$$\begin{cases} du = Audt + F(u)\,dt + u \circ dW, \\ u(0) = u_0 \in \overline{D(A)}, \end{cases} \tag{1}$$

where $A : D(A) \subset X \to X$ is a linear operator and $\overline{D(A)} \neq X$, namely, its domain is non-densely defined. Indeed, W is a one-dimensional Brownian motion and $F : \overline{D(A)} \to X$ is globally Lipschitz continuous. Note that the Cauchy problem (1) is ill-posed if the Hille-Yosida theory for the C_0-semigroup generated by A breaks down. Cauchy problems with a non-dense domain cover several types of differential equations, including delay differential equations, age-structured models and some evolution equations with nonlinear boundary conditions. Since the classical variation of constants formula is not applicative, integrated semigroup theory is used to define a suitable integrated solution. The concept of integrated semigroups was firstly introduced by Arendt [1,2] and was applied to study the existence and uniqueness of solutions to non-homogeneous Cauchy problems in a deterministic setting by Da Prato and Sinestrari [9]. Later on, Thieme [40,39] established the well-posedness of non-autonomous and semilinear Cauchy problems under a prior Hille-Yosida type estimate for the resolvent of non-densely defined operators. By discarding the mentioned estimate, Magal and Ruan [28,30,29,32,31] used the integrated semigroup theory to construct modified variation of constants formula for Cauchy problems with non-dense domain and applied the center manifold theory to study Hopf bifurcation of age-structured models. Liu, Magal and Ruan [23] studied center-unstable manifolds for Cauchy problems with non-dense domain and applications to stability of Hopf bifurcation. There are in general two methods to construct invariant manifolds, one is the graph transform method of Hadamard [15], and the other is Lyapunov-Perron method of Lyapunov [0 TD.0005 Tc(28)Tj0 g.9901 0 unstable manifolds. By using the Wong-Zakai approximation, Shen et al. [26] proved that the invariant manifolds and stable foliations of the Wong-Zakai approximations converge to those of the Stratonovich stochastic evolution equation, respectively.

With the help of the theory of integrated semigroups, we construct a new variation of constants formula by the resolvent operator connecting $\overline{D(A)}$ and X. Unlike the argument using the classical Lyapunov-Perron method, the spectral gap condition contains a crucial constant in the derivation of estimation of random Stieltjes-type convolution. The required conditions for smoothness do not appear to be easily verifiable at first sight. We present a positive answer to the smoothness problem for invariant manifolds and foliations of stochastic evolution equations (1). To be precise, we shall establish the conditions for the existence of C^k-smoothness random center-unstable invariant manifolds and center-stable foliations of (1).

The remaining part of this article is structured as follows. In Section 2, we collect some essential results on random dynamical systems and integrated semigroups, and layout the basic assumptions. Then in Section 3 and Section 4, we prove the existence and smoothness of invariant center-unstable manifolds and invariant center-stable foliations of (1) respectively. Finally in Section 5, we give two illustrative examples about a age-structured model and a stochastic parabolic equation to verify the main results in the present paper.

2. Preliminaries

2.1. Random dynamical systems

Consider flows on the probability space $(\ , \mathcal{F}, \mathbb{P})$, where a flow $\ : \mathbb{R} \times \ \to \ $ of $\{ \ _t\}_{t\in\mathbb{R}}$ is defined on the sample space having following properties:

(i) the mapping $(t, \omega) \mapsto \ _t\omega$ is $(\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}, \mathcal{F})$-measurable.
(ii) $\ _0 = I \ $.
(iii) $\ _{t+s} = \ _t \circ \ _s$ for all $t, s \in \mathbb{R}$.

In addition, we assume that $\mathbb{P}$ is ergodic with respect to $\{ \ _t\}_{t\in\mathbb{R}}$. Then the quadruple $(\ , \mathcal{F}, \mathbb{P}, \{ \ _t\}_{t\in\mathbb{R}})$ is called a metric dynamical system. Usually, let $W(t)$ be a two-sided Wiener process with trajectories in the space $\ := C_0(\mathbb{R}, \mathbb{R})$ of real continuous functions defined on $\mathbb{R}$ which is a set equipped with compact open topology $\mathcal{F} := \mathcal{B}(C_0(\mathbb{R}, \mathbb{R}))$. On this set we consider the measurable flow $\{ \ _t\}_{t\in\mathbb{R}}$ defined by $\ _t\omega(\cdot) = \omega(\cdot + t) - \omega(t)$ which is called the Wiener shift, the associated distribution $\mathbb{P}$ is a Wiener measure defined on $\mathcal{F}$. A set is called $\{ \ _t\}_{t\in\mathbb{R}}$-invariant if $\ _t \ = \ $ for $t \in \mathbb{R}$. For more details, see [3]. Using these notations, we could present the definition of a random dynamical system.

Definition 2.1. Let X be a separable Hilbert space. A continuous random dynamical system on X over a metric dynamical system $(\ , \mathcal{F}, \mathbb{P}, \{ \ _t\}_{t\in\mathbb{R}})$ is a mapping

$$\varphi : \mathbb{R} \times \ \times X \to X, \ (t, \omega, \ ) \mapsto \varphi(t, \omega, \ ),$$

which is $(\mathcal{B}(\mathbb{R}) \otimes \mathcal{F} \otimes \mathcal{B}(X), \mathcal{B}(X))$-measurable and satisfies:

(i) $\varphi(0, \omega, \ ) = \ $, for $\ \in X$ and $\omega \in \ $.
(ii) $\varphi(t + s, \omega, \ ) = \varphi(t, \ _s\omega, \varphi(s, \omega, \ ))$, for all $t, s \in \mathbb{R}$, $\ \in X$ and all $\omega \in \ $.

Definition 2.2. A random set $\mathcal{M}(\omega)$ is called invariant for a random dynamical system $\varphi(t, \omega, \ )$ if $\varphi(t, \omega, \mathcal{M}(\omega)) \subset \mathcal{M}(\ _t\omega)$ for $t \geq 0$.

We will use a coordinate transform to convert the stochastic partial differential equation (1) into a random evolution equation [11,5,10]. To this end, we consider the linear stochastic differential equation

$$d \ + \ dt = dW. \tag{2}$$

We extract the important results on the stationary Ornstein-Uhlenbeck process in the following.

Lemma 2.3. *[10, Lemma 2.1]*

(i) *There exists a* $\{\ _t\}_{t\in\mathbb{R}}$*-invariant set* $\tilde{\ } \subset C_0(\mathbb{R}, \mathbb{R})$ *of full measure satisfying the sublinear growth condition*

$$\lim_{t\to\pm\infty} \frac{|\omega(t)|}{|t|} = 0, \ \omega \in \tilde{\ }.$$

(ii) *The random variable* $\ (\omega) = -\int_{-\infty}^{0} e^s \omega(s)\, ds$ *is well defined for all* $\omega \in \tilde{\ }$ *and generates a unique stationary solution of* (2) *given by*

$$(t, \omega) \mapsto \ (\ _t\omega) = -\int_{-\infty}^{0} e^s \ _t\omega(s)\, ds = -\int_{-\infty}^{0} e^s \omega(s+t)\, ds + \omega(t).$$

The mapping $t \mapsto \ (\ _t\omega)$ *is continuous.*

(iii) *Moreover*

$$\lim_{t\to\pm\infty} \frac{|\ (\ _t\omega)|}{|t|} = 0,$$

for all $\omega \in \tilde{\ }$.

(iv) *In addition*

$$\lim_{t\to\pm\infty} \frac{1}{t} \int_0^t \ (\ _s\omega)\, ds = 0,$$

for all $\omega \in \tilde{\ }$.

Proof. See [43, Lemma 2.2]. $\square$

From now on, we will work on $\left(\tilde{\ }, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, \{\ _t\}_{t\in\mathbb{R}}\right)$, where $\tilde{\mathcal{F}}$ is the trace algebra of $\tilde{\ }$ and the restriction of Wiener measure to this -algebra $\tilde{\mathcal{F}}$ is denoted by $\tilde{\mathbb{P}}$. With a slight abuse of notation, we again denote the metric dynamical system by $\left(\ , \mathcal{F}, \mathbb{P}, \{\ _t\}_{t\in\mathbb{R}}\right)$ in the following context.

2.2. Integrated semigroups

In this subsection, we collect some basic results about non-densely defined operators and integrated semigroups from [28,30,29]. Since A is a non-densely defined linear operator in a Hilbert space X, its resolvent set is denoted by $\ (A) = \{ \ \in \mathbb{C} : \ I - A \text{ is invertible}\}$, and $\ (A) \neq \emptyset$. The spectrum of A is denoted by $\ (A) := \mathbb{C} \setminus \ (A)$. Moreover, let $X_0 = \overline{D(A)}$ equipped with the norm $\|\cdot\|$ and construct the part of A in X_0 denoted by $A_0 : D(A_0) \subset X_0 \to X_0$, which is a linear operator on X_0 defined by

$$A_0 \ = A \ , \forall \ \in D(A_0) := \{ \ \in D(A) : A \ \in X_0\}.$$

Assume that there exists a constant ϑ satisfying $(\vartheta, +\infty) \subset \ (A)$, it is easy to check that for each $\ > \vartheta$,

$$D(A_0) = (\ I - A)^{-1} X_0, \ (\ I - A_0)^{-1} = (\ I - A)^{-1}|_{X_0}.$$

Therefore, it follows from [29, Lemma 2.1] that $\ (A) = \ (A_0)$, and thus $\ (A) = \ (A_0)$.

We next discuss some basic assumptions about operators A and A_0. Assume that A_0 is a Hille-Yosida operator in X_0, i.e., there exist two constants, $\vartheta \in \mathbb{R}$ and $M \geq 1$, such that $(\vartheta, +\infty) \subset \ (A_0)$ and

$$\left\| (\ I - A_0)^{-k} \right\|_{\mathcal{L}(X_0)} \leq \frac{M}{(\ - \vartheta)^k}, \ \forall \ > \vartheta, \ \forall k \geq 1.$$

And we assume

$$\lim_{\ \to +\infty} \ (\ I - A)^{-1} \ = 0, \forall \ \in X.$$

Lemma 2.4. *[35, Theorem 5.3] A linear operator $A_0 : D(A_0) \subset X_0 \to X_0$ is an infinitesimal generator of a C_0-semigroup $(T(t))_{t \geq 0}$ satisfying $\|T(t)\| \leq Me^{\vartheta t}$, if and only if*

(i) *A_0 is densely defined in X_0;*
(ii) *A_0 is a Hille-Yosida operator in X_0.*

Lemma 2.5. *[28, Lemma 2.1] Let $(X, \|\cdot\|)$ be a Banach space and $A : D(A) \subset X \to X$ be a linear operator. Assume that there exists $\vartheta \in \mathbb{R}$, such that $(\vartheta, +\infty) \subset \ (A)$ and*

$$\limsup_{\ \to +\infty} \left\| (\ I - A)^{-1} \right\|_{\mathcal{L}(X_0)} < +\infty.$$

Then the following assertions are equivalent:

(i) $\lim_{\ \to +\infty} \ (\ I - A)^{-1} \ = \ , \forall \ \in X_0$.
(ii) $\lim_{\ \to +\infty} (\ I - A)^{-1} \ = 0, \forall \ \in X$.
(iii) $\overline{D(A_0)} = X_0$.

Now we are ready to introduce the definition of integrated semigroups.

Definition 2.6. Let X be a Banach space. A family of bounded linear operators $\{S(t)\}_{t\geq 0}$ on X is called an integrated semigroup if

(i) $S(0) = 0$.
(ii) The map $t \to S(t)$ is continuous on $[0, +\infty)$ for each $\in X$.
(iii) $S(t)$ satisfies

$$S(s)\,S(t) = \int_0^s (S(r+t) - S(r))\,dr, \forall s, t \geq 0.$$

According to Lemma 2.4 and Lemma 2.5, A_0 generates a C_0-semigroup on X_0 denoted by $(T(t))_{t\geq 0}$. It can also be denoted by $\left(e^{A_0 t}\right)_{t\geq 0}$. Besides, as concluded in [30, Proposition 2.5], A generates an integrated semigroup on X denoted by $(S(t))_{t\geq 0}$ and for each $\in X, t \geq 0$, $(S(t))_{t\geq 0}$ is given by

$$S(t) \quad = \quad \int_0^t T(s)\,(\quad I - A)^{-1} \quad ds + (I - T(t))\,(\quad I - A)^{-1} \quad . \tag{3}$$

Also, the map $t \to S(t)$ is continuously differentiable if and only if $\in X_0$ and

$$\frac{dS(t)}{dt} = T(t) \quad , \forall t \geq 0, \ \forall \quad \in X_0.$$

Next we introduce the definition of an MR operator which is named after Magal and Ruan [28](see Chen [7]).

Definition 2.7 *(MR operator).* Let A be a closed linear operator in a Banach space X with $\overline{D(A)} \neq X$ and A_0 be the part of A in $X_0 = \overline{D(A)}$. A is called an MR operator if

(i) A_0 generates a C_0-semigroup $(T(t))_{t\geq 0}$ on X_0, i.e. A_0 is a Hille-Yosida operator in X_0, and $\overline{D(A_0)} = X_0$. Then A generates an integrated semigroup $(S(t))_{t\geq 0}$ in X.
(ii) Assume that there exists a real number $\ _0 > 0$, and a non-decreasing map $\ : [0, \ _0] \to [0, +\infty)$ such that for each $f \in C^1([0, \ _0]; X)$,

$$\left\| \frac{d}{dt} \int_0^t S(t-s)\,f(s)\,ds \right\| \leq \quad (t) \sup_{s\in[0,t]} \|f(s)\|, \ \forall t \in [0, \ _0],$$

where satisfies

$$\lim_{t\to 0^+} \quad (t) = 0.$$

We assume that A is an MR operator from now on.

Lemma 2.8. *[28, Lemma 2.6] Let* ${}_0 > 0$ *be fixed. For each* $f \in C^1([0, {}_0]; X)$*, define*

$$(S * f)(t) = \int_0^t S(t-s) f(s) ds, \ \forall t \in [0, {}_0].$$

Then we have the following:

(i) *The map* $t \to (S * f)(t)$ *is continuously differentiable on* $[0, {}_0]$.
(ii) $(S * f)(t) \in D(A), \ \forall t \in [0, {}_0]$.
(iii) *if we set* $u(t) = \frac{d}{dt}(S * f)(t)$*, then*

$$u(t) = A \int_0^t u(s) ds + \int_0^t f(s) ds, \ \forall t \in [0, {}_0].$$

(iii) *For each* $\ > \vartheta$*, and each* $t \in [0, {}_0]$*, we have*

$$(\ I - A)^{-1} \frac{d}{dt}(S * f)(t) = \int_0^t T(t-s)(\ I - A)^{-1} f(s) ds.$$

For each $f \in C^1([0, {}_0]; X)$, we denote

$$(S \diamond f)(t) = \frac{d}{dt}(S * f)(t), \ \forall t \in [0, {}_0],$$

which is defined as a Stieltjes convolution. According to Lemma 2.8 and Lemma 2.5(i), we have

$$(S \diamond f)(t) = \lim_{\to +\infty} \int_0^t T(t-s) \ \ (\ I - A)^{-1} f(s) ds, \ \forall t \in [0, {}_0], \ \forall \ > \vartheta.$$

Lemma 2.9. *[29, Proposition 2.13] For* $\ > \vartheta$*, there exists* $C \ > 0$ *such that* $f \in C(\mathbb{R}^+; X)$ *and*

$$\|(S \diamond f)(t)\| \le C \ \sup_{s \in [0,t]} e^{\ (t-s)} \|f(s)\|, \ \forall t \ge 0.$$

Moreover, for each $\varepsilon > 0$*, if* ${}_\varepsilon > 0$ *satisfying* $M \ ({}_\varepsilon) \le \varepsilon$*, it holds*

$$C \ = \frac{2\varepsilon \max(1, e^{- \ \varepsilon})}{1 - e^{(\vartheta - \ )}}$$

To our aim, we define a coordinate transform

$$= \Xi(u,\omega) = ue^{-\ (\ {}_t\omega)},$$

and its reverse transform

$$u = \Xi^{-1}(\ ,\omega) = \ e^{\ (\ {}_t\omega)},$$

for $t \geq 0$. Performing the transformation $\Xi(\ ,\omega)$ into (1), we obtain

$$\begin{cases} d\ \ = A\ \ dt + \ \ (\ {}_t\omega)\ \ dt + G(\ {}_t\omega,\ \ )\,dt, \\ \ (0) = \ \ \in X_0, \end{cases} \tag{4}$$

where $G(\omega,\ \ ) = e^{-\ (\omega)} F\left(e^{\ (\omega)}\ \ \right)$. Obviously, G and F have the same Lipschitz constant.

Definition 2.10. A mapping $\ \in C\left(\mathbb{R}^+; X\right)$ is called an integrated solution of (4) if and only if for $\forall t \geq 0$,

(i) $\int_0^t\ \ (s)\,ds \in D(A)$.
(ii) It holds

$$\ (t) = \ \ + A\int_0^t\ \ (s)\,ds + \int_0^t\ \ (\ {}_s\omega)\ \ (s)\,ds + \int_0^t G(\ {}_s\omega,\ \ (s))\,ds.$$

For the linear part of (4), we define

$${}_{A_0}(t,s) = e^{A_0(t-s)+\int_s^t\ (\ {}_r\omega)dr} = T(t-s)\,e^{\int_s^t\ (\ {}_r\omega)dr}.$$

2.3. Exponential trichotomy

Assumption 2.2. We assume that $T(t)$ satisfies the exponential trichotomy with exponents $\ > \ \geq 0 \geq -\ \ > -\ $ and bound $K \geq 1$, more precisely,

(i) there exist and

(iii) For all $\in X_0$, it holds

$$\|T(t)\Pi_{0c}\ \| \le Ke^{\ |t|}\|\Pi_{0c}\ \|,\ \forall t \in \mathbb{R}, \tag{5}$$

$$\|T(t)\Pi_{0u}\ \| \le Ke^{\ t}\|\Pi_{0u}\ \|,\ \forall t \le 0, \tag{6}$$

$$\|T(t)\Pi_{0s}\ \| \le Ke^{-\ t}\|\Pi_{0s}\ \|,\ \forall t \ge 0, \tag{7}$$

Remark 2.11. Assume $\ >\ \ge 0 \ge -\ \ > -\ $. If X_0 is a finite dimensional space and there exist eigenvalues with real part being zero, less than zero and greater than zero, then the exponential trichotomy holds. If X_0 is an infinite dimensional space and the spectrum of A_0 satisfies $\ (A_0) = \ ^{0s} \cup \ ^{0c} \cup \ ^{0u}$, where $\ ^{0s} = \{\ \in\ (A_0) : \mathrm{Re}(\ ) \le -\ \}$, $\ ^{0c} = \{\ \in\ (A_0) : |\mathrm{Re}(\ )| \le\ \}$, $\ ^{0u} = \{\ \in\ (A_0) : \mathrm{Re}(\ ) \ge\ \}$, and A_0 generates a strong continuous semigroup, then the exponential trichotomy holds too (see [14, page 267]. And if the spectrum of A_0 satisfies $\ (A_0) = \ ^{0s} \cup \ ^{0u}$, where $\ ^{0s}$ and $\ ^{0u}$ are the same as above, then the exponential dichotomy holds. Since $\ (A) = \ (A_0)$, the spectrum of A can be split as A_0.

Since we only require the Hille-Yosida condition on X_0, one needs to extend the mentioned projections from X_0 to X and outline the associated statements below.

Lemma 2.12. *[29, Proposition 3.5] Let $\Pi_0 : X_0 \to X_0$ be a bounded linear projection satisfying*

$$\Pi_0(\ I - A_0)^{-1} = (\ I - A_0)^{-1}\Pi_0,\ \forall\ > \vartheta,$$

and

$$\Pi_0 X_0 \subset D(A_0),\ \textit{and } A_0|_{\Pi_0 X_0} \textit{ is bounded.}$$

Then there exists a unique bounded projection $\Pi : X \to X$ such that

(i) $\Pi|_{X_0} = \Pi_0$;
(ii) $\Pi(X) \subset X_0$;
(iii) $\Pi(\ I - A)^{-1} = (\ I - A)^{-1}\Pi,\ \forall\ > \vartheta$.

With the mentioned information at hand, we are ready to state the decomposition on X. For each $k \in \{c, u, cu\}$, denote by $\Pi_k : X \to X$ the unique extension of Π_{0k} and let $\Pi_s = I_X - (\Pi_c + \Pi_u)$. Then we have for each $p \in \{c, u, s, cu\}$, $\Pi_p(\ I - A)^{-1} = (\ I - A)^{-1}\Pi_p,\ \forall\ > \vartheta$, and $\Pi_p(X) \subset X_0$. Also, $\Pi_p(X_0) = \Pi_{0p}(X_0) = X_{0p}$. Denote $X_p = \Pi_p(X)$, and $A_p = A|_{X_p}$. Then for each $k \in \{c, u\}$, we have $X_{0k} = X_k$, $X_0 = X_{0s} \oplus X_{0c} \oplus X_{0u}$, and $X = X_s \oplus X_c \oplus X_u$. Herein, X_{0c}, X_{0u} and X_{0s} are called center subspace, unstable subspace and stable subspace of X_0 respectively. For more details, refer to [29, page 22].

3. Smooth center-unstable manifolds

This section is dedicated to the existence of invariant manifolds of (1) through the Lyapunov-Perron method.

Definition 3.1. Let $\mathcal{M} = \{\mathcal{M}(\omega) : \omega \in \quad\}$ be a family of X_0, then $\mathcal{M}$ is called a C^k random center-unstable invariant manifold of the random dynamical system φ if $\mathcal{M}$ is invariant under φ and given by the graph of a Lipschitz continuous map $h^{cu}(\cdot, \omega) : X_{0cu} \to X_{0s}$ which is C^k in $\quad$, $k \geq 1$, i.e.,

$$\mathcal{M}(\omega) = \{\quad + h^{cu}(\quad, \omega) : \quad \in X_{0cu}\}.$$

We denote $\mathcal{M}(\omega)$ by $\mathcal{M}^{cu}(\omega)$. Accordingly, a C^k random center (resp. center-stable) invariant manifold $\mathcal{M}^c(\omega)$ (resp. $\mathcal{M}^{cs}(\omega)$) is given by the graph of a Lipschitz map $h^i(\cdot, \omega) : X_{0i} \to X_{0j}$, $i = c$ (resp. cs) and $j = su$ (resp. $j = u$).

We extract some results from [19] about integrated solutions and invariant manifolds of (4). Set

$$(S \diamond G \quad (\quad))(t) = \lim_{\to +\infty} \int_0^t \quad_{A_0}(t,s) \quad (\quad I - A)^{-1} G(\quad_s\omega, \quad (s, \omega, \quad))\, ds.$$

We have the following result:

Proposition 3.2. *[19, Proposition 2.1] Under Assumption 2.2, we have*

$$\Pi_{0s}(S \diamond G \quad (\quad))(t) = \lim_{\to +\infty} \int_0^t \quad_{A_{0s}}(t,s) \quad (\quad I - A_s)^{-1} \Pi_s G(\quad_s\omega, \quad (s, \omega, \quad))\, ds, \tag{8}$$

while for each $k \in \{c, u\}$,

$$\Pi_{0k}(S \diamond G \quad (\quad))(t) = \int_0^t \quad_{A_{0k}}(t,s)\, \Pi_k G(\quad_s\omega, \quad (s, \omega, \quad))\, ds. \tag{9}$$

Moreover, for each $\quad > - \quad$, there exists $C \quad > 0$, such that for $\forall t \geq 0$, we obtain

$$\|\Pi_{0s}(S \diamond G \quad (\quad))(t)\| \leq C \quad \sup_{s \in [0,t]} e^{\quad (t-s)} e^{\int_s^t \quad (\quad_r\omega) dr} \|G(\quad_s\omega, \quad (s, \omega, \quad))\|. \tag{10}$$

Lemma 3.3. *[19, Lemma 2.5, Theorem 3.1] The following assertions are valid:*

(i) *Under Assumption 2.1, (4) possesses a unique integrated solution on X_0 given by*

$$\quad(t) = \quad_{A_0}(t, 0) \quad + (S \diamond G \quad (\quad))(t),$$

which is the so-called variation of constants formula and generates a random dynamical system Φ. Moreover,

$$(t, \omega, \quad) \to \Xi^{-1}(\quad_t\omega, \Phi(t, \omega, \Xi(\omega, \quad))) =: \hat{\Phi}(t, \omega, \quad),$$

is a random dynamical system. Also, this process is a solution of (1) *for* $\in X_0$.

(ii) *Under Assumptions* 2.1-2.2, *if* $\; < \; < \min\{\;,\;\}$ *and* $- \; < \; < -$ *with*

$$KL\left(C \; + \frac{1}{-} + \frac{1}{-}\right) < 1,$$

then there exists a center invariant manifold for (4), *given by*

$$\mathcal{M}^c(\omega) = \{\; + h^c(\;,\omega) : \; \in X_{0c}\},$$

where $h^c(\cdot,\omega) : X_{0c} \to X_{0us}$ *is a Lipschitz continuous map with* $h^c(0,\omega) = 0$ *and*

$$Lip\, h^c(\cdot,\omega) \le \frac{K\left(KL + C\;L(\;-\;)\right)}{(\;-\;)\left(1 - KL\left(C\; + \frac{1}{-} + \frac{1}{-}\right)\right)}.$$

For $_{cu} \in (-\;,-\;)$, define the Banach space

$$\mathcal{C}_{\;cu}((-\infty,0];X_0) = \left\{f \in C((-\infty,0],X_0) : \sup_{t\le 0} e^{-\;_{cu}t - \int_0^t (\;_s\omega)ds}\|f(t)\| < \infty\right\},$$

equipped with the norm

$$\|f(t)\|_{\mathcal{C}_{\;cu}} = \sup_{t\le 0} e^{-\;_{cu}t - \int_0^t (\;_s\omega)ds}\|f(t)\|.$$

And for $_{cs} \in (\;,\min\{\;,\;\})$, define the Banach space

$$\mathcal{C}_{\;cs}([0,+\infty);X_0) = \left\{f \in C([0,+\infty),X_0) : \sup_{t\ge 0} e^{-\;_{cs}t - \int_0^t (\;_s\omega)ds}\|f(t)\| < \infty\right\},$$

equipped with the norm

$$\|f(t)\|_{\mathcal{C}_{\;cs}} = \sup_{t\ge 0} e^{-\;_{cs}t - \int_0^t (\;_s\omega)ds}\|f(t)\|.$$

Define

$$\mathcal{M}^{cu}(\omega) = \{\; \in X_0 : \;(t,\omega,\;) \in \mathcal{C}_{\;cu}((-\infty,0];X_0)\}.$$

Theorem 3.4. *For* $- \quad < \quad < \ _{cu} < -$ *with*

$$KL\left(C \quad - \frac{1}{+ \ _{cu}} - \frac{1}{_{cu} -}\right) < 1, \tag{11}$$

there exists a Lipschitz invariant center-unstable manifold for (4) *which is given by*

$$\mathcal{M}^{cu}(\omega) = \{ \quad + h^{cu}(\ , \omega) : \quad \in X_{0cu}\},$$

where $h^{cu}(\cdot, \omega) : X_{0cu} \to X_{0s}$ *is Lipschitz continuous with* $h^{cu}(0, \omega) = 0$ *and*

$$Lip\ h^{cu}(\cdot, \omega) \leq K_u = \frac{KLC}{1 - KL\left(C \quad - \frac{1}{+ \ _{cu}} - \frac{1}{_{cu} -}\right)}.$$

Proof. We proceed it in three steps.

Step 1. We prove that $\quad \in \mathcal{M}^{cu}$

$$= \quad {}_{A_{0s}}(t,l)\,\Pi_{0s} \quad (l) + \lim_{\to+\infty} \int_l^t {}_{A_{0s}}(t,s) \quad (\quad I - A_s)^{-1}$$
$$\times \Pi_s G(\ {}_s\omega, \quad (s))\,ds. \tag{14}$$

According to (7), for $l \le 0$,

$$\begin{aligned} \left\| {}_{A_{0s}}(t,l)\,\Pi_{0s} \quad (l)\right\| &\le K e^{- \ (t-l)+\int_l^t \ (\ {}_r\omega)dr} \| \quad (l)\| \\ &\le K e^{- \ (t-l)- \ {}_{cu} l+\int_0^t \ (\ {}_r\omega)dr} e^{\ {}_{cu} l-\int_0^l \ (\ {}_r\omega)dr} \| \quad (l)\| \\ &\le K e^{(\ + \ {}_{cu})l- \ t+\int_0^t \ (\ {}_r\omega)dr} \| \quad \|_{\mathcal{C} \ {}_{cu}}. \end{aligned}$$

Let $l \to -\infty$, since $\quad + \ {}_{cu} > 0$, we have

$$\Pi_{0s} \quad (t,\omega, \quad) = \lim_{\to+\infty} \int_{-\infty}^t {}_{A_{0s}}(t,s) \quad (\quad I - A_s)^{-1}\,\Pi_s G(\ {}_s\omega, \quad (s))ds. \tag{15}$$

By letting $l \to -\infty$ and combining with (13), we get (12). The converse can be obtained by a straight computation.

Step 2. We claim that for every $\quad \in X_{0cu}$, (12) has a unique solution in $\mathcal{C} \ {}_{cu}((-\infty,0];X_0)$.

To show this claim, we denote $\mathcal{J}(\ ,\omega, \ )$ the right side of (12), i.e.,

$$\begin{aligned} \mathcal{J}(\ ,\omega, \ ) = \ & {}_{A_{0c}}(t,0) \quad + \int_0^t {}_{A_{0c}}(t,s)\,\Pi_c G(\ {}_s\omega, \quad (s))\,ds \\ & + \int_0^t {}_{A_{0u}}(t,s)\,\Pi_u G(\ {}_s\omega, \quad (s))\,ds \\ & + \lim_{\to+\infty} \int_{-\infty}^t {}_{A_{0s}}(t,s) \quad (\quad I - A_s)^{-1}\,\Pi_s G(\ {}_s\omega, \quad (s))\,ds. \end{aligned} \tag{16}$$

It needs to show that $\mathcal{J}$ maps from $\mathcal{C} \ {}_{cu}((-\infty,0];X_0) \times X_{0cu}$ to $\mathcal{C} \ {}_{cu}((-\infty,0];X_0)$ and it is a uniform contraction. For $\ , \tilde{\ } \in \mathcal{C} \ {}_{cu}((-\infty,0];X_0)$ and $t \le 0$,

$$\left\|\mathcal{J}(\ ,\omega, \ ) - \mathcal{J}(\tilde{\ },\omega, \ )\right\|_{\mathcal{C} \ {}_{cu}} = \sup_{t\le 0} e^{- \ {}_{cu}t-\int_0^t \ (\ {}_s\omega)ds} \left\|\mathcal{J}(\ ,\omega, \ ) - \mathcal{J}(\tilde{\ },\omega, \ )\right\|$$
$$\le KL \sup_{t\le 0} \int_t^0 e^{(\ + \ {}_{cu})(s-t)- \ {}_{cu}s-\int_0^s \ (\ {}_r\omega)dr} \| \quad - \tilde{\ }\|\,ds$$
$$+ KL \sup_{t\le 0} \int_t^0 e^{(\ {}_{cu}- \ )(s-t)- \ {}_{cu}s-\int_0^s \ (\ {}_r\omega)dr} \| \quad - \tilde{\ }\|\,ds + \sup_{t\le 0} e^{- \ {}_{cu}t-\int_0^t \ (\ {}_s\omega)ds} \Delta\mathcal{J}_s$$

$$\leq KL\left(-\frac{1}{+\ _{cu}}-\frac{1}{_{cu}-}\right)\|\ \ -\tilde{\ }\|_{\mathcal{C}\ _{cu}}+\sup_{t\leq 0}e^{-\ _{cu}t-\int_0^t\ (\ _s\omega)ds}\Delta\mathcal{J}_s,$$

where

$$\Delta\mathcal{J}_s=\left\|\lim_{\to+\infty}\int_{-\infty}^{t}\ _{A_{0s}}(t,s)\ \ (\ I-A_s)^{-1}\Pi_s\Big(G(\ _s\omega,\ \ (s))-G(\ _s\omega,\tilde{\ }(s))\Big)ds\right\|.$$

According to Proposition 3.2, for with $-\ <\ <\ _{cu}$, we have

$$\begin{aligned}\Delta\mathcal{J}_s=&\Big\|\lim_{\to+\infty}\lim_{r\to-\infty}\int_r^t\ _{A_{0s}}(t,s)\ \ (\ I-A_s)^{-1}\\&\times\Pi_s\Big(G(\ _s\omega,\ \ (s))-G(\ _s\omega,\tilde{\ }(s))\Big)ds\Big\|\\=&\Big\|\lim_{\to+\infty}\lim_{r\to-\infty}\int_0^{t-r}\ _{A_{0s}}(t,l+r)\ \ (\ I-A_s)^{-1}\\&\times\Pi_s\Big(G(\ _{l+r}\omega,\ \ (l+r))-G(\ _{l+r}\omega,\tilde{\ }(l+r))\Big)dl\Big\|\\\leq&\lim_{r\to-\infty}LC\sup_{l\in[0,t-r]}e^{\ (t-r-l)}e^{\int_{l+r}^t\ (\ \omega)d}\ \|\ \ (l+r)-\tilde{\ }(l+r)\|\\=&LC\sup_{\in(-\infty,t]}e^{\ (t-\ )}e^{\int^t\ (\ \omega)d}\ \|\ (\ )-\tilde{\ }(\ )\|,\end{aligned}$$

implying that

$$\begin{aligned}\sup_{t\leq 0}e^{-\ _{cu}t-\int_0^t\ (\ _s\omega)ds}\Delta\mathcal{J}_s\leq&LC\sup_{t\leq 0}e^{-\ _{cu}t}\sup_{\in(-\infty,t]}e^{\ _{cu}\ +\ (t-\ )}\\&\times e^{-\ _{cu}\ -\int_0\ (\ \omega)d}\ \|\ (\ )-\tilde{\ }(\ )\|\\\leq&LC\sup_{t\leq 0}\sup_{\in(-\infty,t]}e^{(\ _{cu}-\ )\ +(\ -\ _{cu})t}\|\ \ -\tilde{\ }\|_{\mathcal{C}\ _{cu}}\\\leq&LC\ \|\ \ -\tilde{\ }\|_{\mathcal{C}\ _{cu}}.\end{aligned}$$

By combining the above estimates, we have

$$\|\mathcal{J}(\ ,\omega,\ )-\mathcal{J}(\tilde{\ },\omega,\ )\|_{\mathcal{C}\ _{cu}}\leq KL\left(C\ \ -\frac{1}{+\ _{cu}}-\frac{1}{_{cu}-}\right)\|\ \ -\tilde{\ }\|_{\mathcal{C}\ _{cu}}.$$

By (11), we obtain that $\mathcal{J}$ is a uniform contraction with respect to . Furthermore, $\mathcal{J}(\cdot,\omega,\ )$ is well defined from $\mathcal{C}\ _{cu}((-\infty,0];X_0)\times X_{0cu}$ to $\mathcal{C}\ _{cu}((-\infty,0];X_0)$ by setting $\tilde{\ }=0$. From the contraction mapping principle, for any given $\ \in X_{0cu}$, $\mathcal{J}(\cdot,\omega,\ )$ has a unique fixed point

$^{-} \in \mathcal{C}_{\ cu}((-\infty, 0]; X_0)$. That is, for each $t \leq 0$, $\ \in X_{0cu}$, $^{-}(t, \omega, \ )$ is a unique solution to (12) and

$$\mathcal{J}(^{-}, \omega, \ ) = {}^{-}(t, \omega, \ ).$$

Also, for $\forall \ _1, \ _2 \in X_{0cu}$,

$$\begin{aligned}
&\|\mathcal{J}(^{-}, \omega, \ _1) - \mathcal{J}(^{-}, \omega, \ _2)\|_{\mathcal{C}_{\ cu}} \\
&\quad = \|^{-}(t, \omega, \ _1) - {}^{-}(t, \omega, \ _2)\|_{\mathcal{C}_{\ cu}} \\
&\quad \leq K \| \ _1 - \ _2\| + KL\left(C \ - \frac{1}{\ + \ cu} - \frac{1}{\ cu - \ }\right)\|^{-}(t, \omega, \ _1) - {}^{-}(t, \omega, \ _2)\|_{\mathcal{C}_{\ cu}}.
\end{aligned}$$

Thus

$$\|^{-}(t, \omega, \ _1) - {}^{-}(t, \omega, \ _2)\|_{\mathcal{C}_{\ cu}} \leq \frac{K}{1 - KL\left(C \ - \frac{1}{\ + \ cu} - \frac{1}{\ cu - \ }\right)} \| \ _1 - \ _2\|. \tag{17}$$

So $^{-}(t, \omega, \ )$ is Lipschitz continuous from X_{0cu} to $\mathcal{C}_{\ cu}((-\infty, 0]; X_0)$. Since $^{-}(\cdot, \omega, \ )$ can be an ω-wise limit of the iteration of a contraction mapping $\mathcal{J}$ starting at 0 and $\mathcal{J}$ maps a measurable function to a measurable function, $^{-}(\cdot, \omega, \ )$ is measurable. By [6, Lemma III.14], $^{-}$ is measurable with respect to $(\cdot, \omega, \ )$.

Step 3. We will prove that the center-unstable manifold is given by the graph of a Lipschitz continuous map and is invariant.

Let $h^{cu}(\ , \omega) = \Pi_{0s}{}^{-}(0, \omega, \ )$, then by (15),

$$h^{cu}(\ , \omega) = \lim_{\ \to +\infty} \int_{-\infty}^{0} \ _{A_{0s}}(0, s) \ (\ I - A_s)^{-1} \Pi_s G(\ _s\omega, {}^{-}(s, \omega, \ )) \, ds.$$

$\varDelta$

Since $^{-}(0, \omega, \ ) = \ $, we have $h^{cu}(0, \omega) = 0$ and thus h^{cu} is $\mathcal{F}$-measurable. Indeed, by the estimation of $\Delta \mathcal{J}_s$ and (17), it follows that

$$\|hcu$$

Proof. In fact, it follows from Lemma 3.3(i) that

$$\begin{aligned}
u\left(t,\omega,\mathcal{M}^{cu*}(\omega)\right) &= \Xi^{-1}\left({}_t\omega, \quad \left(t,\omega,\Xi\left(\omega,\mathcal{M}^{cu*}(\omega)\right)\right)\right) \\
&= \Xi^{-1}\left({}_t\omega, \quad \left(t,\omega,\mathcal{M}^{cu}(\omega)\right)\right) \subset \Xi^{-1}\left({}_t\omega, \mathcal{M}^{cu}({}_t\omega)\right) \\
&= \mathcal{M}^{cu*}({}_t\omega).
\end{aligned}$$

So $\mathcal{M}^{cu*}(\omega)$ is invariant. Moreover,

$$\begin{aligned}
\mathcal{M}^{cu*}(\omega) &= \Xi^{-1}\left(\omega,\mathcal{M}^{cu}(\omega)\right) \\
&= \left\{u_0 = \Xi^{-1}\left(\omega, \quad + h^{cu}(\ ,\omega)\right) : \quad \in X_{0cu}\right\} \\
&= \left\{u_0 = e^{\ ({}_t\omega)}\left(\quad + h^{cu}(\ ,\omega)\right) : \quad \in X_{0cu}\right\} \\
&= \Big\{u_0 = \quad + e^{\ ({}_t\omega)} h^{cu}\Big(e^{-\ ({}_t\omega)} \quad ,\omega
\end{aligned}$$

$$
\begin{aligned}
{}^{-}(t,\omega,\ ) = {}_{A_{0c}}(t,0)\quad &+ \int_0^t {}_{A_{0c}}(t,s)\,\Pi_c G(\ {}_s\omega, {}^{-}(s,\omega,\ ))\,ds \\
&+ \int_0^t {}_{A_{0u}}(t,s)\,\Pi_u G(\ {}_s\omega, {}^{-}(s,\omega,\ ))\,ds \\
&+ \lim_{\to+\infty} \int_{-\infty}^t {}_{A_{0s}}(t,s)\quad (\ I - A_s)^{-1}\,\Pi_s G(\ {}_s\omega, {}^{-}(s,\omega,\ ))\,ds.
\end{aligned}
$$

By (17), for $\forall\ ,\ {}_0 \in X_{0cu}$ and $i = 1, 2, {}^{-}$ satisfies

$$
\|{}^{-}(t,\omega,\ ) - {}^{-}(t,\omega,\ {}_0)\|_{\mathcal{C}\ {}_{cu+i}} \le \frac{K}{1 - KL\left(C\ {}_{-i}\ - \frac{1}{+\ {}_{cu}+i} - \frac{1}{{}_{cu}-\ +i}\right)} \|\ -\ {}_0\|. \tag{20}
$$

Fix $\ \in (0,\ ^*]$, we first prove that ${}^{-}$ is differentiable from X_{0cu} to $\mathcal{C}\ {}_{cu-}$. For ${}_0 \in X_{0cu}$, $\in \mathcal{C}\ {}_{cu+}$, define $\mathcal{H} : \mathcal{C}\ {}_{cu+} \to \mathcal{C}\ {}_{cu+}$ as follows

$$
\begin{aligned}
\mathcal{H}\ = &\int_0^t {}_{A_{0c}}(t,s)\,\Pi_c D\ G(\ {}_s\omega, {}^{-}(s,\omega,\ {}_0))\quad (s)\,ds \\
&+ \int_0^t {}_{A_{0u}}(t,s)\,\Pi_u D\ G(\ {}_s\omega, {}^{-}(s,\omega,\ {}_0))\quad (s)\,ds \\
&+ \lim_{\to+\infty} \int_{-\infty}^t {}_{A_{0s}}(t,s)\quad (\ I - A_s)^{-1}\,\Pi_s D\ G(\ {}_s\omega, {}^{-}(s,\omega,\ {}_0))\quad (s)\,ds.
\end{aligned}
$$

Using the same arguments as we proved that $\mathcal{J}$ is a contraction and noticing that $\|D\ G(\ {}_s\omega,\ )\| \le Lip\ G = L$, we have $\mathcal{H}$ is a bounded linear operator from $\mathcal{C}\ {}_{cu+}$ to itself with the norm

$$
\|\mathcal{H}\| \le KL\left(C\ {}_{-}\ - \frac{1}{+\ {}_{cu}+} - \frac{1}{{}_{cu}-\ +}\right) < 1.
$$

Thus $id - \mathcal{H}$ is invertible in $\mathcal{C}\ {}_{cu+}$. For $\ ,\ {}_0 \in X_{0cu}$, set

$$
\begin{aligned}
\mathcal{I}(t) = &\int_0^t {}_{A_{0c}}(t,s)\,\Pi_c \overline{\Delta}_G(s)\,ds + \int_0^t {}_{A_{0u}}(t,s)\,\Pi_u \overline{\Delta}_G(s)\,ds \\
&+ \lim_{\to+\infty} \int_{-\infty}^t {}_{A_{0s}}(t,s)\quad (\ I - A_s)^{-1}\,\Pi_s \overline{\Delta}_G(s)\,ds,
\end{aligned}
$$

where

$$\overline{\Delta}_G(s) = G(\ _s\omega, ^-(s,\omega,\ )) - G(\ _s\omega, ^-(s,\omega,\ _0)) \\ - D\ G(\ _s\omega, ^-(s,\omega,\ _0))(^-(s,\omega,\ ) - ^-(s,\omega,\ _0)).$$

We prove $\|\mathcal{I}\|_{\mathcal{C}\ _{cu-}} = o(\|\ -\ _0\|)$ as $\ \to\ _0$, then we have

$$\begin{aligned} ^-(\cdot,\omega,\ ) - ^-(\cdot,\omega,\ _0) - \mathcal{H}(^-(\cdot,\omega,\ ) - ^-(\cdot,\omega,\ _0)) &= \ _{A_0}(t,0)(\ -\ _0) + \mathcal{I} \\ &= \ _{A_0}(t,0)(\ -\ _0) + o(\|\ -\ _0\|), \end{aligned}$$

which implies

$$^-(\cdot,\omega,\ ) - ^-(\cdot,\omega,\ _0) = (id - \mathcal{H})^{-1}\ \ _{A_0}(t,0)(\ -\ _0) + o(\|\ -\ _0\|).$$

We know that $\ _{A_0}(t,0) = T(t)e^{\int_0^t (\ _r\omega)dr}$ is a bounded operator, therefore $^-(\cdot,\omega,\ )$ is differentiable in and $D\ ^-(\cdot,\omega,\ ) \in \mathcal{L}\left(X_{0c}, \mathcal{C}\ _{cu+}\right)$. Next we prove $\|\mathcal{I}\|_{\mathcal{C}\ _{cu+}} = o(\|\ -\ _0\|)$ as $\ \to\ _0$. We write $e^{-(\ _{cu}+\ )t-\int_0^t (\ _r\omega)dr}\mathcal{I}$ as a sum of six terms, i.e., $e^{-(\ _{cu}+\ )t-\int_0^t (\ _r\omega)dr}\mathcal{I} = \sum_{i=1}^6 \mathcal{I}_i$, where for $t < N_1$,

$$\mathcal{I}_1 = e^{-(\ _{cu}+\ )t-\int_0^t (\ _r\omega)dr} \int_{N_1}^{t} \ _{A_{0c}}(t,s)\,\Pi_c\overline{\Delta}_G(s)\,ds,$$

and $\mathcal{I}_1 = 0$ for $t \geq N_1$; For $t < N_1$,

$$\mathcal{I}_2 = e^{-(\ _{cu}+\ )t-\int_0^t (\ _r\omega)dr} \int_{0}^{N_1} \ _{A_{0c}}(t,s)\,\Pi_c\overline{\Delta}_G(s)\,ds,$$

and change N_1 to t if $t \geq N_1$. For $t < N_2$,

$$\mathcal{I}_3 = e^{-(\ _{cu}+\ )t-\int_0^t (\ _r\omega)dr} \int_{N_2}^{t} \ _{A_{0u}}(t,s)\,\Pi_u\overline{\Delta}_G(s)\,ds,$$

and $\mathcal{I}_3 = 0$ for $t \geq N_2$; For $t < N_2$,

$$\mathcal{I}_4 = e^{-(\ _{cu}+\ )t-\int_0^t (\ _r\omega)dr} \int_{0}^{N_2} \ _{A_{0u}}(t,s)\,\Pi_u\overline{\Delta}_G(s)\,ds,$$

and change N_2 to t if $t \geq N_2$. For $t > N_3$,

$$\mathcal{I}_5 = e^{-(\ _{cu}+\ )t-\int_0^t (\ _r\omega)dr} \lim_{\ \to+\infty} \int_{N_3}^{t} \ _{A_{0s}}(t,s)\ \ (\ I - A_s)^{-1}\,\Pi_s\overline{\Delta}_G(s)\,ds,$$

and $\mathcal{I}_5 = 0$ for $t \leq N_3$; For $t > N_3$,

$$\mathcal{I}_6 = e^{-(\ _{cu} +\ )t - \int_0^t (\ _r\omega)dr} \lim_{\to +\infty} \int_{-\infty}^{N_3} \ _{A_{0s}}(t,s) \quad (\ I - A_s)^{-1} \Pi_s \overline{\Delta}_G\ )s)\, ds,$$

and change N_3 to t if $t \leq N_3$. N_1, N_2, N_3 above are negative numbers to be chosen later. By (4) and (17), for $t < N_1$,

$$|\mathcal{I}_1| \leq 2KL \int_t^{N_1} e^{\ |t-s|} e^{-(\ _{cu}+\ )t} e^{(\ _{cu}+2\ )s} ds \, \|^{-}(\cdot,\omega,\ ) - ^{-}(\cdot,\omega,\ _0)\|_{\mathcal{C}\ _{cu}+2}$$

$$\leq 2KL \, \|^{-}(\cdot,\omega,\ ) - ^{-}(\cdot,\omega,\ _0)\|_{\mathcal{C}\ _{cu}+2} \ e^{-(\ +\ _{cu}+\ )t} \int_t^{N_1} e^{(\ +\ _{cu}+2\ )s} ds$$

$$\leq \frac{-2K^2L}{(\ +\ _{cu}+2\ )\left[1 - KL\left(C_{\ -2} - \frac{1}{+\ _{cu}+2} - \frac{1}{_{cu}-\ +2}\right)\right]} e^{\ N_1} \|\ -\ _0\|.$$

For given $\varepsilon > 0$, choose N_1 so negative that

$$\sup_{t\leq 0} |\mathcal{I}_1| \leq \frac{\varepsilon}{6} \|\ -\ _0\|. \tag{21}$$

Fix such N_1, for $t \leq 0$,

$$|\mathcal{I}_2| \leq K \, \|^{-}(\cdot,\omega,\ ) - ^{-}(\cdot,\omega,\ _0)\|_{\mathcal{C}\ _{cu}+} \ e^{-(\ +\ _{cu}+\ )t} \int_{N_1}^{0} e^{(\ +\ _{cu}+\ )s}$$

$$\times \int_0^1 \|D\ G(\ _s\omega,\ ^{-}(s,\omega,\ ) + (1-\ )^{-}(s,\omega,\ _0)) - D\ G(\ _s\omega, ^{-}(s,\omega,\ _0))\| d\ \ ds.$$

Since $G(\omega,\ )$ is C^k, $D\ G(\omega,\ )$ is continuous. For $\varepsilon > 0$, there exists $\ _1 > 0$ such that if $\|\ -\ _0\| <\ _1$,

$$\|D\ G(\ _s\omega,\ ^{-}(s,\omega,\ ) + (1-\ )^{-}(s,\omega,\ _0)) - D\ G(\ _s\omega, ^{-}(s,\omega,\ _0))\|$$

$$\leq \frac{-\varepsilon(\ +\ _{cu}+\ )\left[1 - KL\left(C_{\ -} - \frac{1}{+\ _{cu}+} - \frac{1}{_{cu}-\ +}\right)\right]}{6K^2 e^{(\ +\ _{cu}+\ )N_1}}.$$

Then by (17),

$$\sup_{t\leq 0} |\mathcal{I}_2| \leq \frac{\varepsilon}{6} \|\ -\ _0\| \text{ for } \|\ -\ _0\| <\ _1. \tag{22}$$

Similarly, by choosing N_2 to be sufficiently negative, we have

$$\sup_{t\le 0} |\mathcal{I}_3| \le \frac{\varepsilon}{6} | \quad - \quad _0| . \tag{23}$$

Fix such N_2, there exists $\ _2 > 0$ such that if $\| \quad - \quad _0\| < \ _2$,

$$\sup_{t\le 0} |\mathcal{I}_4| \le \frac{\varepsilon}{6} \| \quad - \quad _0\| . \tag{24}$$

And for $\mathcal{I}_6$, for $t > N_3$,

$$\begin{aligned}
&|\mathcal{I}_6| \\
&= e^{-(\ _{cu}+ \ )t-\int_0^t (\ _r\omega)dr} \left| \lim_{\to+\infty} \lim_{r\to-\infty} \int_0^{N_3-r} \ _{A_{0s}}(t, l+r) \ (\ I - A_s)^{-1} \Pi_s \overline{\Delta}_G (l+r)\, dl \right| \\
&\le 2LC_{\ -} \sup_{h\in(-\infty,N_3]} e^{(\ - \ )(t-h)} e^{-(\ _{cu}+ \ )t} e^{(\ _{cu}+2 \ )h} \|^-(h,\omega, \ ) - ^-(h,\omega, \ _0)\|_{\mathcal{C}_{\ cu+2}} \\
&\le 2LC_{\ -} \sup_{h\in(-\infty,N_3]} e^{(\ -2 \ - \ _{cu})(t-h)} e^{\ h} \|^-(\cdot,\omega, \ ) - ^-(\cdot,\omega, \ _0)\|_{\mathcal{C}_{\ cu+2}} \\
&\le 2LC_{\ -}\, e^{\ N_3} \|^-(\cdot,\omega, \ ) - ^-(\cdot,\omega, \ _0)\|_{\mathcal{C}_{\ cu+2}} \\
&\le \frac{2KLC_{\ -}\, e^{\ N_3}}{1 - KL\left(C_{\ -2 \ } - \frac{1}{\ + \ _{cu+2}} - \frac{1}{\ _{cu-} \ +2}\right)} \| \quad - \quad _0\| .
\end{aligned}$$

Choose N_3 so negative that

$$\sup_{t\le 0} |\mathcal{I}_6| \le \frac{\varepsilon}{6} \| \quad - \quad _0\| . \tag{25}$$

Fix such N_3, for $\mathcal{I}_5$, $t > N_3$,

$$\begin{aligned}
&|\mathcal{I}_5| = e^{-(\ _{cu}+ \ )t-\int_0^t (\ _r\omega)dr} \left| \lim_{\to+\infty} \int_0^{t-N_3} \ _{A_{0s}}(t, l+N_3) \ (\ I - A_s)^{-1} \Pi_s \overline{\Delta}_G (l+N_3)\, dl \right| \\
&\le C_{\ -2} \sup_{h\in[N_3,t]} e^{(\ -2 \ )(t-h)} e^{-(\ _{cu}+ \ )t} e^{(\ _{cu}+2 \ )h} \|^-(h,\omega, \ ) - ^-(h,\omega, \ _0)\|_{\mathcal{C}_{\ cu+2}} \\
&\times \int_0^1 \|D \ G(\ _h\omega, \ ^-(h,\omega, \ ) + (1- \ )^-(h,\omega, \ _0)) - D \ G(\ _h\omega, ^-(h,\omega, \ _0))\| d \\
&\le C_{\ -2} \sup_{h\in[N_3,t]} e^{(\ -4 \ - \ _{cu})(t-h)} e^{\ t} \|^-(\cdot,\omega, \ ) - ^-(\cdot,\omega, \ _0)\|_{\mathcal{C}_{\ cu+2}} \\
&\times \int_0^1 \|D \ G(\ _h\omega, \ ^-(h,\omega, \ ) + (1- \ )^-(h,\omega, \ _0)) - D \ G(\ _h\omega, ^-(h,\omega, \ _0))\| d \quad .
\end{aligned}$$

Similarly, for $\varepsilon > 0$, there exists $_3 > 0$ such that if $\| \ - \ _0\| < \ _3$,

$$\|D\ G(\ _h\omega,\ ^-(h,\omega,\ ) + (1-\ )^-(h,\omega,\ _0)) - D\ G(\ _h\omega,^-(h,\omega,\ _0))\|$$
$$\leq \frac{1 - KL\left(C\ _- \ - \frac{1}{+\ _{cu}+} - \frac{1}{_{cu}-\ +}\right)}{KC\ _{-2}}.$$

Then by (17),

$$\sup_{t\leq 0} |\mathcal{I}_5| \leq \frac{\varepsilon}{6} \| \ - \ _0\| \text{ for } \| \ - \ _0\| < \ _3. \tag{26}$$

By taking $_0 = \min\{\ _1,\ _2,\ _3\}$ and combining (21)-(26), we have

$$\|\mathcal{I}\|_{\mathcal{C}\ _{cu-}} \leq \varepsilon \| \ - \ _0\| \text{ for } \| \ - \ _0\| < \ _0.$$

This implies $\|\mathcal{I}\|_{\mathcal{C}\ _{cu-}} = o(\| \ - \ _0\|)$ as $\ \to \ _0$. Next we prove $D\ ^-(\cdot,\omega,\ )$ is continuous with respect to from X_{0cu} to $\mathcal{L}\left(X_{0cu}, \mathcal{C}\ _{cu}\right)$. By (16), we have

$$\begin{aligned}
D\ ^-(t,\omega,\ ) = &\ _{A_{0c}}(t,0)\,\Pi_{0cu} \\
&+ \int_0^t \ _{A_{0c}}(t,s)\,\Pi_c D\ G(\ _s\omega,^-(s,\omega,\ ))\,D\ ^-(s,\omega,\ )\,ds \\
&+ \int_0^t \ _{A_{0u}}(t,s)\,\Pi_u D\ G(\ _s\omega,^-(s,\omega,\ ))\,D\ ^-(s,\omega,\ )\,ds \\
&+ \lim_{\ \to+\infty} \int_{-\infty}^t \ _{A_{0s}}(t,s)\ (\ I - A_s)^{-1}\,\Pi_s D\ G(\ _s\omega,^-(s,\omega,\ ))\,D\ ^-(s,\omega,\ )\,ds.
\end{aligned}$$

For $\ \in X_{0cu}$, $\ \in \mathcal{C}\ _{cu}$, define the operator $\mathcal{H}' : \mathcal{C}\ _{cu} \to \mathcal{C}\ _{cu}$ as follows

$$\begin{aligned}
\mathcal{H}'\ = &\int_0^t \ _{A_{0c}}(t,s)\,\Pi_c D\ G(\ _s\omega,^-(s,\omega,\ ))\ (s)\,ds \\
&+ \int_0^t \ _{A_{0u}}(t,s)\,\Pi_u D\ G(\ _s\omega,^-(s,\omega,\ ))\ (s)\,ds \\
&+ \lim_{\ \to+\infty} \int_{-\infty}^t \ _{A_{0s}}(t,s)\ (\ I - A_s)^{-1}\,\Pi_s D\ G(\ _s\omega,^-(s,\omega,\ ))\ (s)\,ds.
\end{aligned}$$

Similarly, we have

$$\left\|\mathcal{H}'\right\| \le KL\left(C_{\ -} \ - \frac{1}{+\ \ _{cu}+} - \frac{1}{_{cu}-\ \ +}\right) < 1. \tag{27}$$

For $\ ,\ _0 \in X_{0cu}$,

$$\begin{aligned}
& D\ ^-(t,\omega,\ ) - D\ ^-(t,\omega,\ _0) \\
&= \int_0^t \ _{A_{0c}}(t,s)\,\Pi_c\Big[D\ G(\ _s\omega,\ ^-(s,\omega,\ ))\,D\ ^-(s,\omega,\ ) \\
&- D\ G(\ _s\omega,\ ^-(s,\omega,\ _0))\,D\ ^-(s,\omega,\ _0)\Big]ds \\
&+ \int_0^t \ _{A_{0u}}(t,s)\,\Pi_u\Big[D\ G(\ _s\omega,\ ^-(s,\omega,\ ))\,D\ ^-(s,\omega,\ ) \\
&- D\ G(\ _s\omega,\ ^-(s,\omega,\ _0))\,D\ ^-(s,\omega,\ _0)\Big]ds \\
&+ \lim_{\ \to+\infty}\int_{-\infty}^t \ _{A_{0s}}(t,s)\ \ (\ I - A_s)^{-1}\,\Pi_s\Big[D\ G(\ _s\omega,\ ^-(s,\omega,\ ))\,D\ ^-(s,\omega,\ ) \\
&- D\ G(\ _s\omega,\ ^-(s,\omega,\ _0))\,D\ ^-(s,\omega,\ _0)\Big]ds \\
&= \mathcal{H}'\left(D\ ^-(t,\omega,\ ) - D\ ^-(t,\omega,\ _0)\right) + \mathcal{I}',
\end{aligned} \tag{28}$$

where

$$\begin{aligned}
\mathcal{I}' &= \int_0^t \ _{A_{0c}}(t,s)\,\Pi_c\Big[D\ G(\ _s\omega,\ ^-(s,\omega,\ )) - D\ G(\ _s\omega,\ ^-(s,\omega,\ _0))\Big]D\ ^-(s,\omega,\ _0)\,ds \\
&+ \int_0^t \ _{A_{0u}}(t,s)\,\Pi_u\Big[D\ G(\ _s\omega,\ ^-(s,\omega,\ )) - D\ G(\ _s\omega,\ ^-(s,\omega,\ _0))\Big]D\ ^-(s,\omega,\ _0)\,ds \\
&+ \lim_{\ \to+\infty}\int_{-\infty}^t \ _{A_{0s}}(t,s)\ \ (\ I - A_s)^{-1}\,\Pi_s\Big[D\ G(\ _s\omega,\ ^-(s,\omega,\ )) \\
&- D\ G(\ _s\omega,\ ^-(s,\omega,\ _0))\Big]D\ ^-(s,\omega,\ _0)\,ds.
\end{aligned}$$

By (27), $id - \mathcal{H}'$ has a bounded inverse in $\mathcal{L}\left(X_{0cu}, \mathcal{C}\ _{cu}\right)$. From (28), we have

$$D\ ^-(t,\omega,\ ) - D\ ^-(t,\omega,\ _0) = \left(id - \mathcal{H}'\right)^{-1}\mathcal{I}'.$$

Then by (28),

$$\|D\ ^{-}(t,\omega,\ ) - D\ ^{-}(t,\omega,\ _0)\|_{\mathcal{L}(X_{0cu},\mathcal{C}\ _{cu})} \leq \frac{\|\mathcal{I}'\|_{\mathcal{L}(X_{0cu},\mathcal{C}\ _{cu})}}{1 - KL\left(C\ \ \frac{1}{\ \ _{cu}}\ \ \frac{1}{\ _{cu}}\right)}. \tag{29}$$

Using the same procedure as for $\mathcal{I}$, we have $\|\mathcal{I}'\|_{\mathcal{L}(X_{0cu},\mathcal{C}\ _{cu})} = o(1)$ as $\ \to\ _0$. Then by (29), $D\ ^{-}(\cdot,\omega,\ )$ is continuous with respect to . So we proved that $^{-}(t,\omega,\ )$ is C^1.

Step 2. We prove that $^{-}(t,\omega,\ )$ is C^k.

Make the inductive assumption that $^{-}(\cdot,\omega,\ )$ is *assum* Δ .ΠΔ ΔΦ Ξ ΔΠ ΞΔ . *TD*

4. Smooth center-stable foliations

In this section, we study the existence of invariant foliations of (1). We first introduce the concept of invariant foliations, which relies on the decomposition of the space X as well. Fix $\omega \in$, let $\{\mathcal{W}(\ ,\omega) : \ \in X_0\}$ be a family of submanifolds of X_0 parametrized by $\ \in X_0$.

Definition 4.1. A family of submanifolds $\{\mathcal{W}^{cs}(\ ,\omega) : \ \in X_0\}$ is called a C^k center-stable foliation for X_0 if the following conditions are satisfied:

(i) $\ \in \mathcal{W}^{cs}(\ ,\omega)$ for each $\ \in X_0$.
(ii) $\mathcal{W}^{cs}(\ ,\omega)$ and $\mathcal{W}^{cs}(\tilde{\ },\omega)$ are either disjoint or identical for each $\ ,\tilde{\ } \in X_0$.
(iii) $\mathcal{W}^{cs}(0,\omega)$ is tangent to X_{0cs} at the origin, every leaf $\mathcal{W}^{cs}(\ ,\omega)$ is given by the graph of a C^k function $l^{cs}(\cdot,\omega, \ ) : X_{0cs} \to X_{0u}$, i.e.,

$$\mathcal{W}^{cs} = \{ \ + l^{cs}(\ ,\omega, \ ) : \ \in X_{0cs}\}.$$

Accordingly, a C^k center-unstable (resp. stable, unstable) foliation $\mathcal{W}^{cu}(\omega)$ (resp. $\mathcal{W}^{s}(\omega)$, $\mathcal{W}^{u}(\omega)$) is given by the graph of a Lipschitz map $h^i(\cdot,\omega) : X_{0i} \to X_{0j}$, $i = cu$ (resp. s,u) and $j = s$ (resp. $j = cu, cs$).

Define

$$\mathcal{W}^{cs}(\ ,\omega) = \{\tilde{\ } \in X_0 : \ (t,\omega,\tilde{\ }) - \ (t,\omega, \ ) \in \mathcal{C}_{\ cs}([0,+\infty); X_0)\},$$

where $\ (t,\omega, \ )$ is the solution of (4) with initial data $\ (0) = \ $. We below prove that $\mathcal{W}^{cs}(\ ,\omega)$ is invariant and it is given by the graph of a Lipschitz function.

Theorem 4.2. *Assume that* $- \ < \ < \ _{cs}$ *and* $\ < \ _{cs} < \min\{ \ , \ \}$ *with*

$$KL\left(C \ + \frac{1}{\ _{cs} - \ } + \frac{1}{\ - \ _{cs}}\right) < 1,$$

and assume for $\ ^* > 0$ *such that for* $0 \le \ \le \ ^*$, $\ < \ _{cs} - 2 \ < \ _{cs} - \ < \min\{ \ , \ \}$, $- \ < \ < \ _{cs} - 2 \ < \ _{cs} - \ $ *with*

$$KL\left(C_{\ +i \ } + \frac{1}{\ _{cs} - i \ - \ } + \frac{1}{\ - i \ _{cs} + i \ }\right) < \frac{1}{6},\ i = 1,2, \tag{30}$$

there exists a Lipschitz invariant center-stable foliation for (4) *whose leaf is given by*

$$\mathcal{W}^{cs}(\ ,\omega) = \{ \ + l^{cs}(\ ,\omega, \ ) : \ \in X_{0cs}\},$$

where $\ \in X_0$ *and* $l^{cs}(\cdot,\omega, \ ) : X_{0cs} \to X_{0u}$ *is measurable with respect to* $(\ ,\omega, \ )$ *and Lipschitz continuous in* $\ $ *with*

$$Lip\, l^{cs}(\cdot,\omega, \ ) \le K_s = \frac{K^2 L}{(\ - \ _{cs})\, 1 - KL\left(C \ + \frac{1}{\ _{cs} - \ } + \frac{1}{\ - \ _{cs}}\right)}$$

Moreover, if $K_u K_s < 1$ (K_u is defined in Theorem 3.4), then $\mathcal{W}^{cs}(\ ,\omega) \cap \mathcal{M}^{cu}(\omega)$ contains a unique point.

Proof. We proceed it in five steps.
Step 1. We prove that $\tilde{} \in \mathcal{W}^{cs}(\ ,\omega)$ if and only if there exists a function $(\cdot) \in \mathcal{C}_{\ cs}([0,+\infty); X_0)$ with $(0) = \tilde{} -$ which satisfies

$$
\begin{aligned}
(t) = {}_{A_{0c}}(t,0) \ + \Pi_{0s}(S \diamond \Delta G\ (\ ,\ ))(t) \\
+ \int_0^t {}_{A_{0c}}(t,s)\, \Pi_c \Delta G(\ _s\omega,\ (s,\omega,\ ),\ (s))\, ds \\
+ \int_{+\infty}^t {}_{A_{0u}}(t,s)\, \Pi_u \Delta G(\ _s\omega,\ (s,\omega,\ ),\ (s))\, ds
\end{aligned}
\tag{31}
$$

where $= \Pi_{0cs}(\tilde{} - \ )$ and

$$
\begin{aligned}
\Delta G(\ _s\omega,\ (s,\omega,\ ),\ (s)) = G(\ _s\omega,\ (s,\omega,\ ) + \ (s)) - G(\ _s\omega,\ (s,\omega,\ )), \\
(S \diamond \Delta G\ (\ ,\ ))(t) = (S \diamond G\ (\ +\ ))(t) - (S \diamond G\ (\ ))(t).
\end{aligned}
$$

To this aim, we let $\tilde{} \in \mathcal{W}^{cs}(\ ,\omega)$ and assume that $\in \mathcal{C}_{\ cs}([0,+\infty); X_0)$ is a solution of (4). Let $(t) = \ (t,\omega,\tilde{}) - \ (t,\omega,\ )$. By the variation of constants formula, one further gets

$$
\begin{aligned}
\Pi_{0cs}\ (t,\omega,\ ) = {}_{A_{0c}}(t,0)\,\Pi_{0cs} \ + \Pi_{0s}(S \diamond G\ (\ ))(t) \\
+ \int_0^t {}_{A_{0c}}(t,s)\, \Pi_c G(\ _s\omega,\ (s,\omega,\ ))\, ds,
\end{aligned}
$$

and for all $t, l \in \mathbb{R}$ with $t < l$,

$$
\Pi_{0u}\ (t) = {}_{A_{0u}}(t,l)\,\Pi_{0u}\ (l) + \int_l^t {}_{A_{0s}}(t,s)\, \Pi_u G(\ _s\omega,\ (s))\, ds.
$$

Then we get that

$$
\begin{aligned}
\Pi_{0cs}\ (t) = {}_{A_{0c}}(t,0) \ + \Pi_{0s}(S \diamond \Delta G\ (\ ,\ ))(t) \\
+ \int_0^t {}_{A_{0c}}(t,s)\, \Pi_c \Delta G(\ _s\omega,\ (s,\omega,\ ),\ (s))\, ds.
\end{aligned}
\tag{32}
$$

$$
\Pi_{0u}\ (t) = {}_{A_{0u}}(t,l)\,\Pi_{0u}\ (l) + \int_l^t {}_{A_{0s}}(t,s)\, \Pi_u \Delta G(\ _s\omega,\ (s,\omega,\ ),\ (s))\, ds.
$$

According to (7), for $l > 0$,

$$\begin{aligned}\left\| \ _{A_{0u}}(t,l)\,\Pi_{0u} \quad (l)\right\| &\le K e^{\ (t-l)+\int_l^t (\ _r\omega)dr} \| \quad (l)\| \\ &\le K e^{\ (t-l)- \ _{cs} l+\int_0^t (\ _r\omega)dr} e^{\ _{cs} l-\int_0^l (\ _r\omega)dr} \| \quad (l)\| \\ &\le K e^{(\ _{cs}- \ )l- \ t+\int_0^t (\ _r\omega)dr} \| \quad \|_{\mathcal{C}\ _{cs}}.\end{aligned}$$

Let $l \to +\infty$, since $\ _{cs} - \quad < 0$, we have

$$\Pi_{0u} \quad (t) = \int_{+\infty}^{t} \ _{A_{0u}}(t,s)\,\Pi_u \Delta G(\ _s\omega, \quad (s,\omega, \quad), \quad (s))\,ds. \tag{33}$$

By summing up (32) and (33), we get (31). The converse can be obtained by a straight computation.

Step 2. We claim that (31) has a unique solution in $\mathcal{C}\ _{cs}([0,+\infty); X_0)$ with initial condition $\Pi_{0cs} \quad (0,\omega, \ , \quad) = \quad$ for all $\quad \in X_{0cs}$.

To show this claim, we denote $\mathcal{Z}(\quad, \quad)$ the right side of (31), i.e.

$$\begin{aligned}\mathcal{Z}(\quad,\quad) = &\ _{A_{0c}}(t,0) \quad + \Pi_{0s}(S \diamond \Delta G \ (\quad, \quad))(t) \\ &+ \int_0^t \ _{A_{0c}}(t,s)\,\Pi_c \Delta G(\ _s\omega, \quad (s,\omega, \quad), \quad (s))\,ds \\ &+ \int_{+\infty}^{t} \ _{A_{0u}}(t,s)\,\Pi_u \Delta G(\ _s\omega, \quad (s,\omega, \quad), \quad (s))\,ds\end{aligned} \tag{34}$$

It needs to show that $\mathcal{Z}$ maps from $\mathcal{C}\ _{cs}([0,+\infty); X_0) \times X_{0cs}$ to $\mathcal{C}\ _{cs}([0,+\infty); X_0)$ and it is a uniform contraction. For $\quad, \tilde{\ } \in \mathcal{C}\ _{cs}([0,+\infty); X_0)$, $- \quad < \quad < \ _{cs}$,

$$\begin{aligned}&\left\|\mathcal{Z}(\quad,\quad) - \mathcal{Z}\left(\tilde{\ },\quad\right)\right\|_{\mathcal{C}\ _{cs}} = \sup_{t\ge 0} e^{-\ _{cs}t-\int_0^t (\ _s\omega)ds}\left\|\mathcal{Z}(\quad,\quad) - \mathcal{Z}\left(\tilde{\ },\quad\right)\right\| \\ &\le LC \quad \sup_{s\in[0,t]} e^{(\ -\ _{cs})(t-s)} \left\| \quad - \tilde{\ } \right\|_{\mathcal{C}\ _{cs}} + KL\sup_{t\ge 0}\int_0^t e^{(\ _{cs}- \ )(s-t)- \ _{cs}s-\int_0^s (\ _r\omega)dr} \left\| \quad - \tilde{\ }\right\| ds \\ &+ KL\sup_{t\ge 0}\int_t^{+\infty} e^{(\ _{cs}- \ )(s-t)- \ _{cs}s-\int_0^s (\ _r\omega)dr}\left\| \quad - \tilde{\ }\right\| ds \\ &\le KL\left(C \quad + \frac{1}{\ _{cs} - \ } + \frac{1}{\ - \ _{cs}}\right)\left\| \quad - \tilde{\ }\right\|_{\mathcal{C}\ _{cs}},\end{aligned}$$

which implies that $\mathcal{Z}$ is a uniform contraction with respect to $\quad$. Furthermore, it is straightforward that $\mathcal{Z}(\cdot, \quad)$ is well defined from $\mathcal{C}\ _{cs}([0,+\infty); X_0) \times X_{0cs}$ to $\mathcal{C}\ _{cs}([0,+\infty); X_0)$ by setting

$\tilde{\ } = 0$. From the contraction mapping principle, for any given $\ \in X_{0cs}$, $\mathcal{Z}(\cdot,\ )$ has a unique fixed point $\bar{\ } \in \mathcal{C}_{\ cs}([0,+\infty); X_0)$. That is, for each $t \geq 0$, $\bar{\ }(t,\omega,\ ,\ )$ is a unique solution to (34) and

$$\mathcal{Z}(\bar{\ },\ ) = \bar{\ }(t,\omega,\ ,\ ).$$

Also, for $\forall\ _1,\ _2 \in X_{0cs}$,

$$\begin{aligned}
&\left\|\mathcal{Z}(\bar{\ },\ _1) - \mathcal{Z}(\bar{\ },\ _2)\right\|_{\mathcal{C}_{\ cs}} \\
&\quad = \left\|\bar{\ }(t,\omega,\ _1,\ ) - \bar{\ }(t,\omega,\ _2,\ )\right\|_{\mathcal{C}_{\ cs}} \\
&\quad \leq K\|\ _1 - \ _2\| + KL\left(C\ \ + \frac{1}{\ _{cs} - \ } + \frac{1}{\ - \ _{cs}}\right)\left\|\bar{\ }(t,\omega,\ _1,\ ) - \bar{\ }(t,\omega,\ _2,\ )\right\|_{\mathcal{C}_{\ cs}}.
\end{aligned}$$

Thus

$$\left\|\bar{\ }(t,\omega,\ _1,\ ) - \bar{\ }(t,\omega,\ _2,\ )\right\|_{\mathcal{C}_{\ cs}} \leq \frac{K}{1 - KL\left(C\ \ + \frac{1}{\ _{cs} - \ } + \frac{1}{\ - \ _{cs}}\right)}\|\ _1 - \ _2\|. \tag{35}$$

So $\bar{\ }(t,\omega,\cdot,\ )$ is Lipschitz continuous from X_{0cu} to $\mathcal{C}_{\ cs}([0,+\infty); X_0)$.

Step 3. We prove that $\bar{\ }(\cdot,\omega,\ ,\ )$ is continuous with respect to $(\ ,\ )$. By (30), there is a fixed point in $\mathcal{C}_{\ cs-\ }$ and (35) holds with $\ _{cs}$ replaced by $\ _{cs} - \ $. Then we have

$$\left\|\bar{\ }(t,\omega,\ _1,\ ) - \bar{\ }(t,\omega,\ _2,\ )\right\|_{\mathcal{C}_{\ cs-\ }} \leq \frac{6}{5}K\|\ _1 - \ _2\| \text{ and } \|\bar{\ }(t,\omega,\ _1,\ )\| \leq \frac{6}{5}K\|\ _1\|.$$

For $(\ _1,\ _1), (\ _2,\ _2) \in X_{0cs} \times X_0$, there exists $\bar{\ }(\cdot,\omega,\ _1,\ _1), \bar{\ }(\cdot,\omega,\ _2,\ _2) \in \mathcal{C}_{\ cs-\ } \subset \mathcal{C}_{\ cs}$. For $t \geq 0$,

$$\begin{aligned}
\bar{\ }(\cdot,\omega,\ _1,\ _1) - \bar{\ }(\cdot,\omega,\ _2,\ _2) = {}& \ _{A_{0c}}(t,0)(\ _1 - \ _2) + \int_0^t \ _{A_{0c}}(t,s)\Pi_c\Delta\mathcal{G}(s,\omega)\,ds \\
&+ \int_{+\infty}^t \ _{A_{0u}}(t,s)\Pi_u\Delta\mathcal{G}(s,\omega)\,ds + \lim_{\ \to+\infty}\int_0^t \ _{A_{0s}}(t,s)\ \ (\ I - A_s)^{-1}\Pi_s\Delta\mathcal{G}(s,\omega)\,ds
\end{aligned}$$

where

$$\Delta\mathcal{G}(s,\omega) = \Delta G\left(\ _s\omega,\ \ (s,\omega,\ _1), \bar{\ }(s,\omega,\ _1,\ _1)\right) - \Delta G\left(\ _s\omega,\ \ (s,\omega,\ _2), \bar{\ }(s,\omega,\ _2,\ _2)\right).$$

We prove that $\|\bar{\ }(\cdot,\omega,\ _1,\ _1) - \bar{\ }(\cdot,\omega,\ _2,\ _2)\|_{\mathcal{C}_{\ cs}} = o(1)$ as $(\ _1,\ _1) \to (\ _2,\ _2)$. Set for $t > N_1$,

$$\mathcal{E}_1 = e^{-\ _{cs}t - \int_0^t \ (\ _r\omega)dr}\int_{N_1}^t \ _{A_{0c}}(t,s)\Pi_c\Delta\mathcal{G}(s,\omega)\,ds,$$

and $\mathcal{E}_1 = 0$ for $t \le N_1$; For $t > N_1$,

$$\mathcal{E}_2 = e^{-\ _{cs}t - \int_0^t (\ _r\omega)dr} \int_0^{N_1} \ _{A_{0c}}(t,s)\,\Pi_c \Delta\mathcal{G}(s,\omega)\,ds,$$

and change N_1 to t if $t \le N_1$. For $t > N_2$,

$$\mathcal{E}_3 = -e^{-\ _{cs}t - \int_0^t (\ _r\omega)dr} \int_{N_2}^{t} \ _{A_{0u}}(t,s)\,\Pi_c \Delta\mathcal{G}(s,\omega)\,ds,$$

and $\mathcal{E}_3 = 0$ for $t \le N_2$; For $t > N_2$,

$$\mathcal{E}_4 = -e^{-\ _{cs}t - \int_0^t (\ _r\omega)dr} \int_{-\infty}^{N_2} \ _{A_{0u}}(t,s)\,\Pi_c \Delta\mathcal{G}(s,\omega)\,ds,$$

and change N_2 to t if $t \le N_2$. For $t > N_3$,

$$\mathcal{E}_5 = e^{-\ _{cs}t - \int_0^t (\ _r\omega)dr} \lim_{\ \to +\infty} \int_{N_3}^{t} \ _{A_{0s}}(t,s)\ \ (\ I - A_s)^{-1}\,\Pi_s \Delta\mathcal{G}(s,\omega)\,ds,$$

and $\mathcal{E}_5 = 0$ for $t \le N_3$; For $t > N_3$,

$$\mathcal{E}_6 = e^{-\ _{cs}t - \int_0^t (\ _r\omega)dr} \lim_{\ \to +\infty} \int_{0}^{N_3} \ _{A_{0s}}(t,s)\ \ (\ I - A_s)^{-1}\,\Pi_s \Delta\mathcal{G}(s,\omega)\,ds,$$

and change N_3 to t if $t \le N_3$. N_1, N_2, N_3 above are large positive numbers to be chosen later. For $t > N_1$, and choose $\|\ _1 - \ _2\| \le 1$,

$$\begin{aligned}
|\mathcal{E}_1| \le & KL \int_{N_1}^{t} e^{(\ _{cs} - \ - \ )(s-t)} e^{-\ t} ds [\|\overline{\ }(\cdot,\omega,\ _1,\ _1)\|_{\mathcal{C}_{\ cs-}} + \|\overline{\ }(\cdot,\omega,\ _2,\ _2)\|_{\mathcal{C}_{\ cs-}}] \\
& \le \frac{2K^2 L e^{-\ N_1}}{1 - KL\left(C_{\ +} + \frac{1}{\ _{cs} - \ -} + \frac{1}{-\ _{cs}+}\right)} (2\|\ _1\| + 1) \\
& \le \frac{12}{5} KL e^{-\ N_1} (2\|\ _1\| + 1).
\end{aligned}$$

For given $\varepsilon > 0$, choose N_1 so positive that

$$\sup_{t\ge 0} |\mathcal{E}_1| \le \frac{\varepsilon}{14}. \tag{36}$$

For such N_1, for $t > N_1$,

$$|\mathcal{E}_2| \le e^{- \ cst} \int_0^{N_1} K e^{\ (t-s)-\int_0^s \ (\ _r\omega)dr} \Big[|G\big(\ _s\omega, \ (s,\omega, \ _1) + \overline{\ }(s,\omega, \ _1, \ _1)\big)$$

$$- G\big(\ _s\omega, \ (s,\omega, \ _2) + \overline{\ }(s,\omega, \ _2, \ _2)\big)$$

$$|\mathcal{E}_5| = e^{-\ \ _{cs}t-\int_0^t \ (\ _r\omega)dr} \left| \lim_{\ \to+\infty} \int_0^{t-N_3} \ _{A_{0s}}(t, l+N_3)\ (\ I - A_s)^{-1}\Pi_s \Delta\mathcal{G}(l+N_3)\, dl \right|$$

$$\leq LC_{\ +} \sup_{h\in[N_3,t]} e^{(\ +\ )(t-h)} e^{-\ \ _{cs}t} e^{(\ _{cs}-2\ )h} [\|^{-}(h,\omega,\ _1,\ _1)\|_{\mathcal{C}_{\ _{cs}-2}}$$

$$+ \|^{-}(h,\omega,\ _2,\ _2)\|_{\mathcal{C}_{\ _{cs}-2}}\ ]$$

$$\leq LC_{\ +} \sup_{h\in[N_3,t]} e^{(\ +2\ -\ _{cs})(t-h)} e^{-\ t} [\|^{-}(\cdot,\omega,\ _1,\ _1)\|_{\mathcal{C}_{\ _{cs}-2}} + \|^{-}(\cdot,\omega,\ _2,\ _2)\|_{\mathcal{C}_{\ _{cs}-2}}\ ]$$

$$\leq \frac{2KLC_{\ +}\ e^{-\ N_3}}{1-KL\left(C_{\ +2}\ + \frac{1}{\ _{cs}-2\ -} + \frac{1}{-\ _{cs}+2}\right)} (2\|\ _1\|+1)$$

$$\leq \frac{12}{5} LC_{\ +}\ e^{-\ N_3} (2\|\ _1\|+1).$$

For given $\varepsilon > 0$, choose N_3 so positive that

$$\sup_{t\geq 0} |\mathcal{E}_5| \leq \frac{\varepsilon}{14}. \tag{40}$$

Fix such N_3, for $t > N_3$,

$$|\mathcal{E}_6| \leq LC_{\ +} \sup_{s\in[0,N_3]} e^{(\ +\ -\ _{cs})(t-s)} \Big[|G\big(\ _s\omega,\ \ (s,\omega,\ _1) + ^{-}(s,\omega,\ _1,\ _1)\big)$$

$$- G\big(\ _s\omega,\ \ (s,\omega,\ _2) + ^{-}(s,\omega,\ _2,\ _2)\big)| + |G(\ _s\omega,\ \ (s,\omega,\ _1)) - G(\ _s\omega,\ \ (s,\omega,\ _2))| \Big]$$

$$\leq KLC_{\ +} \Big[\|^{-}(\cdot,\omega,\ _1,\ _1) - ^{-}(\cdot,\omega,\ _2,\ _2)\|_{\mathcal{C}_{\ _{cs}}} + 2\|\ \ (\cdot,\omega,\ _1) - \ \ (\cdot,\omega,\ _2)\|_{\mathcal{C}_{\ _{cs}}} \Big].$$

There exists $\ _3 > 0$ such that if $\|\ _1 - \ _2\| < \ _3$, we have

$$\sup_{0\leq t\leq N_3} \|\ \ (t,\omega,\ _1) - \ \ (t,\omega,\ _2)\| \leq \frac{\varepsilon}{14} \cdot \frac{1}{2KLC_{\ +}}.$$

Then we get that if $t > N_3$ and $\|\ _1 - \ _2\| < \ _3$, we have

$$|\mathcal{E}_6| \leq \frac{1}{6} \|^{-}(\cdot,\omega,\ _1,\ _1) - ^{-}(\cdot,\omega,\ _2,\ _2)\|_{\mathcal{C}_{\ _{cs}}} + \frac{\varepsilon}{14}. \tag{41}$$

If $t \leq N_3$, choose the same $\ _3$ and if $\|\ _1 - \ _2\| < \ _3$, we have

$$\sup_{0\leq s\leq t} \|\ \ (s,\omega,\ _1) - \ \ (s,\omega,\ _2)\| \leq \sup_{0\leq t\leq N_3} \|\ \ (t,\omega,\ _1) - \ \ (t,\omega,\ _2)\| \leq \frac{\varepsilon}{14} \cdot \frac{1}{2KLC_{\ +}},$$

still yielding (41). And there exists $\ _0 > 0$, such that if $\|\ _1 - \ _2\| < \ _0$,

$$\sup_{t\geq 0} e^{-\ \ _{cs}t-\int_0^t \ (\ _r\omega)dr} \|\ \ _{A_{0c}}(t,0)(\ _1 - \ _2)\| \leq \sup_{t\geq 0} e^{(\ -\ _{cs})t} \|\ _1 - \ _2\| \leq \frac{\varepsilon}{14}. \tag{42}$$

Take $\ =\min\{\ _0,\ _1,\ _2,\ _3,1\}$, from (36)-(42), we have if $\|\ _1-\ _2\|<\ $, $\|\ _1-\ _2\|<\ $, then

$$\|\overline{\ }(\cdot,\omega,\ _1,\ _1)-\overline{\ }(\cdot,\omega,\ _2,\ _2)\|_{\mathcal{C}\ _{cs}}\le\frac{1}{2}\|\overline{\ }(\cdot,\omega,\ _1,\ _1)-\overline{\ }(\cdot,\omega,\ _2,\ _2)\|_{\mathcal{C}\ _{cs}}+\frac{\varepsilon}{2},$$

which means

$$\|\overline{\ }(\cdot,\omega,\ _1,\ _1)-\overline{\ }(\cdot,\omega,\ _2,\ _2)\|_{\mathcal{C}\ _{cs}}\le\varepsilon.$$

Therefore $\overline{\ }(\cdot,\omega,\ ,\ )$ is continuous with respect to $(\ ,\ )$. Since $\overline{\ }(\cdot,\omega,\ ,\ )$ can be an ω-wise limit of the iteration of a contraction mapping $\mathcal{Z}$ starting at 0 and $\mathcal{Z}$ maps a measurable function to a measurable function, $\overline{\ }(\cdot,\omega,\ ,\ )$ is measurable. By [6, Lemma III.14], $\overline{\ }$ is measurable with respect to $(\omega,\ ,\ )$.

which implies

$$= l^{cs}\,(\ , \omega, \ ) \text{ and } \ = h^{cu}\,(\ , \omega)\,. \tag{44}$$

By the Lipschitz continuity of

Therefore, $\mathcal{W}^{cs*}(\ ,\omega)$ is a Lipschitz center invariant foliation given by the graph of a Lipschitz continuous function $l^{cs*}(\ ,\omega,\ ) = e^{(\ _t\omega)} l^{cs}\left(e^{-(\ _t\omega)}\ ,\omega,\ \right)$ over X_{0cs}. □

By similar arguments, we could get the existence of a center-unstable foliation $\mathcal{W}^{cu*}(\ ,\omega)$ of (1). Then there is a center foliation $\mathcal{W}^{c}(\ ,\omega)$, which is obtained by an intersection of the other two: $\mathcal{W}^{c} = \mathcal{W}^{cs*} \cap \mathcal{W}^{cu*}$. Next we discuss the smoothness of $\mathcal{W}^{cs*}(\ ,\omega)$. We have the following result.

Theorem 4.4. *Assume that F is C^k in u. For $- \ < \ < k\ _{cs}$ and $\ < k\ _{cs} < \min\{\ ,\ \}$ with*

$$KL\left(C_{\ +(i-1)\ _{cs}} + \frac{1}{i\ _{cs} - \ } + \frac{1}{\ - i\ _{cs}}\right) < 1, \text{ for all } 1 \le i \le k \tag{45}$$

then $l^{cs}(\ ,\omega,\ )$ is C^k in for $\ \in X_0$.

Proof. Step 1. We prove that the fixed point $\overline{\ }(t,\omega,\ ,\ )$ of the Lyapunov-Perron operator $\mathcal{Z}(\cdot,\ )$ defined in Theorem 4.2 is C^1. For notational simplify, we denote $\mathcal{C}_{\ _{cs}}([0,+\infty);X_0)$ by $\mathcal{C}_{\ _{cs}}$ from now on. Notice that for $\ < \ '_{cs} < \ ''_{cs} < \min\{\ ,\ \}$, $\mathcal{C}_{\ '_{cs}} \subset \mathcal{C}_{\ ''_{cs}}$, thus a fixed point in $\mathcal{C}_{\ '_{cs}}$ must be in $\mathcal{C}_{\ ''_{cs}}$. So $\overline{\ }(\cdot,\omega,\ ,\ )$ is independent of $\ _{cs}$ satisfying (45).

Clearly, there exists $\ ^* > 0$ such that for $0 < \ \le \ ^*$, we have $- \ < \ < i\ _{cs} - 2\ < i\ _{cs} - \ $, $\ < i\ _{cs} - 2\ < i\ _{cs} - \ < \min\{\ ,\ \}$ and

$$KL\left(C_{\ +j\ } + \frac{1}{i\ _{cs} - j\ - \ } + \frac{1}{\ - i\ _{cs} + j\ }\right) < 1, \text{ for all } 1 \le i \le k,\ j = 1,2. \tag{46}$$

Then $\mathcal{Z}(\cdot,\ )$ is a uniform contraction in $\mathcal{C}_{\ _{cs}-\ } \subset \mathcal{C}_{\ _{cs}}$. By (34), for $\ \in X_{0cs}$,

$$\begin{aligned}\overline{\ }(t,\omega,\ ,\ ) = {}& _{A_{0c}}(t,0)\ + \Pi_{0s}\left(S \diamond \Delta G\ (\ ,\overline{\ })\right)(t) \\ &+ \int_0^t \ _{A_{0c}}(t,s)\,\Pi_c \Delta G\left(\ _s\omega,\ (s),\overline{\ }(s,\omega,\ ,\ )\right)ds \\ &- \int_t^{+\infty} \ _{A_{0u}}(t,s)\,\Pi_u \Delta G\left(\ _s\omega,\ (s),\overline{\ }(s,\omega,\ ,\ )\right)ds.\end{aligned}$$

By (35), for $\ ,\ _0 \in X_{0cs}$ and $i = 1,2$, $\overline{\ }$ satisfies

$$\left\|\overline{\ }(t,\omega,\ ,\ ) - \overline{\ }(t,\omega,\ _0,\ )\right\|_{\mathcal{C}_{\ _{cs}-i\ }} \le \frac{K\|\ _1 - \ _2\|}{1 - KL\left(C_{\ +i\ } + \frac{1}{\ _{cs}-i\ -\ } + \frac{1}{\ -\ _{cs}+i\ }\right)}.$$

Fix $\ \in (0,\ ^*]$, we first prove that

$$\mathcal{T} \quad = \int_0^t \ _{A_{0c}}(t,s)\,\Pi_c D \ \ \Delta G\big(\ _s\omega, \ \ (s), \overline{\ \ }(s,\omega, \ _0, \ )\big) \ \ (s)\, ds$$

$$- \int_t^{+\infty} \ _{A_{0u}}(t,s)\,\Pi_u D \ \ \Delta G\big(\ _s\omega, \ \ (s), \overline{\ \ }(s,\omega, \ _0, \ )\big) \ \ (s)\, ds$$

$$+ \lim_{\ \to+\infty} \int_0^t \ _{A_{0s}}(t,s) \ \ (\ I - A_s)^{-1}$$

$$\times \Pi_s D \ \ \Delta G\big(\ _s\omega, \ \ (s), \overline{\ \ }(s,\omega, \ _0, \ )\big) \ \ (s)\, ds.$$

Using the same arguments as we proved that $\mathcal{Z}$ is a contraction and $\left\| D \ \ \Delta G(\ _s\omega, \ , \ ) \right\| \le Lip\ G = L$, we have $\mathcal{T}$ is a bounded linear operator from $\mathcal{C}\ _{cs-}$ to itself with the norm

$$\|\mathcal{T}\| \le KL\left(C \ _+ \ + \frac{1}{\ _{cs} - \ \ - \ } + \frac{1}{ - \ \ _{cs} + \ } \right) < 1.$$

Thus $id - \mathcal{T}$ is invertible in $\mathcal{C}\ _{cs-}$. For $\ , \ _0 \in X_{0cs}$, set

$$\mathcal{S}(t) = \int_0^t \ _{A_{0c}}(t,s)\,\Pi_c \overline{\Delta}_{G(\ )}(s)\, ds - \int_t^{+\infty} \ _{A_{0u}}(t,s)\,\Pi_u \overline{\Delta}_{G(\ )}(s)\, ds$$

$$+ \lim_{\ \to+\infty} \int_0^t \ _{A_{0s}}(t,s) \ \ (\ I - A_s)^{-1} \Pi_s \overline{\Delta}_{G(\ )}(s)\, ds$$

where

$$\begin{aligned}
\overline{\Delta}_{G(\ )}(s) &= \Delta G\big(\ _s\omega, \ \ (s), \overline{\ \ }(s,\omega, \ , \ )\big) - \Delta G\big(\ _s\omega, \ \ (s), \overline{\ \ }(s,\omega, \ _0, \ )\big) \\
&\quad - D \ \ \Delta G\big(\ _s\omega, \ \ (s), \overline{\ \ }(s,\omega, \ _0, \ )\big)\big(\overline{\ \ }(s,\omega, \ , \ ) - \overline{\ \ }(s,\omega, \ _0, \ )\big) \\
&= G\big(\ _s\omega, \ \ (s) + \overline{\ \ }(s,\omega, \ , \ )\big) - G\big(\ _s\omega, \ \ (s) + \overline{\ \ }(s,\omega, \ _0, \ )\big) \\
&\quad - D \ \ G\big(\ _s\omega, \ \ (s) + \overline{\ \ }(s,\omega, \ _0, \ )\big)\big(\overline{\ \ }(s,\omega, \ , \ ) - \overline{\ \ }(s,\omega, \ _0, \ )\big).
\end{aligned}$$

We prove $\|\mathcal{S}\|_{\mathcal{C}\ _{cs-}} = o(\| \ - \ _0\|)$ as $\ \to \ _0$, then we have

$$\begin{aligned}
\overline{\ \ }(\cdot,\omega, \ , \ ) - \overline{\ \ }(\cdot,\omega, \ _0, \ ) - \mathcal{T}\big(\overline{\ \ }(\cdot,\omega, \ , \ ) - \overline{\ \ }(\cdot,\omega, \ _0, \ )\big)& \\
= \ _{A_0}(t,0)(\ - \ _0) + \mathcal{S}& \\
= \ _{A_0}(t,0)(\ - \ _0) + o(\| \ - \ _0\|),&
\end{aligned}$$

which implies

$$\overline{\ \ }(\cdot,\omega, \ , \ ) - \overline{\ \ }(\cdot,\omega, \ _0, \ ) = (id - \mathcal{T})^{-1} \ \ _{A_0}(t,0)(\ - \ _0) + o(\| \ - \ _0\|).$$

We know that $_{A_0}(t,0) = T(t)\, e^{\int_0^t (\ _r\omega)dr}$ is a bounded operator, therefore $\overline{\ }(\cdot,\omega,\ ,\ )$ is differentiable in and $D\overline{\ }(\cdot,\omega,\ ,\ ) \in \mathcal{L}\left(X_{0cs}, \mathcal{C}_{\ cs-}\right)$. Next we prove $\|\mathcal{S}\|_{\mathcal{C}_{\ cs-}} = o(\|\ -\ _0\|)$ as $\ \to\ _0$. We write $e^{-(\ _{cs}-\ )t-\int_0^t (\ _r\omega)dr}\mathcal{S} = \sum_{i=1}^{6}\mathcal{S}_i$, where for $t > N_1$,

$$\mathcal{S}_1 = e^{-(\ _{cs}-\ )t-\int_0^t (\ _r\omega)dr} \int_{N_1}^{t} \ _{A_{0c}}(t,s)\, \Pi_c \overline{\Delta}_{G(\ )}(s)\, ds,$$

and $\mathcal{S}_1 = 0$ for $t \le N_1$; For $t > N_1$,

$$\mathcal{S}_2 = e^{-(\ _{cs}-\ )t-\int_0^t (\ _r\omega)dr} \int_{0}^{N_1} \ _{A_{0c}}(t,s)\, \Pi_c \overline{\Delta}_{G(\ )}(s)\, ds,$$

and change N_1 to t if $t \le N_1$. For $t > N_2$,

$$\mathcal{S}_3 = -e^{-(\ _{cs}-\ )t-\int_0^t (\ _r\omega)dr} \int_{t}^{N_2} \ _{A_{0u}}(t,s)\, \Pi_u \overline{\Delta}_{G(\ )}(s)\, ds,$$

and $\mathcal{S}_3 = 0$ for $t \le N_2$; For $t < N_2$,

$$\mathcal{S}_4 = -e^{-(\ _{cs}-\ )t-\int_0^t (\ _r\omega)dr} \int_{N_2}^{+\infty} \ _{A_{0u}}(t,s)\, \Pi_u \overline{\Delta}_{G(\ )}(s)\, ds,$$

and change N_2 to t if $t \ge N_2$. For $t > N_3$,

$$\mathcal{S}_5 = e^{-(\ _{cs}-\ )t-\int_0^t (\ _r\omega)dr} \lim_{\ \to+\infty} \int_{N_3}^{t} \ _{A_{0s}}(t,s)\ \ (\ I - A_s)^{-1} \Pi_s \overline{\Delta}_{G(\ )}(s)\, ds,$$

and $\mathcal{S}_5 = 0$ for $t \le N_3$; For $t > N_3$,

$$\mathcal{S}_6 = e^{-(\ _{cs}-\ )t-\int_0^t (\ _r\omega)dr} \lim_{\ \to+\infty} \int_{0}^{N_3} \ _{A_{0s}}(t,s)\ \ (\ I - A_s)^{-1} \Pi_s \overline{\Delta}_{G(\ )}(s)\, ds,$$

and change N_3 to t if $t \le N_3$. N_1, N_2, N_3 above are large positive numbers to be chosen later. By (5) and (35), for $t > N_1$,

$$|\mathcal{S}_1| \leq 2KL \int_{N_1}^{t} e^{\ (t-s)} e^{-(\ _{cs}-\ )t} e^{(\ _{cs}-2\ )s} \left\| \bar{\ }(\cdot,\omega,\ ,\ ) - \bar{\ }(\cdot,\omega,\ _0,\ ) \right\|_{\mathcal{C}_{\ _{cs}-2}} ds$$

$$\leq 2KL \left\| \bar{\ }(\cdot,\omega,\ ,\ ) - \bar{\ }(\cdot,\omega,\ _0,\ ) \right\|_{\mathcal{C}_{\ _{cs}-2}} e^{(\ -\ _{cs}+\ )t} \int_{N_1}^{t} e^{(\ _{cs}-2\ -\ )s} ds$$

$$\leq \frac{2K^2L}{(\ _{cs}-2\ -\ )\left[1-KL\left(C_{\ +2}\ +\frac{1}{\ _{cs}-2\ -\ }+\frac{1}{\ -\ _{cs}+2\ }\right)\right]} e^{-\ N_1} \|\ -\ _0\|.$$

Choose N_1 so large that

$$\sup_{t\geq 0} |\mathcal{S}_1| \leq \frac{\varepsilon}{6} \|\ -\ _0\|. \tag{47}$$

Fix such N_1,

$$|\mathcal{S}_2| \leq K \left\| \bar{\ }(\cdot,\omega,\ ,\ ) - \bar{\ }(\cdot,\omega,\ _0,\ ) \right\|_{\mathcal{C}_{\ _{cs}-\ }} e^{(\ -\ _{cs}+\ )t} \int_0^{N_1} e^{(\ _{cs}-\ -\ )s}$$

$$\times \int_0^1 \Big\| D\ G\big(\ _s\omega,\ \ (s) + \ \bar{\ }(s,\omega,\ ,\ ) + (1-\ )\bar{\ }(s,\omega,\ _0,\ )\big)$$

$$- D\ G\big(\ _s\omega,\ \ (s) + \bar{\ }(s,\omega,\ _0,\ )\big) \Big\| d\ \ ds.$$

Since $G(\omega,\ )$ is C^k, $D\ G(\omega,\ ,\ )$ is continuous. For $\varepsilon > 0$, there exists $\ _1 > 0$ such that if $\|\ -\ _0\| < \ _1$,

$$\Big\| D\ G\big(\ _s\omega,\ \ (s) + \ \bar{\ }(s,\omega,\ ,\ ) + (1-\ )\bar{\ }(s,\omega,\ _0,\ )\big)$$

$$- D\ G\big(\ _s\omega,\ \ (s) + \bar{\ }(s,\omega,\ _0,\ )\big) \Big\|$$

$$\leq \frac{\varepsilon(\ _{cs}-\ -\ )\left[1-KL\left(C_{\ +\ }\ +\frac{1}{\ _{cs}-\ -\ }+\frac{1}{\ -\ _{cs}+\ }\right)\right]}{6K^2 e^{(\ _{cs}-\ -\ )N_1}}.$$

Then by (35),

$$\sup_{t\geq 0} |\mathcal{S}_2| \leq \frac{\varepsilon}{6} \|\ -\ _0\| \text{ for } \|\ -\ _0\| < \ _1. \tag{48}$$

Similarly, by choosing N_2 to be sufficiently large, we have

$$\sup_{t\geq 0} |\mathcal{S}_3| \leq \frac{\varepsilon}{6} \|\ -\ _0\|. \tag{49}$$

Fix such N_2, there exists $\ _2 > 0$ such that if $\|\ -\ _0\| < \ _2$,

$$\sup_{t\geq 0}|\mathcal{S}_4| \leq \frac{\varepsilon}{6}\| \; - \;_0\| \text{ for } \| \; - \;_0\| < \;_2. \tag{50}$$

For $\mathcal{S}_5$, if $t > N_3$,

$$|\mathcal{S}_5| = e^{-(\;_{cs}-\;)t-\int_0^t \;(\;_r\omega)dr} \lim_{\to+\infty} \int_0^{t-N_3} \;_{A_{0s}}(t, l+N_3) \;(\; I - A_s)^{-1}\Pi_s \overline{\Delta}_{G(\;)}(l+N_3)\, dl$$

$$\leq 2LC_{\;+2} \sup_{h\in[N_3,t]} e^{(\;+4\;-\;_{cs})(t-h)} e^{-\;t} \left\|^{-}(\cdot,\omega,\;,\;) - ^{-}(\cdot,\omega,\;_0,\;)\right\|_{\mathcal{C}_{\;cs-2}}$$

$$\leq \frac{2KLC_{\;+2}\; e^{-\;N_3}}{1-KL\left(C_{\;+2} + \frac{1}{\;_{cs}-2\;-} + \frac{1}{-\;_{cs}+2}\right)} \| \; - \;_0\|.$$

Choose N_3 so large that

$$\sup_{t\geq 0}|\mathcal{S}_5| \leq \frac{\varepsilon}{6}\| \; - \;_0\|. \tag{51}$$

And for the last one $\mathcal{S}_6$,

$$|\mathcal{S}_6| \leq LC_{\;+} \sup_{s\in[0,N_3]} e^{(\;+2\;-\;_{cs})(t-s)} \left\|^{-}(\cdot,\omega,\;,\;) - ^{-}(\cdot,\omega,\;_0,\;)\right\|_{\mathcal{C}_{\;cs-}}$$

$$\times \int_0^1 \Big\|D\; G\big(\;_s\omega,\;(s) + \;^{-}(s,\omega,\;,\;) + (1-\;)^{-}(s,\omega,\;_0,\;)\big)$$

$$- D\; G\big(\;_s\omega,\;(s) + ^{-}(s,\omega,\;_0,\;)\big)\Big\| d\;.$$

For $\varepsilon > 0$, there exists $\;_3 > 0$ such that if $\| \; - \;_0\| < \;_3$,

$$\Big\|D\; G\big(\;_s\omega,\;(s) + \;^{-}(s,\omega,\;,\;) + (1-\;)^{-}(s,\omega,\;_0,\;)\big)$$

$$- D\; G\big(\;_s\omega,\;(s) + ^{-}(s,\omega,\;_0,\;)\big)\Big\|$$

$$\leq \frac{1-KL\left(C_{\;+} + \frac{1}{\;_{cs}-\;-} + \frac{1}{-\;_{cs}+}\right)}{KLC_{\;+}}.$$

Then by (35),

$$\sup_{t\geq 0}|\mathcal{S}_6| \leq \frac{\varepsilon}{6}\| \; - \;_0\| \text{ for } \| \; - \;_0\| < \;_3. \tag{52}$$

By taking $\;_0 = \min\{\;_1,\;_2,\;_3\}$ and combining (47)-(52), we have

$$\|\mathcal{S}\|_{\mathcal{C}_{\;cs-}} \leq \varepsilon\| \; - \;_0\| \text{ for } \| \; - \;_0\| < \;_0.$$

This implies $\|\mathcal{S}\|_{\mathcal{C}_{cs-}} = o(\| - {}_0\|)$ as $\to {}_0$. Next we prove $D^{-}(\cdot,\omega,)$ is continuous with respect to from X_{0cs} to $\mathcal{L}(X_{0cs}, \mathcal{C})$. By (31), we have

$$
\begin{aligned}
& D^{-}(t,\omega, ,) = {}_{A_{0c}}(t,0)\,\Pi_{0cs} \\
& + \int_0^t {}_{A_{0c}}(t,s)\,\Pi_c D\ G\big({}_s\omega, (s) + {}^{-}(s,\omega, ,)\big)\,D^{-}(s,\omega, ,)\,ds \\
& - \int_t^{+\infty} {}_{A_{0u}}(t,s)\,\Pi_u D\ G\big({}_s\omega, (s) + {}^{-}(s,\omega, ,)\big)\,D^{-}(s,\omega, ,)\,ds \\
& + \lim_{\to +\infty} \int_0^t {}_{A_{0s}}(t,s)\ (\ I - A_s)^{-1} \\
& \times \Pi_s D\ G\big({}_s\omega, (s) + {}^{-}(s,\omega, ,)\big)\,D\ (s,\omega, ,)\,ds.
\end{aligned}
$$

For $\in X_{0cs}$ and $\in \mathcal{C}_{cs}$, define the operator $\mathcal{T}' : \mathcal{C}_{cs} \to \mathcal{C}_{cs}$ as follows (replace the ${}_0$ with in the definition of $\mathcal{T}$)

$$
\begin{aligned}
\mathcal{T}' = & \int_0^t {}_{A_{0c}}(t,s)\,\Pi_c D\ \Delta G\big({}_s\omega, (s), {}^{-}(s,\omega, ,)\big)\ (s)\,ds \\
& - \int_t^{+\infty} {}_{A_{0u}}(t,s)\,\Pi_u D\ \Delta G\big({}_s\omega, (s), {}^{-}(s,\omega, ,)\big)\ (s)\,ds \\
& + \lim_{\to +\infty} \int_0^t {}_{A_{0s}}(t,s)\ (\ I - A_s)^{-1} \\
& \times \Pi_s D\ \Delta G\big({}_s\omega, (s), {}^{-}(s,\omega, ,)\big)\ (s)\,ds.
\end{aligned}
$$

Similarly, we have

$$
\|\mathcal{T}'\| \le KL\left(C\ + \frac{1}{{}_{cs} -} + \frac{1}{- {}_{cs}}\right) < 1. \tag{53}
$$

For $, {}_0 \in X_{0cs}$,

$$
\begin{aligned}
& D^{-}(t,\omega, ,) - D^{-}(t,\omega, {}_0,) \\
& = \int_0^t {}_{A_{0c}}(t,s)\,\Pi_c\Big[D\ G\big({}_s\omega, (s) + {}^{-}(s,\omega, ,)\big)\,D^{-}(s,\omega, ,) \\
& - D\ G\big({}_s\omega, (s) + {}^{-}(s,\omega, {}_0,)\big)\,D^{-}(s,\omega, {}_0,)\Big]ds
\end{aligned}
$$

$$+\int_{+\infty}^{t} \quad {}_{A_{0u}}(t,s)\,\Pi_u\Big[D \quad G\big(\ {}_s\omega, \quad (s)+{}^{-}(s,\omega,\ ,\ )\big)\,D^{\,-}(s,\omega,\ ,\ )$$

$$-\,D$$

Step 2. We prove $\overline{\ }(t,\omega,\ ,\ )$ is C^k.

Make the inductive assumption that $\overline{\ }(\cdot,\omega,\ ,\ )$ is C^j from X_{0cs} to $\mathcal{C}_{j\ cs}$ for all $1 \le j \le k-1$ and we prove it for $j = k$. By calculation, the $(k-1)$th-derivative $D^{k-1}\overline{\ }(\cdot,\omega,\ ,\ )$ satisfies the following equation

$$D^{k-1}\overline{\ }(t,\omega,\ ,\ ) = \int_0^t {}_{A_{0c}}(t,s)\,\Pi_c D\ \ G\left(\ _s\omega,\ \ (s)+\overline{\ }(s,\omega,\ ,\ )\right) D^{k-1}\overline{\ }(s,\omega,\ ,\ )\,ds$$

$$-\int_t^{+\infty} {}_{A_{0u}}(t,s)\,\Pi_u D\ \ G\left(\ _s\omega,\ \ (s)+\overline{\ }(s,\omega,\ ,\ )\right) D^{k-1}\overline{\ }(s,\omega,\ ,\ )\,ds$$

$$+\lim_{\ \to+\infty}\int_0^t {}_{A_{0s}}(t,s)\ \ (\ \ I-A_s)^{-1}\Pi_s D\ \ G\left(\ _s\omega,\ \ (s)+\overline{\ }(s,\omega,\ ,\ )\right) D^{k-1}\overline{\ }(s,\omega,\ ,\ )\,ds$$

$$+\int_0^t {}_{A_{0c}}(t,s)\,\Pi_c R(s,\omega,\ ,\ )\,ds - \int_t^{+\infty} {}_{A_{0u}}(t,s)\,\Pi_u R(s,\omega,\ ,\ )\,ds$$

$$+\lim_{\ \to+\infty}\int_0^t {}_{A_{0s}}(t,s)\ \ (\ \ I-A_s)^{-1}\Pi_s R(s,\omega,\ ,\ )\,ds,$$

where

$$R(s,\omega,\ ,\ ) = \sum_{i=1}^{k-3}\binom{k-2}{i} D^{k-2-i}\left(D\ \ G\left(\ _s\omega,\ \ (s)+\overline{\ }(s,\omega,\ ,\ )\right)\right) D^{i+1}\overline{\ }(s,\omega,\ ,\ ).$$

Applying the chain rule to $D^{m-2-i}\left(D\ \ G\left(\ _s\omega,\ \ (s)+\overline{\ }(s,\omega,\ ,\ )\right)\right)$, we observe that each term in $R_{m-1}(s,\omega,\ ,\ )$ contains factors: $D^{i_1}G\left(\ _s\omega,\ \ (s)+\overline{\ }(s,\omega,\ ,\ )\right)$ for some $2 \le i_1 \le m-1$ and at least two derivatives $D^{i_2}\overline{\ }(s,\omega,\ ,\ )$ and $D^{i_3}\overline{\ }(s,\omega,\ ,\ )$ for some $i_2, i_3 \in \{1,...,m-2\}$. Since $D^i\overline{\ }(\cdot,\omega,\ ,\ ) \in \mathcal{C}_{i\ cs}$ for $i = 1,...,m-1$ and F is C^k, $R_{m-1}(\cdot,\omega,\ ,\ ): X_{0cs} \to L^{m-1}(X_{0cs}, X_0)$ are C^1 in . Using the same argument we used in the case $k = 1$, we can show that $D^{m-1}\overline{\ }(\cdot,\omega,\ ,\ )$ is C^1 from X_{0cs} to $L^m\left(X_{0cs}, \mathcal{C}_{m\ cs}\right)$. So $\overline{\ }(t,\omega,\ ,\ )$ is C^k. Thus $l^{cs}(\ ,\omega,\ )$ is C^k, which guarantees the C^k-smoothness of $\mathcal{W}^{cs*}(\ ,\omega)$. □

5. Illustrative examples

In this section, we give two examples about age-structured problems and stochastic parabolic equations. Note that we only verify the main assumptions in this article and do not provide specific results of invariant manifolds and foliations.

5.1. Age-structured problems

Consider the following age-structured model which is a hyperbolic stochastic partial differential equation

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial a} = - \quad u\,(t,a) + u\,(t,a) \circ W\,(t)\,, t \geq t_0,\ a \geq 0, \\ u\,(t,0) = \displaystyle\int_0^{+\infty} h\,(t,a)\,u\,(t,a)\,da, \\ u\,(t_0,\cdot) = \varphi \in L_+^p\,((0,+\infty);\mathbb{R})\,, \end{cases} \tag{56}$$

where a is the age, $\quad > 0$ is the mortality of the population and $h \in C\,(\mathbb{R}, L^q\,((0,+\infty)\,,\mathbb{R}))$ (with $\frac{1}{p} + \frac{1}{q} = 1$). W is a one-dimensional Brownian motion. Set the Banach space $X := \mathbb{R} \times L^p\,((0,+\infty)\,,\mathbb{R})$ equipped with the usual product norm

$$\left\| \begin{pmatrix} r \\ \varphi \end{pmatrix} \right\| = |r| + \|\varphi\|_{L^p((0,+\infty),\mathbb{R})}\,,$$

and set $X_0 := \{0_{\mathbb{R}}\} \times L^p\,((0,+\infty)\,,\mathbb{R})$. Consider a linear operator $\mathcal{A} : D\,(\mathcal{A}) \subset X \to X$ defined by

$$\mathcal{A} \begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} -\varphi\,(0) \\ -\varphi' - \quad \varphi \end{pmatrix}$$

with

$$D\,(\mathcal{A}) = \{0_{\mathbb{R}}\} \times W^{1,p}\,((0,+\infty)\,,\mathbb{R})\,.$$

Notice that $X_0 = \overline{D\,(\mathcal{A})} \neq X$. So $\mathcal{A}$ is non-densely defined on X. Also, $\mathcal{A}$ is Hille-Yosida if and only if $p = 1$. By [30, Lemma 8.1, Lemma 8.3], we have

$$0 < \limsup_{\ \to +\infty} \ ^{1/p} \left\| (\quad I - \mathcal{A})^{-1} \right\|_{\mathcal{L}(X)} < \infty,$$

and we can conclude that $\mathcal{A}_0$, the part of $\mathcal{A}$ in X_0, is Hille-Yosida and generates a C_0-semigroup $\{T_{\mathcal{A}_0}\,(t)\}_{t\geq 0}$. $\mathcal{A}$ generates an integrated semigroup $\{S_{\mathcal{A}}\,(t)\}_{t\geq 0}$. $\mathcal{A}$ is an MR operator.

Consider the family of bounded linear operators $\{\mathcal{F}\,(t)\}_{t\geq 0} \subset \mathcal{L}\,(X_0, X)$ given by

$$\mathcal{F}\,(t) \begin{pmatrix} 0 \\ \varphi \end{pmatrix} := \begin{pmatrix} \int_0^{+\infty} h\,(t,a)\,\varphi\,(a)\,da \\ 0 \end{pmatrix},\ \forall \begin{pmatrix} 0 \\ \varphi \end{pmatrix} \in X.$$

Take

$$u^*\,(t) = \begin{pmatrix} 0_{\mathbb{R}^2} \\ u\,(t,\cdot) \end{pmatrix}, \quad \omega\,(t) = \begin{pmatrix} 0 \\ W\,(t) \end{pmatrix}.$$

Then we can rewrite (56) as

$$du^*\,(t) = (\mathcal{A} + \mathcal{F}\,(t))\,u^*\,(t)\,dt + u^*\,(t) \circ d\omega\,(t)\,,\ t \geq t_0,\ u^*\,(t_0) = \begin{pmatrix} 0_{\mathbb{R}^2} \\ \varphi \end{pmatrix}. \tag{57}$$

Assume additionally

$$\sup_{t\in\mathbb{R}} \|h\,(t,\cdot)\,\|_{L^q} < +\infty.$$

Then Assumption 2.1 can be easily checked by Hölder's inequality.

Definition 5.1. A function $u^* \in C\,(\mathbb{R}, X_0)$ is an integrated solution of (57) if and only if for each $t \geq t_0$,

$$\int_{t_0}^{t} u^*\,(r)\,dr \in D(\mathcal{A}),$$

and

$$u^*\,(t) = u^*\,(t_0) + \int_{t_0}^{t} [\mathcal{A} + \mathcal{F}\,(r)]u^*(r)dr + \int_{t_0}^{t} u^*\,(r) \circ d\omega\,(r)\,.$$

We say that u^* is a t of 57) if

Define

$$\mathcal{C}(\mathbb{R};X) := \left\{ f \in C(\mathbb{R},X) : \|f\|_\infty = \sup_{t\in\mathbb{R}} \|f\|_X < +\infty \right\}.$$

By [32, Theorem 5.13], we have the following result concerning the exponential dichotomy.

Lemma 5.3. *The following assertions are equivalent*

(i) *The evolution family $\{U_{\mathcal{F}}(t,s)\}_{(t,s)\in\Delta}$ has a exponential dichotomy.*
(ii) *For each $f \in \mathcal{C}(\mathbb{R};X)$, there exists a unique integrated solution $\quad \in \mathcal{C}(\mathbb{R};X_0)$ of*

$$\frac{d \quad (t)}{dt} = \mathcal{A} \quad (t) + \mathcal{F}(t) \quad (t) + f(t), \; t \in \mathbb{R}.$$

With the exponential dichotomy verified, we could study stable and unstable foliations of (56) by similar arguments in Theorem 4.2. For the exponential trichotomy, one can refer to Chapter 5 of [29] about the age-structured model with bifurcation parameter.

5.2. Stochastic parabolic equation

We consider the following stochastic parabolic equation with a nonlinear (and nonlocal) boundary condition

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial \quad ^2} + \quad ^2 u(t, \quad) + P(u(t, \quad)) + u(t, \quad) \circ dW(t), t \geq 0, \quad > 0, \\ -\dfrac{\quad u(t,0)}{\partial \quad} = Q(u(t,\cdot)), \\ u(0,\cdot) = u_0 \in L^p((0,+\infty),\mathbb{R}), \end{cases} \tag{59}$$

where $P : L^p((0,+\infty),\mathbb{R}) \to L^p((0,+\infty),\mathbb{R})$ and $Q : L^p((0,+\infty),\mathbb{R}) \to \mathbb{R}$ are arbitrary nonlinear maps which are assumed to be globally Lipschitz continuous and $P(0) = Q(0) = 0$. W is a one-dimensional Brownian motion. Make $X := \mathbb{R} \times L^p((0,+\infty),\mathbb{R})$. Consider a linear operator $A : D(A) \subset X \to X$ defined by

$$A\begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} \varphi'(0) \\ \varphi'' \end{pmatrix}$$

with $D(A) = \{0_{\mathbb{R}}\} \times W^{2,p}((0,+\infty),\mathbb{R})$. Notice that A_0, the part of A in $\overline{D(A)} = \{0_{\mathbb{R}}\} \times L^p((0,+\infty),\mathbb{R})$, is the generator of the strongly continuous semigroup of bounded operators associated to

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial \quad ^2}, \quad t \geq 0, \quad > 0, \\ -\dfrac{\quad u(t,0)}{\partial \quad} = 0, \\ u(0,\cdot) = u_0 \in L^p((0,+\infty),\mathbb{R}). \end{cases}$$

That is

$$A_0 \begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} 0 \\ \varphi'' \end{pmatrix}$$

with

$$D(A_0) = \left\{ \begin{pmatrix} 0 \\ \varphi \end{pmatrix} \in \{0_{\mathbb{R}}\} \times W^{2,p}((0,+\infty), \mathbb{R}) : \varphi'(0) = 0 \right\}.$$

Take

$$F \begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} Q(\varphi) \\ P(\varphi) \end{pmatrix}, \quad u^*(t) = \begin{pmatrix} 0_{\mathbb{R}^2} \\ u(t,\cdot) \end{pmatrix}, \quad \omega(t) = \begin{pmatrix} 0 \\ W(t) \end{pmatrix}.$$

Then we can rewrite (59) as

$$\begin{cases} du^*(t) = \left(\left(A + \ {}^2 I \right) u^*(t) + F\left(u^*(t)\right) \right) dt + u^*(t) \circ d\omega(t), \ t \geq 0, \\ u^*(0,\cdot) = u_0^* \in \overline{D(A)} := X_0. \end{cases}$$

According to [32, Lemma 6.1], A_0 is the infinitesimal generator of a C_0-semigroup $\{T_{A_0}(t)\}_{t\geq 0}$ on X_0. Moreover, $\left(A + \ {}^2 I\right)_0$, the part of $\left(A + \ {}^2 I\right)$, is the infinitesimal generator of a C_0-semigroup on X_0 denoted by $\{T_{(A+\ {}^2 I)_0}(t)\}_{t\geq 0}$. By [28,30, Proposition 2.5], $\left(A + \ {}^2 I\right)$ generates a integrated semigroup $\{S_{(A+\ {}^2 I)}(t)\}_{t\geq 0}$. The spectrum of A_0 is given by

$$(A_0) = \left\{ -(\ k)^2 : k \in \mathbb{N} \right\}.$$

And by [32, Lemma 6.3], we have the following estimations

$$0 < \liminf_{\ \to +\infty} \ {}^{1/p^*} \left\| (\ I - A)^{-1} \right\|_{\mathcal{L}(X)} \leq \limsup_{\ \to +\infty} \ {}^{1/p^*} \left\| (\ I - A)^{-1} \right\|_{\mathcal{L}(X)} < \infty,$$

where $p^* = \frac{2p}{1+p}$. We can obtain that A is non-densely defined and is not a Hille-Yosida operator when $p > 1$. Assumption 2.1 is obviously satisfied by the Lipschitz continuity of P, Q. We make another assumption additionally.

Assumption 5.1. Let $A : D(A) \subset X \to X$ be a linear operator on a Banach space X. Assume that there exist two constants, $\omega_A \in \mathbb{R}$ and $M_A > 0$, such that

(i) $(A_0) \supset \{\ \in \mathbb{C} : \mathrm{Re}(\ ) > \omega_A\}$ and

$$\left\| (\ - \omega_A)(\ I - A_0)^{-1} \right\|_{\mathcal{L}(X_0)} \leq M_A, \ \forall \ \in \mathbb{C}, \ \mathrm{Re}(\ ) > \omega_A;$$

(ii) $(\omega_A, +\infty) \subset \ (A)$ and there exists $p^* > 1$ such that

$$\limsup_{\ \to +\infty} \ {}^{1/p^*} \left\| (\ I - A)^{-1} \right\|_{\mathcal{L}(X)} < \infty.$$

0},

The above assumption can be reformulated by saying that A_0 is sectorial and A is $\frac{1}{p^*}$-almost sectorial. By [32, Lemma 6.1-6.3], Assumption 5.1 is satisfied. Then by [12, Theorem 3.11], for fixed $\ > \omega_A$, $\hat{p} > p^*$ and $\ _0 > 0$, for each $f \in L^{\hat{p}}([0, \ _0], X)$, the following estimate holds

$$\| (S \diamond f)(t) \| \leq M_{\ , \ 0} \int_0^t (t-s)^{-\ } e^{\omega_A (t-s)} \| f(s) \| \, ds, \ \forall t \in [0, \ _0],$$

where $\ \in \left(1 - \frac{1}{p^*}, 1 - \frac{1}{\hat{p}}\right)$ and $M_{\ , \ 0}$ is some positive constant. Therefore

$$\| (S \diamond f)(t) \| \leq \left(M_{\ , \ 0} \int_0^t (t-s)^{-\ } e^{\omega_A (t-s)} ds \right) \sup_{s \in [0,t]} \| f(s) \|, \ \forall t \in [0, \ _0].$$

Then by [32, Lemma 6.4], A is an MR operator. Then we check Assumption 2.2. In fact, by [32, Lemma 6.2] and [29, Lemma 2.1], we have $\ (A_0) = \ (A)$, then

$$\left(A + \ ^2 I\right) = \ \left(A_0 + \ ^2 I\right) = \left\{ -(\ k)^2 + \ ^2 : k \in \mathbb{N} \right\}$$
$$= \left\{ \ ^2, 0, -3\ ^2, -8\ ^2, -15\ ^2, \ldots \right\}$$

and each eigenvalue $\ _k = \left(1 - k^2\right) \ ^2$ corresponding to the eigenfunction

$$\ _k(\ ) = \sin(\ k\ ).$$

By Remark 2.11, we could take\{≥ ∞ ∞^{2} ∕≥ ε*∞ [|≥ ^{2}∞{≥* and [≥∪ †|≥ $\left(0, \infty^2 - \varepsilon^*\right)$ for some $\varepsilon^* \in \left(0, \ ^2\right)$. Thus the spectrum of $A_0 + \ ^2 I$ could be split into three parts $\ _{0s} = \left\{\left(1-k^2\right) \ ^2 : k = 2, 3, \ldots\right\}$, $\ _{0c} = \{0\}$, $\ _{0u} = \left\{\ ^2\right\}$. So $A_0 + \ ^2 I$ satisfies the exponential trichotomy condition and the Assumption 2.2 is satisfied. The center subspace

References

[1] W. Arendt, Resolvent positive operators, Proc. Lond. Math. Soc. s3–54 (2) (Mar 1987) 321–349.
[2] W. Arendt, C.J.K. Batty, M. Hieber, F. Neubrander, Vector-Valued Laplace Transforms and Cauchy Problems, Birkhäuser, Basel, 2001.
[3] L. Arnold, Random Dynamical Systems, Springer, New York, 1998.
[4] T. Caraballo, J. Duan, K. Lu, B. Schmalfuß, Invariant manifolds for random and stochastic partial differential equations, Adv. Nonlinear Stud. 10 (1) (2010) 23–52.
[5] T. Caraballo, J.A. Langa, J.A. Robinson, A stochastic pitchfork bifurcation in a reaction-diffusion equation, Proc. R. Soc. A 457 (2013) (2001) 2041–2061.
[6] C. Castaing, M. Valadier, Convex Analysis and Measurable Multifunctions, Springer, 1997.
[7] D. Chen, Regularity properties of some perturbations of non-densely defined operators with applications, J. Evol. Equ. 20 (2020) 659–702.
[8] X. Chen, A.J. Roberts, J. Duan, Center manifolds for stochastic evolution equations, J. Differ. Equ. Appl. 21 (7) (2015) 606–632.
[9] G. Da Prato, E. Sinestrari, Differential operators with non-dense domain, Ann. Sc. Norm. Super. Pisa, Cl. Sci. 14 (2) (1987) 285–344.
[10] J. Duan, K. Lu, B. Schmalfuß, Invariant manifolds for stochastic partial differential equations, Ann. Probab. 31 (4) (2003) 2109–2135.
[11] J. Duan, K. Lu, B. Schmalfuß, Smooth stable and unstable manifolds for stochastic evolutionary equations, J. Dyn. Differ. Equ. 16 (4) (2004) 949–972.
[12] A. Ducrot, P. Magal, K. Prevost, Integrated semigroups and parabolic equations. Part I: linear perturbation of almost sectorial operators, J. Evol. Equ. 10 (2010) 263–291.
[13] N. Fenichel, Persistence and smoothness of invariant manifolds for flows, Indiana Univ. Math. J. 21 (3) (1971) 193–226.
[14] T. Gallay, A center-stable manifold theorem for differential equations in Banach spaces, Commun. Math. Phys. 152 (2) (1993) 249–268.
[15] J. Hadamard, Sur l'itération et les solutions asymptotiques des équations différentielles, Bull. Soc. Math. Fr. 29 (1901) 224–228.
[16] P.E. Kloeden, T. Lorenz, Mean-square random dynamical systems, J. Differ. Equ. 253 (5) (2012) 1422–1438.
[17] J. Li, K. Lu, P. Bates, Normally hyperbolic invariant manifolds for random dynamical systems: part I - persistence, Trans. Am. Math. Soc. 365 (2013) 5933–5966.
[18] J. Li, K. Lu, P. Bates, Invariant foliations for random dynamical systems, Discrete Contin. Dyn. Syst. 34 (9) (2014) 3639–3666.
[19] Z. Li, C. Zeng, Center manifolds for ill-posed stochastic evolution equations, Discrete Contin. Dyn. Syst., Ser. B 27 (5) (2022) 2483–2499.
[20] Z. Li, C. Zeng, J. Huang, Mean-square manifolds for ill-posed stochastic evolution equations driven by nonlinear noise, J. Differ. Equ. 313 (2022) 382–419.
[21] Z. Lian, K. Lu, Lyapunov exponents and invariant manifolds for random dynamical systems in a Banach space, Mem. Am. Math. Soc. 206 (967) (2010), 106 pp.
[22] P. Liu, M. Qian, Smooth Ergodic Theory of Random Dynamical Systems, Lect. Notes. Math., vol. 1606, Springer-Verlag, Berlin, 1995.
[23] Z. Liu, P. Magal, S. Ruan, Center-unstable manifolds for non-densely defined Cauchy problems and applications to stability of Hopf bifurcation, Can. Appl. Math. Q. 20 (20) (2012) 135–178.
[24] K. Lu, B. Schmalfuß, Invariant manifolds for stochastic wave equation, J. Differ. Equ. 236 (2) (2007) 460–492.
[25] K. Lu, B. Schmalfuß, Invariant foliations for stochastic partial differential

[31] P. Magal, S. Ruan, Theory and Applications of Abstract Semilinear Cauchy Problems, Applied Mathematical Sciences, vol. 201, Springer, New York, 2018.
[32] P. Magal, O. Seydi, Variation of constants formula and exponential dichotomy for non-autonomous non densely defined Cauchy problems, Can. J. Math. 73 (5) (2021) 1347–1389.
[33] S.-E.A. Mohammed, M.K.R. Scheutzow, The stable manifold theorem for stochastic differential equations, Ann. Probab. 27 (2) (1999) 615–652.
[34] A. Neamţu, Random invariant manifolds for ill-posed stochastic evolution equations, Stoch. Dyn. 20 (2) (2020) 2050013.
[35] A. Pazy, Semigroups of Linear Operator and Applications to Partial Differential Equations, Springer, 1983.
[36] O. Perron, Über Stabilität und asymptotisches Verhalten der Integrale von Differentialgleichungssysteme, Math. Z. 29 (1928) 129–160.
[37] Y. Pesin, Characteristic Lyapunov exponents, and smooth ergodic theory, Russ. Math. Surv. 32 (4) (1977) 55–112.
[38] L. Shi, Smooth convergence of random center manifolds for SPDEs in varying phase spaces, J. Differ. Equ. 269 (3) (2020) 1963–2011.
[39] H.R. Thieme, Integrated semigroups and integrated solutions to abstract Cauchy problems, J. Math. Anal. Appl. 152 (2) (1990) 416–447.
[40] H.R. Thieme, Semiflows generated by Lipschitz perturbations of non-densely defined operators, Differ. Integral Equ. 3 (6) (1990) 1035–1066.
[41] B. Wang, Mean-square random invariant manifolds for stochastic differential equations, Discrete Contin. Dyn. Syst., Ser. A 41 (3) (2021) 1449–1468.
[42] C. Zeng, J. Shen, Invariant foliations for stochastic partial differential equations with non-dense domain, Proc. Am. Math. Soc. 150 (2021) 2023–2036.
[43] H. Zhu, Y. Peng, C. Zeng, Y. Li, Forward dynamics and memory effect in a fractional order chemostat minimal model with non-monotonic growth, Discrete Contin. Dyn. Syst., Ser. S 16 (10) (2023) 2749–2764.