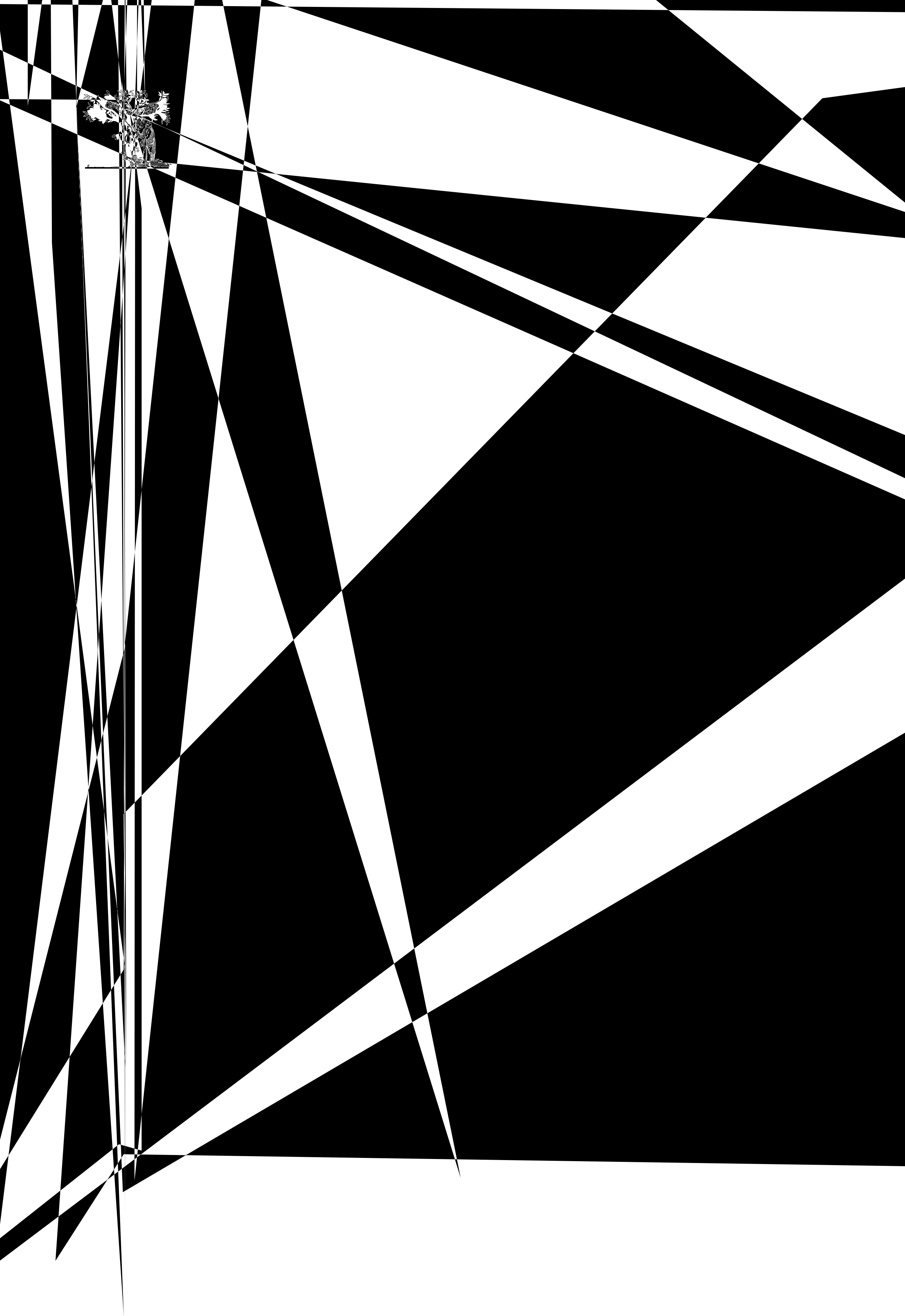

where G is the vector field and the driving signal $W : [0, T] \to \mathbb{R}^d$ is irregular, that is, its paths are only Hölder continuous with possible small Hölder exponent, and in particular not differentiable. Thus the Itô solution map $W \mapsto Y$ is not closable for any reasonable topology on the respective path spaces. Instead, Lyons [1] enriched the path W with some "extra information" $\mathbb{W}$, often called Lévy area, leading to the birth of rough path $\mathbf{W} = (W, \mathbb{W})$. The relevant lifting approaches with certain algebraic and analytical conditions can be found in [2–4], [5, Charpter III], or [6, Chapter 10]. Thereupon, it turns out to be rough differential equations

$$dY_t = G(Y_t)d\mathbf{W}_t, \quad Y_t \in \mathbb{R}^m, \mathbf{W}_t \in \mathbb{R}^d \times \mathbb{R}^{d\otimes d}. \tag{1.1}$$

In this situation, Lyons' universal limit theorem is sure that the Itô-Lyons solution map $\mathbf{W} \mapsto Y$ is continuous under a suitable topology on the space of rough paths. For example, it can be interpreted as Lyons-Davie [1,7,8], Friz-Victoir [5] sense or fractional calculus [9–13].

On the other hand, Gubinelli [14] presented an alternative formulation to interpret the solution of (1.1) as a controlled rough path, which will be elaborated below. In particular, all paths controlled by W composite a Banach space under suitable topology, that provides a lot of convenience for further algebraic consideration, see also [15,16] for an extension to Hölder exponent γ in $(0, 1/3]$.

Natural propagation of rough paths to infinite-dimensional equations has been investigated in distinct ways [17–21]. Thereinto, the employment of semigroup approach can be traced to Gubinelli & Tindel [19], and its variants [20–23]. In this respect, the interesting equation reads

$$dY_t = AY_t dt + G(Y_t)d\mathbf{W}_t,$$

where Y is defined in an infinite-dimensional space . To formulate its mild solution, the well-posedness of rough convolution $\int_0^t S_{t-s}G(Y_s)d\mathbf{W}_s$ should be meaningful, in which $\{S_t\}_{t\geq 0}$ is the C_0-semigroup generated by the linear operator A. Contrary to the finite-dimensional setting, an obstacle to this subject will be encountered due to the time-singularity of $\{S_t\}_{t\geq 0}$ at zero. To be more precise, let $Y \in C^\gamma([0, T]; \ )$ be controlled by $W \in C^\gamma([0, T]; \mathbb{R})$ in Gubinelli's sense [14], namely, there is a $Y' \in C^{2\gamma}([0, T]; \ )$ such that the remainder term R^Y belongs to $C^{2\gamma}([0, T]; \ )$ and satisfies the relation

$$Y_{s,t} = Y'_s W_{s,t} + R^Y_{s,t}.$$

It is known that $S_{t-\cdot}G(Y)$ is controlled with $[S_{t-\cdot}G(Y)]' = S_{t-\cdot}DG(Y)G(Y)$ in finite-dimensional setting. Nevertheless, it is hard to say $S_{t-\cdot}G(Y)$ is also controlled by W regardless of the smoothness of G since $S_\cdot$ is not continuous on . To avoid this embarrassing situation, Gubinelli & Tindel [19] introduced operator-valued rough paths associated to S and weakly controlled rough paths, aiming to compare $Y_t - S_{t-s}Y_s$ rather than $Y_{s,t}$ as mentioned, where $\mathbf{W}$ is assumed to be infinite-dimensional. Recently, Hesse & Neamţu [22,23] reported a different feasible technique to solve the same problem. But if $\mathbf{W}$ is finite-dimensional, rough evolution equation can be handled in a scale of interpolation spaces $\{\ _\alpha\}_{\alpha\in\mathbb{R}}$, on which the semigroup $\{S_t\}_{t\geq 0}$ is a bounded operator family [20,21]. In the sequel, the global-in-time solution for rough partial differential equation has been established [24].

In the framework of rough paths theory, considerable progress has been made understanding the behavior of solutions to rough differential equations (RDEs) over the past two decades. For example, Cass and Friz [25] proved the existence of smooth density for the solution to RDEs of the form (1.1) driven by Gaussian process under Hörmander's condition. Cass, Litterer & Lyons [26] further showed the integrability and explicit tail-estimates for Jacobian of the solution, which is an essential condition in [27,28] for the smoothness of density. Many other references are available for this topic [29–32,20], and [33,34] for other topics.

To our knowledge, there are few works on dynamical properties of RDEs by combining the theory of rough paths [3,17] and random dynamical systems [35]. Compared with Itô calculus, rough paths theory may stimulate more possibility for this subject because the solutions to RDEs are understood in a pathwise sense. No exceptionally zero-measure sets depending on the time and initial values appear, a forbidden situation when generating random dynamical systems in infinite-dimensional spaces, while it is generally inevitable for Itô evolution equations. In this respect, Bailleul, Riedel & Scheutzow [36] first systematically combined rough paths theory and random dynamical system and then proved that the solution to a class of rough ordinary differential equations (RODEs) with drifts can generate a continuous random dynamical system under common conditions. Following it, Duc [37] further proved the existence and upper semi-continuity of the global pullback attractor for rough dissipative systems. Partially different from [36], Neamţu & Kuehn [38] studied rough center manifolds for a controlled RODE in mild sense, see also [39] for Wong-Zakai approximation. As for stochastic rough partial differential equations (RPDEs), the relevant research results on random dynamics are rare. It is worth noting that Neamţu & Kuehn [40] proved the center manifolds for the following semilinear equation

$$dY_t = [AY_t + F(Y_t)]dt + G(Y_t)d\mathbf{W}_t \tag{1.2}$$

on a family of interpolation spaces $\{\ _{\alpha}\}_{\alpha\in\mathbb{R}}$ driven by multiplicative rough noise, where $\mathbf{W}$ is Hölder continuous with exponent in $(1/3, 1/2]$.

The aim of this paper is to develop a theory for random attractors of RPDEs driven by nonlinear multiplicative noise, which goes far beyond the case of a fractional Brownian motion. Due to the difficulty of directly dealing with stochastic (partial) differential equations driven by a nonlinear multiplicative fractional Brownian motion with order $\gamma \in (1/2, 1)$, the stopping time technique was exploited to control the size of driving noise in finite [41] and infinite-dimensional [42] settings respectively. Later on, the same philosophy was further employed in other situations [43–45]. There is little doubt about the possibility to extend the stopping time technique to a low regularity case, but the question is then in what sense that is so. To this aim, a novel sequence of stopping times $\{T_i\}_{i\in\mathbb{Z}}$ is constructed depending on the rough path $\mathbf{W}$ and the Lipschitz parameters of F and G, which has not been employed in rough settings so far. Thereby the γ-Hölder seminorm of $\mathbf{W}$ could be bounded by a constant on each time interval $[T_i, T_{i+1}], i \in \mathbb{Z}$, which allows us to apply a discrete-time Gronwall-like inequality involving an infinite sum. However, another encountering challenge arises from the convergence issue of the infinite sum mentioned. In this regard, analysis for the growth rate and cocycle property of $\{T_i\}_{i\in\mathbb{Z}}$ is conducted such that $T_i - T_j$ could be controlled by $i - j$ in some appropriate sense. Other more properties of $\{T_i\}_{i\in\mathbb{Z}}$ are further discussed to obtain the temperedness and measurability of the required absorbing set.

This paper is structured as follows. Section 2 first provides some basics concerning the theories of rough paths and random dynamical systems, and then reformulate the equations of study. Section 3 concerns the dynamics of RPDEs. Subsection 3.1 derives key estimates for each component of the mild solution. Based on that, a sequence of stopping times is introduced and analyzed in Subsection 3.2. In the rest of this section, with nice properties of stopping times at hand, the existence of a random attractor follows using a random dynamical systems approach.

Finally, an illustrative example of the reaction-diffusion equation with fractional Brownian rough paths is presented in Section 4.

2. Preliminaries

Let us present in this section a short summary on rough paths theory and random dynamical system, which are extracted from [35,5,36,6,21,14] and other references therein. Denote by and Banach spaces, (;) the space of linear operators from to , $\mathbb{E}$ is frequently used for the expectation, $\otimes$ for the tensor product, C^m for the space of m-times differentiable functions, C_b^m for the space of m-times differentiable functions with bounded derivatives, and C^α for α-Hölder functions.

2.1. Rough paths theory

For $\gamma \in (1/3, 1/2]$ and any compact interval $J \subset \mathbb{R}$ containing zero, one can define $\mathbf{W} = (W, \mathbb{W})$ as an γ-Hölder rough path. That is,

$$W \in C^\gamma(J; \ ) \quad \text{and} \quad \mathbb{W} \in C^{2\gamma}(\Delta_J; \ \otimes \ )$$

and the Chen's relation reads

$$\mathbb{W}_{s,t} - \mathbb{W}_{s,u} - \mathbb{W}_{u,t} = W_{s,u} \otimes W_{u,t} \tag{2.1}$$

where $\Delta_J = \{(s,t) \in J^2 : s \le t\}$ and the increment $W_{s,t} = W_t - W_s$. The term $\mathbb{W}$ is referred to as Lévy area or second order process and it can be thought of as the iterated integral

$$\mathbb{W}_{s,t} = \int_s^t W_{s,r} \otimes dW_r.$$

Without loss of generality, assume that $\ = \mathbb{R}$ since the generalization to higher but finite dimensions can be done component-wise and does not require any additional arguments. Denote ${}^{\gamma}(J;\mathbb{R})$ by the space of γ-Hölder continuous rough paths with an appropriate distance between two γ-Hölder rough paths $\mathbf{W} = (W, \mathbb{W})$ and $\tilde{\mathbf{W}} = (\tilde{W}, \tilde{\mathbb{W}})$, defined by

$$\rho_{\gamma,J}\left(\mathbf{W}, \tilde{\mathbf{W}}\right) = \sup_{(s,t)\in\Delta_J} \frac{\left|W_{s,t} - \tilde{W}_{s,t}\right|}{|t-s|^\gamma} + \sup_{(s,t)\in\Delta_J} \frac{\left|\mathbb{W}_{s,t} - \tilde{\mathbb{W}}_{s,t}\right|}{|t-s|^{2\gamma}} \tag{2.2}$$

where $|\cdot|$ denotes the Euclidean norm. Indeed, W_0 is always assumed to be zero and thus we usually write $\mathbf{W}_t = \mathbf{W}_{0,t} = (W_t, \mathbb{W}_{0,t})$ to express the time evolution of rough path $\mathbf{W}$. Also, set $\rho_{\gamma,J}(\mathbf{W}) = \rho_{\gamma,J}(\mathbf{W}, 0)$. Then the mapping $t \mapsto \rho_{\gamma,[0,t]}(\mathbf{W})$ is continuous on J provided that $\mathbf{W} \in {}^{\gamma'}(J;\mathbb{R})$, $\gamma < \gamma' \le 1/2$. In fact, by means of the γ'-Hölder continuity of $\mathbf{W}$, one has

$$\begin{aligned} &\rho_{\gamma,0,T}(\mathbf{W}) - \rho_{\gamma,0,S}(\mathbf{W}) \\ &\le \sup_{S\le s\le t\le T} \frac{|W_{s,t}|}{|t-s|^\gamma} + \sup_{S\le s\le t\le T} \frac{|\mathbb{W}_{s,t}|}{|t-s|^{2\gamma}} + \sup_{0\le s\le t\le S} \frac{|W_{s,t}|}{|t-s|^\gamma} \sup_{S\le s\le t\le T} \frac{|W_{s,t}|}{|t-s|^\gamma} \end{aligned}$$

$$
\begin{aligned}
&\le \rho_{\gamma',[S,T]}(\mathbf{W})(T-S)^{\gamma'-\gamma} + \rho_{\gamma,[0,S]}(\mathbf{W})\rho_{\gamma',[S,T]}(\mathbf{W})(T-S)^{\gamma'-\gamma} \\
&\le \rho_{\gamma',[0,T]}(\mathbf{W})(T-S)^{\gamma'-\gamma} + \rho_{\gamma,[0,T]}(\mathbf{W})\rho_{\gamma',[0,T]}(\mathbf{W})(T-S)^{\gamma'-\gamma},
\end{aligned}
$$

for any $0 \le S \le T$.

Given a separable Banach space $(\ \ , |\cdot|)$, a path $Y \in C^\gamma(J;\ \ )$ is called to be controlled by $W \in C^\gamma(J;\mathbb{R})$ if there exists a $Y' \in C^\gamma(J;\ \ )$ and $R^Y \in C^{2\gamma}(\Delta_J;\ \ )$ such that

$$
Y_{s,t} = Y'_s \otimes W_{s,t} + R^Y_{s,t}, \quad (s,t) \in \Delta_J. \tag{2.3}
$$

Herein, Y' is called Gubinelli derivative of Y, which is unique as long as $W \in C^\gamma \setminus C^{2\gamma}$. All tubes (Y, Y') enjoying (2.3) define a Banach space $\ ^{2\gamma}_{W,J}$ under the norm

$$
\|Y, Y'\|_{\mathcal{D}^{2\gamma}_{W,J}} := |Y_0| + |Y'_0| + \|R^Y\|_{2\gamma,J}.
$$

It is clear that $(G(Y), DG(Y)Y')$ is controlled by W provided that (Y, Y') is a controlled rough path and $G:\ \ \to\ \ $ is in C^2_b. When taking a C_0-semigroup $S.$ into account, however, it is still a challenging work to find the so-called Gubinelli derivative of $S.(G(Y.))$ in infinite-dimensional settings regardless of the smoothness of G. The main tackle lies in the lower regularity of the semigroup on infinite dimensional space. In order to compensate the time regularity of $S.$, one needs the concept of interpolation spaces.

Definition 2.1. [21, Definition 2.1] A family of separable Banach spaces $\{\ _\alpha, |\cdot|_\alpha\}_{\alpha\in\mathbb{R}}$ is called a monotone family of interpolation spaces if the space $\ _{\alpha_2} \subset\ _{\alpha_1}$ with continuous and dense embedding for $\alpha_1 \le \alpha_2$, and meanwhile

$$
|z|^{\alpha_3-\alpha_1}_{\alpha_2} \lesssim |z|^{\alpha_3-\alpha_2}_{\alpha_1} |z|^{\alpha_2-\alpha_1}_{\alpha_3}
$$

holds for $\alpha_1 \le \alpha_2 \le \alpha_3$ and $z \in\ _{\alpha_3}$.

Remark 2.1. Herein $x \lesssim y$ means there is a constant $C > 0$ such that $x \le Cy$. Also, we always let $\|\cdot\|_{\gamma,J,\alpha}$ refer to the γ-Hölder norm in $\ _\alpha$. As a convention, the indexes in $\|\cdot\|_{\gamma,J,\alpha}$ describe the time and space regularities, respectively. Additionally, the symbol $\|\cdot\|_{\infty,J,\alpha}$ stands for the supremum norm in $\ _\alpha$.

It is ready to introduce the concept of a controlled rough path based on a monotone family of interpolation spaces $\{\ _\alpha\}_{\alpha\in\mathbb{R}}$.

Definition 2.2. [21, Definition 4.3] The pair (Y, Y') is called controlled rough path on the family $\{\ _\alpha\}_{\alpha\in\mathbb{R}}$ provided that $Y' \in \big(C(J;\ _{\alpha-\gamma}) \cap C^\gamma(J;\ _{\alpha-2\gamma})\big)$, $Y \in C^\gamma(J;\ _\alpha)$ satisfy (2.3) with the remainder R^Y being in $C^\gamma(\Delta_J;\ _{\alpha-\gamma}) \cap C^{2\gamma}(\Delta_J;\ _{\alpha-2\gamma})$.

Denote the space of controlled rough paths on $\{\ _\alpha\}_{\alpha\in\mathbb{R}}$ by $\ ^{2\gamma}_{W,J,\alpha}$. For simplicity, we may drop the symbol W when it causes no confusion, that is $\ ^{2\gamma}_{J,\alpha}$. It can be shown that $\ ^{2\gamma}_{J,\alpha}$ is also a Banach space equipped with the norm

$$\|Y, Y'\|_{\mathcal{D}^{2\gamma}_{J,\alpha}} := \|Y\|_{\infty,J,\alpha} + \|Y'\|_{\infty,J,\alpha-\gamma} + \|Y'\|_{\gamma,J,\alpha-2\gamma}$$
$$+ \|R^Y\|_{\gamma,J,\alpha-\gamma} + \|R^Y\|_{2\gamma,J,\alpha-2\gamma}.$$

Let $\{S_t\}_{t\geq 0}$ be an analytic C_0-semigroup between $\{\ \ _\alpha\}_{\alpha\in\mathbb{R}}$ satisfies properties that

$$\|S_t\|_{\mathcal{L}(\mathcal{B}_\alpha,\mathcal{B}_{\alpha+\sigma})} \lesssim t^{-\sigma}, \tag{2.4a}$$
$$\|S_t - \mathrm{Id}\|_{\mathcal{L}(\mathcal{B}_{\alpha+\sigma},\mathcal{B}_\alpha)} \lesssim t^{\sigma}, \tag{2.4b}$$

where $\alpha \in \mathbb{R}, \sigma \in [0, 1]$. Then the concept of rough paths integral on $\{\ \ _\alpha\}_{\alpha\in\mathbb{R}}$ is explained as follows:

Lemma 2.1. *[21, Theorem 4.5] For any given $T > 0$, $(U, U') \in\ ^{2\gamma}_{[0,T],\alpha}$, the rough integral*

$$\int_s^t S_{t-r} U_r$$

Remark 2.2. Since the rough integral is defined independently on the concert choice of partitions, let $\Pi(s,\tau)$ and $\Pi(\tau,t)$ be partitions of intervals [

for each $x \in$. If $M(\omega)$ is further compact in almost surely, then M is called a compact random set in , see [46, Chapter III]. In our interest, M is tempered, which means there exists a random variable $r(\omega) \geq 0$ such that $M(\omega)$ is contained in the $r(\omega)$-ball centered at the origin and

$$\lim_{t \to \pm\infty} \frac{\log^+ r(\theta_t \omega)}{|t|} = 0.$$

Let be a random sets family in containing all tempered random sets, namely,

$$= \Big\{ M : M \text{ is a tempered random set in}$$

where $\mathbf{W}$ is a γ'-Hölder rough path cocycle over an ergodic metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$ with $1/3 < \gamma < \gamma' \leq 1/2$. Let $-A$ be a generator of an analytic C_0-semigroup $\{S_t\}_{t\geq 0}$ between $\{\mathcal{B}_\alpha\}_{\alpha\in\mathbb{R}}$ with further property that

$$\|S_t\|_{\mathcal{L}(\mathcal{B}_\alpha)} \leq Ce^{-\lambda t} \tag{2.9a}$$

for some $\lambda > 0$, where $\mathcal{L}(\mathcal{B}_\alpha) = \mathcal{L}(\mathcal{B}_\alpha; \mathcal{B}_\alpha)$, $\alpha \in \mathbb{R}$. By means of (2.5), the mild solution of (2.8) is given by

$$Y_t = S_t\xi + \int_0^t S_{t-s}F(Y_s)ds + \int_0^t S_{t-s}G(Y_s)d\mathbf{W}_s, \quad \xi \in \mathcal{B}. \tag{2.10}$$

The imposed conditions on the regularity of F and G are collected here for $\delta \in [0, 1)$, $\sigma \in [0, \gamma)$:

(H1) There is a constant L_F such that the mapping $F : \mathcal{B}_\alpha \to \mathcal{B}_{\alpha-\delta}$ satisfying

$$|F(X) - F(Y)|_{\alpha-\delta} \leq L_F|X - Y|_\alpha, \quad X, Y \in \mathcal{B}_\alpha.$$

(H2) Let $\vartheta \in \{0, \gamma, 2\gamma\}$, $G \in C_b^3(\mathcal{B}_{\alpha-\vartheta}; \mathcal{B}_{\alpha-\vartheta-\sigma})$, i.e. there is a constant L_G such that for any $X, Y \in \mathcal{B}_{\alpha-\vartheta}$, it holds

$$|G(X) - G(Y)|_{\alpha-\vartheta-\sigma} \leq L_G|X - Y|_{\alpha-\vartheta}; \tag{2.11a}$$

$$\|DG(X) - DG(Y)\|_{\mathcal{L}(\mathcal{B}_{\alpha-\vartheta};\mathcal{B}_{\alpha-\vartheta-\sigma})} \leq L_G|X - Y|_{\alpha-\vartheta}; \tag{2.11b}$$

$$\|D^2G(X) - D^2G(Y)\|_{\mathcal{L}(\mathcal{B}_{\alpha-\vartheta};\mathcal{L}(\mathcal{B}_{\alpha-\vartheta};\mathcal{B}_{\alpha-\vartheta-\sigma}))} \leq L_G|X - Y|_{\alpha-\vartheta}. \tag{2.11c}$$

Moreover, the derivative of $DG(\cdot)G(\cdot) : \mathcal{B}_{\alpha-\vartheta} \to \mathcal{B}_{\alpha-\vartheta-\sigma}$ is assumed to be bounded, i.e.

$$|DG(X)G(X) - DG(Y)G(Y)|_{\alpha-\sigma-\vartheta} \leq L_G|X - Y|_{\alpha-\vartheta}. \tag{2.12}$$

The following is the result regarding to the existence of global-in-time solution and the generation of random dynamical systems, extracted from [24, Theorem 3.9] and [40, Lemma 5.6].

Lemma 2.3. *Let $\mathbf{W} = (W, \mathbb{W})$ be a γ-Hölder rough path cocycle over the metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$ and assume **(H1)** and **(H2)** hold. Then equation (2.8) possesses a unique global-in-time solution $(Y, G(Y))$ of (2.10), which generates a continuous random dynamical system ϕ given by*

$$\phi(t, \omega, \xi) = S_t\xi + \int_0^t S_{t-s}F(Y_s)ds + \int_0^t S_{t-s}G(Y_s)d\mathbf{W}_s(\omega), \quad \omega \in \Omega, t \geq 0.$$

This section is finished by introducing a discrete Gronwall-like lemma that will be used in the next section, and whose proof can be found in [42, Lemma 3.6].

Lemma 2.4. *Let $\lambda, \upsilon_0, k_0, k_1 < 1, k_2$ be positive numbers and let $\{t_i\}_{i \in \mathbb{Z}^+}$ be a sequence of positive numbers, with $t_0 = 0$, such that*

$$t_{i-1} - t_{i-2} \leq -\frac{2}{\lambda} \log k_1, i \geq 2. \tag{2.13}$$

Suppose that for a sequence of positive numbers $\{U_i\}_{i \in \mathbb{N}}$ the following inequalities hold true:

$$U_i \leq k_0 \upsilon_0 e^{-\lambda t_{i-1}} + \sum_{m=1}^{i-1} k_1 U_m e^{-\lambda(t_{i-1} - t_m)} + \sum_{m=1}^{i-1} e^{-\lambda(t_{i-1} - t_m)} k_2 + k_2.$$

Then we have

$$U_i \leq (k_0 \upsilon_0 + k_2)(1 + k_1$$

With the estimates (2.4a) and (2.4b), one has

$$
\begin{aligned}
|(S_{t-s}-\mathrm{Id})S_{s-r}F(Y_r)|_{\alpha-\vartheta} &\leq \|S_{t-s}-\mathrm{Id}\|_{\mathcal{L}(\mathcal{B}_\alpha;\mathcal{B}_{\alpha-\vartheta})}\|S_{s-r}\|_{\mathcal{L}(\mathcal{B}_{\alpha-\delta};\mathcal{B}_\alpha)}|F(Y_r)|_{\alpha-\delta} \\
&\leq C(t-s)^{\vartheta}(s-r)^{-\delta}(L_F\|Y\|_{\infty,\alpha}+|F(0)|_{\alpha-\delta}),
\end{aligned}
$$

where $\vartheta=\{\gamma,2\gamma\}$. Together with the triangle inequality of norm $|\cdot|$, one has

$$
\begin{aligned}
&\left\|\int_0^{\cdot} S_{\cdot-r}F(Y_r)dr\right\|_{\vartheta,[0,T],\alpha-\vartheta} \\
&\leq \sup_{0<s<t\leq T}\frac{\int_0^s\|S_{t-s}-\mathrm{Id}\|_{\mathcal{L}(\mathcal{B}_\alpha;\mathcal{B}_{\alpha-\vartheta})}\|S_{s-r}\|_{\mathcal{L}(\mathcal{B}_{\alpha-\delta};\mathcal{B}_\alpha)}|F(Y_r)|_{\alpha-\delta}dr}{(t-s)^{\vartheta}} \\
&\quad+\sup_{0<s<t\leq T}\frac{\int_s^t\|S_{t-r}\|_{\mathcal{L}(\mathcal{B}_{\alpha-\delta};\mathcal{B}_{\alpha-\vartheta})}|F(Y_r)|_{\alpha-\delta}dr}{(t-s)^{\vartheta}} \\
&\leq C(T^{1-\delta}+T^{(1-\delta)\wedge(1-2\gamma)})(L_F\|Y\|_{\infty,\alpha}+|F(0)|_{\alpha-\delta}),
\end{aligned}
$$

Herein, $T^{(1-\delta)\wedge(1-2\gamma)}$ emerges since we do not presume the difference of δ and 2γ. Hence with previous computations, we have

$$
\left\|\int_0^{\cdot} S_{\cdot-r}F(Y_r)dr,0\right\|_{\mathcal{D}^{2\gamma}_{[0,T],\alpha}} \leq Cc_FT^{1-\tau}\left(1+\|Y,Y'\|_{\mathcal{D}^{2\gamma}_{[0,T],\alpha}}\right).
$$

as required by $\tau=\max\{\delta,2\gamma\}$, $0<T\leq 1$ and $c_F=\max\{L_F,|F(0)|_{\alpha-\delta}\}$. $\square$

Lemma 3.2. *Assume **(H2)** hold, and $(Y,Y')\in {}^{2\gamma}_{[0,T],\alpha}$ solves (2.10). Then for any $T\in[0,1]$, (Y,Y') satisfies*

$$
\begin{aligned}
\|Y,Y'\|_{\mathcal{D}^{2\gamma}_{[0,T],\alpha}} &\leq Cc_G(1+|Y_0|_\alpha)+CT^{1-\tau}c_F\left(1+\|Y,Y'\|_{\mathcal{D}^{2\gamma}_{[0,T],\alpha}}\right) \\
&\quad+Cc_GT^{\gamma-\sigma}\tilde{\rho}_{\gamma,[0,T]}(\mathbf{W})\left(1+\|Y,Y'\|_{\mathcal{D}^{2\gamma}_{[0,T],\alpha}}\right).
\end{aligned}
$$

where

$$
\tilde{\rho}_{\gamma,[0,T]}(\mathbf{W})=(1+\rho_{\gamma,[0,T]})(1+\rho_{\gamma,[0,T]}(\mathbf{W})+\rho^2_{\gamma,[0,T]}(\mathbf{W})).
$$

Proof. Since $(Y,Y')\in {}^{2\gamma}_{[0,T],\alpha}$ is the solution to (2.10), $G'(Y)=DG(Y)Y'=DG(Y)G(Y)$. To our aim, let us denote

$$
\begin{aligned}
I_1 &:= \|G(Y)\|_{\infty,\alpha-\sigma},\\
I_2 &:= \|G'(Y)\|_{\infty,\alpha-\gamma-\sigma},\\
I_3 &:= \|G'(Y)\|_{\gamma,\alpha-2\gamma-\sigma},\\
I_4 &:= \|R^{G(Y)}\|_{\gamma,\alpha-\gamma-\sigma},\\
I_5 &:= \|R^{G(Y)}\|_{2\gamma,\alpha-2\gamma-\sigma},
\end{aligned}
$$

which will be estimated in turn. First, it follows from (2.11a) and (2.12) that

$$
\begin{aligned}
&I_1 \le L_G\|Y\|_{\infty,\alpha} + |G(0)|_{\alpha-\sigma},\\
&I_2 = \|DG(Y)G(Y)\|_{\infty,\alpha-\gamma-\sigma} \le L_G\|Y\|_{\infty,\alpha-\gamma} + |DG(0)G(0)|_{\alpha-\gamma-\sigma}.
\end{aligned}
$$

By the fact that $R^Y_{s,t} = Y_{s,t} - Y'_s W_{s,t}$, it yields

$$
\begin{aligned}
\|Y\|_{\gamma,\alpha-2\gamma} &\le \|R^Y\|_{\gamma,\alpha-2\gamma} + \rho_{\gamma,[0,T]}(\mathbf{W})\|Y'\|_{\infty,\alpha-2\gamma}\\
&\le C\|R^Y\|_{2\gamma,\alpha-2\gamma} + C\rho_{\gamma,[0,T]}(\mathbf{W})\|Y'\|_{\infty,\alpha-\gamma}.
\end{aligned}
$$

Then it further follows from (2.12) that

$$
\begin{aligned}
I_3 &= \sup_{0<s<t\le T} \frac{|DG(Y_t)G(Y_t) - DG(Y_s)G(Y_s)|_{\alpha-2\gamma-\sigma}}{(t-s)^\gamma}\\
&\le L_G \sup_{0<s<t\le T} \frac{|Y_t - Y_s|_{\alpha-2\gamma}}{(t-s)^\gamma}\\
&= L_G\|Y\|_{\gamma,\alpha-2\gamma}\\
&\le CL_G\|R^Y\|_{2\gamma,\alpha-2\gamma} + CL_G\rho_{\gamma,[0,T]}(\mathbf{W})\|Y'\|_{\infty,\alpha-\gamma}.
\end{aligned}
$$

Furthermore, a direct calculation suggests

$$
\begin{aligned}
R^{G(Y)}_{s,t} &= \int_0^1 DG(Y_s + h(Y_t - Y_s))dh(Y_t - Y_s) - DG(Y_s)G(Y_s)W_{s,t}\\
&= \int_0^1 DG(Y_s + h(Y_t - Y_s))dhR^Y_{s,t} + \int_0^1 DG(Y_s + h(Y_t - Y_s))G(Y_s)\\
&\quad - DG(Y_s)G(Y_s)dhW_{s,t}.
\end{aligned}
$$

Letting $\vartheta = \{\gamma, 2\gamma\}$, (2.12) and (2.11b) indicate that

$$
\begin{aligned}
&|DG(Y_s + h(Y_t - Y_s))G(Y_s) - DG(Y_s)G(Y_s)|_{\alpha-\vartheta-\sigma}\\
&\le |DG(Y_s + h(Y_t - Y_s))G(Y_s + h(Y_t - Y_s)) - DG(Y_s)G(Y_s)|_{\alpha-\vartheta-\sigma}\\
&\quad + |DG(Y_s + h(Y_t - Y_s))\,[G(Y_s) - G(Y_s + h(Y_t - Y_s))]\,|_{\alpha-\vartheta-\sigma}
\end{aligned}
$$

$$\leq L_G h|Y_t - Y_s|_{\alpha-\vartheta}.$$

Thus one has

$$\begin{aligned} I_4 &\leq L_G\|R^Y\|_{\gamma,\alpha-\gamma} + L_G T^\gamma \|Y\|_{\gamma,\alpha-\gamma}\rho_{\gamma,[0,T]}(\mathbf{W}) \\ &\leq CL_G\left(1+\rho_{\gamma,[0,T]}(\mathbf{W}) + \rho^2_{\gamma,[0,T]}(\mathbf{W})\right)\|Y, Y'\|_{\mathcal{D}^{2\gamma}_{[0,T],\alpha}}, \end{aligned}$$

and

$$\begin{aligned} I_5 &\leq L_G\|R^Y\|_{2\gamma,\alpha-2\gamma} + L_G\|Y\|_{\gamma,\alpha-2\gamma}\rho_{\gamma,[0,T]}(\mathbf{W}) \\ &\leq CL_G\left(1+\rho_{\gamma,[0,T]}(\mathbf{W}) + \rho^2_{\gamma,[0,T]}(\mathbf{W})\right)\|Y, Y'\|_{\mathcal{D}^{2\gamma}_{[0,T],\alpha}}. \end{aligned}$$

The arrangement of these estimates implies

$$\begin{aligned} \|G(Y), G'(Y)\|_{\mathcal{D}^{2\gamma}_{[0,T],\alpha-\sigma}} &= I_1 + I_2 + I_3 + I_4 + I_5 \\ &\leq CL_G\left(1+\rho_{\gamma,[0,T]}(\mathbf{W}) + \rho^2_{\gamma,[0,T]}(\mathbf{W})\right)\|Y, Y'\|_{\mathcal{D}^{2\gamma}_{[0,T],\alpha}} \\ &\quad + |G(0)|_{\alpha-\sigma} + |DG(0)G(0)|_{\alpha-\gamma-\sigma}. \end{aligned}$$

For the sake of simplicity, we denote $c_G = \max\{L_G, |G(0)|_{\alpha-\sigma}, |DG(0)G(0)|_{\alpha-\gamma-\sigma}\}$, then we have

$$\|G(Y), G'(Y)\|_{\mathcal{D}^{2\gamma}_{[0,T],\alpha-\sigma}} \leq Cc_G\left(1+\rho_{\gamma,[0,T]}(\mathbf{W}) + \rho^2_{\gamma,[0,T]}(\mathbf{W})\right)(1+\|Y, Y'\|_{\mathcal{D}^{2\gamma}_{[0,T],\alpha}}) \tag{3.1}$$

In addition, since $|G(Y_0)|_{\alpha-\sigma} \leq c_G(1+|Y_0|_\alpha)$ and $|G'(Y_0)|_{\alpha-\gamma-\sigma} \leq Cc_G(1+|Y_0|_\alpha)$ (similar to I_2), it follows from (2.7) that

$$\begin{aligned} &\left\|\int_0^\cdot S_{\cdot-s}G(Y_s)d\mathbf{W}_r, G(Y)\right\|_{\mathcal{D}^{2\gamma}_{[0,T],\alpha}} \\ &\leq Cc_G(1+|Y_0|_\alpha) + Cc_G T^{\gamma-\sigma}\tilde{\rho}_{\gamma,[0,T]}(\mathbf{W})\left(1+\|Y, Y'\|_{\mathcal{D}^{2\gamma}_{[0,T],\alpha}}\right). \end{aligned}$$

Owing to the result in Lemma 3.1, one finally has

$$\begin{aligned} \|Y, Y'\|_{\mathcal{D}^{2\gamma}_{[0,T],\alpha}} &\leq Cc_G(1+|Y_0|_\alpha) + CT^{1-\tau}c_F\left(1+\|Y, Y'\|_{\mathcal{D}^{2\gamma}_{[0,T],\alpha}}\right) \\ &\quad + Cc_G T^{\gamma-\sigma}\tilde{\rho}_{\gamma,[0,T]}(\mathbf{W})\left(1+\|Y, Y'\|_{\mathcal{D}^{2\gamma}_{[0,T],\alpha}}\right). \quad \square \end{aligned}$$

Lemma 3.3. *Under the same assumptions as in Lemma 3.2, $Y_T \in {}_{\alpha+\beta}$ for any $0 < \beta < \min\{1 - \delta, \gamma - \sigma\}$.*

Proof. For any $\xi \in {}_{\alpha}$, it follows from (2.4a) that

$$|S_T \xi|_{\alpha+\beta} \leq C T^{-\beta} |\xi|_{\alpha}.$$

Let $0 < \beta < \min\{1 - \delta, \gamma - \sigma\}$. Assumption **(H1)** and (2.6) respectively indicates

$$\left| \int_0^T S_{T-r} F(Y_r) dr \right|_{\alpha+\beta} \leq C T^{1-\beta-\delta} c_F (1 + \|Y\|_{\infty,\alpha}),$$

and

$$\left| \int_0^T S_{T-r} G(Y_r) d\mathbf{W}_r \right|_{\alpha+\beta} \leq C T^{\gamma-\beta-\sigma} \rho_{\gamma,[0,T]} \|G(Y), G'(Y)\|_{\mathcal{D}^{2\gamma}_{[0,T],\alpha-\sigma}}.$$

Making similar deduction to (3.1), one arrives at

$$\begin{aligned} |Y_T|_{\alpha+\beta} &\leq C T^{-\beta} |\xi|_{\alpha} + C c_F T^{1-\beta-\delta} (1 + \|Y\|_{\infty,\alpha}) \\ &\quad + C c_G T^{\gamma-\beta-\sigma} \tilde{\rho}_{\gamma,[0,T]}(\mathbf{W}) \left(1 + \|Y, Y'\|_{\mathcal{D}^{2\gamma}_{[0,T],\alpha}} \right) \\ &< \infty \end{aligned}$$

provided that $(Y, Y') \in {}^{2\gamma}_{[0,T],\alpha}$. The proof is complete. $\square$

3.2. Stopping times analysis

In this subsection, a sequence of stopping times $\{T_i\}_{i \in \mathbb{Z}}$ will be defined in rough paths setting based on previous computations. Its useful properties will be further explored for obtaining our key results in next subsection, the the

$$T_i(\mathbf{W}) := \begin{cases} 0 : i = 0, \\ T_{i-1}(\mathbf{W}) + T(\Theta_{T_{i-1}(\mathbf{W})}\mathbf{W}) : i \in \mathbb{N}, \\ T_{i+1}(\mathbf{W}) + \hat{T}(\Theta_{T_{i+1}(\mathbf{W})}\mathbf{W}) : i \in -\mathbb{N}. \end{cases}$$

Lemma 3.4. $T(\mathbf{W}), -\hat{T}(\mathbf{W}) \in (0, 1)$ *are -measurable and satisfy*

$$T(\mathbf{W}) = -\hat{T}(\Theta_{T(\mathbf{W})}\mathbf{W}), \quad \hat{T}(\mathbf{W}) = -T(\Theta_{\hat{T}(\mathbf{W})}\mathbf{W}). \tag{3.2}$$

Proof. Thanks to the continuity of the mapping $t \mapsto \rho_{\gamma,[0,t]}(\mathbf{W})$, the function

$$f(\mathbf{W}, t) = \mu t^{1-\tau} + c_G t^{\gamma-\sigma} \tilde{\rho}_{\gamma,[0,t]}(\mathbf{W}) - \mu \tag{3.3}$$

is continuous respect to $t \in \mathbb{R}^+$. Hence for any $\kappa > 0$, the set

$$\begin{aligned} \{\omega \in \Omega : T(\mathbf{W}(\omega)) < \kappa\} &= \{\omega \in \Omega : \mu\kappa^{1-\tau} + c_G\kappa^{\gamma-\delta}\tilde{\rho}_{\gamma,[0,\kappa]}(\mathbf{W}) > \mu\} \\ &= \left\{\omega \in \Omega : \tilde{\rho}_{\gamma,[0,\kappa]}(\mathbf{W}) > \frac{\mu - \mu\kappa^{1-\tau}}{c_G\kappa^{\gamma-\delta}}\right\} \end{aligned}$$

is -measurable since $\rho_{\cdot}(\mathbf{W})$ is measurable. Moreover, the mean-value theorem shows that there is a unique $T(\mathbf{W}) \in (0, 1)$ such that

$$T^{1-\tau}(\mathbf{W})\mu + c_G T^{\gamma-\delta}(\mathbf{W})\tilde{\rho}_{\gamma,[0,T(\mathbf{W})]}(\mathbf{W}) = \mu. \tag{3.4}$$

Similarly, $-\hat{T}(\mathbf{W}) \in (0, 1)$ is measurable and

$$\mu(-\hat{T}(\Theta_{T(\mathbf{W})}\mathbf{W}))^{1-\tau} + c_G(-\hat{T}(\Theta_{T(\mathbf{W})}\mathbf{W}))^{\gamma-\sigma}\tilde{\rho}_{\gamma,[\hat{T}(\Theta_{T(\mathbf{W})}\mathbf{W}),0]}(\Theta_{T(\mathbf{W})}\mathbf{W}) = \mu. \tag{3.5}$$

Finally, the proof of (3.2) is proceeded by contradiction. On the one hand, suppose $-\hat{T}(\Theta_{T(\mathbf{W})}\mathbf{W}) < T(\mathbf{W})$. Observing that

$$\rho_{\gamma,[\hat{T}(\Theta_{T(\mathbf{W})}\mathbf{W}),0]}(\Theta_{T(\mathbf{W})}\mathbf{W}) = \rho_{\gamma,[\hat{T}(\Theta_{T(\mathbf{W})}\mathbf{W})+T(\mathbf{W}),T(\mathbf{W})]}(\mathbf{W}),$$

one has

$$\begin{aligned} &\mu(-\hat{T}(\Theta_{T(\mathbf{W})}\mathbf{W}))^{1-\tau} + c_G(-\hat{T}(\Theta_{T(\mathbf{W})}\mathbf{W}))^{\gamma-\sigma}\tilde{\rho}_{\gamma,[\hat{T}(\Theta_{T(\mathbf{W})}\mathbf{W}),0]}(\Theta_{T(\mathbf{W})}\mathbf{W}) \\ &= \mu\left|\hat{T}(\Theta_{T(\mathbf{W})}\mathbf{W})\right|^{1-\tau} + c_G\left|\hat{T}(\Theta_{T(\mathbf{W})}\mathbf{W})\right|^{\gamma-\sigma}\tilde{\rho}_{\gamma,[\hat{T}(\Theta_{T(\mathbf{W})}\mathbf{W})+T(\mathbf{W}),T(\mathbf{W})]}(\mathbf{W}) \\ &< \mu T^{1-\tau}(\mathbf{W}) + c_G T^{\gamma-\sigma}(\mathbf{W})\tilde{\rho}_{\gamma,[\hat{T}(\Theta_{T(\mathbf{W})}\mathbf{W})+T(\mathbf{W}),T(\mathbf{W})]}(\mathbf{W}) \\ &\le \mu, \end{aligned}$$

which is a contradiction since the first formulation equals to μ by (3.5). On the other hand, suppose that $-\hat{T}(\Theta_{T(\mathbf{W})}\mathbf{W}) > T(\mathbf{W})$. Then

$$
\begin{aligned}
&\mu(-\hat{T}(\Theta_{T(\mathbf{W})}\mathbf{W}))^{1-\tau} + L_G(-\hat{T}(\Theta_{T(\mathbf{W})}\mathbf{W}))^{\gamma-\sigma}\tilde{\rho}_{\gamma,[\hat{T}(\Theta_{T(\mathbf{W})}\mathbf{W})+T(\mathbf{W}),T(\mathbf{W})]}(\mathbf{W}) \\
&\quad > T^{1-\tau}(\mathbf{W}) + c_G T^{\gamma-\sigma}(\mathbf{W})\tilde{\rho}_{\gamma,[\hat{T}(\Theta_{T(\mathbf{W})}\mathbf{W})+T(\mathbf{W}),T(\mathbf{W})]}(\mathbf{W}) \\
&\quad \geq \mu,
\end{aligned}
$$

which is also a contradiction with (3.5). Hence it could be concluded that

$$
T(\mathbf{W}) = -\hat{T}(\Theta_{T(\mathbf{W})}\mathbf{W}).
$$

One can obtain the second result by similar steps, so we omit it here. □

Lemma 3.5. *The following statements are true:*

(i) *For any $i, j \in \mathbb{Z}$, $T_i(\mathbf{W}) + T_j(\Theta_{T_i(\mathbf{W})}\mathbf{W}) = T_{i+j}(\mathbf{W})$.*
(ii) *Let $t_1 \leq t_2 \in \mathbb{R}$. Then*

$$
t_1 + T_j(\Theta_{t_1}\mathbf{W}) \leq t_2 + T_j(\Theta_{t_2}\mathbf{W}), \quad j \in \mathbb{Z}^-. \tag{3.6}
$$

If $t_2 + \hat{T}(\Theta_{t_2}\mathbf{W}) \leq t_1 \leq t_2$, then

$$
\begin{aligned}
\cdots &\leq t_1 + \hat{T}(\Theta_{t_1}\mathbf{W}) + \hat{T}(\Theta_{t_1+\hat{T}(\Theta_{t_1}\mathbf{W})}\mathbf{W}) \\
&\leq t_2 + \hat{T}(\Theta_{t_2}\mathbf{W}) + \hat{T}(\Theta_{t_2+\hat{T}(\Theta_{t_2}\mathbf{W})}\mathbf{W}) \\
&\leq t_1 + \hat{T}(\Theta_{t_1}\mathbf{W}) \leq t_2 + \hat{T}(\Theta_{t_2}\mathbf{W}) \leq t_1 \leq t_2.
\end{aligned} \tag{3.7}
$$

Note that it follows from Lemma 3.5 **(i)** that

$$
c_G(T_{i+1}(\mathbf{W}) - T_i(\mathbf{W}))^{\gamma-\sigma}\tilde{\rho}_{\gamma,[T_i(\mathbf{W}),T_{i+1}(\mathbf{W})]}(\mathbf{W}) + \mu(T_{i+1}(\mathbf{W}) - T_i(\mathbf{W}))^{1-\tau} = \mu. \tag{3.8}
$$

Proof. **(i)**It is direct by induction.**(ii)** See [42, Lemma 3.5] for (3.6) when $j = -1$ and (3.7). We prove the case of all $j < -1$ by induction. For $j + 1 \leq i < -1$, let us assume that

$$
t_1 + T_i(\Theta_{t_1}\mathbf{W}) \leq t_2 + T_i(\Theta_{t_2}\mathbf{W}).
$$

Then one has

$$
\begin{aligned}
t_1 + T_j(\Theta_{t_1}\mathbf{W}) &= t_1 + T_{j+1}(\Theta_{t_1}\mathbf{W}) + T_{-1}(\Theta_{T_{j+1}(\Theta_{t_1}\mathbf{W})+t_1}\mathbf{W}) \\
&\leq t_2 + T_{j+1}(\Theta_{t_2}\mathbf{W}) + T_{-1}(\Theta_{T_{j+1}(\Theta_{t_2}\mathbf{W})+t_2}\mathbf{W}) \\
&= t_2 + T_j(\Theta_{t_2}\mathbf{W})
\end{aligned}
$$

as required. □

Let $\nu = \min\{1-\tau, \gamma - \sigma\}$, denote

$$K(\mathbf{W}, \mu) = \left(\frac{\mu + c_G \tilde{\rho}_{\gamma,[-1,0]}(\mathbf{W})}{\mu} \right)^{\frac{1}{\nu}}.$$

Note that $\sup_{r\in[0,1]} \rho_{\gamma,[-1,0]}(\Theta_r \mathbf{W}) \le \rho_{\gamma,[-1,1]}(\mathbf{W})$ and the right-hand side has finite moment by [48, Theorem 1.4.1]. Hence

$$\mathbb{E}\left(\sup_{r\in[0,1]} K(\Theta_r \mathbf{W}, \mu) \right) < \infty.$$

Then it is meaningful to define d as

$$\frac{1}{d} := \mathbb{E}\left(\sup_{r\in[0,1]} K(\Theta_r \mathbf{W}, \mu) \right).$$

Lemma 3.6. *There is an invariant subset (still denoting by Ω) with full measure such that for any $\omega \in \Omega$, we have*

$$d \le \liminf_{k\to-\infty} \frac{T_k(\mathbf{W}(\omega))}{k} < 1. \tag{3.9}$$

Proof. Combining with (3.7) and the definition of d, see the proofs of [42, Lemma 4.4 and Lemma 4.5] or [49, Lemma 4.3]. $\square$

Lemma 3.7. *For small enough c_G, there exist $\mu \in (0, \min\{1/C, 1\})$ and $\eta \in (0, d)$ such that*

$$0 < \frac{1+\eta-d}{2(d-\eta)} < \frac{d-\eta}{2} - \frac{\log(1+C\mu) - \log(1-C\mu)}{\lambda}. \tag{3.10}$$

Proof. Let us first formulate the condition for d with other parameters η, μ by solving (3.10). Let

$$H(\mu) := \frac{\log(1+C\mu) - \log(1-C\mu)}{\lambda}$$

and thus rewrite (3.10) as

$$0 < 1 + \eta - d < (d-\eta)^2 - 2(d-\eta)H(\mu).$$

Adjusting this formula, one has

$$\left(d - \frac{2\eta + 2H(\mu) - 1}{2} \right)^2 + 2\eta H(\mu) + \eta^2 - \eta - 1 - \frac{(2\eta + 2H(\mu) - 1)^2}{4} > 0.$$

This inequality is solved by

$$d > \sqrt{\frac{(2\eta + 2H(\mu) - 1)^2}{4} - 2\eta H(\mu) - \eta^2 + \eta + 1} + \frac{2\eta + 2H(\mu) - 1}{2}$$
$$:= W^+(\eta, \mu)$$

or

$$d < -\sqrt{\frac{(2\eta + 2H(\mu) - 1)^2}{4} - 2\eta H(\mu) - \eta^2 + \eta + 1} + \frac{2\eta + 2H(\mu) - 1}{2}$$
$$:= W^-(\eta, \mu),$$

It is easy to observe that $W^+(\eta, \mu) > 0$, $W^-(\eta, \mu) < 0$ for small enough $\mu, \eta > 0$. Then it is reasonable to consider the positive solution $W^+(\eta, \mu)$ since $d \in (0, 1)$. Note that

$$\lim_{\eta,\mu \to 0} W^+(\eta, \mu) = \frac{\sqrt{5} - 1}{2}$$

due to the continuity of the function $(\eta, \mu) \mapsto W^+(\eta, \mu)$ at $(0, 0)$. Hence there exists an (η, μ) near $(0, 0)$ such that

$$H(\mu) \le \frac{1}{2}, \quad W^+(\eta, \mu) < 1.$$

Hence we complete this proof by requiring $d \ge W^+(\eta, \mu)$. For such given fixed μ, η, by definition of d, one has

$$\frac{1}{d} = \mathbb{E}\left[\sup_{r \in [0,1]} K(\Theta_r \mathbf{W}, \mu)\right] \le \mathbb{E}\left[\left(\frac{\mu + c_G \tilde{\rho}_{\gamma,[-1,1]}(\mathbf{W})}{\mu}\right)^{\frac{1}{\nu}}\right].$$

Then

$$\frac{1}{d} \le \sum_{k=0}^{\lceil \frac{1}{\nu} \rceil} \frac{\lceil \frac{1}{\nu} \rceil !}{k!(\lceil \frac{1}{\nu} \rceil - k)!} \times \frac{c_G^k \mathbb{E}[\tilde{\rho}^k_{\gamma,[-1,1]}(\mathbf{W})]}{\mu^k}$$

by the elementary equality that

$$(1 + x)^{\lceil \frac{1}{\nu} \rceil} = \sum_{k=0}^{\lceil \frac{1}{\nu} \rceil} \frac{\lceil \frac{1}{\nu} \rceil !}{k!(\lceil \frac{1}{\nu} \rceil - k)!} x^k, \quad x \ge 0.$$

For simplicity of notation, we rewrite

$$Q(\gamma, \sigma, \mathbf{W}) = \sum_{k=1}^{\lceil \frac{1}{\nu} \rceil} \frac{\lceil \frac{1}{\nu} \rceil !}{k!(\lceil \frac{1}{\nu} \rceil - k)!} \mathbb{E}[\tilde{\rho}^k_{\gamma,[-1,1]}(\mathbf{W})]. \tag{3.11}$$

Hence the selection of smaller $c_G > 0$ such that

$$1 + \frac{c_G Q(\gamma, \sigma, \mathbf{W})}{\mu^{\lceil \frac{1}{\nu} \rceil}} \leq \frac{1}{W^+(\eta, \mu)},$$

that is

$$c_G \leq \min \left\{ \frac{\mu^{\lceil \frac{1}{\nu} \rceil}(1 - W^+(\eta, \mu))}{Q(\gamma, \sigma, \mathbf{W}) W^+(\eta, \mu)}, 1 \right\}, \tag{3.12}$$

suggests that

$$d \geq W^+(\eta, \mu),$$

which completes this proof. □

Remark 3.1. Between the completion of this article and its publication, Cao and Gao [49] independently obtained the existence of random attractors for the same system in this article and the upper semi-continuity of the approximated systems. The main difference between this article and [49] is that the constructed stopping times are formally different, and thus the different techniques are applied to deduce the needed properties of the respective stopping time sequence and the required compact tempered absorbing set.

3.3. Construction of compact absorbing sets

The technique to derive the existence of random attractors will be based on the sequence of stopping times $\{T_i(\mathbf{W})\}_{i\in\mathbb{Z}}$ in such a way that in any interval $[T_i(\mathbf{W}), T_{i+1}(\mathbf{W})]$ the norm of the noise will be controlled, which allows the construction of a compact absorbing random set. For the sake of simplicity, the abbreviation $\mathbf{W} = \mathbf{W}(\omega)$, $T_i(\mathbf{W}) = T_i$, $T_{i,j} = T_i(\Theta_{T_j}\mathbf{W})$, $i, j \in \mathbb{Z}$ will be frequently used if necessary.

The two auxiliary lemmas below are needed before presenting our main theorem.

Lemma 3.8. *Assume that **(H1)**, **(H2)** hold and $c_F \leq \mu$. For any $i \in \mathbb{Z}^+$, let $t \in [-T_{-i}, -T_{-i-1}]$, $\zeta \in M(\theta_{-t}\omega) \cap \ _\alpha$. Then there is a constant $c > 0$ such that*

$$\sup_{t\in[-T_{-i}, -T_{-i-1}]} |\phi(t + T_{-i}, \theta_{-t}\omega, \zeta)|_\alpha \leq c\,(1 + |\zeta|_\alpha)\,.$$

Proof. Note that $t \in [-T_{-i}, -T_{-i-1}]$. By the definition of ϕ, one has

$$\begin{aligned}\phi(t + T_{-i}, \theta_{-t}\omega, \zeta) = S_{t+T_{-i}}\zeta + \int_0^{t+T_{-i}} S_{t+T_{-i}-r} F(Y_r) dr \\ + \int_0^{t+T_{-i}} S_{t+T_{-i}-r} G(Y_r) d\Theta_{-t}\mathbf{W}_r,\end{aligned}$$

for any $\zeta \in M(\theta_{-t}\omega) \cap \ {}_{\alpha}$. Thanks to Lemma 3.1, (2.6) and (3.1), it follows

$$
\begin{aligned}
&|\phi(t+T_{-i}, \theta_{-t}\omega, \zeta)|_{\alpha} \\
&\quad \leq Ce^{-\lambda(t+T_{-i})}|\zeta|_{\alpha} + C\left(1 + \|Y_{t+\cdot}, Y'_{t+\cdot}\|_{\mathcal{D}^{2\gamma}_{[-t,T_{-i}],\alpha}}\right) \\
&\qquad \times \left[\mu(t+T_{-i})^{1-\tau} + c_G(t+T_{-i})^{\gamma-\sigma}\,\tilde{\rho}_{\gamma,[-t,T_{-i}]}(\mathbf{W})\right] \\
&\quad \leq C|\zeta|_{\alpha} + C\left(1 + \|Y_{t+\cdot}, Y'_{t+\cdot}\|_{\mathcal{D}^{2\gamma}_{[-t,T_{-i}],\alpha}}\right)\left[\mu(T_{-i} - T_{-i-1})^{1-\tau}\right. \\
&\qquad \left. + c_G(T_{-i} - T_{-i-1})^{\gamma-\sigma}\,\tilde{\rho}_{\gamma,[T_{-i-1},T_{-i}]}(\mathbf{W})\right] \\
&\quad \leq C|\zeta|_{\alpha} + C\mu\left(1 + \|Y_{t+\cdot}, Y'_{t+\cdot}\|_{\mathcal{D}^{2\gamma}_{[-t,T_{-i}],\alpha}}\right).
\end{aligned}
$$

Due to Lemma 3.2 and the definition of $\{T_i\}_{i\in\mathbb{Z}}$, one has

$$
\|Y_{t+\cdot}, Y'_{t+\cdot}\|_{\mathcal{D}^{2\gamma}_{[-t,T_{-i}],\alpha}} \leq \frac{Cc_G}{1-C\mu}(1+|\zeta|_{\alpha}) + \frac{C\mu}{1-C\mu}.
$$

Hence

$$
|\phi(t+T_{-i}, \theta_{-t}\omega, \zeta)|_{\alpha} \leq C|\zeta|_{\alpha} + C\mu\left[1 + \frac{Cc_G}{1-C\mu}(1+|\zeta|_{\alpha}) + \frac{C\mu}{1-C\mu}\right].
$$

One further could conclude that there exists $c = c_{\mu}$ such that

$$
\sup_{t\in[-T_{-i},-T_{-i-1}]} |\phi(t+T_{-i}, \theta_{-t}\omega, \zeta)|_{\alpha} \leq c\,(1+|\zeta|_{\alpha}). \quad \square
$$

Remark 3.2. If D is tempered random set in ${}_{\alpha}$, for any $\epsilon > 0$, there is a $T_{\epsilon} > 0$ such that for any $t > T_{\epsilon}$, $|D(\theta_{-t}\omega)|_{\alpha} \leq e^{\epsilon t}$. Hence

$$
\sup_{t\in[-T_{-i},-T_{-i-1}]} \sup_{\zeta\in D(\theta_{-t}\omega)} |\phi(t+T_{-i}, \theta_{-t}\omega, \zeta)|_{\alpha} \leq ce^{\epsilon t}. \tag{3.13}
$$

Lemma 3.9. *Assume that* ***(H1)***, ***(H2)*** *hold and* $c_F \leq \mu$. *For any* $i \in \mathbb{Z}^{+}$, $\xi \in$ m c . c . c. . . .

Proof. The additivity of rough paths integral suggests the following splitting

$$
\begin{aligned}
Y_t &= \phi(t, \theta_{T_{-i}}\omega, \xi) \\
&= S_t\xi + \sum_{k=1}^{i} S_{t-T_{k,-i}} \int_{T_{k-1,-i}}^{T_{k,-i}} S_{T_{k,-i}-r} F(Y_r)dr \\
&\quad + \sum_{k=1}^{i} S_{t-T_{k,-i}} \int_{T_{k-1,-i}}^{T_{k,-i}} S_{T_{k,-i}-r} G(Y_r) d\Theta_{T_{-i}} \mathbf{W}_r \\
&\quad + \int_{T_{i,-i}}^{t} S_{t-r} F(Y_r)dr + \int_{T_{i,-i}}^{t} S_{t-r} G(Y_r) d\Theta_{T_{-i}} \mathbf{W}_r \\
&:= S_t\xi + I_t^F + I_t^G + \int_{T_{i,-i}}^{t} S_{t-r} F(Y_r)dr + \int_{T_{i,-i}}^{t} S_{t-r} G(Y_r) d\Theta_{T_{-i}} \mathbf{W}_r
\end{aligned}
$$

for any $t \in [T_{i,-i}, T_{i+1,-i}]$. (2.9a) gives that

$$\|S_\cdot\xi, 0\|_{\mathcal{D}^{2\gamma}_{[T_{i,-i},T_{i+1,-i}],\alpha}} \le Ce^{-\lambda T_{i,-i}} |\xi|_\alpha.$$

Also, (2.4a) and (2.4b) imply that

$$\left\| S_{t-T_{k,-i}} - S_{s-T_{k,-i}} \right\|_{\mathcal{L}(\mathcal{B}_\alpha;\mathcal{B}_{\alpha-2\gamma})} \le Ce^{-\lambda(s-T_{k,-i})}(t-s)^{2\gamma}. \tag{3.15}$$

Since the family of semigroup $\{S_t\}_{t\ge 0}$ is defined on the family of interpolation space $\{\ \ _\alpha\}_{\alpha\in\mathbb{R}}$, it could be viewed as a bounded linear operator family from one to another. Thus the Gubinelli derivatives of I^F, I^G are 0. Hence only thing needed is to estimate their remainders and themselves.

For I^F, since the estimate of $\|I^F\|_{\infty,[T_{i,-i},T_{i+1,-i}],\alpha}$ is a natural byproduct of $\|R^{I^F}\|_{\Theta,[T_{i,-i},T_{i+1,-i}],\alpha-\vartheta}$, $\vartheta = \{\gamma, 2\gamma\}$, we just state the later one. For that, the following estimates are first observed from Lemma 3.1, (2.9a) and (2.4b):

$$\left| \int_{T_{k-1,-i}}^{T_{k,-i}} S_{T_{k,-i}-r} F(Y_r)dr \right|_\alpha \le C\mu(T_{k,-i} - T_{k-1,-i})^{1-\tau}(1 + \|Y\|_{\infty,[T_{k-1,-i},T_{k,-i}],\alpha})$$

and

$$
\begin{aligned}
\left\| S_{t-T_{k,-i}} - S_{s-T_{k,-i}} \right\|_{\mathcal{L}(\mathcal{B}_\alpha;\mathcal{B}_{\alpha-\vartheta})} &= \left\| S_{s-T_{k,-i}} \right\|_{\mathcal{L}(\mathcal{B}_\alpha)} \|(S_{t-s} - \mathrm{Id})\|_{\mathcal{L}(\mathcal{B}_\alpha;\mathcal{B}_{\alpha-\vartheta})} \\
&\le Ce^{-\lambda(s-T_{k,-i})}(t-s)^\vartheta.
\end{aligned}
$$

Hence for any $s, t \in [T_{i,-i}, T_{i+1,-i}]$, one has

$$
\begin{aligned}
&|I_t^F - I_s^F|_{\alpha-\vartheta} \\
&= \left| \sum_{k=1}^{i} \left(S_{t-T_{k,-i}} - S_{s-T_{k,-i}} \right) \int_{T_{k-1,-i}}^{T_{k,-i}} S_{T_{k,-i}-r} F(Y_r) dr \right|_{\alpha-\vartheta} \\
&\leq \sum_{k=1}^{i} \left\| S_{t-T_{k,-i}} - S_{s-T_{k,-i}} \right\|_{\mathcal{L}(\mathcal{B}_\alpha;\mathcal{B}_{\alpha-\vartheta})} \left| \int_{T_{k-1,-i}}^{T_{k,-i}} S_{T_{k,-i}-r} F(Y_r) dr \right|_{\alpha} \\
&\leq C\mu \sum_{k=1}^{i} e^{-\lambda(s-T_{k,-i})} (T_{k,-i} - T_{k-1,-i})^{1-\tau} (1 + \|Y\|_{\infty,[T_{k-1,-i},T_{k,-i}],\alpha})(t-s)^{\vartheta}.
\end{aligned}
$$

Notice that $R_{s,t}^{I^F} = I_t^F - I_s^F$ for any $s, t \in [T_{i,-i}, T_{i+1,-i}]$, then

$$
\begin{aligned}
&\|R^{I^F}\|_{\vartheta,[T_{i,-i},T_{i+1,-i}],\alpha-\vartheta} \\
&\quad \leq C\mu \sum_{k=1}^{i} e^{-\lambda(T_{i,-i}-T_{k,-i})} (T_{k,-i} - T_{k-1,-i})^{1-\tau} (1 + \|Y\|_{\infty,[T_{k-1,-i},T_{k,-i}],\alpha}).
\end{aligned}
$$

Based on these estimates, one eventually obtains

$$
\begin{aligned}
\|I^F, 0\|_{\mathcal{D}^{2\gamma}_{[T_{i,-i},T_{i+1,-i}],\alpha}} \leq C\mu \sum_{k=1}^{i} e^{-\lambda(T_{i,-i}-T_{k,-i})} \left(T_{k,-i} - T_{k-1,-i} \right)^{1-\tau} \\
\times \left(1 + \|Y, Y'\|_{\mathcal{D}^{2\gamma}_{[T_{k-1,-i},T_{k,-i}],\alpha}} \right).
\end{aligned}
$$

I^G also can be computed by similar induction when taking the following difference into account:

$$
\begin{aligned}
&\left| \int_{T_{k-1,-i}}^{T_{k,-i}} S_{T_{k,-i}-r} G(Y_r) d\Theta_{T_{-i}} \mathbf{W}_r \right|_{\alpha} \\
&\quad \leq C c_G \left(T_{k,-i} - T_{k-1,-i} \right)^{\gamma-\sigma} \tilde{\rho}_{\gamma,[T_{k-1,-i},T_{k,-i}]}(\Theta_{T_{-i}} \mathbf{W}) \\
&\qquad \times \left(1 + \|Y, Y'\|_{\mathcal{D}^{2\gamma}_{[T_{k-1,-i},T_{k,-i}],\alpha}} \right),
\end{aligned}
$$

which is a product of (2.6) and (3.1). Thus one arrives at

$$
\|I^G, 0\|_{\mathcal{D}^{2\gamma}_{[T_{i,-i},T_{i+1,-i}],\alpha}} \leq C c_G \sum_{k=1}^{i} e^{-\lambda(T_{i,-i}-T_{k,-i})} \tilde{\rho}_{\gamma,[T_{k-1,-i},T_{k,-i}]}(\Theta_{T_{-i}} \mathbf{W})
$$

$$\times \left[T_{k,-i} - T_{k-1,-i}\right]^{\gamma-\sigma} \left(1 + \|Y, Y'\|_{\mathcal{D}^{2\gamma}_{[T_{k-1,-i},T_{k,-i}],\alpha}}\right).$$

The remaining two terms follow from Lemma 3.1 and Lemma 3.2 that

$$\left\| \int_{T_{i,-i}}^{\cdot} S_{\cdot-r} F(Y_r) dr, 0 \right\|_{\mathcal{D}^{2\gamma}_{[T_{i,-i},T_{i+1,-i}],\alpha}} \leq C\mu \left(T_{i+1,-i} - T_{i,-i}\right)^{1-\tau} \left(1 + \|Y, Y'\|_{\mathcal{D}^{2\gamma}_{[T_{i,-i},T_{i+1,-i}],\alpha}}\right),$$

and

$$\begin{aligned} &\left\| \int_{T_{i,-i}}^{\cdot} S_{\cdot-r} G(Y_r) d\Theta_{T_{-i}} \mathbf{W}_r, G(Y) \right\|_{\mathcal{D}^{2\gamma}_{[T_{i,-i},T_{i+1,-i}],\alpha}} \\ &\leq Cc_G(1 + |Y_{T_{i,-i}}|_\alpha) + C\left(T_{i+1,-i} - T_{i,-i}\right)^{\gamma-\sigma} \\ &\quad \times \tilde{\rho}_{\gamma,[T_{i,-i},T_{i+1,-i}]}(\Theta_{T_{-i}}\mathbf{W}) \left(1 + \|Y, Y'\|_{\mathcal{D}^{2\gamma}_{[T_{i,-i},T_{i+1,-i}],\alpha}}\right). \end{aligned}$$

Up to now, the estimating result of $\|Y, Y'\|_{\mathcal{D}^{2\gamma}_{[T_{i,-i},T_{i+1,-i}],\alpha}}$ becomes

$$\begin{aligned} &\|Y, Y'\|_{\mathcal{D}^{2\gamma}_{[T_{i,-i},T_{i+1,-i}],\alpha}} \\ &\leq Ce^{-\lambda T_{i,-i}}|\xi|_\alpha + Cc_G(1 + |Y_{T_{i,-i}}|_\alpha) + Cc_G\left(T_{i+1,-i} - T_{i,-i}\right)^{\gamma-\sigma} \\ &\quad \times \tilde{\rho}_{\gamma,[T_{i,-i},T_{i+1,-i}]}(\Theta_{T_{-i}}\mathbf{W}) \left(1 + \|Y, Y'\|_{\mathcal{D}^{2\gamma}_{[T_{i,-i},T_{i+1,-i}],\alpha}}\right) \\ &\quad + C\mu(T_{i+1,-i} - T_{i,-i})^{1-\tau} \left(1 + \|Y, Y'\|_{\mathcal{D}^{2\gamma}_{[T_{i,-i},T_{i+1,-i}],\alpha}}\right) \\ &\quad + C\mu \sum_{k=1}^{i} e^{-\lambda(T_{i,-i}-T_{k,-i})} \left(T_{k,-i} - T_{k-1,-i}\right)^{1-\tau} \left(1 + \|Y, Y'\|_{\mathcal{D}^{2\gamma}_{[T_{k-1,-i},T_{k,-i}],\alpha}}\right) \\ &\quad + Cc_G \sum_{k=1}^{i} e^{-\lambda(T_{i,-i}-T_{k,-i})} \tilde{\rho}_{\gamma,[T_{k-1,-i},T_{k,-i}]}(\Theta_{T_{-i}}\mathbf{W}) \\ &\quad \times \left(T_{k,-i} - T_{k-1,-i}\right)^{\gamma-\sigma} \left(1 + \|Y, Y'\|_{\mathcal{D}^{2\gamma}_{[T_{k-1,-i},T_{k,-i}],\alpha}}\right). \end{aligned}$$

One more thing needed to be done is to compute $|Y_{T_{i,-i}}|_\alpha$. For that, observing the fact that

$$
\begin{aligned}
Y_{T_{i,-i}} = S_{T_{i,-i}}\xi &+ \sum_{k=1}^{i} S_{T_{i,-i}-T_{k,-i}} \int\limits_{T_{k-1,-i}}^{T_{k,-i}} S_{T_{k,-i}-r} F(Y_r)dr \\
&+ \sum_{k=1}^{i} S_{T_{i,-i}-T_{k,-i}} \int\limits_{T_{k-1,-i}}^{T_{k,-i}} S_{T_{k,-i}-r} G(Y_r) d\Theta_{T_{-i}}\mathbf{W}_r,
\end{aligned}
$$

one can immediately obtain that

$$
\begin{aligned}
|Y_{T_{i,-i}}|_\alpha \le Ce^{-\lambda T_{i,-i}}|\xi|_\alpha &+ C\mu \sum_{k=1}^{i} e^{-\lambda(T_{i,-i}-T_{k,-i})} \left(T_{k,-i} - T_{k-1,-i}\right)^{1-\tau} \\
&\times \left(1 + \|Y, Y'\|_{\mathcal{D}^{2\gamma}_{[T_{k-1,-i},T_{k,-i}],\alpha}}\right) \\
&+ Cc_G \sum_{k=1}^{i} e^{-\lambda(T_{i,-i}-T_{k,-i})} \tilde{\rho}_{\gamma,[T_{k-1,-i},T_{k,-i}]}(\Theta_{T_{-i}}\mathbf{W}) \\
&\times \left(T_{k,-i} - T_{k-1,-i}\right)^{\gamma-\sigma} \left(1 + \|Y, Y'\|_{\mathcal{D}^{2\gamma}_{[T_{k-1,-i},T_{k,-i}],\alpha}}\right).
\end{aligned}
$$

Hence the further estimate of $\|Y, Y'\| \mathcal{D}, \mathcal{D} \quad \mathcal{D} \ \mathcal{D}\mathcal{D}$

$$
\begin{aligned}
&\|Y, Y'\|_{\mathcal{D}^{2\gamma}_{[T_{i,-i}, T_{i+1,-i}],\alpha}} \\
&\leq Cc_G + C(1+c_G)e^{-\lambda T_{i,-i}}|\xi|_\alpha + C\mu\left(1 + \|Y, Y'\|_{\mathcal{D}^{2\gamma}_{[T_{i,-i}, T_{i+1,-i}],\alpha}}\right) \\
&\quad + C(1+c_G)\mu\sum_{k=1}^{i} e^{-\lambda(T_{i,-i}-T_{k,-i})}\left(1 + \|Y, Y'\|_{\mathcal{D}^{2\gamma}_{[T_{k-1,-i}, T_{k,-i}],\alpha}}\right).
\end{aligned}
\tag{3.16}
$$

Choosing a proper $\mu \in (0, 1)$ such that $C\mu \leq 1$ and rewriting

$$
y_k = \|Y, Y'\|_{\mathcal{D}^{2\gamma}_{[T_{k-1,-i}, T_{k,-i}],\alpha}},
$$

one can reformulate (3.16) as

$$
y_{i+1} \leq \frac{C(1+c_G)}{1-C\mu}e^{-\lambda T_{i,-i}}|\xi|_\alpha + \frac{C(\mu+c_G)}{1-C\mu} + \frac{C(1+c_G)\mu}{1-C\mu}\sum_{k=1}^{i} e^{-\lambda(T_{i,-i}-T_{k,-i})}(1+y_k).
$$

In order to fit with the form of Lemma 2.4, let $t_k = T_{k,-i}$, $k_0 v_0 = \frac{C(1+c_G)}{1-C\mu}|\xi|_\alpha$, and $k_1 = \frac{C(1+c_G)\mu}{1-C\mu}$, $k_2 = \frac{C(\mu+c_G)}{1-C\mu}$, then by Lemma 3.5**(i)**,

$$
t_{k-1} - t_{k-2} = T_{k-1,-i} - T_{k-2,-i} = T_{1,k-2-i} < 1.
$$

Choosing a proper $\mu > 0$ again such that

$$
1 < -\frac{2}{\lambda}\log\left[\frac{C(1+c_G)\mu}{1-C\mu}\right],
$$

then Lemma 2.4 indicates that

$$
y_{i+1} \leq \frac{C(1+c_G)}{1-C\mu}e^{i\log(1+k_1)-\frac{\lambda}{2}T_{i,-i}}(1+|\xi|_\alpha) + 2k_2\sum_{k=1}^{i} e^{(i-k)\log(1+k_1)-\frac{\lambda}{2}(T_{i,-i}-T_{k,-i})}.
$$

Herein, the property of stopping times in Lemma 3.5**(i)** indicates that $T_{i,-i} - T_{k,-i} = -T_{k-i}$. Hence one further obtains that y_{i+1} is bounded by

$$
\frac{C(1+c_G)}{1-C\mu}e^{i\log(1+k_1)-\frac{\lambda}{2}T_{i,-i}}(1+|\xi|_\alpha) + \frac{C(1+c_G)}{1-C\mu}\sum_{k=1}^{i} e^{(i-k)\log(1+k_1)+\frac{\lambda}{2}T_{k-i}}.
$$

That completes this proof. □

With the above discussions at hand, it is in conditions to prove the main result of this paper.

Theorem 3.1. *Suppose assumptions **(H1)**, **(H2)** hold, $c_F \le \mu$, and c_G satisfies* (3.12)*, then the random dynamical system ϕ possesses a random pullback compact absorbing set $K \in$. Consequently, the random attractor A is given by*

$$A(\omega) = \bigcap_{s\ge 0}\overline{\bigcup_{t\ge s}\phi(t,\theta_{-t}\omega, K(\theta_{-t}\omega))}.$$

Proof. Observe that for any $t \in \mathbb{R}^+$, there exists an $i \in \mathbb{Z}^+$ such that $t \in [T_{i,-i}, T_{i+1,-i-1}]$, that is $-t \in [T_{-i-1}, T_{-i}]$ by Lemma 3.5**(i)**. The cocycle property of random dynamical system ϕ here reads that for any $\zeta \in M(\theta_{-t}\omega) \cap \ _\alpha$,

$$\begin{aligned}\phi(t,\theta_{-t}\omega,\zeta) &= \phi(t,\theta_{-t+T_{-i}-T_{-i}}\omega,\zeta)\\ &= \phi(T_{i,-i},\theta_{T_{-i}},\phi(t - T_{i,-i},\theta_{-t-T_{-i}+T_{-i}}\omega,\zeta)).\end{aligned}$$

Rewriting $\xi = \phi(t + T_{-i},\theta_{-t}\omega,\zeta)$, we first treat the convergence of each term in (3.14) as $i \to \infty$. It follows from Lemma 3.8 that there is a generic constant c such that

$$\begin{aligned}e^{i\log(1+k_1)-\frac{\lambda}{2}T_{i,-i}}(1+|\xi|_\alpha) &\le ce^{\epsilon t+i\log(1+k_1)-\frac{\lambda}{2}T_{i,-i}}\\ &\le ce^{\epsilon T_{i+1,-i-1}+i\log(1+k_1)-\frac{\lambda}{2}T_{i,-i}}\\ &\le ce^{\epsilon(T_{i+1,-i-1}-T_{i,-i})+i\log(1+k_1)+(\epsilon-\frac{\lambda}{2})T_{i,-i}}\\ &\le ce^{i\log(1+k_1)+\left(\epsilon-\frac{\lambda}{2}\right)T_{i,-i}},\end{aligned}$$

for large enough t (consequently, large enough i), where ϵ is assumed to be smaller than $\lambda/2$. In the third inequality, since $T_{i+1,-i-1} - T_{i,-i} = T_{-i} - T_{-i-1} \le 1$, $ce^{\epsilon(T_{i+1,-i-1}-T_{i,-i})}$ is bounded by a constant uniformly with respect to $i \in \mathbb{Z}^+$. Because of (3.9), there exists $d > 0$ such that

$$1 \ge \liminf_{k\to-\infty}\frac{T_k(\mathbf{W})}{k} \ge d, \tag{3.17}$$

which implies for $\eta \in (0,d)$ given in (3.10), there is an $i_\eta \in \mathbb{N}$ such that

$$T_{-i} \le -di + \eta i, \quad -i \le -i_\eta.$$

By means of (3.10) and $c_G \le 1$, there is a $\nu \in [0, d\lambda/2)$ such that

$$\begin{aligned}\frac{\lambda(1+\eta-d)}{2(d-\eta)} < \nu < \frac{\lambda(d-\eta)}{2} - \log\left(\frac{1+C\mu}{1-C\mu}\right)\\ \le \frac{\lambda d}{2} - \log(1+k_1).\end{aligned}$$

Hence

$$
\begin{aligned}
i\log(1+k_1)+\left(\epsilon-\frac{\lambda}{2}\right)T_{i,-i} &\le i\log(1+k_1)+\left(\frac{\lambda}{2}-\epsilon\right)(-di+\eta i)\\
&= i\left[\log(1+k_1)-\frac{(\lambda-2\epsilon)(d-\eta)}{2}\right]\\
&\le \frac{i[d\lambda-2\nu-(\lambda-2\epsilon)(d-\eta)]}{2},
\end{aligned}
$$

where $d\lambda-2\nu-(\lambda-2\epsilon)(d-\eta)<0$ by choosing

$$
0<\epsilon<\frac{2\nu-\lambda\eta}{2(d-\eta)}.
$$

Consequently,

$$
\lim_{i\to+\infty} e^{i\log(1+k_1)-\frac{\lambda}{2}T_{i,-i}}(1+|\xi|_\alpha)=0.
$$

Similarly, one also can obtain

$$
\begin{aligned}
\sum_{k=1}^{i} e^{(i-k)\log(1+k_1)+\frac{\lambda}{2}T_{k-i}(\mathbf{W})} &= \sum_{m=1-i}^{0} e^{-m\log(1+k_1)+\frac{\lambda}{2}T_m(\mathbf{W})}\\
&\le \sum_{m=-\infty}^{0} e^{-m\log(1+k_1)+\frac{\lambda}{2}T_m(\mathbf{W})}.
\end{aligned}
$$

Since for m small enough, $T_m(\mathbf{W})\le m(d-\eta)$,

$$
e^{-m\log(1+k_1)+\frac{\lambda}{2}T_m(\mathbf{W})}\le e^{m(-\log(1+k_1)+\frac{\lambda(d-\eta)}{2})}\le e^{\frac{m(2\nu-d\lambda)}{2}}.
$$

It yields

$$
\lim_{i\to\infty}\sum_{k=1}^{i} e^{(i-k)\log(1+k_1)+\frac{\lambda}{2}T_{k-i}(\mathbf{W})}\le \sum_{m=-\infty}^{0} e^{-m\log(1+k_1)+\frac{\lambda}{2}T_m(\mathbf{W})}<\infty.
$$

Hence it is meaningful to define

$$
R(\omega):=\frac{C}{1-C\mu}\lim_{i\to\infty}\sum_{k=1}^{i} e^{(i-k)\log(1+k_1)+\frac{\lambda}{2}T_{k-i}(\mathbf{W}(\omega))},
$$

and

$$
K(\omega)=\{Y\in \ {}_{\alpha};\ |Y|_\alpha\le R(\omega)\}\quad \omega\in\Omega.
$$

Indeed, $K(\omega)$ absorbs all tempered sets in ${}_{\alpha}$ and measurable since the sequence of stopping times $\{T_i\}_{i\in\mathbb{Z}}$ is measurable. As for the tempered property of $K(\omega)$, that is

$$\lim_{s\to\pm\infty}\frac{\log^+ R(\theta_s\omega)}{|s|}=0,$$

one needs to note that since $(\Omega, \quad, P, \{\theta_t\}_{t\in\mathbb{R}})$ is an ergodic metric dynamical system. It follows from [35, Proposition 4.1.3] that for almost all $\omega\in\Omega$,

$$\lim_{t\to+\infty}\frac{\log^+ R(\theta_t\omega)}{t}=\lim_{t\to-\infty}\frac{\log^+ R(\theta_t\omega)}{|t|}=\{0,\infty\}.$$

Hence, no loss of generalization appears when considering merely the case of negative time. For any $s<0$, there is a $j\in\mathbb{Z}^-$ such that $s\in[T_{j-1},T_j]$, then it follows from (3.6) that

$$s+T_m(\Theta_s\mathbf{W})\le T_j+T_m(\Theta_{T_j}\mathbf{W}),\quad m\in\mathbb{Z}^-.$$

For given $\nu<\frac{\lambda(d-\eta)}{2}-\log\left(\frac{1+C\mu}{1-C\mu}\right)$,

$$\begin{aligned}
e^{\nu s}R(\theta_s\omega)&\le\sum_{m=-\infty}^{0}e^{\nu T_j-m\log(1+k_1)+\frac{\lambda}{2}T_m(\Theta_s\mathbf{W})}\\
&=\sum_{m=-\infty}^{0}e^{\nu T_j-m\log(1+k_1)+\frac{\lambda}{2}[-s+s+T_m(\Theta_s\mathbf{W})]}\\
&\le\sum_{m=-\infty}^{0}e^{\nu T_j-m\log(1+k_1)+\frac{\lambda}{2}[-T_{j-1}+T_j+T_m(\Theta_{T_j}\mathbf{W})]}\\
&=\sum_{m=-\infty}^{0}e^{\nu T_j-m\log(1+k_1)+\frac{\lambda}{2}[T_{m+j}-T_{j-1}]}\\
&\le e^{j\left[\nu(d-\eta)+\frac{\lambda(d-\eta-1)}{2}\right]+\frac{\lambda}{2}}\sum_{m=-\infty}^{0}e^{m\left[\frac{\lambda(d-\eta)}{2}-\log(1+k_1)\right]}
\end{aligned}$$

which converges to 0 as $j\to-\infty$ provided that $\nu>\frac{\lambda(1+\eta-d)}{2(d-\eta)}$. Hence

$$\lim_{s\to-\infty}\frac{\log^+ R(\theta_s\omega)}{|s|}\le\nu<\infty,$$

that completes the proof for tempered property of $K(\omega)$. Up to now, it has been completed to construct a tempered subset $K\subset\ _\alpha$ which absorbs all tempered sets in $\ _\alpha$, $\alpha\in\mathbb{R}$. Particularly, let $\alpha=0$. In this situation, K is an -absorbing set of ϕ. While it has been unknown for the compactness of $K(\omega)$, $\omega\in\Omega$. For that, the absorption of K suggests a distinct -absorbing set $\phi(T_K+1,\theta_{-T_K-1}\omega,K(\theta_{-T_K(\omega)-1}$

$$\mathcal{A}(\omega) = \bigcap_{s\geq 0} \overline{\bigcup_{t\geq s} \phi(t, \theta_{-t}\omega, K(\theta_{-t}\omega))}, \quad \omega \in \Omega. \quad \square$$

4. Application to rough reaction-diffusion equation

In this section, the applicability of Theorem 3.1 for rough reaction-diffusion type equations is discussed, but before proceeding it, let us pause to introduce the fractional Brownian rough paths.

On a probability space $(\Omega, \quad, \mathbb{P})$, a standard fractional Brownian B_t^H is defined to be a continuous centered Gaussian process with

$$\mathbb{E}\left[B_t^H B_s^H\right] = \frac{1}{2}\left(|s|^{2H} + |t|^{2H} - |t-s|^{2H}\right),$$

for $s, t \in \mathbb{R}$, where the Hurst parameter H is supposed to be in $(1/3, 1/2]$. This includes classical Brownian motion as the special case when $H = 1/2$.

Proposition 4.1. *Let B^H be a fractional Brownian motion with $1/3 < H \leq 1/2$, then there is an ergodic dynamical system $(\Omega_H, \quad_H, \mathbb{P}_H, \{\theta_t\}_{t\in\mathbb{R}})$ and a γ-Hölder rough path $\mathbf{B}^H$, $1/3 < \gamma < H \leq 1/2$ such that $\mathbf{B}^H$ enjoys the cocycle property on $(\Omega_H, \quad_H, \mathbb{P}_H, \{\theta_t\}_{t\in\mathbb{R}})$.*

Proof. For $1/3 < \gamma' < H \leq 1/2$, let $\Omega_H = C_0^{\gamma'}(\mathbb{R})$ be the space of continuous functions satisfying $\omega(0) = 0, \omega \in \Omega_H$, $\quad_H$ stands for $\mathcal{B}(C_0^{\gamma'}(\mathbb{R}))$ under compact-open topology, $\mathbb{P}_H$ be the law of B^H and θ be the Wiener shift on Ω_H, then $(\Omega_H, \quad_H, \mathbb{P}_H, \{\theta_t\}_{t\in\mathbb{R}})$ constructs an ergodic metric dynamical system, see [50] for the ergodicity statement. As discussed in [6, Example 10.11] and [23, Lemma 6.2], for any closed interval $[T_1, T_2]$ containing 0 and $\gamma \in (1/3, \gamma')$, the diagonal process B^H could be lifted to a γ-Hölder rough path $\mathbf{B}^H = (B^H, \mathbb{B}^H)$ on $(\Omega_H, \quad_H, \mathbb{P}_H, \{\theta_t\}_{t\in\mathbb{R}})$, and there is a θ-invariant subset (still denoted by Ω_H) of full measure such that $\mathbb{B}_{s,t}^H(\omega)$ is the limit of

$$\mathbb{B}_{s,t}^{H,n}(\omega) = \int_s^t B_r^{H,n}(\omega) dB_r^{H,n}(\omega), \quad [s,t] \subset [T_1, T_2], \omega \in \Omega_H$$

as $n \to \infty$, where $\{B^{H,n}\}_{n\in\mathbb{N}}$ is the piecewise dyadic approximation of B^H, see also [4, Theorem 2]. Moreover, denoting

$$\Theta_s \mathbf{B}_t^H = (B_{s,t+s}^H, \mathbb{B}_{s,t+s}^H),$$

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it was proved in [40, Lemma 5.2] that $\Theta_s \mathbf{B}^H$ is also a rough path on $[T_1 - s, T_2 - s]$ keeping same

$$= \lim_{n\to\infty} \int_0^t B^{H,n}_{s,r+s}(\omega) d B^{H,n}_{r+s}(\omega)$$

$$= \lim_{n\to\infty} \int_0^t B^{H,n}_{r}(\theta_s\omega) d B^{H,n}_{r}(\theta_s\omega)$$

$$= \lim_{n\to\infty} \mathbb{B}^{H,n}_t(\theta_s\omega),$$

where the third equality comes from the result of [36, Lemma 11], we have

$$\mathbf{B}^H_{s,s+t}(\omega) = (B^H_t(\theta_s\omega), \mathbb{B}^H_t(\theta_s\omega)) = \mathbf{B}^H_t(\theta_s\omega)$$

for all $\omega \in \Omega_H$. Thus $\mathbf{B}^H$ defines a rough path cocycle over the ergodic metric dynamical system $(\Omega_H, \quad _H, \mathbb{P}_H, \{\theta_t\}_{t\in\mathbb{R}})$. □

Example 4.1. Let $\mathbb{T}$ stand for the one-dimensional torus, say, $\mathbb{T} = \mathbb{R}/l\mathbb{Z}$ for $l \in \mathbb{R} \setminus \{0\}$ which is a circle of given length. As mentioned in [21, Subsection 6.1], let us introduce the scale of Banach space $\quad _\alpha = H_0^{2\alpha,2}(\mathbb{T})$ which is the closure of $C_c^\infty(\mathbb{T})$ in the Bessel potential space $H^{2\alpha,2}(\mathbb{T})$, $\alpha \in \mathbb{R}$. Consider the rough reaction-diffusion equation with periodic boundary conditions on $\mathbb{T}$:

$$\begin{cases} du_t = [\ \infty\infty[\prime \quad [\{\leq \quad \forall \quad [\quad [\ \leq\ \lceil \quad [\,|\leq \quad \infty\ \prime\ \prime\ \forall\ \infty\ [\\ \end{cases}$$

Data availability

No data was used for the research described in the article.

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