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Mean exit time and escape probability for the Ornstein–Uhlenbeck process

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ABSTRACT

This paper studies the dynamics of the Ornstein–Uhlenbeck process by the deterministic quantities such as the mean exit time and escape probability. By solving the elliptic partial equations, we obtain explicit solutions to both mentioned problems using the special functions. We find that the mean exit time is longer for smaller noise, and the maximum depends on the middle of the given interval. Moreover, the likelihood that the solution orbits exiting the interval from left or right relies on the middle of the interval. The Monte Carlo simulations are carried out to support the obtained results.

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Dynamical systems in engineering and science often occur in a noisy environment. The noise perturbation is a deterministic and periodic function on the associated complex dynamics. Stochastic differential equations are usually considered to be appropriate models for random phenomena. In general, there are two categories of stochastic systems: the Ornstein–Uhlenbeck process and the Fokker–Planck equation distribution. The former describes the probability of a stochastic system, and the latter describes the probability density function of the corresponding process. In this paper, we intend to discuss the stochastic dynamics through the Ornstein–Uhlenbeck process and the escape probability.

escape probability. The former describes how long a stochastic system stays in a given region in the state space, and the latter quantifies the likelihood of a transition of such a system from one regime to another. They are both related to the well-known Fokker–Planck equation and, in fact, satisfy some elliptic partial differential equations under different boundary conditions. The main theme of this paper is to study the complex dynamics using the mentioned deterministic quantities via a simple stochastic system.

As a mean-reverting process, the Ornstein–Uhlenbeck process occupies an important place in finance to model the volatility of the underlying asset price process.² In physics contexts, the associated equation is called the Langevin equation. Precisely, let $\mu > 0, \varepsilon > 0$, consider the Ornstein–Uhlenbeck process,

$$dX_t = -\mu X_t dt + \varepsilon dW_t, \quad X_0 = x_0 \in \mathbb{R}, \quad t \geq 0, \quad (1)$$

where W_t is a one-dimensional standard Brownian motion. Clearly, its solution reads

$$X_t = e^{-\mu t} x_0 + \varepsilon \int_0^t e^{-\mu(t-s)} dW_s, \quad (2)$$

which is Markovian and Gaussian with the mean $\mathbb{E}[X_t] = x_0 e^{-\mu t}$ and covariance

$$\text{Cov}(X_t, X_s) = \frac{\varepsilon^2}{2\mu} (e^{-\mu|t-s|} - e^{-\mu(t+s)}), \quad t \geq 0, s \geq 0.$$

1. INTRODUCTION

The mean, variance, moments, and characteristic functions are often used to quantify random variables and stochastic processes. Accordingly, statistical quantities carrying dynamical information of stochastic differential equations have been developed in recent decades.¹ Among them, we are interested in the mean exit time and

In the following, we turn to state the probability density function for (1). Precisely, the associated Fokker–Planck equation yields

$$\frac{\partial}{\partial t} p(t, x) = \mu \frac{\partial}{\partial x} [xp(t, x)] + \frac{1}{2} \varepsilon^2 \frac{\partial^2}{\partial x^2} p(t, x), \quad (3)$$

with the initial condition $p(0, x) = \delta(x - x_0)$. The corresponding solution is represented as

$$p(t, x) = \frac{1}{\sqrt{\frac{\pi \varepsilon^2}{\mu} (1 - e^{-2\mu t})}} \exp \left(-\frac{(x - x_0 e^{-\mu t})^2}{\frac{\varepsilon^2}{\mu} (1 - e^{-2\mu t})} \right).$$

Besides, one can obtain the stationary property when imposing the specific random initial condition rather than a constant. In fact, if $X_0 \square N(0, \varepsilon^2/(2\mu))$, then the covariance function is

$$\text{Cov}^\square(X_t, X_s) = \frac{\varepsilon^2}{2\mu} e^{-\mu|t-s|}.$$

Therefore, solution to (1) is stationary if the initial distribution is the stationary distribution $N(0, \varepsilon^2/(2\mu))$.

The first-passage problem for the Ornstein–Uhlenbeck process was discussed by Darling and Siegert from a limiting case of an Ehrenfest urn scheme.³ Later on, the mean first-passage time of an Ornstein–Uhlenbeck process to an exponentially decaying barrier was established by a numerical approach.⁴ Moreover, the first-passage problem for the Ornstein–Uhlenbeck process has been studied by Hartich and Godec,⁵ where they reported a solution for the distribution of first-passage times in terms of an infinite series. It is also needed to point out that the formal solution of the first-passage problem for the general class of systems, including the Ornstein–Uhlenbeck process, was presented by Goel and Richter-Dyn.⁶ In physics, there have been many applications on the first-passage problem in recent decades.^{7–12} Recently, the so-called Walk on Moving Spheres (WMS) algorithm was extended to approximate the exit time of the Ornstein–Uhlenbeck process.

Proof. First of all, we denote $u^{\dagger}(x) = v(x)$ and rewrite (4) as

$$-\mu x v(x) + \frac{1}{2} \varepsilon^2 v^{\dagger}(x) = -1.$$

Multiplying both sides of the above equation by the factor $2 \exp(-\mu x^2/\varepsilon^2)/\varepsilon^2$ gives that

$$-\frac{2\mu}{\varepsilon^2} \exp\left(-\frac{\mu x^2}{\varepsilon^2}\right) x v(x) + \exp\left(-\frac{\mu x^2}{\varepsilon^2}\right) v^{\dagger}(x) = -\frac{2}{\varepsilon^2} \exp\left(-\frac{\mu x^2}{\varepsilon^2}\right),$$

which is equivalent to

$$\left(\exp\left(-\frac{\mu x^2}{\varepsilon^2}\right) v(x)\right)^{\dagger} = -\frac{2}{\varepsilon^2} \exp\left(-\frac{\mu x^2}{\varepsilon^2}\right).$$

Thus,

$$\begin{aligned} \exp\left(-\frac{\mu x^2}{\varepsilon^2}\right) v(x) &= -\frac{2}{\varepsilon^2} \int \exp\left(-\frac{\mu x^2}{\varepsilon^2}\right) dx \\ dx &= -\sqrt{\frac{\pi}{\mu \varepsilon^2}} \operatorname{erfi}\left(\frac{\sqrt{\mu}}{\varepsilon} x\right) + C_1, \end{aligned}$$

where C_1 is the constant of integration. Then,

$$v(x) = -\sqrt{\frac{\pi}{\mu \varepsilon^2}} \exp\left(\frac{\mu x^2}{\varepsilon^2}\right) \operatorname{erfi}\left(\frac{\sqrt{\mu}}{\varepsilon} x\right) + C_1 \exp\left(\frac{\mu x^2}{\varepsilon^2}\right).$$

Note that the error function has a series representation,

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \exp(-z^2) \sum_{k=0}^{\infty} \frac{2^k}{(2k+1)!!} z^{2k+1},$$

which implies that

$$\int \exp(z^2) \operatorname{erf}(z) dz = \frac{2}{\sqrt{\pi}} \int \sum_{k=0}^{\infty} \frac{2^k}{(2k+1)!!} z^{2k+1} dz$$

implying that

$$\begin{aligned} C_1 &= \frac{2}{\varepsilon^3} \sqrt{\frac{\mu}{\pi}} \frac{\alpha^2 {}_2F_2\left(1, 1; 3/2, 2; \frac{\mu}{\varepsilon^2} \alpha^2\right) - \beta^2 {}_2F_2\left(1, 1; 3/2, 2; \frac{\mu}{\varepsilon^2} \beta^2\right)}{\operatorname{erfi}\left(\frac{\sqrt{\mu}}{\varepsilon} \alpha\right) - \operatorname{erfi}\left(\frac{\sqrt{\mu}}{\varepsilon} \beta\right)}, \\ C_2 &= \frac{1}{\varepsilon^2} \frac{\beta^2 {}_2F_2\left(1, 1; 3/2, 2; \frac{\mu}{\varepsilon^2} \beta^2\right) \operatorname{erfi}\left(\frac{\sqrt{\mu}}{\varepsilon} \alpha\right) - \alpha^2 {}_2F_2\left(1, 1; 3/2, 2; \frac{\mu}{\varepsilon^2} \alpha^2\right) \operatorname{erfi}\left(\frac{\sqrt{\mu}}{\varepsilon} \beta\right)}{\operatorname{erfi}\left(\frac{\sqrt{\mu}}{\varepsilon} \alpha\right) - \operatorname{erfi}\left(\frac{\sqrt{\mu}}{\varepsilon} \beta\right)}, \end{aligned}$$

which indicates the required statement (5). $\square$

Corollary Assume $D = (-r, r)$ for some $r > 0$. Then, (5) reduces to

$$u(x) = -\frac{x^2}{\varepsilon^2} {}_2F_2\left(1, 1; 3/2, 2; \frac{\mu}{\varepsilon^2} x^2\right) + \frac{r^2}{\varepsilon^2} {}_2F_2\left(1, 1; 3/2, 2; \frac{\mu}{\varepsilon^2} r^2\right). \quad (6)$$

Proof. It is a straightforward consequence by using Theorem II.1 by setting $\alpha = -r$ and $\beta = r$, together with the fact that $\operatorname{erfi}(z) + \operatorname{erfi}(-z) = 0$ for any $z \in \mathbb{R}$. $\square$

$$\begin{aligned} &= \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{2^k}{(2k+2)(2k+1)!!} z^{2k+2} + C_2 \\ &= \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(1)_k (1)_k}{(3/2)_k (2)_k} \frac{(z^2)^k}{k!} + C_2 \\ &= \frac{2}{\sqrt{\pi}} {}_2F_2\left(1, 1; 3/2, 2; z^2\right) + C_2, \end{aligned}$$

where C_2 is another constant of integration. Therefore, we obtain

$$\begin{aligned} u(x) &= \int \left[-\sqrt{\frac{\pi}{\mu \varepsilon^2}} \exp\left(\frac{\mu x^2}{\varepsilon^2}\right) \operatorname{erf}\left(\frac{\sqrt{\mu}}{\varepsilon} x\right) + C_1 \exp\left(\frac{\mu x^2}{\varepsilon^2}\right) \right] dx \\ &= -\frac{\sqrt{\pi}}{\mu} \frac{\mu x^2}{2} {}_2F_2\left(1, 1; 3/2, 2; \frac{\mu}{\varepsilon^2} x^2\right) + \frac{\varepsilon C_1}{2} \sqrt{\frac{\pi}{\mu}} \operatorname{erfi}\left(\frac{\sqrt{\mu}}{\varepsilon} x\right) + C_2 \\ &= -\frac{x^2}{\varepsilon^2} {}_2F_2\left(1, 1; 3/2, 2; \frac{\mu}{\varepsilon^2} x^2\right) + \frac{\varepsilon C_1}{2} \sqrt{\frac{\pi}{\mu}} \operatorname{erfi}\left(\frac{\sqrt{\mu}}{\varepsilon} x\right) + C_2. \end{aligned}$$

Recall that $u(\alpha) = u(\beta) = 0$, one has

$$\begin{aligned} u(\alpha) &= -\frac{\alpha^2}{\varepsilon^2} {}_2F_2\left(1, 1; 3/2, 2; \frac{\mu}{\varepsilon^2} \alpha^2\right) \\ &\quad + \frac{\varepsilon C_1}{2} \sqrt{\frac{\pi}{\mu}} \operatorname{erfi}\left(\frac{\sqrt{\mu}}{\varepsilon} \alpha\right) + C_2 = 0, \\ u(\beta) &= -\frac{\beta^2}{\varepsilon^2} {}_2F_2\left(1, 1; 3/2, 2; \frac{\mu}{\varepsilon^2} \beta^2\right) \\ &\quad + \frac{\varepsilon C_1}{2} \sqrt{\frac{\pi}{\mu}} \operatorname{erfi}\left(\frac{\sqrt{\mu}}{\varepsilon} \beta\right) + C_2 = 0, \end{aligned}$$

We are next concerned about the impact of noise on the mean exit time. Without loss of generality, we consider u as the function of ε in (6) and observe that

$$\begin{aligned} \frac{du(\varepsilon)}{d\varepsilon} &= \sqrt{\frac{\pi}{\mu}} \frac{1}{\varepsilon^2} \left[x \exp\left(\frac{\mu}{\varepsilon^2} x^2\right) \operatorname{erf}\left(\frac{\sqrt{\mu}}{\varepsilon} x\right) \right. \\ &\quad \left. - r \exp\left(\frac{\mu}{\varepsilon^2} r^2\right) \operatorname{erf}\left(\frac{\sqrt{\mu}}{\varepsilon} r\right) \right] < 0 \end{aligned}$$

due to the monotonicity of the related functions. This leads to the result that the mean exit time is longer for smaller noise with the

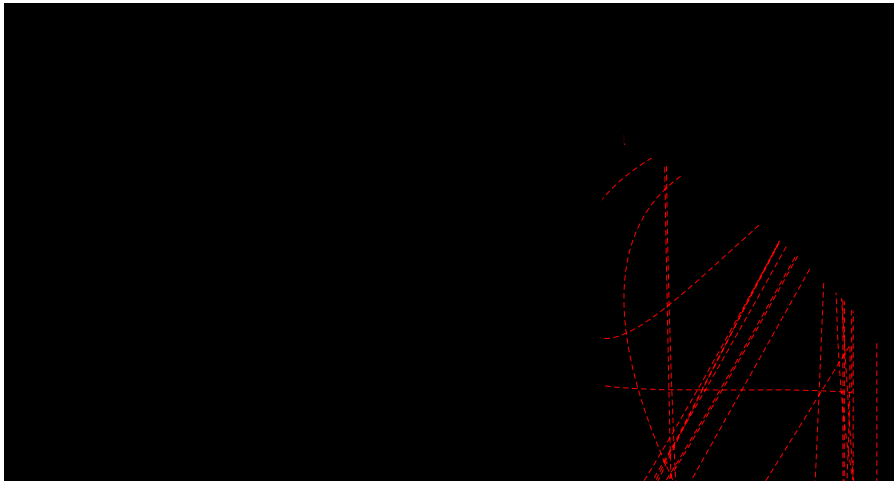




TABLE III. Monte Carlo value (MCV) and the theoretical value (TV) for escape probability starting from $x_0 = 0$.

	$\alpha = -3, \beta = 1, \mu = 1$		$\alpha = -1, \beta = 1, \mu = 1$		$\alpha = -1, \beta = 2, \mu = 1$	
	MCV	TV	MCV	TV	MCV	TV
$\varepsilon = 1.0$	0.9991	0.9990	0.4983	0.5000	0.0802	0.0816
$\varepsilon = 1.5$	0.9554	0.9547	0.4994	0.5000	0.2158	0.2174
$\varepsilon = 2.0$	0.8814	0.8817	0.5004	0.5000	0.2712	0.2715

We now turn to verify (8). For given parameters, the responding results on the escape probability are shown in Fig. 2, which are consistent with the results in Sec. III.

Applying the Monte Carlo method presented in the Ref. 14, we next carry out the simulation of the mean exit time and escape probability. More precisely, we take the number of samples paths ($N = 100\,000$) and sufficiently large time ($T = 100$) with the time step ($dt = 0.0005$) for $\varepsilon = 1, 1.5$, and 2 under the domains $D = (-3, 1)$, $(-1, 1)$, and $(-1, 2)$, respectively, to calculate the mean exit time from D of sample paths starting from $x_0 = 0$ and the escape probability through the right terminal. By taking $\mu = 1$, the corresponding results are listed in Tables II and III.

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DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

REFERENCES

- ¹J. Duan, *An Introduction to Stochastic Dynamics* (Cambridge University Press, 2015).
- ²G. Uhlenbeck and L. Ornstein, *Phys. Rev.* **82**, 823 (1930).
- ³D. Darling and A. Siebert, *Ann. Math. Stat.* **4**, 624 (1953).
- ⁴H. Tuckwell and F. Wan, *J. Appl. Prob.* **21**, 695709 (1984).
- ⁵D. Hartich and A. Godec, *J. Stat. Mech.* **2016**, 024002.
- ⁶N. S. Goel and N. Richter-Dyn, *Stochastic Models in Biology* (Elsevier, 2016).
- ⁷K. Malakar et al., *J. Stat. Mech.* **2017**, 043215.
- ⁸B. Derrida et al., *Phys. Rev. Lett.* **120**, 751 (2018).
- ⁹I. Neri et al., *Phys. Rev. X* **7**, 011019 (2017).
- ¹⁰F. Sedlmeier et al., *J. Stat. Phys.* **143**, 240 (2011).
- ¹¹Y. von Hansen et al., *Phys. Rev. E* **83**, 051501 (2011).
- ¹²R. Belousov et al., *Soft Matter* (published online 2020).
- ¹³S. Herrmann and N. Massin, *Discrete Cont. Dyn. Syst.: Ser. B* **27**, 3199 (2020).
- ¹⁴E. Allen, *Modeling with Itô Stochastic Differential Equations* (Springer, 2007).