,1}(\bar{\Omega} \times (0; \infty))$ satisfying (5). Moreover, if

$$\alpha > \frac{2 - 2(\frac{\alpha}{\beta})^2}{8 - 2}; \quad (6)$$

then the classical solution $(u; v; w)$ of system (3) satisfies

$$\|u(\cdot; t)\|_{L^1} - \|k_{L^1}\| + \|v(\cdot; t)\|_{L^1} - \|k_{L^1}\| + \|w(\cdot; t)\|_{L^1} - \|k_{L^1}\| \leq ce^{-\lambda t} \text{ for all } t > 0;$$

where c and λ are positive constants independent of t .

Remark 1. When $\beta = 1$, from (6) we know that the solution will converge to the unique non-trivial constant state $(1; \frac{\alpha}{\beta}; \frac{\gamma}{\delta})$ for any $\alpha > 0$. We conjecture that the same asymptotic stability results hold for $\beta \neq 1$, which however is left as an open problem due to the technical reasons.

Outline of main approaches: Inspired by the ideas in [3, 49], we first establish the boundedness criterion for the solution of the system (3). More precisely, by combining the semigroup theory, Gagliardo-Nirenberg inequality, the L^p energy estimate and Moser-Alikakos iteration, we show that when repulsion cancels attraction (i.e. $\beta = \delta$), the uniform boundedness of L^r -norm of $u(\cdot; t)$ for some $r > \frac{n}{4}$ can rule out the blow-up of solutions for the system (3) (see Lemma 2.3). With the boundedness criterion established in Lemma 2.3 in hand, we use the coupled energy estimate as in [38] together with the method of heat Neumann semigroup to study the boundedness of the solution to the system (3) in higher dimensions. The relations of dampening parameter δ and the space dimension n are found to ensure the boundedness of solution for system (3). Specially, our results show that when $1 \leq n \leq 9$ and repulsion cancels attraction the global classical solution with uniform-in-time bound exist for the prototype logistic source $f(u) = u(1 - u)$ with $\delta > 0$, which is substantially different from the classical chemotaxis model with logistic source. Moreover, based on the ideas in [2, 39], we show that under $\delta > \frac{\chi^2 \alpha^2 (\beta - \delta)^2}{8\delta \beta^2}$, the functional $F(t)$ defined as

$$F(t) := \int (\frac{1}{2} u^2 - \ln u) + \frac{1}{2} \int \left(s - \frac{(\chi \cdot \nabla u)^2}{2\delta} \right)^2 + \frac{2}{2} \int (v - \frac{1}{2})^2$$

for all $t > 0$, act as a Lyapunov functional for the system (3) with $\beta = \delta$ and appropriate choices of the positive constant α_1 and α_2 , which will be used to study the large time behavior of solutions.

The remainder of the paper is organized as follows. In section 2, we establish the boundedness criterion for the solution of the system (3) in the case of repulsion cancels attraction. With the aid of the boundedness criterion in Lemma 2.3, we show the existence of globally boundedness classical solutions to the system (3) for arbitrary dimension in section 3. In section 4, The global dynamic of solutions to the system (3) with $f(u) = u(1 - u)$ will be studied.

2. Local existence and boundedness criterion.

2.1. Local existence and preliminaries. In what follows, without confusion, we shall abbreviate $\int f dx$ as $\int f$ for simplicity. Moreover, we shall use c_i or $C_i (i = 1; 2; 3; \dots)$ to denote generic constants which may vary in the context. The existence of local solutions of the problem (3) can be proved by the fixed point theorem and the maximum principle along the same line shown in [22, 40, 42].

Lemma 2.1. *Assume that $(u_0; v_0; w_0) \in [W^{1,\infty}(\Omega)]^3$ and $\Omega \subset \mathbb{R}^n (n \geq 1)$ is a bounded domain with smooth boundary. Then there exists $T_{max} \in (0; \infty]$ such that the system (3) with (4) has a unique nonnegative classical solution $(u; v; w) \in C^0([0; T_{max}); \mathbb{R}^3) \cap C^{2,1}([0; T_{max}); \mathbb{R}^3)$. Moreover $u > 0$ in $(0; T_{max})$ and*

$$\text{if } T_{max} < \infty; \text{ then } \|u(\cdot; t)\|_{L^1} \rightarrow 0 \text{ as } t \rightarrow T_{max}^-.$$

Furthermore, the L^1 -norm of u is uniformly bounded, i.e. there exists a constant M_0 independent of t such that $\|u(\cdot; t)\|_{L^1} \leq M_0$.

Proof. The proof of local-in-time existence of classical solutions to the system (3) is quite standard, see [22, 40, 42] for details. Since $f(0) = 0$, using the maximum principle we can derive u, v, w are nonnegative, as shown in [22, 40]. Integrating the first equation of the system (3) and using (4), one can derive that

$$\frac{d}{dt} \int u = \int f(u) - \int (a - bu^\theta) - \int u + c_j j;$$

where $c = \max_{f(a-bu^\theta+u): u \geq 0} < 1$ due to (1). The L^1 -norm of u is uniformly bounded by using the standard Gronwall's inequality. $\square$

The following Gagliardo-Nirenberg inequality will be frequently used later.

Lemma 2.2 (Gagliardo-Nirenberg inequality). *Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary. Assume that $1 < p, q < \infty$ and satisfying $(n-kq)p < nq$ for some $k > 0$ and $r \geq 2(0; p)$. Then, for any $u \in W^{k,q}(\Omega) \cap L^r(\Omega)$, there exist two constants c_1 and c_2 depending only on Ω, q, k, r and n such that*

$$\|u\|_{L^p} \leq c_1 \|u\|_{L^q}^a \|u\|_{L^r}^{1-a} + c_2 \|u\|_{L^r};$$

where $a \in (0, 1)$ fulfilling

$$\frac{1}{p} = a \left(\frac{1}{q} - \frac{k}{n} \right) + (1-a) \frac{1}{r}.$$

We should remark that the original Gagliardo-Nirenberg inequality (e.g. see [32]) is stated only for $r = 1$, but this condition can be easily relaxed to $r \geq 2(0; p)$ by using Hölder's inequality (cf. [43, Lemma 3.2]).

2.2. Boundedness criterion. Inspiring by the works in [3, 49], we will show the boundedness criterion of solutions for the system (3) as follows.

Lemma 2.3 (Criterion for boundedness). *Suppose the conditions in Lemma 2.1 hold. Let $(u; v; w)$ be the solution of system (3) defined on its maximal existence time interval $[0; T_{max})$. If Ω is bounded and there exists a constant $M > 0$ such that for all $p > \frac{n}{4}$*

$$\|u(\cdot; t)\|_{L^p} \leq M; \text{ for all } t \in (0; T_{max});$$

then one can obtain a constant $C > 0$ independent of t such that

$$\|u(\cdot; t)\|_{L^1} + \|v(\cdot; t)\|_{W^{1,1}} + \|w(\cdot; t)\|_{W^{1,1}} \leq C \text{ for all } t \in (0; T_{max}); \quad (7)$$

Next, we will prove Lemma 2.3. Before that we first present some basic estimates of solutions. Letting $s := w - v$ and noting $\nabla \cdot u = 0$, then the system (3) can be transformed into

$$\begin{cases} u_t = -u + r(u \cdot s) + f(u); & x \in \Omega; t > 0; \\ s_t = -s + (u \cdot \nabla) v; & x \in \Omega; t > 0; \\ v_t = -v + (u \cdot \nabla) v; & x \in \Omega; t > 0; \\ \frac{\partial u}{\partial \nu} = \frac{\partial s}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0; & x \in \partial \Omega; t > 0; \\ u(x; 0) = u_0(x); s(x; 0) := w_0(x) - v_0(x); v(x; 0) = v_0(x); & x \in \Omega. \end{cases} \quad (8)$$

Then for the transformed system (8), we have the following results:

Lemma 2.4. *Let $(u; s; v)$ be a solution of (8) defined on its maximal existence time interval $[0; T_{max})$. Suppose $p \geq 1$ and for $i = 1, 2$*

$$\begin{cases} q_i \in [1; \frac{np}{n-ip}); & \text{if } p \leq \frac{n}{i}; \\ q_i \in [1; 1]; & \text{if } p > \frac{n}{i}. \end{cases} \quad (9)$$

If there exists a constant $M > 0$ such that for some $T \in (0; T_{max})$, it holds that

$$\|u(\cdot; t)\|_{L^p} \leq M \text{ for all } t \in (0; T); \quad (10)$$

then for all $t \geq (0; T)$, one has

$$k r v(\cdot; t) k_{L^{q_1}} \leq C_v(p; q_1; M); \quad k v(\cdot; t) k_{L^{q_2}} \leq C_v(p; q_2; M); \quad (11)$$

Proof. Suppose that there exists a constant $K > 0$ such that

$$k k_{L^q} \leq K \quad \text{for all } q > p - 1. \quad (12)$$

Then using the Hölder's inequality and (12), for all $r \geq [1; p]$ one has

$$\begin{aligned} k k_{L^r} &= \left(\int \cdot^r \right)^{\frac{1}{r}} = \left[\left(\int \cdot^q \right)^{\frac{r}{q}} \left(\int 1^{\frac{q-r}{r}} \right)^{\frac{q-r}{q}} \right]^{\frac{1}{r}} \\ &= k k_{L^q} j \cdot^{\frac{q-r}{rq}} \leq K j \cdot^{\frac{q-r}{rq}}. \end{aligned}$$

Hence, we may assume that $q_i > p (i = 1; 2)$ in the proof of this lemma for convenience.

Using the variation of constants representation of v , then from the third equation of system (8) we have

$$v(\cdot; t) = e^{t(-\beta)} v_0 + \int_0^t e^{(t-\tau)(-\beta)} u(\cdot; \tau) d\tau \quad \text{for all } t \geq (0; T)$$

which together with (10) gives

$$\begin{aligned} k v(\cdot; t) k_{L^{q_2}} &\leq e^{-\beta t} k e^t v_0 k_{L^{q_2}} + \int_0^t e^{-\beta(t-\tau)} k e^{(t-\tau)} u(\cdot; \tau) k_{L^{q_2}} d\tau \\ &\leq c_1 k v_0 k_{L^1} + c_2 M \int_0^t e^{-\beta(t-\tau)} (1 + (t-\tau)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q_2})}) d\tau \\ &\leq c_1 k v_0 k_{L^1} + c_2 M \int_0^\infty e^{-\beta\sigma} (1 + \sigma^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q_2})}) d\sigma; \end{aligned} \quad (13)$$

and

$$\begin{aligned} k r v(\cdot; t) k_{L^{q_1}} &\leq e^{-\beta t} k r e^t v_0 k_{L^{q_1}} + \int_0^t e^{-\beta(t-\tau)} k r e^{(t-\tau)} u(\cdot; \tau) k_{L^{q_1}} d\tau \\ &\leq c_3 e^{-\beta t} k v_0 k_{W^{1,1}} + c_4 M \int_0^t e^{-\beta(t-\tau)} (1 + (t-\tau)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{p}-\frac{1}{q_1})}) d\tau \\ &\leq c_3 e^{-\beta t} k v_0 k_{W^{1,1}} + c_4 M \int_0^\infty e^{-\beta\sigma} (1 + \sigma^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{p}-\frac{1}{q_1})}) d\sigma; \end{aligned} \quad (14)$$

where the smoothing properties of $(e^\tau \cdot)_{\tau \geq 0}$ have been used (see [6, Lemma 3.3] or [46, Lemma 1.3]). Thanks to the conditions (9) of q_1 and q_2 , we know that $c_5 := \int_0^\infty e^{-\beta\sigma} (1 + \sigma^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q_2})}) d\sigma < 1$ and $c_6 := \int_0^\infty e^{-\beta\sigma} (1 + \sigma^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{p}-\frac{1}{q_1})}) d\sigma < 1$. Hence from (13) and (14), we can derive that

$$k v(\cdot; t) k_{L^{q_2}} \leq c_1 k v_0 k_{L^1} + c_2 c_5 M \quad \text{for all } t \geq (0; T);$$

and

$$k r v(\cdot; t) k_{L^{q_1}} \leq c_3 k v_0 k_{W^{1,1}} + c_4 c_6 M \quad \text{for all } t \geq (0; T);$$

and thereby prove (11). Hence the proof of this lemma is completed. $\square$

Lemma 2.5. Let $q_2 \geq 1$ and

$$\begin{cases} q_3 \geq [1; \frac{nq_2}{n-q_2}); & \text{if } q_2 \leq n; \\ q_3 \geq [1; 1]; & \text{if } q_2 > n; \end{cases} \quad (15)$$

If for all $M > 0$ there exists a constant $C_s(q_2; q_3; M) > 0$ such that for some $T \in (0; T_{max})$, we have

$$kv(\cdot; t)k_{L^{q_2}} \leq M \text{ for all } t \in (0; T); \quad (16)$$

then

$$krs(\cdot; t)k_{L^{q_3}} \leq C_s(q_2; q_3; M); \quad (17)$$

Proof. Without loss of generality, we assume that $q_3 > q_2$ for simplification. Using the variation of constants representation of s , from the second equation of the system (8), we can derive that

$$s(\cdot; t) = e^{(-\delta)t}s_0 + \int_0^t e^{(-\delta)(t-\tau)}v(\cdot; \tau)d\tau;$$

Then using again the smoothing properties of $(e^\tau)_{\tau \geq 0}$ and noting (16), one has

$$\begin{aligned} krsk_{L^{q_3}} &\leq e^{-\delta t}kre^{-t}s_0k_{L^{q_3}} + \int_0^t \int_0^\tau e^{-\delta(t-\tau)}kre^{-(t-\tau)}v(\cdot; \tau)k_{L^{q_3}}d\tau d\tau \\ &\leq c_7ks_0k_{W^{1,1}} + c_8 \int_0^t e^{-\delta(t-\tau)} \left(1 + (t-\tau)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{q_2}-\frac{1}{q_3})}\right) kv(\cdot; \tau)k_{L^{q_2}}d\tau \\ &\leq c_7ks_0k_{W^{1,1}} + c_9 \int_0^t e^{-\delta(t-\tau)} \left(1 + (t-\tau)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{q_2}-\frac{1}{q_3})}\right) d\tau; \end{aligned} \quad (18)$$

Since q_3 satisfies (15), we have $\frac{1}{2} - \frac{n}{2}(\frac{1}{q_2} - \frac{1}{q_3}) > -1$, which implies

$$c_{10} := \int_0^t e^{-\delta(t-\tau)} \left(1 + (t-\tau)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{q_2}-\frac{1}{q_3})}\right) d\tau < 1;$$

Then (17) follows from (18). Hence the proof of this Lemma 2.5 is completed. $\square$

Next, we will give the proof of Lemma 2.3.

Proof of Lemma 2.3. If $p > \frac{n}{3}$, from (9) we can choose $q_2 > n$ such that $kvk_{L^{q_2}}$ is uniformly bounded. Then from Lemma 2.5, one has

$$krsk_{L^1} \leq c_1 \text{ for all } t > 0; \quad (19)$$

which together with the well-known Moser-Alikakos iteration technique (cf. [1, 40]) gives the L^∞ -bound of u . Since $ku(\cdot; t)k_{L^1}$ (i.e. $p = 1$) is uniformly bounded (see Lemma 2.1), when $n \geq 2$, we have (19) and hence the L^∞ -bound of u .

Next, we will consider the case $\frac{n}{4} < p \leq \frac{n}{3}$ with $n \geq 3$, which gives $n \frac{np}{n-2p} > \frac{n}{2}$. Then choosing $\frac{n}{2} < q_2 < \frac{np}{n-2p} \leq n$, one has $\frac{nq_2}{n-q_2} > n$. Hence from (17) and (15), we can obtain

$$krsk_{L^r} \leq c_2 \text{ for some } n < r < \frac{nq_2}{n-q_2}; \quad (20)$$

Multiplying the first equation of the system (8) by u^{k-1} ($k > \frac{n}{3} - 1$) and integrating by parts over $\mathbb{R}^n$, then using Young's inequality,

$$\begin{aligned} & (k-1) \int u^{k-2} j r u j^2 - (k-1) \int u^{k-1} r u - r s + a \int u^{k-1} - b \int u^{k+\theta-1} \\ & \frac{k-1}{4} \int u^{k-2} j r u j^2 + \frac{k-1}{3} \int u^k j r s j^2 - \frac{b}{k} \int u^{k+\theta-1} + c_3; \end{aligned}$$

that is

$$\frac{d}{dt} \int u^k + \frac{(k-1)}{k} \int j r u^{\frac{k}{2}} j^2 + b \int u^{k+\theta-1} - \frac{k(k-1)}{3} \int u^k j r s j^2 + c_4; \quad (21)$$

Using Hölder's inequality, (20) and Gagliardo-Nirenberg inequality (see Lemma 2.2), one has

$$\begin{aligned} \int u^k j r s j^2 & \leq k u^k k_{L^{\frac{r}{r-2}}} k j r s j^2 k_{L^{\frac{r}{2}}} \leq c_2^2 k u^{\frac{k}{2}} k_{L^{\frac{2r}{r-2}}}^2 \\ & \leq c_5 \left\| r u^{\frac{k}{2}} \right\|_{L^2}^{2\sigma_1} \left\| u^{\frac{k}{2}} \right\|_{L^{\frac{2p}{k}}}^{2(1-\sigma_1)} + c_6 \left\| u^{\frac{k}{2}} \right\|_{L^{\frac{2p}{k}}}^2 \end{aligned} \quad (22)$$

where $\sigma_1 = \frac{\frac{k}{2p} - \frac{r-2}{2r}}{\frac{k}{2p} + \frac{1}{n} - \frac{1}{2}} \in (0, 1)$ due to $r > n$ and $k > p$. Noting that $\left\| u^{\frac{k}{2}} \right\|_{L^{\frac{2p}{k}}}^2 = k u^k k_{L^p}^k \leq M^k$, hence from (22) together with Young's inequality, we can derive that

$$\int u^k j r s j^2 \leq c_7 \left\| r u^{\frac{k}{2}} \right\|_{L^2}^{2\sigma_1} + c_8 \int j r u^{\frac{k}{2}} j^2 + c_9; \quad (23)$$

Using Gagliardo-Nirenberg inequality again, we can find $\sigma_2 = \frac{\frac{k}{2p} - \frac{1}{2}}{\frac{k}{2p} + \frac{1}{n} - \frac{1}{2}} \in (0, 1)$, such that

$$\begin{aligned} \int u^k & \leq k u^{\frac{k}{2}} k_{L^2}^2 \leq c_{10} k r u^{\frac{k}{2}} k_{L^2}^{2\sigma_2} k u^{\frac{k}{2}} k_{L^{\frac{2p}{k}}}^{2(1-\sigma_2)} + c_{10} k u^{\frac{k}{2}} k_{L^{\frac{2p}{k}}}^2 \\ & \leq c_{11} \int j r u^{\frac{k}{2}} j^2 + c_{11}; \end{aligned} \quad (24)$$

Substituting (23) and (24) into (21) and choosing σ_1 and σ_2 small enough, we have

$$\frac{d}{dt} \int u^k + \int u^k \leq c_{12};$$

which together with Grönwall's inequality yields

$$k u(\cdot; t) k_{L^k} \leq c_{13}; \text{ for all } k > \frac{n}{3}.$$

Then using Lemma 2.4 again, we can derive that (19) holds for all $n \geq 3$, which together with the well-known Moser iteration gives

$$k u(\cdot; t) k_{L^1} \leq c_{14}; \text{ for all } t \in (0; T_{max}); \quad (25)$$

Moreover, through a straightforward reasoning involving standard parabolic regularity theory ([19]), we have

$$k v(\cdot; t) k_{W^{1,1}} + k w(\cdot; t) k_{W^{1,1}} \leq c_{15}; \text{ for all } t \in (0; T_{max});$$

which combines with (25) gives (7). Then the proof of Lemma 2.3 is completed. $\square$

3. Proof of Theorem 1.1. In this section, we are devoted to proving Theorem 1.1 based on the boundedness criterion of solutions for system (3) (see Lemma 2.3). From Lemma 2.1, one has $k u k_{L^1} \leq c_1$ for all $t \geq 1$. Then using the boundedness criterion established in Lemma 2.3, we know that the system (3) has a global

classical solution with uniform-in-time bound for $n \geq 3$. Hence to complete the proof of Theorem 1.1, we only need to consider the case $n = 4$.

3.1. Parameter conditions. Before proving our main results in Theorem 1.1, we first introduce some notations that will be used later. For $n = 4$, $k > \frac{n}{4}$, $\kappa_1 > 1$, $\kappa_2 > 1$, $\theta > 1$, we define

$$\begin{cases} \ell_1 := \frac{2n\lambda}{n+q}, & \ell_2 := \frac{2(k+\theta-1)(\kappa_1-1)}{k+\theta-3}, \\ \ell_3 := \frac{2(k+\theta-1)}{\theta-1}, & \ell_4 := \frac{2(\kappa_2-1)\lambda}{\lambda-1}, \end{cases} \quad (26)$$

and

$$\begin{cases} \theta_i := \theta_i(k; \ell_1; r; q; \epsilon) = \frac{2\kappa_1}{\ell_i} \frac{\frac{\ell_i}{r}-1}{\frac{2\kappa_1}{r}+\frac{2}{n}-1}; & \text{for } i = 1, 2; \\ \theta_i := \theta_i(k; \ell_2; r; q; \epsilon) = \frac{2\kappa_2}{\ell_i} \frac{\frac{\ell_i}{r}-1}{\frac{2\kappa_2}{r}+\frac{2}{n}-1}; & \text{for } i = 3, 4 \end{cases} \quad (27)$$

as well as

$$\begin{cases} f_i := f_i(k; \ell_1; r; q; \epsilon) = \frac{\ell_i \theta_i}{2\kappa_1} = \frac{\frac{\ell_i}{r}-1}{\frac{2\kappa_1}{r}+\frac{2}{n}-1}; & \text{for } i = 1, 2; \\ f_i := f_i(k; \ell_2; r; q; \epsilon) = \frac{\ell_i \theta_i}{2\kappa_2} = \frac{\frac{\ell_i}{r}-1}{\frac{2\kappa_2}{r}+\frac{2}{n}-1}; & \text{for } i = 3, 4; \end{cases} \quad (28)$$

Next, we will show some results on the parameters which will be used in the proof of the boundedness of global solutions based on some ideas in [4, 51].

Lemma 3.1. *Let*

The combination of (33) and (34) implies that (32) holds for $i = 1; 2$ in the case of

$$\begin{cases} \gamma > \frac{n}{2(n-2)}; \\ \frac{n-2}{n-1} \frac{1}{n-1} < \gamma_1 < \frac{n}{2(n-1)}(k + \gamma_1) \frac{1}{n-1}; \end{cases} \quad (35)$$

Similarly, since $\gamma_3 = \frac{2(k+\theta-1)}{\theta-1}$ and $\gamma_4 = \frac{2(\kappa_2-1)\lambda}{\lambda-1}$, one can check that (32) for $i = 3; 4$ will be satisfied if

$$\begin{cases} k > \frac{6-n}{2(n-3)}(\gamma_1); & \gamma_2 > \frac{k+\theta-1}{\theta-1} \frac{1}{n-3}; \\ 1 + \frac{n}{2(n-3)} < \gamma_2 < \frac{n-2}{n-3} \frac{1}{n-3}; \end{cases}$$

which yields

$$\begin{cases} k > \frac{n+2}{2(n-3)}(\gamma_1); \\ \frac{k+\theta-1}{\theta-1} \frac{1}{n-3} < k_2 < \frac{n-2}{n-3} \frac{1}{n-3}. \end{cases} \quad (36)$$

Thus, the combination of (35) and (36) implies that γ_1 and γ_2 exist if

$$k > \frac{n+2}{2(n-3)}(\gamma_1)$$

and

$$\frac{n-3}{n-2} \frac{k+\gamma_1}{1} < \gamma_2 < \frac{n}{2(n-2)}(k + \gamma_1); \quad (37)$$

One can check that $\frac{n-3}{n-2} \frac{k+\gamma_1}{1} < \frac{n}{2(n-2)}(k + \gamma_1)$ if $\gamma_1 > 3 \frac{6}{n}$. Hence there exists some $\gamma_1 > 1$ such that (37) holds. In summary, when $1 < \gamma_1 < 2$ and $\gamma_1 > 3 \frac{6}{n}$, we can find a constant $K_1 = \max\{f_{\frac{n}{4}}; \frac{n+2}{2(n-3)}(\gamma_1)g\}$ such that when $k > K_1$ there exist $\gamma_1 > 1$, $\gamma_2 > 1$, $\gamma_3 > 1$ such that (32) holds.

Case 2. 2. In this case, letting $q = \frac{2n}{n-2}$, $r = 2$, $r = \frac{2n}{n-4}$ and hence $\gamma_1 = \frac{2n}{n+2}$ and $\gamma_2 = \frac{2(k+\theta-1)(\kappa_1-1)}{k+\theta-3}$, then one can check that (32) holds for $i = 1; 2$ under the following conditions:

$$\gamma > \frac{n}{n-2}, \quad \gamma_1 > \frac{n-2}{n} \frac{2}{n} \quad (38)$$

and

$$2 < \gamma_1 < \frac{n+2}{2n}(k + \gamma_1) \frac{2}{n}; \quad (39)$$

Furthermore, noting $\gamma_3 = \frac{2(k+\theta-1)}{\theta-1}$ and $\gamma_4 = 2(\gamma_2 - 1) \gamma' = \frac{2(\kappa_2-1)\lambda}{\lambda-1}$, we can derive that (32) with $i = 3; 4$ will be hold if

$$k > \frac{4}{n-4}(\gamma_1), \quad \gamma_2 > \frac{k+\gamma_1}{1} \frac{2}{n-4} \quad (40)$$

and

$$1 + \frac{n}{n-4} < \gamma_2 < \frac{n-2}{n-4} \frac{2}{n-4}; \quad (41)$$

Combining (38)-(41), we can derive that (32) will hold for all $i = 1; 2; 3; 4$ if there exist positive constant $k > \frac{n}{4}$ and $\gamma_1 > 1$ such that

$$\begin{cases} k > \frac{n+6}{n-4}(\gamma_1); \\ \frac{n-4}{n-2} \frac{k+\gamma_1}{\theta-1} < \gamma_2 < \frac{n+2}{2(n-2)}(k + \gamma_1); \end{cases}$$

which yields

$$\begin{cases} k > K_2 = \max\left\{\frac{n}{4}; \frac{n+6}{n-4}(\gamma_1)\right\}; \\ \gamma_1 > 1 + \frac{2(n-4)}{n+2}; \end{cases} \quad (42)$$

Moreover, one can easily check that

$$\begin{cases} 2 > 1 + \frac{2(n-4)}{n+2}; & \text{if } 1 \leq n \leq 9; \\ 2 > 1 + \frac{2(n-4)}{n+2}; & \text{if } n \geq 10; \end{cases} \quad (43)$$

Hence the combination of (42) and (43) implies (32) hold when $k > K_2$ and

$$\begin{cases} = 2; & \text{for } 1 \leq n \leq 9; \\ > 1 + \frac{2(n-4)}{n+2}; & \text{for } n \geq 10; \end{cases}$$

Then the proof of this lemma is completed. $\square$

3.2. A priori estimates. Next, we will show some basic energy estimates of the solution for the system (8).

Lemma 3.2. Assume that $(u_0; v_0; w_0) \in [W^{1,\infty}(\Omega)]^3$ and $\Omega \subset \mathbb{R}^n (n \geq 4)$ is a bounded domain with smooth boundary. Suppose $\gamma = \frac{n-4}{n-2}$ and $f(u)$ satisfy (4). Then there exist $q \in [1; \frac{n}{n-2})$, $r \in [1; \frac{n}{n-1})$ and $\theta \in [1; \frac{nq}{n-q})$ and a constant $C > 0$ independent of t such that the solution $(u; v; s)$ of the system (8) satisfies

$$\|k v\|_{L^q} + \|k r v\|_{L^r} + \|k r s\|_{L^{\theta}} \leq C. \quad (44)$$

Furthermore, if $n \geq 2$ then (44) holds with $q \in [1; \frac{2n}{n-2})$, $r = 2$ and $\theta \in [1; \frac{nq}{n-q})$.

Proof. Note $\|k u\|_{L^1}$ is uniformly bounded (see Lemma 2.1) for all $t \geq 0$. Hence from Lemma 2.4 and Lemma 2.5, we can obtain (44) directly.

Next, we show that the regularity can be improved if $n \geq 2$. Multiplying the third equation of (8) by $2v$ and $2 - v$ respectively, and then integrating them with respect to x , we end up with

$$\frac{d}{dt} \int v^2 + 2 \int j r v j^2 + 2 \int v^2 = 2 \int u v - \int v^2 + \frac{2}{n-2} \int u^2; \quad (45)$$

and

$$\frac{d}{dt} \int j r v j^2 + 2 \int j - v j^2 + 2 \int j r v j^2 = -2 \int u - v - \int j - v j^2 + \frac{2}{n-2} \int u^2. \quad (46)$$

The combination of (45) and (46) gives

$$\frac{d}{dt} \int (v^2 + j r v j^2) + c_1 \int (v^2 + j r v j^2) = c_2 \int u^2. \quad (47)$$

From the first equation of the system (3), we have

$$\frac{d}{dt} \int u + b \int u^\theta = c_3. \quad (48)$$

Combining (47) and (48) and using the facts $\|k u\|_{L^1} \leq c_4$ and $n \geq 2$, one has

$$\begin{aligned} & \frac{d}{dt} \int \left(\frac{c_2}{b} u + v^2 + j r v j^2 \right) + c_1 \int \left(\frac{c_2}{b} u + v^2 + j r v j^2 \right) \\ & \leq c_2 \int (u^\theta - u^2) + c_5 - c_6; \end{aligned}$$

which together with Gronwall's inequality gives

$$\|k v\|_{L^2} + \|k r v\|_{L^2} \leq c_7. \quad (49)$$

Hence using the Sobolev inequality, from (49) we have for all $1 \leq q < \frac{2n}{n-2}$ that

$$\|k v\|_{L^q} \leq c_8 \|k v\|_{W^{1,2}} \leq c_9. \quad (50)$$

From (50), using the similar argument as (18) one has $k r s k_{L^{\frac{r}{r-2}}} \leq C$ with $r \geq 2$ $[1; \frac{nq}{n-q})$, which together with (49) and (50) gives (44) with $q \geq [1; \frac{2n}{n-2})$, $r = 2$ and $r \geq 2 [1; \frac{nq}{n-q})$. Then the proof of this lemma is completed. $\square$

With the results obtained in Lemma 3.1 and Lemma 3.2 in hand, we will establish the following key lemma in the proof of Theorem 1.1.

Lemma 3.3. *Assume that the conditions in Lemma 3.2 hold. If (30) holds, then for sufficiently large $k \geq (\frac{n}{4} + 1)$, there exists a constant $c_1 > 0$ independent of t such that the solution of the system (8) satisfies*

$$k u(\cdot; t) k_{L^k} \leq c_1. \quad (51)$$

Proof. Using a similar argument as in proof of (21), one has

$$\frac{d}{dt} \int u^k + \frac{k}{k-1} \int j r u^{\frac{k}{2}} j^2 + b \int u^{k+\theta-1} - \frac{k(k-1)}{3} \int u^k j r s j^2 + c_2. \quad (52)$$

Differentiating the third equation of (8) and then multiplying it with $r v$, applying the identity $\frac{1}{2} j r v j^2 = r v - r v + j D^2 v j^2$ and $\frac{1}{n} j v j^2 = j D^2 v j^2$, we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} j r v j^2 &= r v - r v + j D^2 v j^2 \\ &= \frac{1}{2} j r v j^2 - \frac{1}{n} j v j^2 + r u - r v - j r v j^2. \end{aligned} \quad (53)$$

We multiply (53) with $2^{-1} j r v j^{2(\kappa_1-1)}$ ($\kappa_1 > 1$) and integrate it to get

$$\begin{aligned} \frac{d}{dt} \int j r v j^{2\kappa_1} + \frac{2}{n} \int j v j^2 j r v j^{2(\kappa_1-1)} + \int j r v j^{2\kappa_1} \\ - \int j r v j^{2(\kappa_1-1)} j r v j^2 + 2^{-1} \int j r v j^{2(\kappa_1-1)} r u - r v \\ + (1 - 2^{-1}) \int j r v j^{2\kappa_1}. \end{aligned} \quad (54)$$

Noting $j r v j^{2(\kappa_1-2)} |j r v j^2|^2 = \frac{4}{\kappa_1^2} j r j r v j^{\kappa_1} j^2$, then the first term of (54) can be rewritten as

$$\begin{aligned} & \int j r v j^{2(\kappa_1-1)} j r v j^2 \\ &= \int_{\partial} j r v j^{2(\kappa_1-1)} \frac{\partial j r v j^2}{\partial \nu} - (1 - 1) \int j r v j^{2(\kappa_1-2)} |j r v j^2|^2 \\ &= \int_{\partial} j r v j^{2(\kappa_1-1)} \frac{\partial j r v j^2}{\partial \nu} - \frac{4(1-1)}{1} \int j r j r v j^{\kappa_1} j^2. \end{aligned} \quad (55)$$

Using the trace inequality [37, Remark 52.9] that for any $\epsilon > 0$:

$$k' k_{L^2(\partial)} \leq \epsilon k' k_{L^2(\Omega)} + C_\epsilon k' k_{L^2(\Omega)};$$

and the inequality $\frac{\partial |\nabla v|^2}{\partial \nu} \leq 2m j r v j^2$ on ∂ for some constants $m > 0$ (cf. [30, Lemma 4.2] and [13]), we can estimate the first term on the right hand of (55) as

follows

$$\begin{aligned} & \int_{\partial} jrvj^{2(\kappa_1-1)} \frac{\partial jrvj^2}{\partial} - 2m \int_{\partial} jrvj^{2\kappa_1} \\ & = 2m \int_{\partial} k jrvj^{\kappa_1} k_{L^2(\partial)}^2 \\ & \quad - \frac{1}{1} k r jrvj^{\kappa_1} k_{L^2}^2 + c_3 k jrvj^{\kappa_1} k_{L^2}^2. \end{aligned} \quad (56)$$

Substituting (55), (56) into (54), we have

$$\begin{aligned} & \frac{d}{dt} \int jrvj^{2\kappa_1} + \frac{2}{n} \int jrvj^2 jrvj^{2(\kappa_1-1)} + \int jrvj^{2\kappa_1} + \frac{3(1-1)}{1} \int |r jrvj^{\kappa_1}|^2 \\ & \quad - 2 \int jrvj^{2(\kappa_1-1)} r u - r v + c_4 \int jrvj^{2\kappa_1}. \end{aligned} \quad (57)$$

Since $\kappa_1 > 1$ and $k r v k_{L^1} \in C$, then using the Gagliardo-Nirenberg inequality we can find $\theta = \frac{\kappa_1 - \frac{1}{2}}{\kappa_1 + \frac{1}{n} - \frac{1}{2}} \in (0, 1)$ such that

$$\begin{aligned} c_4 k jrvj^{\kappa_1} k_{L^2}^2 & \leq c_5 k r jrvj^{\kappa_1} k_{L^2}^{2\theta} k jrvj^{\kappa_1} k_{L^{\frac{1}{1-\theta}}}^{2(1-\theta)} + c_5 k jrvj^{\kappa_1} k_{L^{\frac{1}{\kappa_1}}}^2 \\ & \leq c_6 k r jrvj^{\kappa_1} k_{L^2}^{2\theta} + c_7 \\ & \quad - \frac{1}{1} k r jrvj^{\kappa_1} k_{L^2}^2 + c_8. \end{aligned} \quad (58)$$

Moreover, we can estimate the first item on the right hand side of (57) as follow

$$\begin{aligned} & \int_{\partial} jrvj^{2(\kappa_1-1)} r u - r v \\ & = \int_{\partial} u jrvj^{2(\kappa_1-1)} - v \int_{\partial} jrvj^{2(\kappa_1-2)} r v - r jrvj^2 \\ & \quad - \frac{2}{n} \int jrvj^2 jrvj^{2(\kappa_1-1)} + \frac{1}{1} \int jrvj^2 jrvj^{\kappa_1} j^2 + c_9 \int u^2 jrvj^{2(\kappa_1-1)}. \end{aligned} \quad (59)$$

Combining (58), (59) with (57), one has

$$\frac{d}{dt} \int jrvj^{2\kappa_1} + \frac{1}{1} \int jrvj^2 jrvj^{\kappa_1} j^2 + \int jrvj^{2\kappa_1} - c_9 \int u^2 jrvj^{2(\kappa_1-1)} + c_8. \quad (60)$$

Similarly, we differentiate the second equation of (8) and multiply it with $r s$ to have

$$\frac{1}{2} \frac{d}{dt} j r s j^2 = r s - r s + (r v - r s) j r s j^2.$$

Multiplying the above identity with $2 \int j r s j^{2(\kappa_2-1)}$ and integrating it with respect to x , we end up with

$$\frac{d}{dt} \int j r s j^{2\kappa_2} + \frac{2}{2} \int j r j r s j^{\kappa_2} j^2 + \int j r s j^{2\kappa_2} - c_{10} \int v^2 j r s j^{2(\kappa_2-1)} + c_{11}. \quad (61)$$

Using Hölder's inequality and Young's inequality, we have

$$\begin{aligned} & \frac{k(k-1)}{3} \int u^k j r s j^2 + c_9 \int u^2 j r v j^{2(\kappa_1-1)} \\ & \leq \frac{k(k-1)}{3} \left(\int u^{k+\theta-1} \right)^{\frac{k}{k+\theta-1}} \left(\int j r s j^{\frac{2(k+\theta-1)}{\theta-1}} \right)^{\frac{\theta-1}{k+\theta-1}} \end{aligned}$$

$$\begin{aligned}
& + c_9 \left(\int u^{k+\theta-1} \right)^{\frac{2}{k+\theta-1}} \left(\int jrvj^{\frac{2(k+\theta-1)(\kappa_1-1)}{k+\theta-3}} \right)^{\frac{k+\theta-3}{k+\theta-1}} \\
& \frac{b}{2} \int u^{k+\theta-1} + c_{12} \int jrsj^{\frac{2(k+\theta-1)}{\theta-1}} + c_{12} \int jrvj^{\frac{2(k+\theta-1)(\kappa_1-1)}{k+\theta-3}}.
\end{aligned} \quad (62)$$

Similarly, using Hölder's inequality and Young's inequality together with Gagliardo-Nirenberg inequality, we can find a $\gamma > \frac{q}{2}$ satisfying $\frac{1}{\lambda} + \frac{1}{\lambda^0} = 1$ and $\gamma = \frac{\frac{1}{q} - \frac{1}{2\lambda}}{\frac{1}{q} + \frac{1}{n} - \frac{n+q}{2n\lambda}} \in (0, 1)$ such that

$$\begin{aligned}
& c_{10} \int v^2 jrvj^{2(\kappa_2-1)} \leq c_{10} \left(\int v^{2\lambda} \right)^{\frac{1}{\lambda}} \left(\int jrsj^{2(\kappa_2-1)\lambda^0} \right)^{\frac{1}{\lambda^0}} \\
& c_{13} \int v^{2\lambda} + c_{13} \int jrvj^{2(\kappa_2-1)\lambda^0} \\
& c_{14} k r v k_{L^{\frac{2n\lambda}{n+q}}}^{2\lambda\theta} k v k_{L^q}^{2\lambda(1-\theta)} + c_{14} k v k_{L^q}^{2\lambda} + c_{13} \int jrvj^{2(\kappa_2-1)\lambda^0} \\
& c_{15} \left(\int jrvj^{\frac{2n\lambda}{n+q}} \right)^{\frac{\theta(n+q)}{n}} + c_{13} \int jrvj^{2(\kappa_2-1)\lambda^0} + c_{16} \\
& = c_{15} \int jrvj^{\frac{2n\lambda}{n+q}} + c_{13} \int jrvj^{2(\kappa_2-1)\lambda^0} + c_{16},
\end{aligned} \quad (63)$$

where the last identity holds by noting that $\frac{\theta(n+q)}{n} = 1$. Then letting $\gamma_1 = \frac{2n\lambda}{n+q}$, $\gamma_2 = \frac{2(k+\theta-1)(\kappa_1-1)}{k+\theta-3}$, $\gamma_3 = \frac{2(k+\theta-1)}{\theta-1}$, $\gamma_4 = 2(\kappa_2-1)$, $\gamma_5 = \frac{2(\kappa_2-1)\lambda}{\lambda-1}$, $c_{17} = \max\{c_{12}, c_{13}, c_{15}\}$ and combining (52), (60) and (61) and using (62) and (63), one has

$$\begin{aligned}
& \frac{d}{dt} \int (u^k + jrvj^{2\kappa_1} + jrsj^{2\kappa_2}) + b \int u^{k+\theta-1} + \int jrvj^{2\kappa_1} + \int jrsj^{2\kappa_2} \\
& + \frac{k}{k-1} \int jru^{\frac{k}{2}} j^2 + \frac{1}{k-1} \int |rjrvj^{\kappa_1}|^2 + \frac{2}{k-2} \int |rjrsj^{\kappa_2}|^2 \\
& \frac{k(k-1)}{3} \int u^k jrsj^2 + c_9 \int u^2 jrvj^{2(\kappa_1-1)} + c_{10} \int v^2 jrvj^{2(\kappa_2-1)} + c_{18} \\
& \frac{b}{2} \int u^{k+\theta-1} + c_{17} \left(\int jrvj^{\gamma_1} + \int jrvj^{\gamma_2} + \int jrsj^{\gamma_3} + \int jrsj^{\gamma_4} \right) + c_{19}.
\end{aligned} \quad (64)$$

Combining Lemma 3.1 and Lemma 3.2, using Gagliardo-Nirenberg inequality and Young's inequality, we have for $i = 1, 2$

$$\begin{aligned}
c_{17} \int jrvj^{\gamma_i} &= c_{17} \left\| jrvj^{\kappa_1} \right\|_{L^{\frac{\gamma_i}{\kappa_1}}}^{\frac{\gamma_i}{\kappa_1}} \\
& c_{20} \left(k r jrvj^{\kappa_1} k_{L^2}^{\theta_i} k jrvj^{\kappa_1} k_{L^{\frac{r}{\kappa_1}}}^{(1-\theta_i)} + k jrvj^{\kappa_1} k_{L^{\frac{r}{\kappa_1}}} \right)^{\frac{\gamma_i}{\kappa_1}} \\
& c_{21} \left(\int |rjrvj^{\kappa_1}|^2 \right)^{\frac{\gamma_i \theta_i}{2\kappa_1}} + c_{22} \\
& \frac{2}{k-2} \int |rjrvj^{\kappa_1}|^2 + c_{23}
\end{aligned} \quad (65)$$

and for $i = 3, 4$

$$c_{17} \int jrsj^{\gamma_i} \leq \frac{2}{k-2} \int |rjrsj^{\kappa_2}|^2 + c_{24}. \quad (66)$$

Substituting (65) and (66) into (64), one has

$$\begin{aligned} & \frac{d}{dt} \int (u^k + jrvj^{2\kappa_1} + jrsj^{2\kappa_2}) + b \int u^{k+\theta-1} + \int jrvj^{2\kappa_1} + \int jrsj^{2\kappa_2} \\ & + \frac{k}{k} \frac{1}{1} \int jru^{\frac{k}{2}}j^2 + \frac{1}{1} \frac{1}{1} \int |rjrvj^{\kappa_1}|^2 + \frac{2}{2} \frac{1}{2} \int |rjrsj^{\kappa_2}|^2 \\ & \frac{b}{2} \int u^{k+\theta-1} + \frac{1}{1} \frac{1}{1} \int |rjrvj^{\kappa_1}|^2 + \frac{2}{2} \frac{1}{2} \int |rjrsj^{\kappa_2}|^2 + c_{25}; \end{aligned}$$

which implies

$$\frac{d}{dt} \int (u^k + jrvj^{2\kappa_1} + jrsj^{2\kappa_2}) + \frac{b}{2} \int u^{k+\theta-1} + \int jrvj^{2\kappa_1} + \int jrsj^{2\kappa_2} \leq c_{25}; \quad (67)$$

Using Young's inequality, we have

$$\int u^k \leq \frac{b}{2} \int u^{k+\theta-1} + c_{26}. \quad (68)$$

Substituting (68) into (67), we have

$$\frac{d}{dt} \int (u^k + jrvj^{2\kappa_1} + jrsj^{2\kappa_2}) + \int u^k + \int jrvj^{2\kappa_1} + \int jrsj^{2\kappa_2} \leq c_{27};$$

which together with Gronwall's inequality gives (51). $\square$

Remark 2. According to Lemma 2.4, when $1 < 2$ we only obtain $q \geq \left[1; \frac{n}{n-2}\right)$, $r \geq \left[1; \frac{n}{n-1}\right)$ and $\kappa \geq \left[1; \frac{nq}{n-q}\right)$. However, if k, κ_1, κ_2 and θ are fixed, (31) is true for $i = 1; 2; 3; 4$ with $q = \frac{n}{n-2}$, $r = \frac{n}{n-1}$ and $\kappa = \frac{nq}{n-q} = \frac{n}{n-3}$. Then by continuity argument (see also [4, 51] for details), we can choose q, r, κ sufficiently close to $\frac{n}{n-2}$, $\frac{n}{n-1}$ and $\frac{n}{n-3}$ respectively such that (31) holds. Similar arguments also hold for the case $\frac{n}{n-1} \geq 2$.

3.3. Proof of Theorem 1.1.

Proof of Theorem 1.1. Since the L^1 -norm of u is uniformly bounded for all $t \geq 0$ (see Lemma 2.1), then we can apply Lemma 2.3 with $p = 1$ to find a constant $c_1 > 0$ independent of t such that $ku(\cdot; t)k_{L^1} + k(v; w)(\cdot; t)k_{W^{1,1}} \leq c_1$ for all $t \geq 0; T_{max}$ when $n \geq 3$. This along with Lemma 2.1 proves Theorem 1.1 in the case of $n \geq 3$. When $n = 4$, Theorem 1.1 is a direct result from the combination of Lemma 2.3, Lemma 3.3 and Lemma 2.1. $\square$

4. Proof of Theorem 1.2. In this section, we are devoting to prove Theorem 1.2. When $f(u) = u(1-u)$, noting $\Delta u = 0$ and using the transformation $s = w - v$, the system (3) becomes

$$\begin{cases} u_t = -u + r(u - s) + u(1-u); & x \in \mathbb{R}; t > 0; \\ s_t = -s + s + (-v)_t; & x \in \mathbb{R}; t > 0; \\ v_t = -v + u - v; & x \in \mathbb{R}; t > 0; \\ \frac{\partial u}{\partial \nu} = \frac{\partial s}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0; & x \in \partial \Omega; t > 0; \\ u(x; 0) = u_0(x); s(x; 0) := w_0(x) - v_0(x); v(x; 0) = v_0(x); & x \in \mathbb{R}; \end{cases} \quad (69)$$

The existence of global classical solution can be obtained directly from Theorem 1.1 by noting $\Delta u = 0$ in this case. Hence in the following subsections, we will focus

on studying the large time behavior of solutions to complete the proof of Theorem 1.2.

4.1. Construction of an energy functional. Next, based on the idea in [39], we will construct a Lyapunov functional by studying the time evolution of each of the integrals therein.

Lemma 4.1. *Let $(u; s; v)$ be the global classical solution of the system (69). Then we have*

$$\frac{d}{dt} \int (u - 1 - \ln u) = \int (u - 1)^2 + \frac{1}{4} \int j r s j^2 \quad (70)$$

for all $t > 0$.

Proof. Motivated by the ideas from [39], we multiply the first equation of the system (69) by $(1 - \frac{1}{u})$ to obtain

$$\begin{aligned} \frac{d}{dt} \int (u - 1 - \ln u) &= \int u^2 + 2 \int u - j j - \int \frac{j r u j^2}{u^2} - \int \frac{r u - r s}{u} \\ &= \int (u - 1)^2 - \int \frac{j r u j^2}{u^2} - \int \frac{r u - r s}{u}. \end{aligned} \quad (71)$$

Using Young's inequality, one has

$$\int \frac{r u - r s}{u} \leq \int \frac{j r u j^2}{u^2} + \frac{1}{4} \int j r s j^2,$$

which together with (71) gives (70). $\square$

Lemma 4.2. *Suppose $(u; s; v)$ is a global classical solution of the system (69). Then it has*

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int \left(s - \frac{(\quad)}{2} \right)^2 &= \int j r s j^2 - \frac{1}{2} \int \left(s - \frac{(\quad)}{2} \right)^2 \\ &\quad + \frac{2(\quad)^2}{2} \int (v - \quad)^2; \end{aligned} \quad (72)$$

Proof. Testing the second equation of (69) by $s - \frac{\chi \alpha (\beta - \delta)}{\delta \beta}$, we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int \left(s - \frac{(\quad)}{2} \right)^2 &= \int j r s j^2 + (\quad) \int v s - \int s^2 + \frac{(\quad)}{2} \int s - \frac{2(\quad)^2}{2} \int v \\ &\quad - \int j r s j^2 - \frac{1}{2} \int s^2 + \frac{(\quad)}{2} \int s + \frac{2(\quad)^2}{2} \int v^2 - \frac{2(\quad)^2}{2} \int v \\ &= \int j r s j^2 - \frac{1}{2} \int \left(s - \frac{(\quad)}{2} \right)^2 + \frac{2(\quad)^2}{2} \int (v - \quad)^2; \end{aligned}$$

which yields (72), where we have used

$$(\quad) \int v s - \frac{1}{2} \int s^2 + \frac{2(\quad)^2}{2} \int v^2 =$$

$\square$

Lemma 4.3. Assume that $(u; s; v)$ is a global classical solution of the system (69). Then we have

$$\frac{1}{2} \frac{d}{dt} \int (v - 1)^2 - \int j r v j^2 - \frac{1}{2} \int \left(v - 1 \right)^2 + \frac{1}{2} \int (u - 1)^2; \quad (73)$$

Proof. We multiply the third equation of the system (69) by $v - \frac{\alpha}{\beta}$, and then use Young's inequality to derive that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int (v - 1)^2 &= \int j r v j^2 - \int v^2 + \int u v - \frac{1}{2} \int u + \int v \\ &\quad - \int j r v j^2 - \frac{1}{2} \int v^2 + \frac{1}{2} \int u^2 - \frac{1}{2} \int u + \int v \\ &= \int j r v j^2 - \frac{1}{2} \int \left(v - 1 \right)^2 + \frac{1}{2} \int (u - 1)^2; \end{aligned}$$

which yields (73). $\square$

Lemma 4.4. Let $(u; s; v)$ be the global classical solution of the system (69). Suppose

$$> \frac{1}{8} \frac{1}{2} \left(\frac{1}{2} \right)^2;$$

then there exist positive constants ϵ_1, ϵ_2 such that for all $t > 0$, the function

$$F(t) := \int (u - 1 - \ln u) + \frac{1}{2} \int \left(s - \frac{1}{2} \right)^2 + \frac{1}{2} \int (v - 1)^2$$

satisfies

$$F'(t) \leq D(t); \quad (74)$$

where

$$D(t) := D \left\{ \int (u - 1)^2 + \int \left(s - \frac{1}{2} \right)^2 + \int (v - 1)^2 \right\}$$

with the constant $D > 0$ defined by (77).

Proof. Since $> \frac{\chi^2 \alpha^2 (\beta - \delta)^2}{8 \delta \beta^2}$ and hence $\frac{2 \beta \mu}{\alpha^2} > \frac{\chi^2 (\beta - \delta)^2}{4 \delta \beta}$, then we can find a positive constant ϵ_2 satisfying

$$\frac{2}{2} > \epsilon_2 > \frac{1}{4} \left(\frac{1}{2} \right)^2. \quad (75)$$

The combination of Lemma 4.1, 4.2 and 4.3 gives

$$\begin{aligned} F'(t) &= \int (u - 1)^2 + \frac{1}{4} \int j r s j^2 \\ &\quad + \epsilon_1 \left\{ \int j r s j^2 - \frac{1}{2} \int \left(s - \frac{1}{2} \right)^2 + \frac{1}{2} \left(\frac{1}{2} \right)^2 \int (v - 1)^2 \right\} \\ &\quad + \epsilon_2 \left\{ \int j r v j^2 - \frac{1}{2} \int \left(v - 1 \right)^2 + \frac{1}{2} \int (u - 1)^2 \right\} \\ &= \left(\frac{2}{2} \right) \int (u - 1)^2 - \frac{1}{2} \int \left(s - \frac{1}{2} \right)^2 \\ &\quad + \left(\epsilon_1 - \frac{1}{4} \right) \int j r s j^2 - \left(\frac{2}{2} - \frac{1}{2} \left(\frac{1}{2} \right)^2 \right) \int (v - 1)^2; \end{aligned} \quad (76)$$

Letting $\alpha = \frac{1}{4}$ and noting (75), then from (76), we have (74) by taking

$$D := \min f \left(\frac{2}{2}, \frac{2}{8}; \frac{2}{2}, \frac{2}{8} \right) \frac{2}{8} g > 0: \quad (77)$$

□

4.2. Convergence of solutions. Next, we will study the convergence of the solution. Before that, we first give a basic lemma which will be used later to study the properties of the function $F(t)$.

Lemma 4.5. *Assume that $\phi(z) := z - 1 - \ln z; z > 0$. Then $\phi(z)$ is nonnegative with $\phi(1) = \phi'(1) = 0$, and there exists a constant $c > 0$ such that*

$$\frac{1}{c} \phi(z)^2 \leq \phi(z) \leq c \phi(z)^2; \quad \text{for all } z \geq \left(\frac{1}{2}, \frac{3}{2}\right):$$

Proof. Since $\phi'(z) = 1 - \frac{1}{z}$ and $\phi''(z) = \frac{1}{z^2} > 0$ for all $z > 0$, we can obtain all the statements immediately. □

Lemma 4.6. *Let $\alpha > \frac{\alpha^2 \chi^2 (\beta - \delta)^2}{8\delta\beta^2}$ and $(u; s; v)$ be the global classical solution of the system (69). Then one has*

$$\int_1^t \int \left[(u - 1)^2 + \left(s - \frac{(\quad)}{\quad} \right)^2 + \left(v - \quad \right)^2 \right] < 1: \quad (78)$$

Proof. Using Lemma 4.5 and the definition of $F(t)$ in Lemma 4.4, one can derive that $F(t)$ is nonnegative. Hence from (74), we have

$$\int_1^t D(\cdot) d\tau \leq F(1) - F(t) \leq F(1); \quad \text{for all } t > 1:$$

Using the definition of $D(t)$, one can derive (78) directly. □

Next, we will use (78) to show the convergence of solution $(u; v; w)$ in L^∞ -norm. Before that, we first improve the regularities of solution $(u; v; w)$.

Lemma 4.7. *Let $(u; v; w)$ be the solution obtained in Theorem 1.1. Then there exists a constant $\alpha \in (0, 1)$ such that*

$$\|u\|_{C^{\sigma, \frac{\sigma}{2}}(\times[t, t+1])} + \|v\|_{C^{2+\sigma, 1+\frac{\sigma}{2}}(\times[t, t+1])} + \|w\|_{C^{2+\sigma, 1+\frac{\sigma}{2}}(\times[t, t+1])} \leq C \quad (79)$$

for all $t \geq 1$.

Proof. From Theorem 1.1, we know that u, v, w and $u(1 - u)$ is bounded in $L^\infty(\times(0, 1))$. Then applying the standard parabolic regularity theory (e.g. see [35, Theorem 1.3] and [39, Lemma 3.2]), from the first equation of system (3), we can find a constant $\alpha \in (0, 1)$ such that

$$\|u\|_{C^{\sigma, \frac{\sigma}{2}}(\times[t, t+1])} \leq c_1 \quad \text{for all } t > 1: \quad (80)$$

Moreover, from the second and third equation of the system (3), we can use the standard parabolic Schauder theory [19] to obtain

$$\|v; w\|_{C^{2+\sigma, 1+\frac{\sigma}{2}}(\times[t, t+1])} \leq c_2 \quad \text{for all } t > 1: \quad (81)$$

The combination of (80) and (81) yields (79). □

Using the similar argument as in [39, Lemma 3.10], we will show the uniform convergence of solution without convergence rate.

Lemma 4.8. Under $\gamma > \frac{\alpha^2 \chi^2 (\beta - \delta)^2}{8\delta\beta^2}$, let $(u; s; v)$ be the global classical solution of the system (69). Then it follows that

$$ku(\cdot; t) - 1k_{L^1} \rightarrow 0; \text{ as } t \rightarrow 1; \quad (82)$$

and

$$ks(\cdot; t) - \frac{1}{2}k_{L^1} \rightarrow 0; \quad kv(\cdot; t) - k_{L^1} \rightarrow 0; \text{ as } t \rightarrow 1; \quad (83)$$

Proof. Since $(u; s; v)$ is the global classical solution of the system (69), we can study the convergence properties of solution with the C^0 -norm. If one can show that

$$ku(\cdot; t) - 1k_{C^0} \rightarrow 0; \text{ as } t \rightarrow 1; \quad (84)$$

then (82) follows directly.

Next, we will show the statement (84) is right. In fact, suppose that (84) is wrong, then for some $c_1 > 0$, there exist some sequences $(x_j)_{j \in \mathbb{N}}$ and $(t_j)_{j \in \mathbb{N}} \subset (0; 1)$ satisfying $t_j \rightarrow 1$ as $j \rightarrow \infty$ such that

$$ju(x_j; t_j) - 1j \geq c_1; \text{ for all } j \in \mathbb{N}.$$

From Lemma 4.7, we know that $u - 1$ is uniformly continuous in $(1; 1)$, then there exists $r > 0$ and $T_1 > 0$ such for any $j \in \mathbb{N}$,

$$ju(x; t) - 1j \leq \frac{c_1}{2} \text{ for all } x \in B_r(x_j) \setminus \{x_j\} \text{ and } t \in (t_j; t_j + T_1); \quad (85)$$

Because of the smoothness of ϕ , we can get a constant $c_2 > 0$ such that

$$jB_r(x_j) \setminus \{x_j\}j \leq c_2; \text{ for all } x_j \in \mathbb{R}^n. \quad (86)$$

Using (85) and (86), then for all $j \in \mathbb{N}$, we have

$$\begin{aligned} \int_{t_j}^{t_j + T_1} \int ju(x; t) - 1j^2 dx dt &= \int_{t_j}^{t_j + T_1} \int_{B_r(x_j) \cap \mathbb{R}^n} ju(x; t) - 1j^2 dx dt \\ &\leq \int_{t_j}^{t_j + T_1} jB_r(x_j) \setminus \{x_j\}j \left(\frac{c_1}{2}\right)^2 dt \\ &\leq \frac{c_1^2 c_2 T_1}{4}, \end{aligned}$$

which however contradict the fact that as $j \rightarrow \infty$, $t_j \rightarrow 1$

$$\int_{t_j}^{t_j + T_1} \int (u(x; t) - 1)^2 dx dt = \int_{t_j}^{\infty} \int (u(x; t) - 1)^2 dx dt \neq 0 \text{ as } j \rightarrow \infty.$$

Then (84) and (82) actually are true. We can derive (83) by using the similar argument. Hence the proof of Lemma 4.8 is completed. $\square$

4.3. Exponential convergence.

Lemma 4.9. Let $\gamma > \frac{\alpha^2 \chi^2 (\beta - \delta)^2}{8\delta\beta^2}$ and $(u; s; v)$ be the global classical solution of the system (69). Then there exist $t_0 > 0$ and $\gamma_3 > 0$ such that

$$u(x; t) \in \left(\frac{1}{2}, \frac{3}{2}\right) \text{ for all } x \in \mathbb{R}^n \text{ and } t > t_0; \quad (87)$$

and such that the functions $F(t)$ and $D(t)$ defined in Lemma 4.4 satisfy

$$D(t) \leq \gamma_3 F(t) \text{ for all } t > t_0; \quad (88)$$

Proof. From (82), we can get a $t_0 > 0$ such that for all $t > t_0$

$$ku - 1k_{L^1} < \frac{1}{2};$$

which gives (87) immediately. From Lemma 4.5, we can get two constants $c_1, c_2 > 0$ such that

$$c_1(z - 1)^2 \leq z - 1 \leq \ln z \leq c_2(z - 1)^2 \text{ for all } z \geq \left(\frac{1}{2}, \frac{3}{2}\right); \quad (89)$$

Hence, using (87) and (89) and choosing $\frac{1}{\gamma_3} := \frac{\min\{c_2, \frac{\gamma_1}{2}, \frac{\gamma_2}{2}\}}{D}$, we have

$$F(t) \leq c_2 \int (u - 1)^2 + \frac{1}{2} \int \left(s - \frac{(\quad)}{\quad}\right)^2 + \frac{2}{2} \int (v - \quad)^2 - \frac{1}{3} D(t);$$

which yields (88). The proof of this lemma is completed. $\square$

Lemma 4.10. Assume that $\alpha > \frac{\alpha^2 \chi^2 (\beta - \delta)^2}{8\delta\beta^2}$, and suppose $(u; s; v)$ is the global classical solution of the system (69). Then there exists a constant $c > 0$ such that for all $t > 0$

$$ku - 1k_{L^2} \leq ce^{-\frac{\gamma_3}{2}t}; \quad (90)$$

and

$$ks - \frac{(\quad)}{\quad}k_{L^2} \leq ce^{-\frac{\gamma_3}{2}t}; \quad kv - \quad k_{L^2} \leq ce^{-\frac{\gamma_3}{2}t}; \quad (91)$$

where γ_3 is given in Lemma 4.9.

Proof. The combination of Lemma 4.9 and (74) show that there exists $t_0 > 0$ such that for all $t > t_0$, one has $u(x; t) \geq \frac{1}{2}$ and $\frac{3}{2}$ and

$$F'(t) \leq -D(t) - \gamma_3 F(t); \quad (92)$$

After some calculations, from (92), we have

$$F(t) \leq F(t_0)e^{-\gamma_3(t-t_0)}; \text{ for all } t > t_0;$$

which together with the fact that

$$F(t) \leq c_1 \int (u - 1)^2 + \frac{1}{2} \int \left(s - \frac{(\quad)}{\quad}\right)^2 + \frac{2}{2} \int (v - \quad)^2$$

gives (90) and (91). The proof of Lemma 4.10 is completed. $\square$

Next, we will use the interpolation inequality to derive the uniform exponential stabilization property.

Lemma 4.11. Suppose $\alpha > \frac{\alpha^2 \chi^2 (\beta - \delta)^2}{8\delta\beta^2}$, and let $(u; s; v)$ be the global classical solution of the system (69). Then there exist two constants $c > 0$ and $\gamma_4 > 0$ such that for all $t > 0$

$$ku(\cdot; t) - 1k_{L^1} \leq ce^{-\gamma_4 t}; \quad (93)$$

and

$$ks(\cdot; t) - \frac{(\quad)}{\quad}k_{L^1} \leq ce^{-\gamma_4 t}; \quad kv(\cdot; t) - \quad k_{L^1} \leq ce^{-\gamma_4 t}; \quad (94)$$

Proof. Based on (79), one can readily get a constant $c_3 > 0$ (e.g. see [39, Lemma 3.14]) such that

$$ku(; t)k_{W^{1,1}} \leq c_3; \text{ for all } t > 1;$$

This, along with (90) and the Gagliardo-Nirenberg inequality, yields

$$ku(; t) \leq k_{L^1} \leq c_1 ku(; t) \leq k_{W^{1,1}}^{\frac{n}{n+2}} k_{L^2}^{\frac{2}{n+2}} \leq ce^{-\gamma_4 t}; \text{ for all } t > t_0;$$

which gives (93). Similarly, using the boundedness of $ks(; t)k_{W^{1,1}}$ and $kv(; t)k_{W^{1,1}}$, and Gagliardo-Nirenberg inequality, we can derive (94). $\square$

Proof of Theorem 1.2. The existence of global classical solution is a direct result of Theorem 1.1 by noting $\gamma = 2$ if $f(u) = u(1 - u)$. From Lemma 4.11, we only need to prove the convergence rate of w to complete the proof of Theorem 1.2. In fact, since $s = w - v$ and $u = w + v$, then using (94), we have

$$\begin{aligned} w - u &= \frac{s}{2} + \frac{v}{2} - u \\ &= \frac{1}{2} \left(s - \frac{u^2}{2} \right) + - \left(v - \frac{u^2}{2} \right); \end{aligned}$$

and hence

$$\begin{aligned} kw - u &\leq k_{L^1} \leq \frac{1}{2} ks - \frac{u^2}{2} k_{L^1} + -kv - k_{L^1} \\ &\leq c_1 e^{-\gamma_4 t}; \end{aligned}$$

which together with the convergence rate of u, v in Lemma 4.11 finish the proof of Theorem 1.2. $\square$

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