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$$\begin{cases} du = Audt + F(u)\,dt + \sigma(u)dW, \\ u(0) = u_0 \in \overline{D(A)} = X_0, \end{cases} \tag{1}$$

where $A : D(A) \subset X \to X$ is a linear operator whose domain is non-densely defined. Indeed, W is a two-sided cylindrical Wiener process on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathbb{P})$. While F and σ are nonlinear functions with some natural imposed conditions (specified later). Note that the Cauchy problem to (1) is ill-posed since the Hille-Yosida theory for C_0-semigroup generated by A breaks down. Cauchy problems with a non-dense domain cover several types of differential equations, including delay differential equations, age-structured models and some evolution equations with nonlinear boundary conditions. To overcome the embarrassment, the integrated semigroup theory allows us to define a suitable mild solution, or so-called integrated solution. The concept of integrated semigroups was first introduced by Arendt [1,2] and it was applied to study the existence and uniqueness of solutions to such non-homogeneous Cauchy problems in a deterministic setting by Da Prato and Sinestrari [3]. Later on, Thieme [4,5] established the well-posedness of non-autonomous and semilinear Cauchy problems under a prior Hille-Yosida type estimate for the resolvent of non-densely defined operators. By discarding the mentioned estimate, Magal and Ruan [6–8] improved the above results and applied the obtained center manifold theorem to study Hopf bifurcation of age-structured models. Recently, Neamţu [9] extended the mentioned theory of integrated semigroups to ill-posed stochastic evolution equations (1) driven by linear white noise ($\sigma(u) = u$) and established the existence of random stable/unstable manifolds based on the Lyapunov-Perron method. Following this direction, Zeng and Shen [10] obtained the existence of the invariant foliations of (1). Moreover, Li and Zeng [11] studied the center manifold issue of (1) with exponential trichotomy. Therefore, the mentioned work brings much interest to discern the long-time dynamics of (1).

On the other hand, the above-mentioned results are mainly about the pathwise random invariant manifolds of the equation. A pathwise random invariant manifolds is a set-valued random variable that is given by a graph of a Lipschitz map. To study such manifolds, we need to define pathwise random dynamical systems for the stochastic evolution equation. There are several results regarding invariant manifolds in the framework of random dynamical systems. Mohammed and Scheutzow [12] studied the existence of local stable and unstable manifolds of stochastic differential equations driven by semimartingales. Duan et al. [13], Lu and Schmalfuss [14] and Caraballo et al. [15] studied stable and unstable manifolds for stochastic partial differential equations. Lian and Lu [16] proved a multiplicative ergodic theorem and then use this theorem to establish the stable and unstable manifold theorem for nonuniformly hyperbolic random invariant sets. Li et al. [17] proved the persistence of smooth normally hyperbolic invariant manifolds for dynamical systems under random perturbations. As for center manifolds, Chen et al. [18] studied center manifolds for stochastic partial differential equations under an assumption of exponential trichotomy. Shi [19] studied the limiting behavior of center manifolds for a class of singularly perturbed stochastic partial differential equations in terms of the phase spaces.

To our knowledge, the existence of pathwise random dynamical systems for stochastic partial differential equations is mainly restricted to additive white noise and linear multiplicative white noise. In this case, the key step is to transform the stochastic partial differential equations into random ones. For instance, a coordinate transform based on the stationary Ornstein-Uhlenbeck process was used to transform (1) driven by linear noise to a deterministic equation with random coefficients [20,21,13]. Also, there were a few shots for the case of nonlinear multiplicative noise. To this aspect, Garrido-Atienza et al. [22] studied the local unstable manifold for stochastic PDEs with a fractional Brownian motion with $1/2 < H < 1$ using fractional calculus technique and constructing a stopping time sequence. By means of rough paths theory, Neamţu and Kuehn [23] established the existence and regularity of local center manifolds for rough differential equations.

Our aim here is to develop a theory of mean-square random invariant manifolds of (1) based on the so-called mean-square random dynamical system, presented by Lorenz and Kloeden [24]. Indeed, we adopt Wang's approach in [25] and integrated semigroup theory. Precisely, we will define a backward and forward stochastic equation, respectively, and both of them involve the conditional expectation $\mathbb{E}(\cdot|\mathcal{F}_t)$ since every term of them is required to be $\mathcal{F}_t$-adapted. Then we use the Lyapunov-Perron method to construct mean-square random unstable and stable invariant manifolds, respectively. By solving these two equations in the space of stochastic processes defined on backward time and forward time respectively and defining random sets of these solutions, we can prove the existence of the invariant manifold which is given by a graph of a Lipchitz map. But it turns out that the invariance of random set defined by solutions of forward one can not be proved. Alternatively, we establish the existence of mean-square random stable sets which is given by a graph of Lipschitz map which maps from a subspace instead of the whole space.

It also needs to emphasize that the standard variation of constants formula is not applicable due to the non-densely defined operator A. Even worse, Young's convolution inequality is not available such that the Gronwall-type lemma fails to deduce estimates of the solutions. For the former, we will construct a new variation of constants formula by the resolvent operator connecting $\overline{D(A)}$ and X. While for the latter, we will impose an additional condition to complete the required estimates.

The rest of this paper is organized as follows. In Section 2, we collect some basic definitions of mean-square random dynamical systems and integrated semigroups and lay out the basic assumptions. Then in Section 3 and Section 4, we prove the existence of mean-square random unstable invariant manifolds and stable invariant sets for (1), respectively. Finally in Section 6, we summarize the conclusions obtained and discuss several possible extensions.

2. Preliminaries

2.1.

Suppose V is a Banach space with norm $\|\cdot\|_V$. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathbb{P})$ be a complete filtered probability space satisfying the usual condition, that is, $\{\mathcal{F}_t\}_{t\in\mathbb{R}}$ is an increasing right continuous family of sub-σ-algebras of $\mathcal{F}$ that contains all $\mathbb{P}$-null sets. Given $p \in (1, +\infty)$ and $t \in \mathbb{R}$, denote by $L^p(\Omega, \mathcal{F}_t; V)$ the subspace of $L^p(\Omega, \mathcal{F}; V)$, which consists of all strongly $\mathcal{F}_t$-measurable functions ψ in $L^p(\Omega, \mathcal{F}; V)$. We now present the definition of mean dynamical system over $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathbb{P})$ (see [24,25]).

Definition 2.1. A family $\Phi = \left\{\Phi(t,\tau) : t \in \mathbb{R}^+, \tau \in \mathbb{R}\right\}$ of mappings is called a mean-square random dynamical system on $L^2(\Omega, \mathcal{F}; V)$ over the space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathbb{P})$ if $\forall\, \tau \in \mathbb{R}$, $t, s \in \mathbb{R}^+$, it follows that

(i) $\Phi(t,\tau)$ maps $L^2(\Omega, \mathcal{F}_\tau; V)$ to $L^2(\Omega, \mathcal{F}_{t+\tau}; V)$;
(ii) $\Phi(0,\tau)$ is the identity operator on $L^2(\Omega, \mathcal{F}_\tau; V)$;
(iii) $\Phi(t+s,\tau) = \Phi(t, \tau+s) \circ \Phi(s,\tau)$.

Then we present the definition of the invariance property.

Definition 2.2. Suppose $\Phi = \left\{\Phi(t,\tau) : t \in {}^{+}, \tau \in \right\}$ is a mean-square random dynamical system on $L^2(\Omega, \mathcal{F}; V)$ over $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathbb{P})$ and $\mathcal{M} = \{\mathcal{M}(\tau) \subseteq L^2(\Omega, \mathcal{F}_\tau; V) : \tau \in \}$ is a family of subsets of $L^2(\Omega, \mathcal{F}; V)$. Then $\mathcal{M}$ is called a mean-square random invariant set of Φ in V if for all $t \in {}^{+}$,

$$\Phi(t,\tau)\mathcal{M}(\tau) \subseteq \mathcal{M}(\tau + t).$$

We list a part of basic assumptions here. Let Y be another separable Hilbert space. The norm of X is denoted by $\|\cdot\|$. Let $\mathcal{L}(Y, X)$ be the space of continuous linear operators from Y to X. Given a symmetric and nonnegative operator $Q \in \mathcal{L}(Y, X)$, we write $Y_0 = Q^{1/2}Y$. The space of Hilbert-Schmidt operators from Y_0 to X is denoted by $\mathcal{L}_2(Y_0, X)$ with norm $\|\cdot\|_{\mathcal{L}_2(Y_0,X)}$. From now on we assume that W is a two-sided Y-valued Q-Wiener process defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathbb{P})$. For simplicity, we denote $L^2(\Omega, \mathcal{F}; X)$ and its norm as $L^2(\Omega, X)$ and $\|\cdot\|_{L^2(\Omega,X)}$, respectively.

Assumption 2.1. Assume that $F : L^2(\Omega, X_0) \to L^2(\Omega, X)$ is globally Lipschitz continuous, i.e.

$$\|F(u_1) - F(u_2)\|_{L^2(\Omega,X)} \leq L_1\|u_1 - u_2\|_{L^2(\Omega,X_0)}, \tag{2}$$

for all $u_1, u_2 \in L^2(\Omega, X_0)$, where L_1 is the Lipschitz constant and $F(0) = 0$. Thus

$$\|F(u)\|_{L^2(\Omega,X)} \leq L_1\|u\|_{L^2(\Omega,X_0)}, \tag{3}$$

for all $u \in L^2(\Omega, X_0)$. For the nonlinear diffusion term, we assume that $\sigma : L^2(\Omega, X_0) \to L^2(\Omega, \mathcal{L}_2(Y_0, X))$ is globally Lipschitz continuous, i.e.

$$\|\sigma(u_1) - \sigma(u_2)\|_{L^2(\Omega,\mathcal{L}_2(Y_0,X))} \leq L_2\|u_1 - u_2\|_{L^2(\Omega,X_0)}, \tag{4}$$

for all $u_1, u_2 \in L^2(\Omega, X_0)$, where L_2 is the Lipschitz constant and $\sigma(0) = 0$. Thus

$$\|\sigma(u)\|_{L^2(\Omega,\mathcal{L}_2(Y_0,X))} \leq L_2\|u\|_{L^2(\Omega,X_0)}, \tag{5}$$

for all $u \in L^2(\Omega, X_0)$.

2.2. [illegible]

In this subsection, we collect some basic results about non-densely defined operator and integrated semigroup from [6,8,7,26]. The resolvent set of A is denoted by $\rho(A) = \{\lambda \in \mathbb{C} : \lambda I - A \text{ is invertible}\}$ and we write $R_\lambda(A) := (\lambda I - A)^{-1}$. The spectrum of A is denoted by $\sigma(A) := \mathbb{C} \backslash \rho(A)$. And we construct the part of A denoted by $A_0 : D(A_0) \subset X_0 \to X_0$, which is a linear operator on X_0 defined by

$$A_0 x = Ax, \forall x \in D(A_0) := \{y \in D(A) : Ay \in X_0\}.$$

Assume that there exists a constant ϑ satisfying $(\vartheta, +\infty) \subset \rho(A)$, which means $\rho(A) \neq \emptyset$. Then it follows from [7, Lemma 2.1] that $\rho(A) = \rho(A_0)$, from which we get $\sigma(A) = \sigma(A_0)$ and for each $\lambda > \vartheta$,

$$D(A_0) = R_\lambda(A)X_0,\ R_\lambda(A_0) = R_\lambda(A)|_{X_0}.$$

Recall that A is said to be a Hille-Yosida operator if there exist two constants, $\vartheta \in$ and $M \geq 1$, such that $(\vartheta, +\infty) \subset \rho(A)$ and

$$\left\| (\lambda I - A)^{-k} \right\|_{\mathcal{L}(X)} \leq \frac{M}{(\lambda - \vartheta)^k},\ \forall \lambda > \vartheta,\ \forall k \geq 1.$$

In the following, we assume A satisfies some weaker conditions:

Assumption 2.2. Let $(X, \|\cdot\|)$ be a Banach space and let $A : D(A) \subset X \to X$ be a linear operator. Assume that

(a) A is a Hille-Yosida operator on X_0;
(b) $\lim_{\lambda \to +\infty} R_\lambda(A)x = 0,\ \forall x \in X$.

Lemma 2.1. 6, 2.1 $(X, \|\cdot\|)$ $A : D(A) \subset X \to X$. A $\vartheta \in$, $(\vartheta, +\infty) \subset \rho(A)$

$$\limsup_{\lambda \to +\infty} \lambda \|R_\lambda(A)\|_{\mathcal{L}(X_0)} < +\infty.$$

:

(i) $\lim_{\lambda \to +\infty} \lambda R_\lambda(A)x = x, \forall x \in X_0$;
(ii) $\lim_{\lambda \to +\infty} R_\lambda(A)x = 0, \forall x \in X$;
(iii) $\overline{D(A_0)} = X_0$.

Lemma 2.2. 27, 5.3 A $A : D(A) \subset X \to X$ - C_0- $(T(t))_{t \geq 0}$ $\|T(t)\|_{\mathcal{L}(X)} \leq Me^{\vartheta t}$,

(i) A X;
(ii) A - .

Then according to Assumption 2.2 and Lemma 2.2, we infer that A_0 generates a C_0-semigroup on X_0 denoted by $(T(t))_{t \geq 0}$, which can also be denoted by $\left(e^{A_0 t}\right)_{t \geq 0}$. Now we give the definition of integrated semigroups.

Definition 2.3. Let $(X, \|\cdot\|)$ be a Banach space. A family of bounded linear operators $\{S(t)\}_{t \geq 0}$ on X is called an integrated semigroup if

(i) $S(0) = 0$;
(ii) the map $t \to S(t)x$ is continuous on $[0, +\infty)$ for each $x \in X$;
(iii) $S(t)$ satisfies

$$S(s)S(t) = \int_0^s (S(r+t) - S(r))\,dr, \forall s, t \geq 0.$$

An integrated semigroup $\{S(t)\}_{t\geq 0}$ is said to be non-degenerate if whenever $S(t)x = 0$, $\forall t \geq 0$, then $x = 0$. According to Thieme [4], we say that a linear operator $A : D(A) \subset X \to X$ is the generator of a non-degenerate integrated semigroup $\{S(t)\}_{t\geq 0}$ on X if and only if

$$x \in D(A),\ y = Ax \Leftrightarrow S(t)x - tx = \int_0^t S(s)\, y ds,\ \forall t \geq 0.$$

Lemma 2.3. *6, … 2.5 … A … 2.2 … A … $\{S(t)\}_{t\geq 0}$. … $x \in X, t \geq 0$ … $\mu > \vartheta$, $S(t)x$ …*

$$S(t)x = \mu \int_0^t T(s)\, R_\mu(A) ds + (I - T(t))\, R_\mu(A)x. \tag{6}$$

A … $t \to S(t)x$ … $x \in X_0$ …

$$\frac{dS(t)x}{dt} = T(t)x,\ \forall t \geq 0,\ \forall x \in X_0.$$

From Lemma 2.3, we can deduce that for $\mu > \vartheta$, $S(t)$ commutes with $(\mu I - A)^{-1}$ and

$$S(t)x = \int_0^t T(s)x ds,\ \forall x \in X_0.$$

Hence, for each $x \in X_0,\ t \geq 0$ and each $\mu > \vartheta$,

$$R_\mu(A)S(t)x = S(t)\, R_\mu(A)x = \int_0^t T(s)\, R_\mu(A)x ds. \tag{7}$$

In addition, by Arendt [2, Lemma 3.2.2 b),d)], $S(t)x \in X_0$ for $x \in X$. Thus by (7) and Lemma 2.1(i), we have for each $x \in X_0,\ t \geq 0$ and each $\mu > \vartheta$,

$$S(t)x = \lim_{\mu \to +\infty} \int_0^t T(s) \mu R_\mu(A)x ds.$$

2.3.

We need to find the integrated solution (or mild solution) of (1). Recall the non-homogeneous Cauchy problem discussed in [2,6,8,7,26]

$$\frac{du}{dt} = Au(t) + f(t),\ t \geq 0,\ u(0) = x \in X_0, \tag{8}$$

with $f \in L^1((0,\tau_0), X)$. If A is a generator of a C_0-semigroup $(T(t))_{t\geq 0}$, then the variation of constants formula indicates that (8) has a unique mild solution $u(t)$ given by

$$u(t) = T(t)x + \int_s^t T(t-s) f(s)\, ds.$$

Now we consider (8) under the assumptions we set up for A in this article.

Lemma 2.4. *6, 2.6 A 2.2 $\tau_0 > 0$. $f \in C^1([0,\tau_0], X)$,*

$$(S * f)(t) = \int_0^t S(s) f(t-s) ds, \ \forall t \in [0,\tau_0].$$

:

(i) $t \to (S * f)(t)$ $[0,\tau_0]$;
(ii) $(S * f)(t) \in D(A), \ \forall t \in [0,\tau_0]$;
(iii) $u(t) = \frac{d}{dt}(S * f)(t)$,

$$u(t) = A\int_0^t u(s) ds + \int_0^t f(s) ds, \ \forall t \in [0,\tau_0].$$

(iv) $\lambda > \vartheta$, $t \in [0,\tau_0]$,

$$R_\lambda(A) \frac{d}{dt}(S * f)(t) = \int_0^t T(t-s) R_\lambda(A) f(s)\, ds.$$

Definition 2.4. A continuous map $u \in C([0,\tau_0], X)$ is called an integrated solution (or mild solution) of (8) if and only if

$$\int_0^t u(s)\, ds \in D(A), \ \forall t \in [0,\tau_0], \tag{9}$$

and

$$u(t) = x + A\int_0^t u(s) ds + \int_0^t f(s) ds, \ \forall t \in [0,\tau_0].$$

We assume additionally A is a closed operator, from (9) we know that if u is an integrated solution of (9), then $u(t) \in X_0$, $\forall t \in [0,\tau_0]$. And we need the following assumption:

Assumption 2.3. Assume that there exists a non-decreasing map $\delta : [0, \tau_0] \to [0, +\infty)$ satisfying

$$\lim_{t\to 0^+} \delta(t) = 0,$$

such that for each $f \in C([0, \tau_0]; X)$,

$$\left\| \frac{d}{dt}(S * f)(t) \right\| \le \delta(t) \sup_{s\in[0,t]} \|f(s)\|, \ \forall t \in [0, \tau_0].$$

Lemma 2.5. 6, 2.12 A 2.2 A 2.3 . $x \in X_0$ $f \in L^1((0, \tau_0), X)$, (8) $u \in C([0, \tau_0], X_0)$

$$u(t) = T(t)x + \frac{d}{dt}(S * f)(t), \ \forall t \in [0, \tau_0].$$

Since $\frac{d}{dt}(S * f)(t) \in X_0$ for all $t \in [0, \tau_0]$, denote $\frac{d}{dt}(S * f)(t)$ as $(S \diamond f)(t)$, by Lemma 2.4(iv) and Lemma 2.1(i), we have

$$(S \diamond f)(t) = \lim_{\lambda\to+\infty} \int_0^t T(t-s)\lambda R_\lambda(A) f(s)\, ds, \ \lambda > \vartheta.$$

Similarly, we could define the integrated solution of (1).

Definition 2.5. For $\tau \in$, if $u \in C\left([\tau, +\infty), L^2(\Omega, X)\right)$ is a $\mathcal{F}_t$-progressively measurable process, then it's called an integrated solution (or mild solution) of (1) if and only if

$$\int_\tau^t u(s)\, ds \in D(A), \ \forall t \in [\tau, +\infty)$$

and

$$u(t) = u_0 + A \int_\tau^t u(s) ds + \int_\tau^t F(u(s)) ds + \int_\tau^t \sigma(u(s)) dW(s), \ \forall t \in [\tau, +\infty).$$

Since we shall consider (1) with $t \in$, we give the following modified assumption

Assumption 2.4. Assume that there exists a non-decreasing map $\delta :$ $\to [0, +\infty)$ satisfying

$$\lim_{t\to 0} \delta(t) = 0,$$

such that for each $f \in C($ $; X)$,

$$\|(S \diamond f)(t)\| \le \delta(t) \sup_{s\in[0,t]} \|f(s)\|, \ \forall t \in \ .$$

Lemma 2.6. 7, 2.13 A 2.2 A 2.4 . $\kappa > \vartheta$, $C_\kappa > 0$ $f \in C(\ ; X)$

$$\|(S \diamond f)(t)\| \leq C_\kappa \sup_{s\in[0,t]} e^{\kappa(t-s)} \|f(t)\|, \ \forall t \in \ .$$

, $\varepsilon > 0$, $\rho_\varepsilon > 0$ $M\delta(\rho_\varepsilon) \leq \varepsilon$,

$$C_\kappa = \frac{2\varepsilon \max\left(1, e^{-\kappa\rho_\varepsilon}\right)}{1 - e^{(\vartheta-\kappa)\rho_\varepsilon}}.$$

We now are ready to present the existence of solution to (1).

Theorem 2.1. A 2.1, A 2.2 A 2.4 . $\tau \in$ $u_0 \in L^2(\Omega, \mathcal{F}_\tau; X_0)$, (1) $L^2(\Omega, X_0)$ $[\tau, +\infty)$

$$\begin{aligned} u(t, \tau, u_0) =& T(t-\tau)u_0 + \lim_{\lambda\to+\infty} \int_\tau^t T(t-s)\lambda R_\lambda(A)F(u(s))\,ds \\ &+ \lim_{\lambda\to+\infty} \int_\tau^t T(t-s)\lambda R_\lambda(A)\sigma(u(s))\,dW(s). \end{aligned} \tag{10}$$

Proof. Note that (10) is the so-called modified variation of constants formula. By applying the resolvent operator $(\lambda I - A)^{-1}$ which maps X to X_0 and regarding that $D(A_0) = (\lambda I - A)^{-1} X_0$, we could turn (1) to a evolution equation on X_0 on which A_0 generates a C_0-semigroup as follows

$$dR_\lambda(A)u(t) = A_0 R_\lambda(A)u(t)\,dt + R_\lambda(A)F(u(t))\,dt + R_\lambda(A)\sigma(u(t))\,dW(t). \tag{11}$$

Then by the variation of constants formula, we could obtain the mild solution of (11) given by

$$\begin{aligned} R_\lambda(A)u(t) =& T(t-\tau)R_\lambda(A)u_0 + \int_\tau^t T(t-s)R_\lambda(A)F(u(s))\,ds \\ &+ \int_\tau^t T(t-s)R_\lambda(A)\sigma(u(s))\,dW(s). \end{aligned}$$

By Lemma 2.1(i), we obtain (10). Rewrite (10) as

$$u(t) = T(t-\tau)u_0 + (S \diamond (F(u) + \sigma(u)\,dW))(s). \tag{12}$$

By Lemma 2.4(iii) and Definition 2.5, (12) is an integrated solution of (1). For given $\kappa_0 > \vartheta$, we denote by $\mathcal{C}_{\kappa_0,\tau}$ the space of all X_0-valued $\mathcal{F}_t$-progressively measurable processes $f(t), t \in [\tau, +\infty)$ such that $f : [\tau, +\infty) \to L^2(\Omega, X_0)$ is continuous and

$$\sup_{t\geq\tau}\left(e^{-\kappa t}\left\|f(t)\right\|_{L^2(\Omega,X_0)}\right)<\infty,$$

with norm

$$\left\|f(t)\right\|_{\mathcal{C}_{\kappa_0,\tau}}=\sup_{t\geq\tau}\left(e^{-\kappa_0 t}\left\|f(t)\right\|_{L^2(\Omega,X_0)}\right),\ \forall f\in\mathcal{C}_{\kappa_0,\tau}.$$

Denote $\mathcal{G}_{u_0}(u)(t)$ the right side of (10), i.e.

$$\begin{aligned}\mathcal{G}_{u_0}(u)(t)=&T(t-\tau)u_0+\lim_{\lambda\to+\infty}\int_\tau^t T(t-s)\lambda R_\lambda(A)F(u(s))\,ds\\&+\lim_{\lambda\to+\infty}\int_\tau^t T(t-s)\lambda R_\lambda(A)\sigma(u(s))\,dW(s).\end{aligned}$$

For $u_1,u_2\in C\left([\tau,+\infty),L^2(\Omega,X_0)\right)$ be two X_0-valued $\mathcal{F}_t$-progressively measurable processes, we have, for $t\geq\tau$,

$$\begin{aligned}&\left\|\mathcal{G}_{u_0}(u_1)(t)-\mathcal{G}_{u_0}(u_2)(t)\right\|_{L^2(\Omega,X_0)}\\&\leq\lim_{\lambda\to+\infty}\int_0^{t-\tau}\left\|T(t-g)\lambda R_\lambda(A)\left(F(u_1(g))-F(u_2(g))\right)\right\|_{L^2(\Omega,X_0)}dl\\&+\left\|\lim_{\lambda\to+\infty}\int_0^{t-\tau}T(t-g)\lambda R_\lambda(A)\left(\sigma(u_1(g))-\sigma(u_2(g))\right)dW(l)\right\|_{L^2(\Omega,X_0)}\end{aligned}$$

where $g=l+\tau$. By Lemma 2.6, (2) and (4), for $\varepsilon>0$, $\rho_\varepsilon>0$ with $M\delta(\rho_\varepsilon)\leq\varepsilon$ and $\kappa_0>\vartheta$, there exists a constant

$$C_{\kappa_0}=\frac{2\varepsilon\max\left(1,e^{-\kappa_0\rho_\varepsilon}\right)}{1-e^{(\vartheta-\kappa_0)\rho_\varepsilon}},$$

such that

$$\begin{aligned}&e^{-\kappa_0 t}\left\|\mathcal{G}_{u_0}(u_1)(t)-\mathcal{G}_{u_0}(u_2)(t)\right\|_{L^2(\Omega,X_0)}\\&\leq L_1C_{\kappa_0}\sup_{l\in[0,t-\tau]}e^{-\kappa_0 g}\left\|u_1(g)-u_2(g)\right\|_{L^2(\Omega,X_0)}\\&+C_{\kappa_0}\sup_{l\in[0,t-\tau]}e^{-\kappa_0 g}\mathbb{E}\left[\left\|\sigma(u_1(g))-\sigma(u_2(g))\right\|^2_{\mathcal{L}_2(Y_0,X)}\right]^{\frac{1}{2}}\\&\leq(L_1+L_2)C_{\kappa_0}\left\|u_1-u_2\right\|_{\mathcal{C}_{\kappa_0,\tau}}.\end{aligned}\tag{13}$$

By (13), we get that $\mathcal{G}_{u_0}(\cdot)(t):\mathcal{C}_{\kappa_0,\tau}\to\mathcal{C}_{\kappa_0,\tau}$ is well-defined. Let $\varepsilon>0$ such that $\varepsilon\max((L_1+L_2),1)<1/8$ and $\kappa>\max(0,\vartheta)$ such that,

$$\frac{1}{1-e^{(\vartheta-\kappa_0)\rho_\varepsilon}} < 2, \ \forall \kappa_0 \geq \kappa.$$

Then by (13), we get that

$$\left\| \mathcal{G}_{u_0}(u_1)(t) - \mathcal{G}_{u_0}(u_2)(t) \right\|_{\mathcal{C}_{\kappa_0,\tau}} \leq 4(L_1+L_2)\varepsilon \|u_1-u_2\|_{\mathcal{C}_{\kappa_0,\tau}} \leq \frac{1}{2}\|u_1-u_2\|_{\mathcal{C}_{\kappa_0,\tau}}.$$

So $\mathcal{G}_{u_0}$ is a contraction. From the contraction mapping principle, there exists a unique integrated solution of (1) in $\mathcal{C}_{\kappa_0,\tau}$ for any given $\tau \in$. Thus the uniqueness of the solution is proved. □

For further details, please refer to [2,26,9]. Now we can define a mean-square random dynamical system for (1). Given $t \in {}^{+}$ and $\tau \in$, let $\Phi(t,\tau)$ be a mapping from $L^2(\Omega, \mathcal{F}_\tau; X_0)$ to $L^2(\Omega, \mathcal{F}_{\tau+t}; X_0)$ given by

$$\Phi(t,\tau)(u_0) = u(t+\tau, \tau, u_0),$$

where $u_0 \in L^2(\Omega, \mathcal{F}_\tau; X_0)$. Due to the uniqueness of solutions, for $\forall t, s \geq 0$ and $\tau \in$,

$$\Phi(t+s,\tau) = \Phi(t, s+\tau) \circ \Phi(s,\tau).$$

As a consequence, Φ is a mean-square random dynamical system generated by solutions of (1) in $L^2(\Omega, \mathcal{F}; X)$. Based on the forward solutions (10), we could define the backward solutions on $(-\infty, \tau]$. Given $\tau \in$, an X_0-valued $\mathcal{F}_t$-progressively measurable process $\xi(t), t \in (-\infty, \tau]$, is called an integrated solution of (1) on $(-\infty, \tau]$ if $\xi(r) \in L^2(\Omega, \mathcal{F}_r; X_0)$ for all $r \in (-\infty, \tau]$ and $u(t, r, \xi(r)) = \xi(t)$ for all $r \leq t \leq \tau$, where $u(t, r, \xi(r)) = \xi(t)$ is the unique solution of (1) with initial value $\xi(r)$ at initial time r.

2.4.

Assumption 2.5. We assume $T(t)$ satisfies the pseudo exponential dichotomy with exponents $\beta < \alpha$ and bound K. That's, there exists a continuous linear projection operator with finite rank $\Pi_{0u} \in \mathcal{L}(X_0)$ such that

$$\Pi_{0u} T(t) = T(t)\Pi_{0u}, \ \forall t \in \ .$$

Denote $\Pi_{0s} = I_{X_0} - \Pi_{0u}$, $X_{0u} = \Pi_{0u}X_0$ and $X_{0s} = \Pi_{0s}X_0$, it holds

$$\left\| T_{A_{0u}}(t)\Pi_{0u} \right\| \leq K e^{\alpha t}, \ \forall t \leq 0, \tag{14}$$

$$\left\| T_{A_{0s}}(t)\Pi_{0s} \right\| \leq K e^{\beta t}, \ \forall t \geq 0, \tag{15}$$

where $A_{0p} = A_0|_{X_{0p}}$ and $T_{A_{0p}}(t)$ is the C_0-semigroup generated by A_{0p}, $\forall p \in \{u, s\}$.

Remark 2.1. Assume $\alpha > \gamma \geq 0 \geq -\gamma > -\beta$. If X_0 is an infinite dimensional space and the spectrum of A_0 satisfies $\sigma(A_0) = \sigma^{0s} \cup \sigma^{0c} \cup \sigma^{0u}$, where $\sigma^{0s} = \{\lambda \in \sigma(A_0) : \mathrm{Re}(\lambda) \leq -\beta\}$, $\sigma^{0c} = \{\lambda \in \sigma(A_0) : |\mathrm{Re}(\lambda)| \leq \gamma\}$, $\sigma^{0u} = \{\lambda \in \sigma(A_0) : \mathrm{Re}(\lambda) \geq \alpha\}$, and A_0 generates a strong continuous semigroup, then the exponential trichotomy holds, too (see [28, page 267]. And if the spectrum of A_0 satisfies $\sigma(A_0) = \sigma^{0s} \cup \sigma^{0u}$, where σ^{0s} and σ^{0u} are the same as above, then the exponential dichotomy holds. Since $\sigma(A) = \sigma(A_0)$, the spectrum of A can be split as A_0.

Since we only require the Hille-Yosida condition on X_0, one needs to extend the mentioned projections from X_0 to X and outline the associated statement below (see [7, Proposition 3.5].

Lemma 2.7. $\Pi_0 : X_0 \to X_0$

$$\Pi_0 R_\lambda (A_0) = R_\lambda (A_0) \Pi_0, \ \forall \lambda > \vartheta,$$

$$\Pi_0 X_0 \subset D(A_0), \quad A_0|_{\Pi_0 X_0}.$$

$\Pi : X \to X$

(i) $\Pi|_{X_0} = \Pi_0$;
(ii) $\Pi(X) \subset X_0$;
(iii) $\Pi R_\lambda (A) = R_\lambda (A) \Pi$, $\forall \lambda > \vartheta$.

$x \in X$,

$$\Pi x = \lim_{\lambda \to +\infty} \Pi_0 \lambda R_\lambda (A) x = \lim_{h \to 0^+} \frac{1}{h} \Pi_0 S(h) x,$$

S A.

We use Lemma 2.7 to state the decomposition on X. Denote $\Pi_u : X \to X$ the unique extension of Π_{0u} and let $\Pi_s = I_X - \Pi_u$. Then we have for each $p \in \{u, s\}$,

$$\Pi_p R_\lambda (A) = R_\lambda (A) \Pi_p, \ \forall \lambda > \vartheta,$$

and

$$\Pi_p (X_0) \subset X_0.$$

Set

$$X_{0p} = \Pi_p (X_0), \ X_p = \Pi_p (X), \ \text{and } A_p = A|_{X_p}.$$

Then we have $X_u = X_{0u}$. Moreover, we have

$$X_0 = X_{0s} \oplus X_{0u}, \ \text{and } X = X_s \oplus X_u.$$

Herein, X_{0u} and X_{0s} are called unstable subspace and stable subspace of X_0 respectively. For more details, see [7, page 22]. Then we give the definition of mean-square random invariant manifolds and mean-square random invariant sets.

Definition 2.6. Let $\mathcal{M} = \left\{\mathcal{M}(\tau) \subseteq L^2(\Omega, \mathcal{F}_\tau; X_0) : \tau \in \quad \right\}$ be a family of subsets of the space $L^2(\Omega, \mathcal{F}; X_0)$.

(i) $\mathcal{M}$ is called a mean-square random unstable invariant set of Φ in X_0 if $\mathcal{M}$ is invariant under Φ and there is a family $h^u = \{h^u(\cdot, \tau) : \tau \in \quad \}$ of mappings such that for every $\tau \in \quad$, $h^u(x, \tau) : D(h^u(x, \tau)) \subseteq L^2(\Omega, \mathcal{F}_\tau; X_{0u}) \to L^2(\Omega, \mathcal{F}_\tau; X_{0s})$ is Lipschitz continuous and

$$\mathcal{M}(\tau) = \left\{x + h^u(x, \tau) : x \in D\left(h^u(x, \tau)\right)\right\};$$

(ii) $\mathcal{M}$ is called a mean-square random unstable invariant manifold of Φ in X_0 if $\mathcal{M}$ is invariant under Φ and there is a family $h^u = \{h^u(\cdot, \tau) : \tau \in \quad \}$ of mappings such that for every $\tau \in \quad$, $h^u(x, \tau) : L^2(\Omega, \mathcal{F}_\tau; X_{0u}) \to L^2(\Omega, \mathcal{F}_\tau; X_{0s})$ is Lipschitz continuous and

$$\mathcal{M}(\tau) = \left\{x + h^u(x, \tau) : x \in L^2(\Omega, \mathcal{F}_\tau; X_{0u})\right\};$$

(iii) $\mathcal{M}$ is called a mean-square random stable invariant set of Φ in X_0 if $\mathcal{M}$ is invariant under Φ and there is a family $h^s = \{h^s(\cdot, \tau) : \tau \in \quad \}$ of mappings such that for every $\tau \in \quad$, $h^s(x, \tau) : D(h^s(x, \tau)) \subseteq L^2(\Omega, \mathcal{F}_\tau; X_{0s}) \to L^2(\Omega, \mathcal{F}_\tau; X_{0u})$ is Lipschitz continuous and

$$\mathcal{M}(\tau) = \left\{x + h^s(x, \tau) : x \in D\left(h^s(x, \tau)\right)\right\};$$

(iv) $\mathcal{M}$ is called a mean-square random stable invariant set of Φ in X_0 if $\mathcal{M}$ is invariant under Φ and there is a family $h^s = \{h^s(\cdot, \tau) : \tau \in \quad \}$ of mappings such that for every $\tau \in \quad$, $h^s(x, \tau) : L^2(\Omega, \mathcal{F}_\tau; X_{0s}) \to L^2(\Omega, \mathcal{F}_\tau; X_{0u})$ is Lipschitz continuous and

$$\mathcal{M}(\tau) = \left\{x + h^s(x, \tau) : x \in L^2(\Omega, \mathcal{F}_\tau; X_{0s})\right\}.$$

3. Mean-square random unstable invariant manifolds

This section is dedicated to the existence of mean-square random unstable invariant manifolds of (1) through the Lyapunov-Perron method. Given $\tau \in \quad$, denote by $\mathcal{C}_\tau^-$ the space of all X_0-valued $\mathcal{F}_t$-progressively measurable processes $\xi(t), t \in (-\infty, \tau]$ such that $\xi : (-\infty, \tau] \to L^2(\Omega, X_0)$ is continuous and

$$\sup_{t \le \tau} \left(e^{-\gamma t} \|\xi(t)\|_{L^2(\Omega, X_0)}\right) < \infty, \tag{16}$$

with norm

$$\|\xi(t)\|_{\mathcal{C}_\tau^-} = \sup_{t \le \tau} \left(e^{-\gamma t} \|\xi(t)\|_{L^2(\Omega, X_0)}\right), \ \forall \xi \in \mathcal{C}_\tau^-,$$

where $\gamma \in (\beta, \alpha)$. Notice that $\mathcal{C}_\tau^-$ is a Banach space. To construct mean-square random unstable invariant manifolds of (1), we need to find all solutions of (1) in the space $\mathcal{C}_\tau^-$.

Lemma 3.1. A 2.1, A 2.2, A 2.4 A 2.5 . $\xi \in \mathcal{C}_\tau^-$ $\tau \in$. ξ (1) $(\infty, \tau]$ $x \in L^2(\Omega, \mathcal{F}_\tau; X_{0u})$ $t \le \tau$,

$$\begin{aligned}
\xi(t) = & T_{A_{0u}}(t-\tau)x - \int_t^\tau T_{A_{0u}}(t-r)\Pi_u F(\xi(r))\,dr \\
& - \int_t^\tau T_{A_{0u}}(t-r)\Pi_u \sigma(\xi(r))\,dW(r) \\
& + \lim_{\lambda\to+\infty} \int_{-\infty}^t T_{A_{0s}}(t-r)\lambda R_\lambda(A_s)\Pi_s F(\xi(r))\,dr \\
& + \lim_{\lambda\to+\infty} \int_{-\infty}^t T_{A_{0s}}(t-r)\lambda R_\lambda(A_s)\Pi_s \sigma(\xi(r))\,dW(r),
\end{aligned} \tag{17}$$

$L^2(\Omega, X_0)$.

Proof. Step 1: necessity. Suppose $\xi \in \mathcal{C}_\tau^-$ with $\tau \in$ is an integrated solution of (1). We prove that (17) holds true. Since $\xi \in \mathcal{C}_\tau^-$, by (14) and Assumption 2.1, the first two integrals on the right side of (8) are well-defined in $L^2(\Omega, X_0)$. For the third integral, according to Lemma 2.6, (3) and (15), for $\zeta \in (\beta, \gamma)$,

$$\begin{aligned}
& \lim_{\lambda\to+\infty} \int_{-\infty}^t \left\| T_{A_{0s}}(t-r)\lambda R_\lambda(A_s)\Pi_s F(\xi(r)) \right\|_{L^2(\Omega,X_0)} dr \\
& = \lim_{\lambda\to+\infty} \lim_{g\to-\infty} \int_0^{t-g} \left\| T_{A_{0s}}(t-g-l)\lambda R_\lambda(A_s)\Pi_s F(\xi(l+g)) \right\|_{L^2(\Omega,X_0)} dl \\
& \le \lim_{g\to-\infty} L_1 C_\zeta \sup_{l\in[0,t-g]} e^{\zeta(t-g-l)} \|\xi(l+g)\|_{L^2(\Omega,X_0)} \\
& = L_1 C_\zeta \sup_{q\in(-\infty,t]} e^{\zeta(t-q)} \|\xi(q)\|_{L^2(\Omega,X_0)} \\
& \le L_1 C_\zeta \sup_{q\in(-\infty,t]} e^{\zeta(t-q)} e^{\gamma q} \|\xi(q)\|_{\mathcal{C}_\tau^-} \\
& = L_1 C_\zeta \sup_{q\in(-\infty,t]} e^{(\gamma-\zeta)q} e^{(\zeta-\gamma)t} e^{\gamma t} \|\xi(q)\|_{\mathcal{C}_\tau^-} \\
& = L_1 C_\zeta e^{\gamma t} \|\xi\|_{\mathcal{C}_\tau^-} < \infty.
\end{aligned} \tag{18}$$

By (18), for $t \le \tau$,

$$\lim_{\lambda\to+\infty}\int_{-\infty}^{t} T_{A_{0u}}(t-r)\,\lambda R_\lambda(A_s)\Pi_s F(\xi(r))dr$$

is well-defined in $L^2(\Omega, X_0)$. For the fourth integral, according to Lemma 2.6, (5) and (14), for $\zeta\in(\beta,\gamma)$,

$$\begin{aligned}
&\left\|\lim_{\lambda\to+\infty}\int_{-\infty}^{t} T_{A_{0s}}(t-r)\,\lambda R_\lambda(A_s)\Pi_s\sigma(\xi(r))\,dW(r)\right\|_{L^2(\Omega,X_0)}\\
&=\mathbb{E}\left[\left(\lim_{\lambda\to+\infty}\lim_{g\to-\infty}\int_{0}^{t-g} T_{A_{0s}}(t-l-g)\,\lambda R_\lambda(A_s)\Pi_s\sigma(\xi(l+g))dW(l)\right)^2\right]^{\frac{1}{2}}\\
&\le\lim_{g\to-\infty}C_\zeta\sup_{l\in[0,t-g]}e^{\zeta(t-g-l)}\mathbb{E}\left[\|\sigma(\xi(l+g))\|^2_{\mathcal{L}_2(Y_0,X)}\right]^{\frac{1}{2}}\\
&\le L_2C_\zeta\sup_{q\in(-\infty,t]}e^{\zeta(t-q)}e^{\gamma q}\|\xi(q)\|_{\mathcal{C}_\tau^-}\\
&=L_2C_\zeta\sup_{q\in(-\infty,t]}e^{(\gamma-\zeta)q}e^{(\zeta-\gamma)t}e^{\gamma t}\|\xi(q)\|_{\mathcal{C}_\tau^-}\\
&=L_2C_\zeta e^{\gamma t}\|\xi\|_{\mathcal{C}_\tau^-}\\
&<\infty.
\end{aligned}\tag{19}$$

By (19), for $t\le\tau$,

$$\lim_{\lambda\to+\infty}\int_{-\infty}^{t} T_{A_{0s}}(t-r)\,\lambda R_\lambda(A_s)\Pi_s\sigma(\xi(r))\,dW(r),$$

is well-defined in $L^2(\Omega, X_0)$.

Since $\xi\in\mathcal{C}_\tau^-$ is a solution of (1) on $(-\infty,\tau]$, by (10), for all $s\le t\le\tau$,

$$\begin{aligned}
\xi(t)=&T(t-s)\xi(s)+\lim_{\lambda\to+\infty}\int_{s}^{t} T(t-r)\,\lambda R_\lambda(A)F(\xi(r))\,dr\\
&+\lim_{\lambda\to+\infty}\int_{s}^{t} T(t-r)\,\lambda R_\lambda(A)\sigma(\xi(r))\,dW(r).
\end{aligned}\tag{20}$$

Then we apply projections Π_{0u} and Π_{0s} to (20). Note

Also, $A|_{\Pi_u(X)} = A|_{\Pi_u(X)} = A|_{X_u} = A_u$ satisfies Assumption 2.2 in $\Pi_u(X) = X_u$. Moreover, we know that Π_u has a finite rank and $\Pi_u(X) \subset X_0$, which further indicates that $A_0|_{\Pi_u(X_0)} = A_0|_{X_{0u}} = A_{0u}$ and $A|_{\Pi_u(X)} = A|_{X_u} = A_u$. Thus if for each $x \in X_0$ and each $t \le \tau$, then the map $t \to \Pi_u\xi(t)$ solves

$$d\Pi_u\xi(t) = A_{0u}\Pi_u\xi(t)dt + \Pi_u F(\xi(t))dt + \Pi_u\sigma(\xi(t))dW(t)$$

in $\Pi_u(X_0) = X_{0u}$, then for all $s \le t \le \tau$,

$$\begin{aligned}
\Pi_u\xi(t) =& T_{A_0|_{\Pi_s(X_0)}}(t-s)\,\Pi_u\xi(s) + \left(S_{A|_{\Pi_u(X)}} \diamond \Pi_u(F(\xi) + \sigma(\xi)\,dW)\right)(r) \\
=& T_{A_{0s}}(t-s)\,\Pi_u\xi(s) + \left(S_{A_u} \diamond \Pi_u(F(\xi) + \sigma(\xi)\,dW)\right)(r) \\
=& T_{A_{0u}}(t-s)\,\Pi_u\xi(s) + \lim_{\lambda\to+\infty}\int_s^t T_{A_{0u}}(t-r)\,\lambda R_\lambda(A_u)\Pi_u F(\xi(r))dr \\
&+ \lim_{\lambda\to+\infty}\int_s^t T_{A_{0u}}(t-r)\,\lambda R_\lambda(A_u)\Pi_u\sigma(\xi(r))dW(r)\,.
\end{aligned} \tag{21}$$

Since Π_u is a extension of Π_{0u} from X_0 to X and $\Pi_u(X) \subset X_0$, by (21) and Lemma 2.1(i), we have

$$\begin{aligned}
\Pi_{0u}\xi(t) =& \Pi_u\xi(t) = T_{A_{0u}}(t-s)\,\Pi_{0u}\xi(s) + \int_s^t T_{A_{0u}}(t-r)\,\Pi_u F(\xi(r))dr \\
&+ \int_s^t T_{A_{0u}}(t-r)\,\Pi_u\sigma(\xi(r))dW(r)\,.
\end{aligned} \tag{22}$$

Through similar arguments, we have

$$\begin{aligned}
\Pi_{0s}\xi(t) =& T_{A_{0s}}(t-s)\,\Pi_{0s}\xi(s) + \lim_{\lambda\to+\infty}\int_s^t T_{A_{0s}}(t-r)\,\lambda R_\lambda(A_s)\Pi_s F(\xi(r))dr \\
&+ \lim_{\lambda\to+\infty}\int_s^t
\end{aligned}$$

$$\Pi_{0u}\xi(\tau) = T_{A_{0u}}(\tau - s)\Pi_{0u}\xi(s) + \int_s^\tau T_{A_{0u}}(\tau - r)\Pi_u F(\xi(r))dr + \int_s^\tau T_{A_{0u}}(\tau - r)\Pi_u \sigma(\xi(r))dW(r).$$

Hence for all $s \le t \le \tau$,

$$T_{A_{0u}}(t-\tau)\Pi_{0u}\xi(\tau) = T_{A_{0u}}(t - s)\Pi_{0u}\xi(s) + \int_s^\tau T_{A_{0u}}(t - r)\Pi_u F(\xi(r))dr + \int_s^\tau T_{A_{0u}}(t - r)\Pi_u \sigma(\xi(r))dW(r). \tag{24}$$

By (22) and (24),

$$\begin{aligned}\Pi_{0u}\xi(t) = {} & T_{A_{0u}}(t-\tau)\Pi_{0u}\xi(\tau) - \int_s^\tau T_{A_{0u}}(t - r)\Pi_u F(\xi(r))dr \\ & - \int_s^\tau T_{A_{0u}}(t - r)\Pi_u \sigma(\xi(r))dW(r) + \int_s^t T_{A_{0u}}(t - r)\Pi_u F(\xi(r))dr \\ & + \int_s^t T_{A_{0u}}(t - r)\Pi_u \sigma(\xi(r))dW(r).\end{aligned}$$

Thus

$$\Pi_{0u}\xi(t) = T_{A_{0u}}(t-\tau)\Pi_{0u}\xi(\tau) + \int_\tau^t T_{A_{0u}}(t - r)\Pi_u F(\xi(r))dr + \int_\tau^t T_{A_{0u}}(t - r)\Pi_u \sigma(\xi(r))dW(r). \tag{25}$$

By (23) and (25), for all $s \le t \le \tau$, we have

$$\xi(t) = T_{A_{0u}}(t-\tau)\Pi_{0u}\xi(\tau) + \int_\tau^t T_{A_{0u}}(t - r)\Pi_u F(\xi(r))dr + \int_\tau^t T_{A_{0u}}(t - r)\Pi_u \sigma(\xi(r))dW(r) + T_{A_{0s}}(t-s)\Pi_{0s}\xi(s) \tag{26}$$

$$+\lim_{\lambda\to+\infty}\int_s^t T_{A_{0s}}(t-r)\lambda R_\lambda(A_s)\Pi_s F(\xi(r))dr$$

$$+\lim_{\lambda\to+\infty}\int_s^t T_{A_{0s}}(t-r)\lambda R_\lambda(A_s)\Pi_s \sigma(\xi(r))dW(r).$$

Since for all $s\le t\le\tau$, let $s\to-\infty$ in (26), then by (15),

$$\begin{aligned}\left\|T_{A_{0s}}(t-s)\Pi_{0s}\xi(s)\right\|_{L^2(\Omega,X_0)} &\le Ke^{\beta(t-s)}\|\xi(s)\|_{L^2(\Omega,X_0)}\\ &\le Ke^{\beta t}e^{(\gamma-\beta)s}\|\xi(s)\|_{\mathcal{C}_\tau^-}\to 0.\end{aligned}$$

Thus (17) is fulfilled as wished with $x=\Pi_{0u}\xi(\tau)$.

Step 2: sufficiency. Suppose $\xi\in\mathcal{C}_\tau^-$ satisfies (17) for $t\le\tau$. Since $\xi\in\mathcal{C}_\tau^-$, we know ξ is an X_0-valued $\mathcal{F}_t$-progressively measurable process and $\xi\in C\left((-\infty,\tau],L^2(\Omega,X_0)\right)$. And from (17), one can verify that for all $r\le t\le\tau$,

$$\begin{aligned}\xi(t)=&T(t-r)\xi(r)+\lim_{\lambda\to+\infty}\int_r^t T(t-s)\lambda R_\lambda(A)F(\xi(s))ds\\ &+\lim_{\lambda\to+\infty}\int_r^t T(t-s)\lambda R_\lambda(A)\sigma(\xi(s))dW(s).\end{aligned}\tag{27}$$

From (27), we get that for all $r\le t\le\tau$,

$$\xi(t)=u(t,r,\xi(r)),$$

where u is the integrated solution of (1) with initial condition $\xi(r)$ at initial time r. Then by the definition of integrated solutions of (1) on $(-\infty,\tau]$, ξ is an integrated solution of (1) on $(-\infty,\tau]$. □

Now we are ready to construct mean-square unstable manifolds for (1). By Lemma 3.1, if $\xi\in\mathcal{C}_\tau^-$ is an integrated solution of (1), then ξ must satisfy (17). Take the conditional expectation $\mathbb{E}(\cdot|\mathcal{F}_t)$ on (17), we get for $t\le\tau$,

$$\begin{aligned}\xi(t)=&\mathbb{E}\left[T_{A_{0u}}(t-\tau)x-\int_t^\tau T_{A_{0u}}(t-r)\Pi_u F(\xi(r))d(r)\,|\mathcal{F}_t\right]\\ &+\lim_{\lambda\to+\infty}\int_{-\infty}^t T_{A_{0s}}(t-r)\lambda R_\lambda(A_s)\Pi_s F(\xi(r))dr\\ &+\lim_{\lambda\to+\infty}\int_{-\infty}^t T_{A_{0s}}(t-r)\lambda R_\lambda(A_s)\Pi_s\sigma(\xi(r))dW(r),\end{aligned}\tag{28}$$

in $L^2(\Omega, X_0)$ with $x \in L^2(\Omega, \mathcal{F}_\tau; X_{0u})$. For $\tau \in$ and $t \leq \tau$, set

$$\mathcal{M}^u(\tau) = \left\{\xi(\tau) \in L^2(\Omega, \mathcal{F}_\tau; X_0) : \xi(t) \in \mathcal{C}_\tau^- \text{and satisfies (28)}\right\}.$$

We below prove that $\mathcal{M}^u(\tau)$ is invariant under Φ and it is given by a graph of Lipschitz function.

Theorem 3.1. *A … A … 2.1, A … 2.2, A … 2.4 … A … 2.5 … $\beta < \zeta < \gamma < \alpha$ …*

$$K\left(L_1(\alpha-\gamma)^{-1} + L_1 C_\zeta + L_2 C_\zeta\right) < 1, \tag{29}$$

… (1) …

$$\mathcal{M}^u(\tau) = \left\{x + h^u(x,\tau) : x \in L^2(\Omega, \mathcal{F}_\tau; X_{0u})\right\},$$

… $h^u(\cdot,\tau) : L^2(\Omega, \mathcal{F}_\tau; X_{0u}) \to L^2(\Omega, \mathcal{F}_\tau; X_{0s})$ … .

Proof. We proceed it in three steps.
Step 1. For every $\tau \in$ and $x \in L^2(\Omega, \mathcal{F}_\tau; X_{0u})$, we prove that (28) has a unique solution in $\mathcal{C}_\tau^-$. Denote $\mathcal{J}(\xi, x, \tau)(t)$ the right side of (28), i.e.

$$\begin{aligned}\mathcal{J}(\xi,x,\tau)(t) =& \mathbb{E}\left[T_{A_{0u}}(t-\tau)x - \int_t^\tau T_{A_{0u}}(t-r)\Pi_u F(\xi(r))\,dr \middle| \mathcal{F}_t\right] \\ &+ \lim_{\lambda\to+\infty}\int_{-\infty}^t T_{A_{0s}}(t-r)\lambda R_\lambda(A_s)\Pi_s F(\xi(r))\,dr \\ &+ \lim_{\lambda\to+\infty}\int_{-\infty}^t T_{A_{0s}}(t-r)\lambda R_\lambda(A_s)\Pi_s \sigma(\xi(r))\,dW(r).\end{aligned} \tag{30}$$

We could find that ξ is a solution to (28) in $\mathcal{C}_\tau^-$ if and only if ξ is a fixed point of $\mathcal{J}(\xi, x, \tau)(t)$. It needs to show that $\mathcal{J}(\cdot, x, \tau)(t) : \mathcal{C}_\tau^- \to \mathcal{C}_\tau^-$ is well defined. First we prove the process $\mathcal{J}(\cdot, x, \tau)(t)$, $t \leq \tau$ is predictable and continuous in t in $L^2(\Omega, X_0)$. Denote $L^2_{\mathcal{F}}((-\infty,\tau], X_0)$ the set of all X_0-valued $\mathcal{F}_t$-progressively measurable processes. By [29, Lemma 2.1], there exists a unique pair $(y_1, y_2) \in L^2_{\mathcal{F}}\left((-\infty,\tau], L^2(\Omega, X_0)\right) \times L^2_{\mathcal{F}}((-\infty,\tau], \mathcal{L}_2(Y_0, X))$ such that for all $t \leq \tau$,

$$\begin{aligned}&y_1(t) + \int_t^\tau T_{A_{0u}}(t-r)\Pi_u F(\xi(r))\,dr + \int_t^\tau T(t-r)y_2(r)\,dW(r) \\ &= T_{A_{0u}}(t-\tau)x.\end{aligned} \tag{31}$$

From (31), we find that $y_1(t)$ is continuous in t in $L^2(\Omega, X_0)$. Take the conditional expectation of (31), we get that for $t \leq \tau$,

$$y_1(t) = \mathbb{E}\left[T_{A_{0u}}(t-\tau)x - \int_t^\tau T_{A_{0u}}(t-r)\Pi_u F(\xi(r))\,dr|\mathcal{F}_t\right]. \tag{32}$$

From (32), the first term of the right side of (30) is a X_0-valued $\mathcal{F}_t$-progressively measurable process and continuous in t in $L^2(\Omega, X_0)$. The last two terms are both $\mathcal{F}_t$-adapted and continuous in t in $L^2(\Omega, X_0)$, so they are both $\mathcal{F}_t$-progressively measurable. Then we get that $\mathcal{J}(\xi, x, \tau)(t),\ t \le \tau$, is $\mathcal{F}_t$-progressively measurable and continuous in t in $L^2(\Omega, X_0)$. Secondly, we prove that $\mathcal{J}(\xi, x, \tau)(t)$ satisfies (16). We have, for $t \le \tau$,

$$\begin{aligned}
&\|\mathcal{J}(\xi, x, \tau)(t)\|_{L^2(\Omega, X_0)} \\
&\le \left\|T_{A_{0u}}(t-\tau)x\right\|_{L^2(\Omega, X_0)} + \int_t^\tau \left\|T_{A_{0u}}(t-r)\Pi_u F(\xi(r))\right\|_{L^2(\Omega, X_0)} dr \\
&+ \lim_{\lambda\to+\infty} \int_{-\infty}^t \left\|T_{A_{0s}}(t-r)\lambda R_\lambda(A_s)\Pi_s F(\xi(r))\right\|_{L^2(\Omega, X_0)} dr \\
&+ \left\|\lim_{\lambda\to+\infty} \int_{-\infty}^t T_{A_{0s}}(t-r)\lambda R_\lambda(A_s)\Pi_s \sigma(\xi(r))\,dW(r)\right\|_{L^2(\Omega, X_0)}.
\end{aligned} \tag{33}$$

For the first term on the right side of (33), by (14), we have, for $t \le \tau$,

$$\left\|T_{A_{0u}}(t-\tau)x\right\|_{L^2(\Omega, X_0)} \le Ke^{\alpha(t-\tau)}\|x\|_{L^2(\Omega, X_0)} \le Ke^{\gamma(t-\tau)}\|x\|_{L^2(\Omega, X_0)}. \tag{34}$$

For the second term on the right side of (33), by (14) and (3), we have, $t \le \tau$,

$$\begin{aligned}
\int_t^\tau \left\|T_{A_{0u}}(t-r)\Pi_u F(\xi(r))\right\|_{L^2(\Omega, X_0)} dr &\le KL_1 \int_t^\tau e^{\alpha(t-r)}\|\xi(r)\|_{L^2(\Omega, X_0)} dr \\
&\le KL_1\|\xi\|_{\mathcal{C}_\tau^-} \int_t^\tau e^{\alpha t} e^{(\gamma-\alpha)r} dr \\
&\le KL_1(\alpha-\gamma)^{-1} e^{\gamma t}\|\xi\|_{\mathcal{C}_\tau^-}.
\end{aligned} \tag{35}$$

Then it follows from (18), (19), (33)-(35) that

$$\begin{aligned}
e^{-\gamma t}\|\mathcal{J}(\xi, x, \tau)(t)\|_{L^2(\Omega, X_0)} &\le Ke^{-\gamma\tau}\|x\|_{L^2(\Omega, X_0)} \\
&\quad + K\|\xi\|_{\mathcal{C}_\tau^-}\left(L_1(\alpha-\gamma)^{-1} + L_1 C_\zeta + L_2 C_\zeta\right).
\end{aligned}$$

So we proved that $\mathcal{J}(\xi, x, \tau)(t)$ satisfies (16), which means $\mathcal{J}(\xi, x, \tau)(t) \in \mathcal{C}_\tau^-$ and it further indicates that $\mathcal{J}(\cdot, x, \tau)(t) : \mathcal{C}_\tau^- \to \mathcal{C}_\tau^-$ is well defined. Then we prove $\mathcal{J}(\cdot, x, \tau)(t) : \mathcal{C}_\tau^- \to \mathcal{C}_\tau^-$ is a contraction. Given $\xi_1, \xi_2 \in \mathcal{C}_\tau^-$, we have, for all $t \le \tau$,

$$
\begin{aligned}
&\left\| \mathcal{J}\left(\xi_1, x, \tau\right)(t) - \mathcal{J}\left(\xi_2, x, \tau\right)(t) \right\|_{L^2(\Omega, X_0)} \\
&\leq \int_t^\tau \left\| T_{A_{0u}}(t-r)\Pi_u \left(F\left(\xi_1(r)\right) - F\left(\xi_2(r)\right)\right) \right\|_{L^2(\Omega, X_0)} dr \\
&\quad + \lim_{\lambda \to +\infty} \int_{-\infty}^t \left\| T_{A_{0s}}(t-r)\lambda R_\lambda(A_s)\Pi_s \left(F\left(\xi_1(r)\right) - F\left(\xi_2(r)\right)\right) \right\|_{L^2(\Omega, X_0)} dr \\
&\quad + \left\| \lim_{\lambda \to +\infty} \int_{-\infty}^t T_{A_{0s}}(t-r)\lambda R_\lambda(A_s)\Pi_s \left(\sigma\left(\xi_1(r)\right) - \sigma\left(\xi_2(r)\right)\right) dW(r) \right\|_{L^2(\Omega, X_0)} \\
&= \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3.
\end{aligned}
\tag{36}
$$

By (2) and (14), we have, for $t \leq \tau$,

$$
\begin{aligned}
\mathcal{J}_1 &\leq K L_1 \int_t^\tau e^{\alpha(t-r)} \left\| \xi_1(r) - \xi_2(r) \right\|_{L^2(\Omega, X_0)} dr \\
&\leq K L_1 \left\| \xi_1 - \xi_2 \right\|_{\mathcal{C}_\tau^-} \int_t^\tau e^{\alpha t} e^{(\gamma-\alpha) r} dr \\
&\leq K L_1 (\alpha-\gamma)^{-1} e^{\gamma t} \left\| \xi_1 - \xi_2 \right\|_{\mathcal{C}_\tau^-}.
\end{aligned}
\tag{37}
$$

By Lemma 2.6, (2) and (15), we have, for $t \leq \tau$ and $\zeta \in (\beta, \gamma)$,

$$
\begin{aligned}
\mathcal{J}_2 &\leq \lim_{g \to -\infty} L_1 C_\zeta \sup_{l \in [0, t-g]} e^{\zeta(t-g-l)} \left\| \xi_1(l+g) - \xi_2(l+g) \right\|_{L^2(\Omega, X_0)} \\
&= L_1 C_\zeta \sup_{q \in (-\infty, t]} e^{\zeta(t-q)} \left\| \xi_1(q) - \xi_2(q) \right\|_{L^2(\Omega, X_0)} \\
&\leq L_1 C_\zeta \sup_{q \in (-\infty, t]} e^{\zeta(t-q)} e^{\gamma q} \left\| \xi_1(q) - \xi_2(q) \right\|_{\mathcal{C}_\tau^-} \\
&= L_1 C_\zeta \sup_{q \in (-\infty, t]} e^{(\gamma-\zeta) q} e^{(\zeta-\gamma) t} e^{\gamma t} \left\| \xi_1(q) - \xi_2(q) \right\|_{\mathcal{C}_\tau^-} \\
&= L_1 C_\zeta e^{\gamma t} \left\| \xi_1 - \xi_2 \right\|_{\mathcal{C}_\tau^-}.
\end{aligned}
\tag{38}
$$

By Lemma 2.6 and (4), we have, for $t \leq \tau$ and $\zeta \in (\beta, \gamma)$,

$$
\begin{aligned}
\mathcal{J}_3 &= \left\| \lim_{\lambda \to +\infty} \int_{-\infty}^t T_{A_{0s}}(t-r)\lambda R_\lambda(A_s)\Pi_s \left(\sigma\left(\xi_1(r)\right) - \sigma\left(\xi_2(r)\right)\right) dW(r) \right\|_{L^2(\Omega, X_0)} \\
&\leq \lim_{g \to -\infty} C_\zeta \sup_{l \in [0, t-g]} e^{\zeta(t-g-l)} \mathbb{E}\left[\left\| \sigma\left(\xi_1(l+g)\right) - \sigma\left(\xi_2(l+g)\right) \right\|_{\mathcal{L}_2(Y_0, X)}^2 \right]^{\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
&\leq L_2 C_\zeta \sup_{q\in(-\infty,t]} e^{\zeta(t-q)} e^{\gamma q} \|\xi_1(q)-\xi_2(q)\|_{\mathcal{C}_\tau^-} \\
&= L_2 C_\zeta \sup_{q\in(-\infty,t]} e^{(\gamma-\zeta)q} e^{(\zeta-\gamma)t} e^{\gamma t} \|\xi_1(q)-\xi_2(q)\|_{\mathcal{C}_\tau^-} \\
&= L_2 C_\zeta e^{\gamma t} \|\xi_1-\xi_2\|_{\mathcal{C}_\tau^-}.
\end{aligned} \tag{39}
$$

By combining (37), (38) and (39), we have, for $t \leq \tau$,

$$
\begin{aligned}
&\|\mathcal{J}(\xi_1,x,\tau)(t)-\mathcal{J}(\xi_2,x,\tau)(t)\|_{L^2(\Omega,X_0)} \\
&\leq K\left(L_1(\alpha-\gamma)^{-1}+L_1C_\zeta+L_2C_\zeta\right)e^{\gamma t}\|\xi_1-\xi_2\|_{\mathcal{C}_\tau^-}.
\end{aligned}
$$

Then we have, for $t \leq \tau$,

$$
\begin{aligned}
&\|\mathcal{J}(\xi_1,x,\tau)(t)-\mathcal{J}(\xi_2,x,\tau)(t)\|_{\mathcal{C}_\tau^-} \\
&\leq K\left(L_1(\alpha-\gamma)^{-1}+L_1C_\zeta+L_2C_\zeta\right)\|\xi_1-\xi_2\|_{\mathcal{C}_\tau^-}.
\end{aligned} \tag{40}
$$

By the conditions of the theorem, $\mathcal{J}(\cdot,x,\tau)(t):\mathcal{C}_\tau^- \to \mathcal{C}_\tau^-$ is a contraction. From the contraction mapping principle, for any given $\tau \in$ and $x \in L^2(\Omega,\mathcal{F}_\tau;X_{0u})$, $\mathcal{J}(\cdot,x,\tau)(t)$ has a unique fixed point $\overline{\xi}(x)$ in $\mathcal{C}_\tau^-$. That is, for every $t \leq \tau$, $\overline{\xi}(x)$ is the unique solution of (30) and

$$
\mathcal{J}\left(\overline{\xi},x,\tau\right)=\overline{\xi}(x).
$$

Let $K\left(L_1(\alpha-\gamma)^{-1}+L_1C_\zeta+L_2C_\zeta\right)=\eta$. By (40), for $x_1,x_2 \in L^2(\Omega,\mathcal{F}_\tau;X_{0u})$, we have, for $t \leq \tau$,

$$
\begin{aligned}
&\left\|\mathcal{J}\left(\overline{\xi},x_1,\tau\right)(t)-\mathcal{J}\left(\overline{\xi},x_2,\tau\right)(t)\right\|_{\mathcal{C}_\tau^-} \\
&=\left\|\overline{\xi}(x_1)-\overline{\xi}(x_2)\right\|_{\mathcal{C}_\tau^-} \\
&\leq \sup_{t\leq\tau} e^{-\gamma t}\left\|T_{A_{0u}}(t-\tau)(x_1-x_2)\right\|_{L^2(\Omega,X_0)}+\eta\left\|\overline{\xi}(x_1)-\overline{\xi}(x_2)\right\|_{\mathcal{C}_\tau^-} \\
&\leq Ke^{-\alpha\tau}\sup_{t\leq\tau} e^{(\alpha-\gamma)t}\|x_1-x_2\|_{L^2(\Omega,X_0)}+\eta\left\|\overline{\xi}(x_1)-\overline{\xi}(x_2)\right\|_{\mathcal{C}_\tau^-} \\
&\leq Ke^{-\gamma\tau}\|x_1-x_2\|_{L^2(\Omega,X_0)}+\eta\left\|\overline{\xi}(x_1)-\overline{\xi}(x_2)\right\|_{\mathcal{C}_\tau^-}.
\end{aligned} \tag{41}
$$

Thus by (41), for all $x_1,x_2 \in L^2(\Omega,\mathcal{F}_\tau;X_{0u})$, we have

$$
\left\|\overline{\xi}(x_1)(\tau)-\overline{\xi}(x_2)(\tau)\right\|_{L^2(\Omega,X_0)} \leq K(1-\eta)^{-1}\|x_1-x_2\|_{L^2(\Omega,X_0)}. \tag{42}
$$

Step 2. We prove that the mean-square unstable invariant manifold is given by a graph of a Lipschitz continuous map. Let $h^u(x,\tau)=\Pi_{0s}\overline{\xi}(x)(\tau)$. Let $s \to -\infty$ and $t=\tau$ in (23), we have

$$h^u(x,\tau) = \lim_{\lambda\to+\infty} \int_{-\infty}^{\tau} T_{A_{0s}}(\tau - r)\,\lambda R_\lambda(A_s)\Pi_s F\left(\overline{\xi}(x)(r)\right)dr$$
$$+ \lim_{\lambda\to+\infty} \int_{-\infty}^{\tau} T_{A_{0s}}(\tau - r)\,\lambda R_\lambda(A_s)\Pi_s \sigma\left(\overline{\xi}(x)(r)\right)dW(r)\,. \tag{43}$$

Indeed, by (38), (39) and (42), it follows that for all $x_1, x_2 \in L^2(\Omega, \mathcal{F}_\tau; X_{0u})$,

$$\left\| h^u(x_1,\tau) - h^u(x_2,\tau) \right\|_{L^2(\Omega,X_0)} \le KC_\zeta(L_1 + L_2)(1-\eta)^{-1}\left\|x_1 - x_2\right\|_{L^2(\Omega,X_0)}. \tag{44}$$

By the definition of $h^u(\cdot,\tau)$ and (44), we get that $h^u(\cdot,\tau) : L^2(\Omega, \mathcal{F}_\tau; X_{0u}) \to L^2(\Omega, \mathcal{F}_\tau; X_{0s})$ is a Lipschitz continuous map. Then by (30) and (43), we have for all $\tau \in$ and $x \in L^2(\Omega, \mathcal{F}_\tau; X_{0u})$,

$$\overline{\xi}(x)(\tau) = x + h^u(x,\tau)\,. \tag{45}$$

By the definition of $\mathcal{M}^u(\tau)$ and Lemma 3.1, $\xi(\tau) \in \mathcal{M}^u(\tau)$ if and only if there exists $x \in L^2(\Omega, \mathcal{F}_\tau; X_{0u})$ such that $\xi(\tau) = x + h^u(x,\tau)$. Therefore, we have

$$\mathcal{M}^u(\tau) = \left\{ x + h^u(x,\tau) : x \in L^2(\Omega, \mathcal{F}_\tau; X_{0u}) \right\}.$$

Step 3. We prove that $\mathcal{M}^u(\tau)$ is invariant under Φ. By Definition 2.2, we need to show that for each $\xi(\tau) \in \mathcal{M}^u(\tau)$ and $t_0 > 0$, $\Phi(t_0,\tau)\xi(\tau) \in \mathcal{M}^u(\tau + t_0)$. By the definition of $\mathcal{M}^u(\tau)$, for $\xi(\tau) \in \mathcal{M}^u(\tau)$, $\xi \in \mathcal{C}_\tau^-$ and satisfies (28) on $(-\infty,\tau]$. Let

$$\tilde{\xi}(t) = \begin{cases} u(t,\tau,\xi(\tau)), & \tau \le t \le \tau + t_0, \\ \xi(t), & t < \tau. \end{cases} \tag{46}$$

We show that $\tilde{\xi}(\tau + t_0) \in \mathcal{M}^u(\tau + t_0)$. Since $\xi \in \mathcal{C}_\tau^-$, it is easy to get that $\tilde{\xi} \in \mathcal{C}_{\tau+t_0}^-$. Then we prove that ξ satisfies (28) on $(-\infty, \tau + t_0]$, which means, for all $t \le t_0 + \tau$,

$$\tilde{\xi}(t) = \mathbb{E}\left[T_{A_{0u}}(t - \tau - t_0)\,x - \int_t^{\tau+t_0} T_{A_{0u}}(t - r)\Pi_u F\left(\tilde{\xi}(r)\right)d(r)\,|\mathcal{F}_t \right]$$
$$+ \lim_{\lambda\to+\infty} \int_{-\infty}^{t} T_{A_{0s}}(t - r)\,\lambda R_\lambda(A_s)\Pi_s F\left(\tilde{\xi}(r)\right)dr \tag{47}$$
$$+ \lim_{\lambda\to+\infty} \int_{-\infty}^{t} T_{A_{0s}}(t - r)\,\lambda R_\lambda(A_s)\Pi_s \sigma\left(\tilde{\xi}(r)\right)dW(r)$$

in $L^2(\Omega, X_0)$ with $x = \Pi_{0u}\tilde{\xi}(\tau + t_0) \in L^2\left(\Omega, \mathcal{F}_{\tau+t_0}; X_{0u}\right)$. By (46) and (10), we have

$$
\begin{aligned}
\tilde{\xi}(\tau+t_0) =& T(t_0)\xi(\tau) + \lim_{\lambda\to+\infty}\int_{\tau}^{\tau+t_0} T(\tau+t_0-r)\lambda R_\lambda(A)F\left(\tilde{\xi}(r)\right)dr \\
&+ \lim_{\lambda\to+\infty}\int_{\tau}^{\tau+t_0} T(\tau+t_0-r)\lambda R_\lambda(A)\sigma\left(\tilde{\xi}(r)\right)dW(r).
\end{aligned}
\tag{48}
$$

Then we use the same method of getting (22) to apply Π_{0u} to (48), we have

$$
\begin{aligned}
x =& \Pi_{0u}\tilde{\xi}(\tau+t_0) = T(t_0)\Pi_{0u}\xi(\tau) + \int_{\tau}^{\tau+t_0} T(\tau+t_0-r)\Pi_u F\left(\tilde{\xi}(r)\right)dr \\
&+ \int_{\tau}^{\tau+t_0} T(\tau+t_0-r)\Pi_u\sigma\left(\tilde{\xi}(r)\right)dW(r).
\end{aligned}
\tag{49}
$$

Then (47) can be rewritten as, for $t \le \tau + t_0$,

$$
\begin{aligned}
\tilde{\xi}(t) =& \mathbb{E}\left[T_{A_{0u}}(t-\tau)\Pi_{0u}\xi(\tau) - \int_t^{\tau} T_{A_{0u}}(t-r)\Pi_u F\left(\tilde{\xi}(r)\right)d(r)\,|\mathcal{F}_t\right] \\
&- \mathbb{E}\left[\int_t^{\tau} T_{A_{0u}}(t-r)\Pi_u\sigma\left(\tilde{\xi}(r)\right)dW(r)\,|\mathcal{F}_t\right] \\
&+ \lim_{\lambda\to+\infty}\int_{-\infty}^{t} T_{A_{0s}}(t-r)\lambda R_\lambda(A_s)\Pi_s F\left(\tilde{\xi}(r)\right)dr \\
&+ \lim_{\lambda\to+\infty}\int_{-\infty}^{t} T_{A_{0s}}(t-r)\lambda R_\lambda(A_s)\Pi_s\sigma\left(\tilde{\xi}(r)\right)dW(r).
\end{aligned}
\tag{50}
$$

Since $\xi \in \mathcal{C}_\tau^-$ satisfies (28) on $(-\infty, \tau]$, then $\tilde{\xi}$ satisfies (50) for $t \le \tau$. It remains to show that $\tilde{\xi}$ satisfies (50) for $t > \tau$. By (46) and (10), for $t > \tau$,

$$
\begin{aligned}
\tilde{\xi}(t) =& T(t-\tau)\xi(\tau) + \lim_{\lambda\to+\infty}\int_{\tau}^{t} T(t-r)\lambda R_\lambda(A)F\left(\tilde{\xi}(r)\right)dr \\
&+ \lim_{\lambda\to+\infty}\int_{\tau}^{t} T(t-r)\lambda R_\lambda(A)\sigma\left(\tilde{\xi}(r)\right)dW(r).
\end{aligned}
\tag{51}
$$

Since $\xi \in \mathcal{C}_\tau^-$ satisfies (28) with $x = \Pi_{0u}\xi(\tau)$ on $(-\infty, \tau]$, let $t = \tau$ in (28), we have,

$$
\begin{aligned}
\xi(\tau) =& \Pi_{0u}\xi(\tau) + \lim_{\lambda\to+\infty}\int_{-\infty}^{\tau} T_{A_{0s}}(\tau-r)\,\lambda R_\lambda(A_s)\Pi_s F(\xi(r))\,dr \\
&+ \lim_{\lambda\to+\infty}\int_{-\infty}^{\tau} T_{A_{0s}}(\tau-r)\,\lambda R_\lambda(A_s)\Pi_s\sigma(\xi(r))\,dW(r) \\
=& \Pi_{0u}\xi(\tau) + \lim_{\lambda\to+\infty}\int_{-\infty}^{\tau} T_{A_{0s}}(\tau-r)\,\lambda R_\lambda(A_s)\Pi_s F\left(\tilde{\xi}(r)\right)dr \\
&+ \lim_{\lambda\to+\infty}\int_{-\infty}^{\tau} T_{A_{0s}}(\tau-r)\,\lambda R_\lambda(A_s)\Pi_s\sigma\left(\tilde{\xi}(r)\right)dW(r).
\end{aligned}
\tag{52}
$$

By (51), (52) and Lemma 2.1(i), we have, for $t>\tau$,

$$
\begin{aligned}
\tilde{\xi}(t) =& T(t-\tau)\,\Pi_{0u}\xi(\tau) + \lim_{\lambda\to+\infty}\int_{-\infty}^{\tau} T_{A_{0s}}(t-r)\,\lambda R_\lambda(A_s)\Pi_s F\left(\tilde{\xi}(r)\right)dr \\
&+ \lim_{\lambda\to+\infty}\int_{-\infty}^{\tau} T_{A_{0s}}(t-r)\,\lambda R_\lambda(A_s)\Pi_s\sigma\left(\tilde{\xi}(r)\right)dW(r) \\
&+ \lim_{\lambda\to+\infty}\int_{\tau}^{t} T(t-r)\,\lambda R_\lambda(A)F\left(\tilde{\xi}(r)\right)dr \\
&+ \lim_{\lambda\to+\infty}\int_{\tau}^{t} T(t-r)\,\lambda R_\lambda(A)\sigma\left(\tilde{\xi}(r)\right)dW(r) \\
=& T(t-\tau)\,\Pi_{0u}\xi(\tau) + \lim_{\lambda\to+\infty}\int_{-\infty}^{t} T_{A_{0s}}(t-r)\,\lambda R_\lambda(A_s)\Pi_s F\left(\tilde{\xi}(r)\right)dr \\
&+ \lim_{\lambda\to+\infty}\int_{-\infty}^{t} T_{A_{0s}}(t-r)\,\lambda R_\lambda(A_s)\Pi_s\sigma\left(\tilde{\xi}(r)\right)dW(r) \\
&+ \int_{\tau}^{t} T_{A_{0u}}(t-r)\,\Pi_u F\left(\tilde{\xi}(r)\right)dr + \int_{\tau}^{t} T_{A_{0u}}(t-r)\,\Pi_u\sigma\left(\tilde{\xi}(r)\right)dW(r).
\end{aligned}
\tag{53}
$$

Since $\tilde{\xi}(t)=u(t,\tau,\xi(\tau))$ is $\mathcal{F}_t$-adapted for $t>\tau$, all terms of (53) are $\mathcal{F}_t$-measurable for $t>\tau$, so for $t>\tau$, take $\mathbb{E}(\cdot|\mathcal{F}_t)$ on (53), we get (50). So far we proved $\tilde{\xi}$ satisfies (50), which means it satisfies (47) and (28) as well for all $t\le\tau+t_0$. So we get $\tilde{\xi}\in\mathcal{C}^{-}_{\tau+t_0}$ satisfies (28) on $(-\infty,\tau+t_0]$, by the definition of $\mathcal{M}^u(\tau)$, we get $\tilde{\xi}(\tau+t_0)\in\mathcal{M}^u(\tau+t_0)$. By (46),

$$\Phi\left(t_0,\tau\right)\left(\xi\left(\tau\right)\right)=u\left(\tau+t_0,\tau,\xi\left(\tau\right)\right)=\tilde{\xi}\left(\tau+t_0\right).$$

So $\Phi\left(t_0,\tau\right)\left(\xi\left(\tau\right)\right)\in\mathcal{M}^u\left(\tau+t_0\right)$, implying $\Phi\left(t_0,\tau\right)\mathcal{M}^u\left(\tau\right)\subseteq\mathcal{M}^u\left(\tau+t_0\right)$ for all $t_0>0$. So $\mathcal{M}^u\left(\tau\right)$ is invariant under Φ. □

4. Mean-square random stable invariant sets

This section is dedicated to the existence of mean-square random stable sets of (1) through the same method. Given $\tau\in$, denote by $\mathcal{C}_\tau^+$ the space of all X_0-valued $\mathcal{F}_t$-progressively measurable processes $\xi\left(t\right):t\in[\tau,+\infty)$ such that $\xi:[\tau,+\infty)\to L^2\left(\Omega,X_0\right)$ is continuous and

$$\sup_{t\geq\tau}\left(e^{-\gamma t}\|\xi\left(t\right)\|_{L^2\left(\Omega,X_0\right)}\right)<\infty,\tag{54}$$

with norm

$$\|\xi\left(t\right)\|_{\mathcal{C}_\tau^+}=\sup_{t\geq\tau}\left(e^{-\gamma t}\|\xi\left(t\right)\|_{L^2\left(\Omega,X_0\right)}\right),\ \forall\xi\in\mathcal{C}_\tau^+,$$

where $\gamma\in\left(\beta,\alpha\right)$. Notice that $\mathcal{C}_\tau^+$ is a Banach space. To construct mean-square random stable invariant sets of (1), we need to find all solutions of (1) in the space $\mathcal{C}_\tau^+$.

Lemma 4.1. *A … 2.1, A … 2.2, A … 2.4 … A … 2.5 … $\xi\in\mathcal{C}_\tau^+$ … $\tau\in$ … ξ … (1) … $[\tau,+\infty)$ … $x\in L^2\left(\Omega,\mathcal{F}_\tau;X_{0s}\right)$ … $t\geq\tau$,*

$$\begin{aligned}\xi\left(t\right)=&T_{A_{0s}}\left(t-\tau\right)x-\int_t^\infty T_{A_{0u}}\left(t-r\right)\Pi_uF\left(\xi\left(r\right)\right)dr\\&-\int_t^\infty T_{A_{0u}}\left(t-r\right)\Pi_u\sigma\left(\xi\left(r\right)\right)dW\left(r\right)\\&+\lim_{\lambda\to+\infty}\int_\tau^t T_{A_{0s}}\left(t-r\right)\lambda R_\lambda\left(A_s\right)\Pi_sF\left(\xi\left(r\right)\right)dr\\&+\lim_{\lambda\to+\infty}\int_\tau^t T_{A_{0s}}\left(t-r\right)\lambda R_\lambda\left(A_s\right)\Pi_s\sigma\left(\xi\left(r\right)\right)dW\left(r\right)\end{aligned}\tag{55}$$

… $L^2\left(\Omega,X_0\right)$.

Proof. Step 1: necessity. Suppose $\xi\in\mathcal{C}_\tau^+$ with $\tau\in$ is an integrated solution of (1). We prove that (55) holds true. Since $\xi\in\mathcal{C}_\tau^+$, by Lemma 2.6 and Assumption 2.2, the last two integrals on the right side of (55) are well-defined in $L^2\left(\Omega,X_0\right)$. For the first integral, according to (3) and (14),

$$\int_t^{\infty} \left\| T_{A_{0u}}(t-r)\Pi_u F(\xi(r)) \right\|_{L^2(\Omega,X_0)} dr \le KL_1 \int_t^{\infty} e^{\alpha(t-r)} \|\xi(r)\|_{L^2(\Omega,X_0)} dr$$
$$\le KL_1 \|\xi(r)\|_{\mathcal{C}_\tau^+} e^{\alpha t} \int_t^{\infty} e^{-(\alpha-\gamma)r} dr \tag{56}$$
$$\le KL_1 (\alpha-\gamma)^{-1} e^{\gamma t} \|\xi\|_{\mathcal{C}_\tau^+}$$
$$< \infty.$$

By (56), for $t \ge \tau$,

$$\int_t^{\infty} T_{A_{0u}}(t-r)\Pi_u F(\xi(r))\, dr$$

is well-defined in $L^2(\Omega, X_0)$. For the second integral, according to (5) and (14),

$$\left\| \int_t^{\infty} T_{A_{0u}}(t-r)\Pi_u \sigma(\xi(r))\, dW(r) \right\|_{L^2(\Omega,X_0)}$$
$$\le \left(\mathbb{E}\left[\int_t^{\infty} \left\| T_{A_{0u}}(t-r)\Pi_u \sigma(\xi(r)) \right\|^2_{\mathcal{L}_2(Y_0,X)} dr \right] \right)^{\frac{1}{2}}$$
$$\le KL_2 \left(\int_t^{\infty} e^{2\alpha(t-r)} \mathbb{E}\left[\|\xi(r)\|^2_{L^2(\Omega,X_0)} \right] dr \right)^{\frac{1}{2}} \tag{57}$$
$$\le KL_2 \|\xi(r)\|_{\mathcal{C}_\tau^+} e^{\alpha t} \left(\int_t^{\infty} e^{-2(\alpha-\gamma)r} dr \right)^{\frac{1}{2}}$$
$$\le \frac{1}{\sqrt{2}} KL_2 (\alpha-\gamma)^{-\frac{1}{2}} e^{\gamma t} \|\xi\|_{\mathcal{C}_\tau^+}$$
$$< \infty.$$

By (57), for $t \ge \tau$,

$$\int_t^{\infty} T_{A_{0u}}(t-r)\Pi_u \sigma(\xi(r))\, dW(r)$$

is well-defined in $L^2(\Omega, X_0)$. Since $\xi \in \mathcal{C}_\tau^+$ is a solution of (1) on $[\tau, +\infty)$, by (10), for all $t \ge \tau$,

$$
\begin{aligned}
\xi(t) = T(t-\tau)\xi(\tau) + \lim_{\lambda\to+\infty}\int_{\tau}^{t} T(t-r)R_{\lambda}(A)F(\xi(r))\,dr \\
+ \lim_{\lambda\to+\infty}\int_{\tau}^{t} T(t-r)\lambda R_{\lambda}(A)\sigma(\xi(r))\,dW(r).
\end{aligned}
\tag{58}
$$

Then we apply projections Π_{0u} and Π_{0s} to (58). Note that $\Pi_u : X \to X$ is a bounded linear projection satisfying

$$
\Pi_u R_{\lambda}(A) = R_{\lambda}(A)\Pi_u, \ \forall \lambda > \vartheta.
$$

Also, $A|_{\Pi_u(X)} = A|_{\Pi_u(X)} = A|_{X_u} = A_u$ satisfies Assumption 2.2 in $\Pi_u(X) = X_u$. Moreover, we know that Π_u has a finite rank and $\Pi_u(X) \subset X_0$, which further indicates that $A_0|_{\Pi_u(X_0)} = A_0|_{X_{0u}} = A_{0u}$ and $A|_{\Pi_u(X)} = A|_{X_u} = A_u$. Thus if for each $x \in X_0$ and each $t \ge \tau$, then the map $t \to \Pi_u \xi(t)$ solves

$$
d\Pi_u\xi(t) = A_{0u}\Pi_u\xi(t)dt + \Pi_u F(\xi(t))dt + \Pi_u\sigma(\xi(t))dW(t)
$$

in $\Pi_u(X_0) = X_{0u}$, then for all $t \ge \tau$,

$$
\begin{aligned}
\Pi_u\xi(t) =& T_{A_0|_{\Pi_s(X_0)}}(t-\tau)\Pi_u\xi(\tau) + \left(S_{A|_{\Pi_u(X)}} \diamond \Pi_u(F(\xi) + \sigma(\xi)\,dW)\right)(r) \\
=& T_{A_{0s}}(t-\tau)\Pi_u\xi(\tau) + \left(S_{A_u} \diamond \Pi_u(F(\xi) + \sigma(\xi)\,dW)\right)(r) \\
=& T_{A_{0u}}(t-\tau)\Pi_u\xi(\tau) + \lim_{\lambda\to+\infty}\int_{\tau}^{t} T_{A_{0u}}(t-r)\lambda R_{\lambda}(A_u)\Pi_u F(\xi(r))dr \\
& + \lim_{\lambda\to+\infty}\int_{\tau}^{t} T_{A_{0u}}(t-r)\lambda R_{\lambda}(A_u)\Pi_u\sigma(\xi(r))dW(r).
\end{aligned}
\tag{59}
$$

Since Π_u is a extension of Π_{0u} from X_0 to X and $\Pi_u(X) \subset X_0$, by (59) and Lemma 2.1(i), we have

$$
\begin{aligned}
\Pi_{0u}\xi(t) =& \Pi_u\xi(t) = T_{A_{0u}}(t-\tau)\Pi_{0u}\xi(\tau) + \int_{\tau}^{t} T_{A_{0u}}(t-r)\Pi_u F(\xi(r))dr \\
& + \int_{\tau}^{t} T_{A_{0u}}(t-r)\Pi_u\sigma(\xi(r))dW(r).
\end{aligned}
\tag{60}
$$

Through similar arguments, we have

$$\Pi_{0s}\xi(t) = T_{A_{0s}}(t-\tau)\Pi_{0s}\xi(\tau) + \lim_{\lambda\to+\infty}\int_{\tau}^{t} T_{A_{0s}}(t-r)\lambda R_{\lambda}(A_s)\Pi_s F(\xi(r))dr$$
$$+ \lim_{\lambda\to+\infty}\int_{\tau}^{t} T_{A_{0s}}(t-r)\lambda R_{\lambda}(A_s)\Pi_s \sigma(\xi(r))dW(r). \tag{61}$$

Let $t = s$ in (60), we have, for all $s \geq \tau$,

$$\Pi_{0u}\xi(s) = T_{A_{0u}}(s-\tau)\Pi_{0u}\xi(\tau) + \int_{\tau}^{s} T_{A_{0u}}(s-r)\Pi_u F(\xi(r))dr$$
$$+ \int_{\tau}^{s} T_{A_{0u}}(s-r)\Pi_u \sigma(\xi(r))dW(r). \tag{62}$$

Hence for all $s \geq t \geq \tau$,

$$T_{A_{0u}}(t-s)\Pi_{0u}\xi(s) = T_{A_{0u}}(t-\tau)\Pi_{0u}\xi(\tau) +$$

$$+\lim_{\lambda\to+\infty}\int_{\tau}^{t} T_{A_{0s}}(t-r)\,\lambda R_{\lambda}(A_s)\Pi_s\sigma(\xi(r))dW(r)\,.$$

Since for all $s\geq t\geq\tau$, let $s\to+\infty$ in (65), then by (14),

$$\begin{aligned}\left\|T_{A_{0u}}(t-s)\,\Pi_{0u}\xi(s)\right\|_{L^2(\Omega,X_0)} &\leq Ke^{\alpha(t-s)}\|\xi(s)\|_{L^2(\Omega,X_0)}\\ &\leq Ke^{\alpha t}e^{(\gamma-\alpha)s}\|\xi(s)\|_{\mathcal{C}_\tau^-}\to 0.\end{aligned}$$

Thus (55) is fulfilled as wished with $x=\Pi_{0s}\xi(\tau)$.

Step 2: sufficiency. Suppose $\xi\in\mathcal{C}_\tau^+$ satisfies (55) for $t\geq\tau$. Since $\xi\in\mathcal{C}_\tau^+$, we know ξ is an X_0-valued $\mathcal{F}_t$-progressively measurable process and $\xi\in C\left([\tau,+\infty),L^2(\Omega,X_0)\right)$. And from (55), one can verify that for all $t\geq\tau$,

$$\begin{aligned}\xi(t)=&T(t-\tau)\xi(\tau)+\lim_{\lambda\to+\infty}\int_{\tau}^{t\,t}T(t-s)\,\lambda R_\lambda(A)F(\xi(s))\,ds\\ &+\lim_{\lambda\to+\infty}\int_{\tau}^{t\,t}T(t-s)\,\lambda R_\lambda(A)\sigma(\xi(s))\,dW(s)\,.\end{aligned}\tag{66}$$

From (66), we get that for all $t\geq\tau$,

$$\xi(t)=u(t,\tau,\xi(\tau))\,,$$

where u is the integrated solution of (1) with initial condition $\xi(\tau)$ at initial time τ. Then by the definition of integrated solutions of (1) on $[\tau,+\infty)$, ξ is an integrated solution of (1) on $[\tau,+\infty)$. □

Now we are ready to construct the mean-square random stable invariant sets of (1). By Lemma 4.1, if $\xi\in\mathcal{C}_\tau^+$ is an integrated solution of (1), then ξ must satisfy (55). Take the conditional expectation $\mathbb{E}(\cdot|\mathcal{F}_t)$ on (55), we get for $t\geq\tau$,

$$\xi(t)=T_{A_{0s}}(t-\tau)\,x-\mathbb{E}\left[\int_t^\infty T_{A_{0u}}(t-r)\Pi_u F(\xi(r))\,dr\quad(\right.$$

in $L^2(\Omega, X_0)$ with $x \in L^2(\Omega, \mathcal{F}_\tau; X_{0s})$. Similar to what we did when constructing unstable manifolds, given $\tau \in$, set

$$\mathcal{M}^{s*}(\tau) = \left\{ \xi(\tau) \in L^2(\Omega, \mathcal{F}_\tau; X_0) : \xi \in \mathcal{C}_\tau^+ \text{and satisfies (67) on } [\tau, +\infty) \right\}.$$

But it turns out that the invariance property of $\mathcal{M}^{s*}(\tau)$ cannot be proved due to the $\mathcal{F}_t$-adaptedness issues, which means we cannot give the existence of mean-square random stable invariant manifolds yet. We shall consider the mean-square random stable invariant sets instead. For every $\tau \in$ and $t \geq \tau$, set

$$\mathcal{M}^s(\tau) = \left\{ \xi(\tau) \in L^2(\Omega, \mathcal{F}_\tau; X_0) : \xi(t) \in \mathcal{C}_\tau^+ \text{ and is an integrated solution of (1)} \right\}. \tag{68}$$

We shall prove that $\mathcal{M}^s(\tau)$ is a mean-square random stable invariant set of (1). By Lemma 4.1, we need to find all solutions of (55) which belongs to $\mathcal{C}_\tau^+$ on $[\tau, +\infty)$. Given $\tau \in$, define a subset of $L^2(\Omega, \mathcal{F}_\tau; X_{0s})$ as X_{0s}^τ which is given by

$$X_{0s}^\tau = \left\{ x \in L^2(\Omega, \mathcal{F}_\tau; X_{0s}) : \text{there exists } \xi \in \mathcal{C}_\tau^+ \text{ satisfying (55)} \right\}.$$

X_{0s}^τ is nonempty since $0 \in X_{0s}^\tau$. By (68) and Lemma 4.1, we get that, for every given $\tau \in$ and $t \geq \tau$,

$$\mathcal{M}^s(\tau) = \left\{ \xi(\tau) \in L^2(\Omega, \mathcal{F}_\tau; X_0) : \exists\, x \in X_{0s}^\tau \text{ such that } \xi(t) \in \mathcal{C}_\tau^+ \text{ satisfies (55)} \right\}. \tag{69}$$

We below prove that $\mathcal{M}^s(\tau)$ is a invariant set of Φ and it is given by a graph of Lipschitz function.

Theorem 4.1. *A A 2.1, A 2.2, A 2.4 A 2.5* *$\beta < \zeta < \gamma < \alpha$*

$$K\left(L_1(\alpha-\gamma)^{-1} + L_2(2\alpha-2\gamma)^{-\frac{1}{2}} + L_1 C_\zeta + L_2 C_\zeta\right) < 1, \tag{70}$$

(1)

$$\mathcal{M}^s(\tau) = \left\{ x + h^s(x, \tau) : x \in L^2(\Omega, \mathcal{F}_\tau; X_{0s}) \right\},$$

$h^s(\cdot, \tau) : X_{0s}^\tau \to L^2(\Omega, \mathcal{F}_\tau; X_{0u})$.

Proof. We proceed it in three steps.

Step 1. By the definition of X_{0s}^τ, if $x \in X_{0s}^\tau$, then there exists at least one $\xi \in \mathcal{C}_\tau^+$ which satisfies (55). We prove that when (70) is satisfied, such ξ is unique. Suppose $\xi_1, \xi_2 \in \mathcal{C}_\tau^+$ satisfy (55) with $x \in X_{0s}^\tau$, then by (55), we have, for $t \geq \tau$,

$$
\begin{aligned}
&\left\| \xi_1(t) - \xi_2(t) \right\|_{L^2(\Omega, X_0)} \\
&\leq \int_t^{\infty} \left\| T_{A_{0u}}(t-r)\Pi_u \left(F(\xi_1(r)) - F(\xi_2(r)) \right) \right\|_{L^2(\Omega, X_0)} dr \\
&\leq \left\| \int_t^{\infty} T_{A_{0u}}(t-r)\, \Pi_u \left(\sigma(\xi_1(r)) - \sigma(\xi_2(r)) \right) dW(r) \right\|_{L^2(\Omega, X_0)} \\
&+ \lim_{\lambda \to +\infty} \int_\tau^t \left\| T_{A_{0s}}(t-r) \lambda R_\lambda(A_s) \Pi_s \left(F(\xi_1(r)) - F(\xi_2(r)) \right) \right\|_{L^2(\Omega, X_0)} dr \\
&+ \left\| \lim_{\lambda \to +\infty} \int_\tau^t T_{A_{0s}}(t-r)\, \lambda R_\lambda(A_s) \Pi_s \left(\sigma(\xi_1(r)) - \sigma(\xi_2(r)) \right) dW(r) \right\|_{L^2(\Omega, X_0)} \\
&= \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4.
\end{aligned}
\tag{71}
$$

By (2) and (14), we have, for $t \geq \tau$,

$$
\begin{aligned}
\mathcal{I}_1 &\leq K L_1 \int_t^{\infty} e^{\alpha(t-r)} \left\| \xi_1(r) - \xi_2(r) \right\|_{L^2(\Omega, X_0)} dr \\
&\leq K L_1 \left\| \xi_1(r) - \xi_2(r) \right\|_{\mathcal{C}_\tau^+} e^{\alpha t} \int_t^{\infty} e^{-(\alpha-\gamma)r} dr \\
&\leq K L_1 (\alpha - \gamma)^{-1} e^{\gamma t} \left\| \xi_1 - \xi_2 \right\|_{\mathcal{C}_\tau^+}.
\end{aligned}
\tag{72}
$$

By (4) and (14), we have, for $t \geq \tau$,

$$
\begin{aligned}
\mathcal{I}_2 &= \mathbb{E}\left[\left(\int_t^{\infty} T_{A_{0u}}(t-r)\, \Pi_u \left(\sigma(\xi_1(r)) - \sigma(\xi_2(r)) \right) dW(r) \right)^2 \right]^{\frac{1}{2}} \\
&\leq K L_2 \left(\int_t^{\infty} e^{2\alpha(t-r)} \mathbb{E}\left[\left\| \xi_1(r) - \xi_2(r) \right\|^2_{L^2(\Omega, X_0)} \right] dr \right)^{\frac{1}{2}} \\
&\leq K L_2 \left\| \xi_1(r) - \xi_2(r) \right\|_{\mathcal{C}_\tau^+} e^{\alpha t} \left(\int_t^{\infty} e^{-2(\alpha-\gamma)r} dr \right)^{\frac{1}{2}} \\
&\leq K L_2 (2\alpha - 2\gamma)^{-\frac{1}{2}} e^{\gamma t} \left\| \xi_1 - \xi_2 \right\|_{\mathcal{C}_\tau^+}.
\end{aligned}
\tag{73}
$$

By Lemma 2.6 and (2), we have, for $t \geq \tau$ and $\zeta \in (\beta, \gamma)$,

$$
\begin{aligned}
\mathcal{I}_3 &= \lim_{\lambda\to+\infty} \int_{\tau}^{t} \left\| T_{A_{0s}}(t-r)\lambda R_\lambda(A_s)\Pi_s\left(F(\xi_1(r)) - F(\xi_2(r))\right)\right\|_{L^2(\Omega,X_0)} dr \\
&\le L_1 C_\zeta \sup_{s\in[0,t-\tau]} e^{\zeta(t-\tau-s)} \|\xi_1(s+\tau) - \xi_2(s+\tau)\|_{L^2(\Omega,X_0)} \\
&\le L_1 C_\zeta e^{\gamma t} \|\xi_1 - \xi_2\|_{\mathcal{C}_\tau^+}.
\end{aligned} \tag{74}
$$

By Lemma 2.6 and (4), we have, for $t \ge \tau$ and $\zeta \in (\beta, \gamma)$,

$$
\begin{aligned}
\mathcal{I}_4 &= \left\| \lim_{\lambda\to+\infty} \int_{\tau}^{t} T_{A_{0s}}(t-r)\lambda R_\lambda(A_s)\Pi_s\left(\sigma(\xi_1(r)) - \sigma(\xi_2(r))\right) dW(r) \right\|_{L^2(\Omega,X_0)} \\
&\le C_\zeta \sup_{s\in[0,t-\tau]} e^{\zeta(t-\tau-s)} \mathbb{E}\left[\|\sigma(\xi_1(s+\tau)) - \sigma(\xi_2(s+\tau))\|^2_{\mathcal{L}_2(Y_0,X)} \right]^{\frac{1}{2}} \\
&\le L_2 C_\zeta e^{\gamma t} \|\xi_1 - \xi_2\|_{\mathcal{C}_\tau^+}.
\end{aligned} \tag{75}
$$

By combining (71)-(75), we have, for $t \ge \tau$,

$$
\begin{aligned}
&\|\xi_1(t) - \xi_2(t)\|_{\mathcal{C}_\tau^+} \\
&\le K\left(L_1(\alpha-\gamma)^{-1} + L_2(2\alpha - 2\gamma)^{-\frac{1}{2}} + L_1 C_\zeta + L_2 C_\zeta\right) \|\xi_1(t) - \xi_2(t)\|_{\mathcal{C}_\tau^+}.
\end{aligned} \tag{76}
$$

By (70) and (76), we get that for $t \ge \tau$, $\xi_1(t) = \xi_2(t)$. So such ξ is unique. Given $\tau \in$ and $x \in X_{0s}^\tau$, denote by $\hat{\xi}(x)$ the unique solution of (55) through the above steps. Let $K\left(L_1(\alpha-\gamma)^{-1} + L_2(2\alpha-2\gamma)^{-\frac{1}{2}} + L_1 C_\zeta + L_2 C_\zeta\right) = \delta$. By (76), for $x_1, x_2 \in X_{0s}^\tau$, we have, for $t \ge \tau$,

$$
\begin{aligned}
&\left\| \hat{\xi}(x_1) - \hat{\xi}(x_2) \right\|_{\mathcal{C}_\tau^+} \\
&\le \sup_{t\ge\tau} e^{-\gamma t} \left\| T_{A_{0s}}(t-\tau)(x_1 - x_2) \right\|_{L^2(\Omega,X_0)} + \delta \left\| \hat{\xi}(x_1) - \hat{\xi}(x_2) \right\|_{\mathcal{C}_\tau^+} \\
&\le K e^{-\beta\tau} \sup_{t\ge\tau} e^{(\beta-\gamma)t} \|x_1 - x_2\|_{L^2(\Omega,X_0)} + \delta \left\| \hat{\xi}(x_1) - \hat{\xi}(x_2) \right\|_{\mathcal{C}_\tau^+} \\
&\le K e^{-\gamma\tau} \|x_1 - x_2\|_{L^2(\Omega,X_0)} + \delta \left\| \hat{\xi}(x_1) - \hat{\xi}(x_2) \right\|_{\mathcal{C}_\tau^+}.
\end{aligned} \tag{77}
$$

Thus by (77), for all $x_1, x_2 \in X_{0s}^\tau$, we have

$$
\left\| \hat{\xi}(x_1)(\tau) - \hat{\xi}(x_2)(\tau) \right\|_{L^2(\Omega,X_0)} \le K(1-\delta)^{-1} \|x_1 - x_2\|_{L^2(\Omega,X_0)}. \tag{78}
$$

Step 2. We prove that the mean-square stable invariant set is given by a graph of a Lipschitz continuous map. Let $h^s(x,\tau) = \Pi_{0u}\hat{\xi}(x)(\tau)$. Let $s \to +\infty$ and $t = \tau$ in (64), we have

$$
\begin{aligned}
h^{s}(x,\tau) = & -\int_{\tau}^{\infty} T_{A_{0u}}(\tau - r)\,\Pi_u F\left(\hat{\xi}(x)(r)\right)dr \\
& -\int_{\tau}^{\infty} T_{A_{0u}}(\tau - r)\,\Pi_u \sigma\left(\hat{\xi}(x)(r)\right)dW(r)\,.
\end{aligned}
\tag{79}
$$

Indeed, by (72), (73) and (78), it follows that for all $x_1, x_2 \in X_{0s}^{\tau}$,

$$
\begin{aligned}
& \left\| h^{s}(x_1,\tau) - h^{s}(x_2,\tau) \right\|_{L^2(\Omega, X_0)} \\
& \leq K^2 \left(L_1 (\alpha - \gamma)^{-1} + L_2 (2\alpha - 2\gamma)^{-\frac{1}{2}} \right) (1-\delta)^{-1} \left\| x_1 - x_2 \right\|_{L^2(\Omega, X_0)}\,.
\end{aligned}
\tag{80}
$$

By the definition of $h^{s}(\cdot,\tau)$ and (44), we get that $h^{s}(\cdot,\tau) : X_{0s}^{\tau} \to L^2(\Omega, \mathcal{F}_{\tau}; X_{0u})$ is a Lipschitz continuous map. Then by (55) and (79), we have for all $\tau \in$ and $x \in X_{0s}^{\tau}$,

$$
\hat{\xi}(x)(\tau) = x + h^{s}(x,\tau)\,. \tag{81}
$$

By the definition of $\mathcal{M}^{s}(\tau)$ (69) and Lemma 4.1, $\xi(\tau) \in \mathcal{M}^{s}(\tau)$ if and only if there exists $x \in X_{0s}^{\tau}$ such that $\xi(\tau) = x + h^{s}(x,\tau)$. Therefore, we have

$$
\mathcal{M}^{s}(\tau) = \left\{ x + h^{s}(x,\tau) : x \in X_{0s}^{\tau} \right\}.
$$

Step 3. Given $\tau \in$, we prove that $\mathcal{M}^{s}(\tau)$ is invariant under Φ. By Definition 2.2, we need to show that for each $\xi(\tau) \in \mathcal{M}^{s}(\tau)$ and $t > 0$, $\Phi(t,\tau)\xi(\tau) \in \mathcal{M}^{s}(\tau + t)$. By (68), for $\xi(\tau) \in \mathcal{M}^{s}(\tau)$, ξ is an integrated solution of (1) on $[\tau, +\infty)$. It is easy to prove that ξ also belongs to $\mathcal{C}_{\tau+t}^{+}$ and is an integrated solution of (1) on $[\tau + t, +\infty)$. So by (68), $\xi(\tau + t) \in \mathcal{M}^{s}(\tau + t)$. Since ξ is an integrated solution of (1) on $[\tau, +\infty)$, we find that

$$
\Phi(t,\tau)(\xi(\tau)) = u(t+\tau, \tau, \xi(\tau)) = \xi(t+\tau)\,.
$$

So $\Phi(t,\tau)(\xi(\tau)) \in \mathcal{M}^{s}(\tau + t)$, which implies $\Phi(t,\tau)\mathcal{M}^{s}(\tau) \subseteq \mathcal{M}^{s}(\tau + t)$. So $\mathcal{M}^{s}(\tau)$ is invariant under Φ. $\square$

5. Example

Consider the following stochastic Stratonovich parabolic partial equation

$$
\begin{cases}
\dfrac{\partial u(t,x)}{\partial t} = \left(\dfrac{\partial^2 u(t,x)}{\partial x^2} + \dfrac{\pi^2}{2} u(t,x) + g_0(u(t,x)) \right) dt + \sigma(u(t,x))\,dW(t)\,, \\
-\dfrac{\partial u(t,0)}{\partial x} = g_1(u(t,\cdot))\,, \\
\dfrac{\partial u(t,1)}{\partial x} = g_2(u(t,\cdot))\,, \\
u(0,\cdot) = u_0 \in L^2(0,1)\,,
\end{cases}
\tag{82}
$$

where $x \in (0,1)$, $g_0 : L^2(0,1) \to L^2(0,1)$, $g_1, g_2 : L^2(0,1) \to$ are arbitrary nonlinear functions, which are assumed to be globally Lipschitz continuous and $g_0(0) = g_1(0) = g_2(0) = 0$. Herein, W is a two-sided cylindrical Wiener process on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathbb{P})$ and $L^2(0,1)$ is the usual L^2-space on the interval $(0,1)$. σ is a nonlinear function. In order to incorporate the boundary conditions, we define

$$\mathcal{Y} = \quad \times \quad \times L^2(0,1) \quad \text{and} \quad \mathcal{Y}_0 = \{0\} \times \{0\} \times L^2(0,1),$$

which are Banach spaces equipped with the usual product norm. Let us denote

$$\mathcal{A}\begin{pmatrix} 0 \\ 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} \varphi'(0) \\ -\varphi'(1) \\ \varphi'' \end{pmatrix}$$

with domain $D(\mathcal{A}) = \{0\} \times \{0\} \times W^{2,2}(0,1)$ and its part

$$\mathcal{A}_0\begin{pmatrix} 0 \\ 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \varphi'' \end{pmatrix}$$

with domain $D(\mathcal{A}_0) = \{0\} \times \{0\} \times \left\{\varphi \in W^{2,2}(0,1) : \varphi'(0) = \varphi'(1) = 0\right\}$, where $W^{2,2}(0,1)$ is the usual Sobolev space. Notice that

$$\mathcal{Y}_0 = \overline{D(\mathcal{A})} = \{0\} \times \{0\} \times L^2(0,1) \neq \mathcal{Y}$$

and take

$$F\begin{pmatrix} 0 \\ 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} g_0(\varphi) \\ g_1(\varphi) \\ g_2(\varphi) \end{pmatrix}, \quad u^*(t,\cdot) = \begin{pmatrix} 0 \\ 0 \\ u(t,\cdot) \end{pmatrix}, \quad \omega(t) = \begin{pmatrix} 0 \\ 0 \\ W(t) \end{pmatrix}.$$

Then we can rewrite (82) as

$$\begin{cases} du^*(t) = \left(\left(\mathcal{A} + \dfrac{1}{2}\pi^2 I\right) u^*(t) + F\left(u^*(t)\right)\right) dt + \sigma\left(u^*(t)\right) d\omega(t), \\ u^*(0) = u_0^* \in \mathcal{Y}_0. \end{cases}$$

Then $F : L^2(\Omega, \mathcal{Y}_0) \to L^2(\Omega, \mathcal{Y})$ is globally Lipschitz continuous by the Lipschitz continuity of $g_i, i = 0,1,2$. Assume σ is C_b^1 and satisfies $\sigma(0) = 0$. Then Assumption 2.1 is satisfied. According to [26, Lemma 6.1 & Lemma 6.3] that the linear operator $\mathcal{A}_0$ is the infinitesimal generator of $\left(T_{\mathcal{A}_0}(t)\right)_{t\geq 0}$ a C_0-semigroup on $\mathcal{Y}_0$ and the non-densely defined operator $\mathcal{A}$ is 3/4-almost sectorial, namely, its resolvent satisfies

$$0 < \liminf_{\lambda\to\infty} \lambda^{\frac{3}{4}} \left\|(\lambda I - \mathcal{A})^{-1}\right\|_{\mathcal{L}(x)} \leq \limsup_{\lambda\to\infty} \lambda^{\frac{3}{4}} \left\|(\lambda I - \mathcal{A})^{-1}\right\|_{\mathcal{L}(x)} < \infty.$$

Then [8, Lemma 8.5] guarantees Assumption 2.3. Moreover, the spectrum of $\mathcal{A}_0$ is given by

$$\sigma\left(\mathcal{A}_0\right)=\left\{-(\pi k)^2: k \in \mathbb{N}\right\}.$$

Moreover, $\left(\mathcal{A}+\frac{1}{2}\pi^2 I\right)_0$, the part of $\left(\mathcal{A}+\frac{1}{2}\pi^2 I\right)$, is the infinitesimal generator of a C_0-semigroup on $\mathcal{Y}_0$ denoted by $\left(T_{\left(\mathcal{A}+\frac{1}{2}\pi^2 I\right)_0}(t)\right)_{t\geq 0}$. Also, according to [26, Lemma 6.4] and [6,8, Proposition 2.5], we get that Assumption 2.2 is satisfied, which means $\mathcal{A}$ does not satisfy the Hille-Yosida condition but $\left(\mathcal{A}+\frac{1}{2}\pi^2 I\right)$ generates a integrated semigroup $\left(S_{\left(\mathcal{A}+\frac{1}{2}\pi^2 I\right)}(t)\right)_{t\geq 0}$. So we next check Assumption 2.5. In fact, by [26, Lemma 6.2] and [7, Lemma 2.1], we have $\sigma\left(\mathcal{A}_0\right)=\sigma\left(\mathcal{A}\right)$, then

$$\begin{aligned}
\sigma\left(\mathcal{A}+\frac{1}{2}\pi^2 I\right) &= \sigma\left(\mathcal{A}_0+\frac{1}{2}\pi^2 I\right) \\
&= \left\{-(\pi k)^2+\frac{1}{2}\pi^2 : k \in \mathbb{N}\right\} \\
&= \left\{\frac{1}{2}\pi^2, -\frac{1}{2}\pi^2, -\frac{7}{2}\pi^2, -\frac{17}{2}\pi^2, \ldots\right\},
\end{aligned}$$

and each eigenvalue $\lambda_k=\left(\frac{1}{2}-k^2\right)\pi^2$ corresponding to the eigenfunction

$$\psi_k(x)=\sin(\pi k x).$$

By Remark 2.1, we could take $\alpha=\beta=\frac{1}{2}\pi^2-\varepsilon^*$ for any $\varepsilon^* \in \left(0, \frac{1}{2}\pi^2\right)$. Thus the spectrum of $\mathcal{A}_0+\frac{1}{2}\pi^2 I$ could be split into two parts $\sigma^{0s}=\{\left(\frac{1}{2}-k^2\right)\pi^2 : k=1,2,\ldots\}$, $\sigma^{0u}=\{\frac{1}{2}\pi^2\}$. So $\mathcal{A}_0+\frac{1}{2}\pi^2 I$ satisfies the exponential dichotomy condition and the Assumption 2.5 is satisfied. The stable subspace $\mathcal{Y}_{0s}$ and unstable subspace $\mathcal{Y}_{0u}$ of $\mathcal{Y}_0$ are span$\{\sin(\pi n x), n=1,2,\ldots\}$, and $\{0\}$, respectively.

Therefore, the results obtained in Theorems 3.1 and 4.1 are applicable and thus justify the existence of mean-square random unstable manifolds and stable sets for (82).

6. Concluding remarks

Under the suitable assumptions, we have established the existence of mean-square random unstable invariant manifolds and mean-square random stable invariant set of (1) that are given by graphs of Lipschitz maps.

We conclude this paper by discussing several possible extensions of our result. The first ex- is

(2) We need to point out that the usual assumptions $F : L^2(\Omega, X_0) \to L^2(\Omega, X)$ and $\sigma : L^2(\Omega, X_0) \to L^2(\Omega, \mathcal{L}_2(Y_0, X))$ are C^k (for $k \geq 1$) can not guarantee the smoothness of random invariant manifolds or invariant sets. The main difficulty that one has to face arises from the fact that the required gap condition contains some C_κ like constant since Young's convolution inequality is not applicable. There is no conceptual obstruction to the use of the method of proof presented

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