

Smoothness of invariant manifolds for stochastic evolution equations with non-dense domain

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With the crucial estimation of random Stieltjes-type convolution, the Lyapunov–Perron method allows us to establish the existence of invariant dynamics for stochastic evolution equation with non-dense domain. In this paper, we develop the conditions for C^k

where $A : D(A) \rightarrow X$ (X is a linear operator and $f \in L^1((0, T), X)$), the Cauchy problem has been extensively studied in [17] if $\overline{D(A)} = X$. However, there are many situations in which the density condition is not satisfied, we refer to the Cauchy problem covering several types of differential equations, including delay differential equations, age-structured models and some evolution equations with nonlinear boundary conditions. In this respect, Da Prato and Sinestrari [5] studied the existence and uniqueness of solutions to (1.1) using the theory of integrated semigroup introduced by Arendt [1, 2]. Later on, Thieme [18, 19] proved the well-posedness of non-autonomous and semilinear Cauchy problems under a prior Hille–Yosida-type estimate for the resolvent of non-densely defined operators. Without the mentioned estimate, Magal and Ruan studied the center manifolds of semilinear Cauchy problems with non-dense domain and applied the results to study Hopf bifurcation of age-structured models without the prior Hille–Yosida-type estimate for the resolvent of non-densely defined operators [12–14].

Since the real world is inevitably affected by internal and external noise sources, we are interested to study the long time behavior of (1.1) under the influence of noise. The first result goes to the work by Neamtău [16], in which she considered the following Stratonovich stochastic evolution equation:

$$\begin{aligned} du &= Audt + F(u)dt + u \circ dW, \\ u(0) &= u_0 \in \overline{D(A)}, \end{aligned} \tag{1.2}$$

where F is some nonlinear function with some natural imposed conditions (specified later) and W is a Brownian motion. By extending the mentioned theory of integrated semigroups to (1.2), she established the existence of random stable/unstable manifolds based on the Lyapunov–Perron method. Following this philosophy, Zeng and Shen [21] obtained the existence of the invariant foliations of (1.2). Furthermore, Li and Zeng [10] studied the existence of center manifold of (1.2) with suitable exponential trichotomy condition. It is worth pointing out that the result on the smoothness of the obtained invariant manifolds for (1.2) is still open. We also noticed that one possible approach was mentioned in concluding discussion in [16] using the implicit function theorem.

On the other hand, little seems to be known about the long-time dynamics the following Itô stochastic evolution equation:

$$\begin{aligned} du &= Audt + F(u)dt + \sigma(u)dW, \\ u(0) &= u_0 \in \overline{D(A)}, \end{aligned} \tag{1.3}$$

where σ is another nonlinear function with some natural imposed conditions (speci-

proved the existence of mean-square random unstable invariant manifolds and stable sets of (1.3) under the assumption of exponential trichotomy. However, the smoothness result is also still open.

Unlike the argument using the classical Lyapunov-Perron method, the spectral gap condition contains a crucial constant in the derivation of estimation of random Stieltjes-type convolution. The required conditions for smoothness do not appear to be easily verifiable at first sight. This situation inspires us to present a positive answer to the smoothness problem for invariant manifolds of stochastic evolution equations (1.2) and (1.3). To be precise, we shall establish the conditions for C^k -smoothness of center manifolds of (1.2) proved in [10], and mean-square random unstable invariant manifolds and stable sets of (1.3) presented in [11]. The rest of this paper is organized as follows. In Sec. 2, we collect some basic definitions of mean-square random dynamical systems and integrated semigroups and lay out the main results we obtained in [10, 11]. Then, in Sec. 3, we prove the C^k -smoothness of center invariant manifolds of (1.2). Finally, in Secs. 4 and 5, we prove the C^k -smoothness of mean-square random unstable manifolds and random stable sets, respectively. Finally, in Sec. 6, we give an example about stochastic parabolic equation to verify the part of main results in this paper.

2. Preliminaries

In general, we verified the smoothness of invariant manifolds of stochastic evolution equations with non-dense domain in two situations. The first one is the case driven by linear multiplicative noise, we used the standard transformation with the Ornstein-Uhlenbeck process and pathwise methods. The second one is the case driven by nonlinear noise, we considered the problem under the framework of mean-square random dynamical systems instead of the classical random dynamical systems.

2.1. Random dynamical systems

We look at flows on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where a flow $\theta : \mathbb{R} \times \Omega \rightarrow \Omega$ of $\{\theta_t\}_{t \in \mathbb{R}}$ is defined on the sample space Ω having the following properties:

- (i) the mapping $(t, \omega) \mapsto \theta_t \omega$ is $(\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}, \mathcal{F})$ -measurable. $\rightarrow$
- (ii) $\theta_0 = \text{Id}$.
- (iii) $\theta_{t+s} = \theta_t \circ \theta_s$ for all $t, s \in \mathbb{R}$.

In addition, we assume that $\mathbb{P}$ is ergodic with respect to $\{\theta_t\}_{t \in \mathbb{R}}$. Then the quadruple $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t \in \mathbb{R}})$ is called a metric dynamical system. Usually, let $B(t)$ be a two-sided Brownian motion with trajectories in the space $\Omega := C_0(\mathbb{R}, \mathbb{R})$ of real continuous functions defined on $\mathbb{R}$ which is a set equipped with compact open topology $\mathcal{F} := \mathcal{B}(C_0(\mathbb{R}, \mathbb{R}))$. On this set, we consider the measurable flow $\{\theta_t\}_{t \in \mathbb{R}}$ defined by $\theta_t \omega(\bullet) = \omega(\bullet + t) - \omega(t)$ which is called the Wiener shift, the associated

distribution $\mathbb{P}$ is a Wiener measure defined on $\mathbf{F}$. A set Ω is called $\{\theta_t\}_{t \in \mathbb{R}}$ -invariant if $\theta_t \Omega = \Omega$ for $t \in \mathbb{R}$. For more details, see [3]. Using these notations, we could present the definition of random dynamical system.

Definition 2.1. Let X be a separable Hilbert space. A continuous random dynamical system on X over a metric dynamical system $(\Omega, \mathbf{F}, \mathbb{P}, \{\theta_t\}_{t \in \mathbb{R}})$ is a mapping

$$\phi : \mathbb{R} \times \Omega \times X \rightarrow X, \quad (t, \omega, x) \mapsto \phi(t, \omega, x), \quad \rightarrow$$

which is $(\mathcal{B}(\mathbb{R}) \otimes \mathbf{F} \otimes \mathcal{B}(X), \mathcal{B}(X))$ -measurable and satisfies:

- (i) $\phi(0, \omega, x) = x$, for $x \in X$ and $\omega \in \Omega$.
- (ii) $\phi(t+s, \omega, x) = \phi(t, \theta_s \omega, \phi(s, \omega, x))$, for all $t, s \in \mathbb{R}$, $x \in X$ and all $\omega \in \Omega$.

We will use a coordinate transform to convert the stochastic partial differential equation (1.2) into a random evolution equation [4, 6, 7]. To do this, we consider the linear stochastic differential equation

$$dz + \mu z dt = dW, \quad (2.1)$$

where μ is a positive parameter and W is a one-dimensional two-sided Brownian motion. We extract the important results on the stationary Ornstein–Uhlenbeck process from [4, 7] in the following lemma.

Lemma 2.1. (i) *There exists a $\{\theta_t\}_{t \in \mathbb{R}}$ -invariant set $\tilde{\Omega} \subset C_0(\mathbb{R}, \mathbb{R})$ of full measure satisfying the sublinear growth condition*

$$\lim_{t \rightarrow \pm\infty} \frac{|\omega(t)|}{|t|} = 0, \quad \omega \in \tilde{\Omega}.$$

- (ii) *The random variable $z(\omega) = -\mu \int_{-\infty}^0 e^{\mu s} \omega(s) ds$ is well defined for all $\omega \in \tilde{\Omega}$ and generates a unique stationary solution of (2.1) given by*

$$(t, \omega) \mapsto z(\theta_t \omega) = -\mu \int_{-\infty}^0 e^{\mu s} \theta_t \omega(s) ds = -\mu \int_{-\infty}^0 e^{\mu s} \theta_t \omega(s+t) ds + \mu \omega(t).$$

The mapping $t \mapsto z(\theta_t \omega)$ is continuous. $\rightarrow$

- (iii) *Moreover*

$$\lim_{t \rightarrow \pm\infty} \frac{|z(\theta_t \omega)|}{|t|} = 0,$$

for all $\omega \in \tilde{\Omega}$.

- (iv) *In addition*

$$\lim_{t \rightarrow \pm\infty} \frac{1}{t} \int_0^t z(\theta_s \omega) ds = 0,$$

for all $\omega \in \tilde{\Omega}$.

From now on, we will work on $(\tilde{\Omega}, \tilde{\mathbf{F}}, \tilde{\mathbb{P}}, \{\theta_t\}_{t \in \mathbb{R}})$, where $\tilde{\mathbf{F}}$ is the trace algebra of $\tilde{\Omega}$ and the restriction of Wiener measure to this σ -algebra $\tilde{\mathbf{F}}$ is denoted by $\tilde{\mathbb{P}}$. With a slight abuse of notation, we again denote the metric dynamical system by $(\Omega, \mathbf{F}, \mathbb{P}, \{\theta_t\}_{t \in \mathbb{R}})$ in the following context.

2.2. Mean-square random dynamical systems

Suppose V is a Banach space with norm $\|\cdot\|_V$. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in \mathbb{R}}, \mathbb{P})$ be a complete filtered probability space satisfying the usual condition, that is, $\{\mathcal{F}_t\}_{t \in \mathbb{R}}$ is an increasing right continuous family of sub- σ -algebras of $\mathcal{F}$ that contains all $\mathbb{P}$ -null sets. Given $p \in (1, +\infty)$ and $t \in \mathbb{R}$, denote by $L^p(\Omega, \mathcal{F}_t; V)$ the subspace of $L^p(\Omega, \mathcal{F}; V)$, which consists of all strongly $\mathcal{F}_t$ -measurable functions ψ in $L^p(\Omega, \mathcal{F}; V)$. We now present the definition of mean dynamical system over $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in \mathbb{R}}, \mathbb{P})$ (see [9, 20]).

Definition 2.2. If for all $\tau \in \mathbb{R}$, $t, s \in \mathbb{R}^+$, the following conditions are satisfied:

- (i) $\Phi(t, \tau)$ maps $L^2(\Omega, \mathcal{F}_\tau; V)$ to $L^2(\Omega, \mathcal{F}_{t+\tau}; V)$;
- (ii) $\Phi(0, \tau)$ is the identity operator on $L^2(\Omega, \mathcal{F}_\tau; V)$;
- (iii) $\Phi(t + s, \tau) = \Phi(t, \tau + s) \circ \Phi(s, \tau)$.

then a family of mappings $\Phi = \{\Phi(t, \tau) : t \in \mathbb{R}^+, \tau \in \mathbb{R}\}$ is called a mean-square random dynamical system on $L^2(\Omega, \mathcal{F}; V)$ over $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in \mathbb{R}}, \mathbb{P})$.

Then, we present the definition of the invariance property.

Definition 2.3. Suppose $\Phi = \{\Phi(t, \tau) : t \in \mathbb{R}^+, \tau \in \mathbb{R}\}$ is a mean-square random dynamical system on $L^2(\Omega, \mathcal{F}; V)$ over $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in \mathbb{R}}, \mathbb{P})$ and $M = \{M(\tau) \subset L^2(\Omega, \mathcal{F}_\tau; V) : \tau \in \mathbb{R}\}$ is a family of subsets of $L^2(\Omega, \mathcal{F}; V)$. Then M is called a mean-square random invariant set of Φ in V if for all $t \in \mathbb{R}^+$,

$$\Phi(t, \tau)M(\tau) \subset M(\tau + t).$$

We list a part of basic assumptions here. Let Y be another separable Hilbert space. The norm of X is denoted by $\|\cdot\|_X$. Let $L(Y, X)$ be the space of continuous linear operators from Y to X . Given a symmetric and nonnegative operator $Q \in L(Y, X)$, we write $Y_0 = Q^{1/2}Y$. The space of Hilbert-Schmidt operators from Y_0 to X is denoted by $L_2(Y_0, X)$ with norm $\|\cdot\|_{L_2(Y_0, X)}$. From now on, we assume that W is a two-sided Y -valued Q -Wiener process defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in \mathbb{R}}, \mathbb{P})$. For simplicity, we denote $L^2(\Omega, \mathcal{F}; X)$ and its norm as $L^2(\Omega, X)$ and $L^2(\cdot, X)$, respectively.

Assumption 1. For the nonlinear drift term, we assume that $F : L^2(\Omega, X_0) \rightarrow L^2(\Omega, X)$ is globally Lipschitz continuous, i.e.

$$\|F(u_1) - F(u_2)\|_{L^2(\cdot, X)} \leq L_1 \|u_1 - u_2\|_{L^2(\cdot, X_0)} \quad (2.2)$$

for all $u_1, u_2 \in L^2(\Omega, X_0)$, where L_1 is the Lipschitz constant and $F(0) = 0$. Thus

$$\|F(u)\|_{L^2(\cdot, X)} \leq L_1 \|u\|_{L^2(\cdot, X_0)} \quad (2.3)$$

for all $u \in L^2(\Omega, X_0)$. For the nonlinear diffusion term, we assume $\sigma : L^2(\Omega, X_0) \rightarrow L^2(\Omega, L_2(Y_0, X))$ is globally Lipschitz continuous, i.e.

$$\|\sigma(u_1) - \sigma(u_2)\|_{L^2(\cdot, L_2(Y_0, X))} \leq L_2 \|u_1 - u_2\|_{L^2(\cdot, X_0)}, \quad (2.4)$$

for all $u_1, u_2 \in L^2(\Omega, X_0)$, where L_2 is the Lipschitz constant and $\sigma(0) = 0$. Thus

$$\sigma(u)_{L^2(\cdot, \mathcal{L}_2(Y_0, X))} \leq L_2 \|u\|_{L^2(\cdot, X_0)}, \quad \square \quad \square \quad (2.5)$$

for all $u \in L^2(\Omega, X_0)$.

2.3. Integrated semigroups

In this subsection, we collect some basic results about non-densely defined operator and integrated semigroups from [12–15]. The resolvent set of A is denoted by $\rho(A) = \{\lambda \in \mathbb{C} : \lambda I - A \text{ is invertible}\}$ and we write $R_\lambda(A) := (\lambda I - A)^{-1}$. The spectrum of A is denoted by $\sigma(A) := \mathbb{C} \setminus \rho(A)$. And we construct the part of A denoted by $A_0 : D(A_0) \subset X_0 \rightarrow X_0$, which is a linear operator on X_0 defined by

$$A_0 x = Ax, \quad x \in D(A_0) := \{y \in D(A) : Ay \in X_0\}.$$

Assume that there exists a constant ϑ satisfying $(\vartheta, +\infty) \subset \rho(A)$, which means $\rho(A) \neq \emptyset$. Then it follows from [14, Lemma 2.1] that $\bar{\rho}(A) = \rho(A_0)$, from which we get $\sigma(A) = \sigma(A_0)$.

Lemma 2.3. A linear operator $A : D(A) \subset X \rightarrow X$ is an infinitesimal generator of a C_0 -semigroup $(T(t))_{t \geq 0}$ satisfying $\|T(t)\|_{\mathcal{L}(X)} \leq Me^{\vartheta t}$, if and only if

- (i) A is densely defined in X ;
- (ii) A is a Hille–Yosida operator.

Then according to Assumption 2 and Lemma 2.3, we infer that A_0 generates a C_0 -semigroup on X_0 denoted by $(T(t))_{t \geq 0}$, which can also be denoted by $(e^{A_0 t})_{t \geq 0}$. Now, we give the definition of integrated semigroups.

Definition 2.4. Let $(X, \|\cdot\|)$ be a Banach space. A family of bounded linear operators $\{S(t)\}_{t \geq 0}$ on X is called an integrated semigroup if

- (i) $S(0) = 0$;
- (ii) the map $t \mapsto S(t)x$ is continuous on $[0, +\infty)$ for each $x \in X$;
- (iii) $S(t)$ satisfies

$$S(s)S(t) = \int_0^s (S(r+t) - S(r))dr, \quad s, t \geq 0. \quad \square$$

An integrated semigroup $\{S(t)\}_{t \geq 0}$ is said to be non-degenerate if whenever $S(t)x = 0$, $t \geq 0$, then $x = 0$. According to Thieme [19], we say that a linear operator $A : D(A) \subset X \rightarrow X$ is the generator of a non-degenerate integrated semigroup $\{S(t)\}_{t \geq 0}$ on X if and only if

$$x \in D(A), \quad y = Ax \Leftrightarrow S(t)x - tx = \int_0^t S(s)yds, \quad t \geq 0. \quad \square$$

The following result is from [12, Proposition 2.5].

Lemma 2.4. Let Assumption 1 be satisfied. Then A generate a uniquely determined non-degenerate integrated semigroup $\{S(t)\}_{t \geq 0}$. Moreover, for each $x \in X$, $t \geq 0$ and each $\mu > \vartheta$, $S(t)x$ is given by

$$S(t)x = \mu \int_0^t T(s)R_\mu(A)ds + (I - T(t))R_\mu(A)x.$$

Also, the map $t \mapsto S(t)x$ is continuously differentiable if and only if $x \in X_0$ and

$$\frac{dS(t)x}{dt} = T(t)x, \quad t \geq 0, \quad x \in X_0. \quad \square \quad \square$$

From Lemma 2.4, we can deduce that for $\mu > \vartheta$, $S(t)$ commutes with $(\mu I - A)^{-1}$ and

$$S(t)x = \int_0^t T(s)xds, \quad x \in X_0. \quad \square$$

Hence, for each $x \in X_0, t \geq 0$ and each $\mu > \vartheta$,

$$R_\mu(A)S(t)x = S(t)R_\mu(A)x = \int_0^t T(s)R_\mu(A)x ds. \quad (2.6)$$

In addition, by [2, Lemma 3.2.2(b), (d)], $S(t)x \in X_0$ for $x \in X$. Thus by (2.6) and Lemma 2.2(i), we have for each $x \in X_0, t \geq 0$ and each $\mu > \vartheta$,

$$S(t)x = \lim_{\mu \rightarrow +\infty} \int_0^t T(s)\mu R_\mu(A)x ds.$$

The following result is from [12, Lemma 2.6].

Lemma 2.5. *Let Assumption 1 be satisfied and let $\tau_0 > 0$ be fixed. Denote for each $f \in C^1([0, \tau_0], X)$,*

$$(S \square f)(t) = \int_0^t S(s)f(t-s)ds, \quad t \in [0, \tau_0]. \quad \square$$

Then we have the following:

- (i) *The map $t \mapsto (S \square f)(t)$ is continuously differentiable on $[0, \tau_0]$;*
- (ii) *$(S \square f)(t) \in D(A)$, $t \in [0, \tau_0]$;* $\square$
- (iii) *if we set $u(t) = \frac{d}{dt}(S \square f)(t)$, then*

$$u(t) = A \int_0^t u(s)ds + \int_0^t f(s)ds, \quad t \in [0, \tau_0]. \quad \square$$

- (iv) *For each $\lambda > \vartheta$, and each $t \in [0, \tau_0]$, we have*

$$R_\lambda(A) \frac{d}{dt}(S \square f)(t) = \int_0^t T(t-s)R_\lambda(A)f(s)ds.$$

Assumption 3. *Assume that there exists real number $T > 0$, and a non-decreasing map $\delta : [0, \tau_0] \rightarrow [0, +\infty)$ such that for each $f \in C^1([0, \tau_0]; X)$,*

$$\frac{d}{dt}(S \square f)(t) \leq \delta(t) \sup_{s \in [0, t]} f(s), \quad t \in [0, \tau_0]. \quad \square \quad \square \quad \square$$

The following result is from [14, Proposition 2.13].

Lemma 2.6. *Let Assumptions 2 and 3 be satisfied. Then for $\kappa > \vartheta$, there exists $C_\kappa > 0$ such that $f \in C(\mathbb{R}; X)$ and*

$$\frac{d}{dt}(S \square f)(t) \leq C_\kappa \sup_{s \in [0, t]} e^{\kappa(t-s)} f(s), \quad t \in \mathbb{R}. \quad \square \quad \square \quad \square$$

Moreover, for each $\varepsilon > 0$, if $\rho_\varepsilon > 0$ satisfying $M\delta(\rho_\varepsilon) \leq \varepsilon$, it holds

$$C_\kappa = \frac{2\varepsilon \max(1, e^{-\kappa \rho_\varepsilon})}{1 - e^{(\vartheta - \kappa) \rho_\varepsilon}}.$$

2.4. Integrated solutions

Herein, we will not describe the specific definition of integrated solutions to (1.2) and (1.3). Details about this can be found in [10, 11].

Assumption 4. $F : X_0 \rightarrow X$ is globally Lipschitz continuous, i.e. $\|F(u) - F(v)\|_X \leq L \|u - v\|_{X_0}$ for all $u, v \in X_0$. $\rightarrow$

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such that for each $f \in C(\mathbb{R}; X)$,

$$\frac{d}{dt}(S \circ f)(t) \leq \delta(t) \sup_{s \in [0, t]} f(s), \quad t \in \mathbb{R}. \quad \square$$

Then, we have the following result from [11, Theorem 2.1]

Theorem 2.1. *Let Assumptions 1, 2 and 5 be satisfied. Given $\tau \in \mathbb{R}$ and $u_0 \in L^2(\Omega, F_\tau; X_0)$, then (1.3) possesses a unique integrated solution in $L^2(\Omega, X_0)$ on $[\tau, +\infty)$ given by*

$$\begin{aligned} u(t, \tau, u_0) = & T(t - \tau)u_0 + \lim_{\lambda \rightarrow +\infty} \int_{\tau}^t T(t - s)\lambda R_\lambda(A)F(u(s))ds \\ & + \lim_{\lambda \rightarrow +\infty} \int_{\tau}^t T(t - s)\lambda R_\lambda(A)\sigma(u(s))dW(s). \end{aligned} \quad (2.8)$$

Now, we can define a mean-square random dynamical system for (1.3). Given $t \in \mathbb{R}^+$ and $\tau \in \mathbb{R}$, let $\Phi(t, \tau)$ be a mapping from $L^2(\Omega, F_\tau; X_0)$ to $L^2(\Omega, F_{\tau+t}; X_0)$ given by

$$\Phi(t, \tau)(u_0) = u(t + \tau, \tau, u_0).$$

where $u_0 \in L^2(\Omega, F_\tau; X_0)$. Due to the uniqueness of solutions, for $t, s \geq 0$ and $\tau \in \mathbb{R}$,

$$\Phi(t + s, \tau) = \Phi(t, s + \tau) \circ \Phi(s, \tau).$$

As a consequence, Φ is a mean-square random dynamical system generated by solutions of (1.3) in $L^2(\Omega, F; X)$. Based on the forward solutions (2.8), we could define the backward solutions on $(-\infty, \tau]$. Given $\tau \in \mathbb{R}$, an X_0 -valued predictable process $\xi(t), t \in (-\infty, \tau]$, is called a integrated solution of (1.3) on $(-\infty, \tau]$ if $\xi(r) \in L^2(\Omega, F_r; X_0)$ for all $r \in (-\infty, \tau]$ and $u(t, r, \xi(r)) = \xi(t)$ for all $r \leq t \leq \tau$, where $u(t, r, \xi(r)) = \xi(t)$ is the unique solution of (1.3) with initial value $\xi(r)$ at initial time r .

2.5. Exponential dichotomy and exponential trichotomy

Assumption 6. *We assume $T(t)$ satisfies the pseudo exponential dichotomy with exponents $\beta < \alpha$ and bound K , that is, there exists a continuous linear projection operator with finite rank $\Pi_{0u} \in L(X_0)$ such that*

$$\Pi_{0u}T(t) = T(t)\Pi_{0u}, \quad t \in \mathbb{R}. \quad \square$$

The restriction $T(t)|_{R(\Pi_{0u})}, t \geq 0$ is an isomorphism of the range $R(\Pi_{0u})$ of Π_{0u} onto itself, and we define $T(t)$ for $t < 0$ as the inverse map of $T(-t)|_{R(\Pi_{0u})}$; Denote $\Pi_{0s} = I_{X_0} - \Pi_{0u}$, $X_{0u} = \Pi_{0u}X_0$ and $X_{0s} = \Pi_{0s}X_0$, it holds

$$T_{A_{0u}}(t)\Pi_{0u}x = Ke^{\alpha t}x, \quad t \leq 0, \quad \square \quad (2.9) \quad \square$$

$$T_{A_{0s}}(t)\Pi_{0s}x = Ke^{\beta t}x, \quad t \geq 0, \quad \square \quad (2.10) \quad \square$$

where $A_{0p} = A_0|_{X_{0p}}$ and $T_{A_{0p}}(t)$ is the C_0 -semigroup generated by A_{0p} , $p \in \{u, s\}$. $\square$

Assumption 7. We assume $T(t)$ satisfies the exponential trichotomy with exponents $\alpha > \gamma \geq 0 \geq -\gamma > -\beta$ and bound K , that's there exist two continuous linear projection operators with finite-rank $\Pi_{0c} \in L(X_0) \setminus \{0\}$ and $\Pi_{0u} \in L(X_0)$ such that

$$\Pi_{0u}\Pi_{0c} = \Pi_{0c}\Pi_{0u} = 0,$$

and

$$\Pi_{0k}T(t) = T(t)\Pi_{0k}, \quad t \geq 0, \quad k \in \{c, u\}. \quad \square \quad \square$$

For $p \in \{c, u, s\}$, the restriction $T(t)|_{R(\Pi_{0p})}$, $t \geq 0$ is an isomorphism of the range $R(\Pi_{0p})$ of Π_{0p} onto itself, and we define $T(t)$ for $t < 0$ as the inverse map of $T(-t)|_{R(\Pi_{0p})}$; Denote

$$\Pi_{0s} = I_{X_0} - (\Pi_{0c} + \Pi_{0u}), \quad X_{0c} = \Pi_{0c}X_0, \quad X_{0u} = \Pi_{0u}X_0, \quad X_{0s} = \Pi_{0s}X_0.$$

and for all $x \in X_0$, it holds

$$T_{A_{0c}}(t)\Pi_{0c}x = Ke^{\gamma|t|}x, \quad t \in \mathbb{R}, \quad \square \quad (2.11) \quad \square$$

$$T_{A_{0u}}(t)\Pi_{0u}x = Ke^{\alpha t}x, \quad t \leq 0, \quad \square \quad (2.12) \quad \square$$

$$T_{A_{0s}}(t)\Pi_{0s}x = Ke^{-\beta t}x, \quad t \geq 0, \quad \square \quad (2.13) \quad \square$$

where $A_{0p} = A_0|_{X_{0p}}$ and $T_{A_{0p}}(t)$ is the C_0 -semigroup generated by A_{0p} , $p \in \{c, u, s\}$. $\square$

Since we only require the Hille–Yosida condition on X_0 , one needs to extend the mentioned projections from X_0 to X and outline the associ

We use Lemma 2.8 to state the decomposition on X . Denote $\Pi_u : X \rightarrow X$ the unique extension of Π_{0u} and let $\Pi_s = I_X - \Pi_u$. Then we have for each $p \in \{u, s\}$,

$$\Pi_p R_\lambda(A) = R_\lambda(A) \Pi_p, \quad \lambda > \vartheta, \quad \square$$

and

$$\Pi_p(X_0) \subset X_0.$$

And we have

$$\Pi_p(X_0) = \Pi_{0p}(X_0) = X_{0p}.$$

Set

$$X_p = \Pi_p(X), \quad \text{and} \quad A_p = A|_{X_p}.$$

Then we have $X_u = X_{0u}$. Moreover, we have

$$X_0 = X_{0s} \oplus X_{0u}, \quad \text{and} \quad X = X_s \oplus X_u.$$

Herein, X_{0u} and X_{0s} are called unstable subspace and stable subspace of X_0 , respectively. Similarly, For each $k \in \{c, u\}$, denote by $\Pi_k : X \rightarrow X$ the unique extension of Π_{0k} and

$$\Pi_s = I_X - (\Pi_c + \Pi_u).$$

Then we have for each $p \in \{c, u, s\}$,

$$\Pi_p(\lambda I - A)^{-1} = (\lambda I - A)^{-1} \Pi_p, \quad \lambda > \vartheta, \quad \square$$

and

$$\Pi_p(X_0) = \Pi_{0p}(X_0) = X_{0p}.$$

And for each $p \in \{c, u\}$,

$$\Pi_p(X_0) \subset X_0.$$

Then for each $p \in \{c, u, s\}$, set

$$X_p = \Pi_p(X), \quad \text{and} \quad A_p = A|_{X_p}.$$

Then for each $k \in \{c, u\}$,

$$X_{0k} = X_k, \quad X_0 = X_{0s} \oplus X_{0c} \oplus X_{0u}, \quad \text{and} \quad X = X_s \oplus X_c \oplus X_u.$$

Herein, X_{0c} , X_{0u} and X_{0s} are called center subspace, unstable subspace and stable subspace of X_0 , respectively. For more details, see [14, p. 22].

3. Smoothness of Center Manifolds

This section is dedicated to the $\mathbf{C}^k$ -smoothness of random center invariant manifolds of (1.2). For each $\eta > 0$ and a Banach space $(X, \|\cdot\|_X)$, we define the Banach space

$$\mathbf{C}_\eta(\mathbb{R}; X) = \{f \in \mathbf{C}(\mathbb{R}; X) : \sup_{t \in \mathbb{R}} e^{-\eta|t| - \int_0^t z(\theta_\tau \omega) d\tau} \|f(t)\|_X < \infty\}$$

with the norm

$$\|f\|_{\mathbf{C}_\eta(\mathbb{R}; X)} = \sup_{t \in \mathbb{R}} e^{-\eta|t| - \int_0^t z(\theta_\tau \omega) d\tau} \|f(t)\|_X. \quad \square \quad \square$$

Set

$$\mathbf{M}^c(\omega) := \{v_0 \in X_0 : v(t, \omega, v_0) \in \mathbf{C}_\eta(\mathbb{R}; X_0)\},$$

where $v(t, \omega, v_0)$ is the solution of (2.7) with the initial data $v(0) = v_0$. We will use $\mathbf{C}_\eta$ to denote $\mathbf{C}_\eta(\mathbb{R}; X_0)$ for simplicity later. We have the following result from [10, Theorems 3.1 and 3.2].

Lemma 3.1. *Let Assumptions 2-4 and 7 be satisfied. If $\gamma < \eta < \min\{\alpha, \beta\}$ and $-\beta < \zeta < -\eta$ with*

$$KL \quad \mathbf{C}_\zeta + \frac{1}{\eta - \gamma} + \frac{1}{\alpha - \eta} < 1, \quad (3.1)$$

then there exists a center invariant manifold for (2.7), given by

$$\mathbf{M}^c(\omega) = \{\xi + h^c(\xi, \omega) : \xi \in X_{0c}\},$$

where $h^c(\cdot, \omega) : X_{0c} \rightarrow X_{0u} \oplus X_{0s}$ is a Lipschitz continuous map and $h^c(0, \omega) = 0$. $\mathbf{M}^{c*}(\omega) = \Xi^{-1}(\omega, \mathbf{M}^c(\omega))$ is a center invariant manifold of (1.2).

We prove that $\mathbf{M}^c(\omega)$ is a $\mathbf{C}^k$ invariant center manifold for (2.7) by induction.

Theorem 3.1. *Let Assumptions 2-4 and 7 be satisfied. Assume F is $\mathbf{C}^k$ in u . If $\gamma < k\eta < \min\{\alpha, \beta\}$ and $-\beta < \zeta < -k\eta$ with*

$$KL \quad \mathbf{C}_{\zeta+(i-1)\eta} + \frac{1}{i\eta - \gamma} + \frac{1}{\alpha - i\eta} < 1, \quad \text{for all } 1 \leq i \leq k, \quad (3.2)$$

then $\mathbf{M}^c(\omega)$ is a $\mathbf{C}^k$ center invariant manifold of (1.2), i.e. $h^c(\xi, \omega)$ is $\mathbf{C}^k$ in ξ .

Proof. Step 1. We prove that the fixed point $\nabla(t, \omega, \xi)$ of the Lyapunov-Perron operator $\mathbf{J}(\cdot, \xi)$ defined in [10, Theorem 3.1] is $\mathbf{C}^1$. Note that for $\gamma < \eta' < \eta'' < \min\{\alpha, \beta\}$, $\mathbf{C}_{\eta'} \subset \mathbf{C}_{\eta''}$, thus a fixed point in $\mathbf{C}_{\eta'}$ must be in $\mathbf{C}_{\eta''}$. So $\nabla(\cdot, \omega, \xi)$ is independent of η satisfying (3.1).

Clearly, there exists $\nu^* > 0$ such that for $0 < \nu \leq \nu^*$, we have

$$-\beta < \zeta < -i\eta - 2\nu < -i\eta - \nu \quad (3.3)$$

$$\gamma < i\eta - 2\nu < i\eta - \nu < \min\{\alpha, \beta\} \quad (3.4)$$

$$K L C_{\zeta + j\nu} + \frac{1}{i\eta - j\nu - \gamma} + \frac{1}{\alpha - i\eta + j\nu} < 1, \quad \text{for all } 1 \leq i \leq k, j = 1, 2. \quad (3.5)$$

Then $J(\bullet, \xi)$ is a uniform contraction in $C_{\eta-\nu} \square C_\eta$. By [10, (3.8)], for $\xi \square X_{0c}$,

$$\begin{aligned} \nabla(t, \omega, \xi) &= \varphi_{A_{0c}}(t, 0)\xi + \int_0^t \varphi_{A_{0c}}(t, r)\Pi_c G(\theta_r \omega, \nabla(r, \omega, \xi))dr \\ &\quad - \int_t^{+\infty} \varphi_{A_{0u}}(t, r)\Pi_u G(\theta_r \omega, \nabla(r, \omega, \xi))dr \\ &\quad + \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^t \varphi_{A_{0s}}(t, r)\lambda(\lambda I - A_s)^{-1}\Pi_s G(\theta_r \omega, \nabla(r, \omega, \xi))dr. \end{aligned}$$

By [10, (3.13)], for $\xi, \xi_0 \square X_{0c}$ and $i = 1, 2$, ∇ satisfies

$$\|\nabla(t, \omega, \xi) - \nabla(t, \omega, \xi_0)\|_{C_{\eta-i\nu}} \leq \frac{K}{1 - K L C_{\zeta + i\nu} + \frac{1}{\eta - i\nu - \gamma} + \frac{1}{\alpha - \eta + i\nu}} \|\xi - \xi_0\|. \quad (3.6)$$

Fix $\nu \square (0, \nu^*]$, we first prove that ∇ is differentiable from X_{0c} to $C_{\eta-\nu}$. For $\xi \square X_{0c}$, $\nu \square C_{\eta-\nu}$, define $H : C_{\eta-\nu} \rightarrow C_{\eta-\nu}$ as follows:

$$\begin{aligned} H\nu &= \int_0^t \varphi_{A_{0c}}(t, r)\Pi_c D_v G(\theta_r \omega, \nabla(r, \omega, \xi_0))\nu(r)dr \\ &\quad - \int_t^{+\infty} \varphi_{A_{0u}}(t, r)\Pi_u D_v G(\theta_r \omega, \nabla(r, \omega, \xi_0))\nu(r)dr \\ &\quad + \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^t \varphi_{A_{0s}}(t, r)\lambda R_\lambda(A_s)\Pi_s D_v G(\theta_r \omega, \nabla(r, \omega, \xi_0))\nu(r)dr. \end{aligned}$$

Using the same arguments as we proved that J is a contraction and noticing that $D_v G(\theta_s \omega, \nu) \leq \text{Lip } G = L$, we have H is a bounded linear operator from $C_{\eta-\nu}$ to itself with the norm

$$K L C_{\zeta + \nu} + \frac{1}{\eta - \nu - \gamma} + \frac{1}{\alpha - \eta + \nu} < 1.$$

Thus $\text{id} + H$ is invertible in $C_{\eta-\nu}$. For $\xi, \xi_0 \square X_{0c}$, set

$$\begin{aligned} I_c(t) &= \int_0^t \varphi_{A_{0c}}(t, r)\Pi_c [G(\theta_r \omega, \nabla(r, \omega, \xi)) - G(\theta_r \omega, \nabla(r, \omega, \xi_0))] \\ &\quad - D_v G(\theta_r \omega, \nabla(r, \omega, \xi_0))(\nabla(r, \omega, \xi) - \nabla(r, \omega, \xi_0))]dr \end{aligned}$$

$$\begin{aligned}
& + \int_{+\infty}^t \varphi_{A_{0u}}(t, r) \Pi_u [G(\theta_r \omega, \nabla(r, \omega, \xi)) - G(\theta_r \omega, \nabla(r, \omega, \xi_0))] \\
& - D_v G(\theta_r \omega, \nabla(r, \omega, \xi_0)) (\nabla(r, \omega, \xi) - \nabla(r, \omega, \xi_0))] dr \\
& + \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^t \varphi_{A_{0s}}(t, r) \lambda R_\lambda(A_s) \Pi_s [G(\theta_r \omega, \nabla(r, \omega, \xi)) \\
& - G(\theta_r \omega, \nabla(r, \omega, \xi_0)) - D_v G(\theta_r \omega, \nabla(r, \omega, \xi_0)) (\nabla(r, \omega, \xi) - \nabla(r, \omega, \xi_0))] dr.
\end{aligned}$$

We prove $\|I_c\|_{C_{\eta-\nu}} = o(|\xi - \xi_0|)$ as $\xi \rightarrow \xi_0$, then we have $\square \square \rightarrow$

$$\begin{aligned}
& \nabla(\bullet, \omega, \xi) - \nabla(\bullet, \omega, \xi_0) + (\nabla(\bullet, \omega, \xi) - \nabla(\bullet, \omega, \xi_0)) \\
& = \varphi_{A_0}(t, 0)(\xi - \xi_0) + I_c \\
& = \varphi_{A_0}(t, 0)(\xi - \xi_0) + o(|\xi - \xi_0|),
\end{aligned}$$

which implies

$$\nabla(\bullet, \omega, \xi) - \nabla(\bullet, \omega, \xi_0) = (\text{id} + I_c)^{-1} \varphi_{A_0}(t, 0)(\xi - \xi_0) + o(|\xi - \xi_0|).$$

We know that $\varphi_{A_0}(t, 0) = T(t)e^{\int_0^t z(\theta_\tau \omega) d\tau}$ is a bounded operator, therefore $\nabla(\bullet, \omega, \xi)$ is differentiable in ξ and $D_\xi \nabla(\bullet, \omega, \xi) \in L(X_{0c}, C_{\eta-\nu})$. Next, we prove $\|I_c\|_{C_{\eta-\nu}} = o(|\xi - \xi_0|)$ as $\xi \rightarrow \xi_0$. We write $e^{-(\eta-\nu)|t| - \int_0^t z(\theta_\tau \omega) d\tau} I_c$ as a sum of six terms, i.e. $e^{-(\eta-\nu)|t| - \int_0^t z(\theta_\tau \omega) d\tau} I_c = \sum_{i=1}^6 I_{ci}$, where for $t > N_1$,

$$\begin{aligned}
I_{c1} &= e^{-(\eta-\nu)|t| - \int_0^t z(\theta_\tau \omega) d\tau} \int_{N_1}^t \varphi_{A_{0c}}(t, r) \Pi_c [G(\theta_r \omega, \nabla(r, \omega, \xi)) \\
& - G(\theta_r \omega, \nabla(r, \omega, \xi_0)) - D_v G(\theta_r \omega, \nabla(r, \omega, \xi_0)) (\nabla(r, \omega, \xi) - \nabla(r, \omega, \xi_0))] dr,
\end{aligned}$$

and $I_{c1} = 0$ for $t \leq N_1$; For $t > N_1$,

$$\begin{aligned}
I_{c2} &= e^{-(\eta-\nu)|t| - \int_0^t z(\theta_\tau \omega) d\tau} \int_0^{N_1} \varphi_{A_{0c}}(t, r) \Pi_c [G(\theta_r \omega, \nabla(r, \omega, \xi)) \\
& - G(\theta_r \omega, \nabla(r, \omega, \xi_0)) - D_v G(\theta_r \omega, \nabla(r, \omega, \xi_0)) (\nabla(r, \omega, \xi) - \nabla(r, \omega, \xi_0))] dr,
\end{aligned}$$

and change N_1 to t if $t \leq N_1$. For $t < N_2$,

$$\begin{aligned}
I_{c3} &= -e^{-(\eta-\nu)|t| - \int_0^t z(\theta_\tau \omega) d\tau} \int_t^{N_2} \varphi_{A_{0u}}(t, r) \Pi_u [G(\theta_r \omega, \nabla(r, \omega, \xi)) \\
& - G(\theta_r \omega, \nabla(r, \omega, \xi_0)) - D_v G(\theta_r \omega, \nabla(r, \omega, \xi_0)) (\nabla(r, \omega, \xi) - \nabla(r, \omega, \xi_0))] dr,
\end{aligned}$$

and $I_{c3} = 0$ for $t \geq N_2$; For $t < N_2$,

$$\begin{aligned}
I_{c4} &= -e^{-(\eta-\nu)|t| - \int_0^t z(\theta_\tau \omega) d\tau} \int_{N_2}^{+\infty} \varphi_{A_{0u}}(t, r) \Pi_u [G(\theta_r \omega, \nabla(r, \omega, \xi)) \\
& - G(\theta_r \omega, \nabla(r, \omega, \xi_0)) - D_v G(\theta_r \omega, \nabla(r, \omega, \xi_0)) (\nabla(r, \omega, \xi) - \nabla(r, \omega, \xi_0))] dr,
\end{aligned}$$

and change N_2 to t if $t \geq N_2$. For $t > N_3$,

$$I_{c5} = -e^{-(\eta-\nu)|t|-\int_0^t z(\theta_\tau \omega) d\tau} \lim_{\lambda \rightarrow +\infty} \int_{N_3}^t \varphi_{A_{0s}}(t, r) \lambda R_\lambda(A_s) \Pi_s [G(\theta_r \omega, \nabla(r, \omega, \xi)) \\ - G(\theta_r \omega, \nabla(r, \omega, \xi_0)) - D_v G(\theta_r \omega, \nabla(r, \omega, \xi_0))(\nabla(r, \omega, \xi) - \nabla(r, \omega, \xi_0))] dr,$$

and $I_{c5} = 0$ for $t \leq N_3$; For $t > N_3$,

$$I_{c6} = -e^{-(\eta-\nu)|t|-\int_0^t z(\theta_\tau \omega) d\tau} \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^{N_3} \varphi_{A_{0s}}(t, r) \lambda R_\lambda(A_s) \Pi_s [G(\theta_r \omega, \nabla(r, \omega, \xi)) \\ - G(\theta_r \omega, \nabla(r, \omega, \xi_0)) - D_v G(\theta_s \omega, \nabla(r, \omega, \xi_0))(\nabla(r, \omega, \xi) - \nabla(r, \omega, \xi_0))] dr,$$

and change N_3 to t if $t \leq N_3$. N_1, N_2, N_3 above are large positive numbers to be chosen later. By (2.11) and (3.6), for $t > N_1$,

$$|I_{c1}| \leq 2KL \int_{N_1}^t e^{\gamma|t-r|} e^{-(\eta-\nu)|t|} e^{(\eta-2\nu)|r|} dr |\nabla(\bullet, \omega, \xi) - \nabla(\bullet, \omega, \xi_0)|_{C_{\eta-2\nu}} \\ \leq 2KL |\nabla(\bullet, \omega, \xi) - \nabla(\bullet, \omega, \xi_0)|_{C_{\eta-2\nu}} e^{(\gamma-\eta+\nu)|t|} \int_{N_1}^t e^{(\eta-2\nu-\gamma)|r|} dr \\ \leq \frac{2K^2L}{(\eta-2\nu-\gamma) \left(1 - KL \left(C_{\zeta+2\nu} + \frac{1}{\eta-2\nu-\gamma} + \frac{1}{\alpha-\eta+2\nu}\right)\right)} e^{-\nu N_1} |\xi - \xi_0|.$$

Choose N_1 so large that

$$\sup_{t \in \mathbb{R}} |I_{c1}| \leq \frac{\varepsilon}{6} |\xi - \xi_0|. \quad (3.7)$$

Fix such N_1 ,

$$|I_{c2}| \leq K |\nabla(\bullet, \omega, \xi) - \nabla(\bullet, \omega, \xi_0)|_{C_{\eta-\nu}} e^{(\gamma-\eta+\nu)|t|} \int_0^{N_1} e^{(\eta-\nu-\gamma)|r|} dr \\ \times |D_v G(\theta_r \omega, b\nabla(r, \omega, \xi) + (1-b)\nabla(r, \omega, \xi_0)) - D_v G(\theta_r \omega, \nabla(r, \omega, \xi_0))| db dr.$$

Since $G(\omega, v)$ is C^k , $D_v G(\omega, v)$ is continuous. For $\varepsilon > 0$, there exists $\rho_1 > 0$ such that if $|\xi - \xi_0| < \rho_1$,

$$D_v G(\theta_r \omega, b\nabla(r, \omega, \xi) + (1-b)\nabla(r, \omega, \xi_0)) - D_v G(\theta_r \omega, \nabla(r, \omega, \xi_0)) \\ \leq \frac{\varepsilon(\eta-\nu-\gamma) \left(1 - KL \left(C_{\zeta+\nu} + \frac{1}{\eta-\nu-\gamma} + \frac{1}{\alpha-\eta+\nu}\right)\right)}{6K^2 e^{(\eta-\nu-\gamma)N_1}}.$$

Then by (3.6),

$$\sup_{t \in \mathbb{R}} |I_{c2}| \leq \frac{\varepsilon}{6} |\xi - \xi_0| \quad \text{for } |\xi - \xi_0| < \rho_1. \quad (3.8)$$

Similarly, by choosing N_2 to be sufficiently large, we have

$$\sup_{t \in \mathbb{R}} |I_{c3}| \leq \frac{\varepsilon}{6} |\xi - \xi_0|. \quad (3.9)$$

Fix such N_2 , there exists $\rho_2 > 0$ such that if $|\xi - \xi_0| < \rho_1$,

$$\sup_{t \in \mathbb{R}} |I_{c4}| \leq \frac{\varepsilon}{6} |\xi - \xi_0|. \quad (3.10)$$

For I_{c5} , by Lemma 2.6

$$\begin{aligned} |I_{c5}| &= e^{-(\eta-\nu)|t| - \int_0^t z(\theta_\tau \omega) d\tau} \lim_{\lambda \rightarrow +\infty} \lim_{p \rightarrow -\infty} \int_0^{t-N_3} \varphi_{A_{0s}}(t, l + N_3) \lambda R_\lambda(A_s) \\ &\quad \times \Pi_s[G(\theta_{l+N_3} \omega, \nabla(l + N_3, \omega, \xi)) - G(\theta_{l+N_3} \omega, \nabla(l + N_3, \omega, \xi_0)) \\ &\quad - D_v G(\theta_{l+N_3} \omega, \nabla(l + N_3, \omega, \xi_0))(\nabla(l + N_3, \omega, \xi) - \nabla(l + N_3, \omega, \xi_0))] dl \\ &\leq 2LC_{\zeta+2\nu} \sup_{h \in [N_3, t]} e^{(\zeta+2\nu)(t-h)} e^{-(\eta-\nu)|t|} e^{(\eta-2\nu)|h|} |\nabla(h, \omega, \xi) - \nabla(h, \omega, \xi_0)|_{C_{\eta-2\nu}} \\ &\leq 2LC_{\zeta+2\nu} \sup_{h \in [N_3, t]} e^{(\zeta+4\nu-\eta)(t-h)} e^{-\nu t} |\nabla(h, \omega, \xi) - \nabla(h, \omega, \xi_0)|_{C_{\eta-2\nu}} \quad \square \end{aligned}$$

From (3.3) and (3.4), we have

$$\zeta + 2\nu + \eta < 0, \quad 2\nu - 2\eta < 0.$$

Then we have

$$\zeta + 4\nu - \eta < 0.$$

So by (3.6),

$$\begin{aligned} |I_{c5}| &\leq 2LC_{\zeta+2\nu} e^{-\nu t} |\nabla(h, \omega, \xi) - \nabla(h, \omega, \xi_0)|_{C_{\eta-2\nu}} \quad \square \\ &\leq \frac{2KLC_{\zeta+2\nu} e^{-\nu N_3}}{1 - KLC_{\zeta+2\nu} + \frac{1}{\eta-2\nu-\gamma} + \frac{1}{\alpha-\eta+2\nu}} |\xi - \xi_0|. \end{aligned}$$

And for the last one I_{c6} , for $t > N_3$,

$$\begin{aligned} |I_{c6}| &= e^{-(\eta-\nu)|t| - \int_0^t z(\theta_\tau \omega) d\tau} \lim_{\lambda \rightarrow +\infty} \lim_{p \rightarrow -\infty} \int_0^{N_3-p} \varphi_{A_{0s}}(t, l + p) \lambda R_\lambda(A_s) \\ &\quad \times \Pi_s[G(\theta_{l+p} \omega, \nabla(l + p, \omega, \xi)) - G(\theta_{l+p} \omega, \nabla(l + p, \omega, \xi_0)) \\ &\quad - D_v G(\theta_{l+p} \omega, \nabla(l + p, \omega, \xi_0))(\nabla(l + p, \omega, \xi) - \nabla(l + p, \omega, \xi_0))] dl \\ &\leq 2LC_{\zeta+2\nu} \sup_{h \in (-\infty, N_3]} e^{(\zeta+2\nu)(t-h)} e^{-(\eta-\nu)|t|} e^{(\eta-2\nu)|h|} |\nabla(h, \omega, \xi) \\ &\quad - \nabla(h, \omega, \xi_0)|_{C_{\eta-2\nu}}. \quad \square \end{aligned}$$

If $h > 0$,

$$\begin{aligned} |I_{c6}| &\leq 2LC_{\zeta+2\nu} \sup_{h \in (-\infty, N_3]} e^{(\zeta+4\nu-\eta)(t-h)} e^{-\nu t} |\nabla(h, \omega, \xi) - \nabla(h, \omega, \xi_0)|_{C_{\eta-2\nu}} \\ &\leq \frac{2KLC_{\zeta+2\nu} e^{-\nu N_3}}{1 - KL C_{\zeta+2\nu} + \frac{1}{\eta-2\nu-\gamma} + \frac{1}{\alpha-\eta+2\nu}} |\xi - \xi_0|. \end{aligned}$$

And if $h \leq 0$,

$$\begin{aligned} |I_{c6}| &\leq 2LC_{\zeta+2\nu} \sup_{h \in (-\infty, N_3]} e^{(\zeta+3\nu-\eta)t} e^{-(\zeta+\eta)h} |\nabla(h, \omega, \xi) - \nabla(h, \omega, \xi_0)|_{C_{\eta-2\nu}} \\ &= 2LC_{\zeta+2\nu} e^{(\zeta+\eta)(t-N_3)} e^{(3\nu-2\eta)t} |\nabla(h, \omega, \xi) - \nabla(h, \omega, \xi_0)|_{C_{\eta-2\nu}} \\ &\leq \frac{2KLC_{\zeta+2\nu} e^{(3\nu-2\eta)N_3}}{1 - KL C_{\zeta+2\nu} + \frac{1}{\eta-2\nu-\gamma} + \frac{1}{\alpha-\eta+2\nu}} |\xi - \xi_0|. \end{aligned}$$

Note that $3\nu - 2\eta < 0$. Choose N_3 so large that

$$\sup_{t \in \mathbb{R}} |I_{c5}| \leq \frac{\varepsilon}{6} |\xi - \xi_0|. \quad (3.11)$$

$$\sup_{t \in \mathbb{R}} |I_{c6}| \leq \frac{\varepsilon}{6} |\xi - \xi_0|. \quad (3.12)$$

By taking $\rho_0 = \min\{\rho_1, \rho_2\}$ and combining (3.7)–(3.12), we have

$$|I_c|_{C_{\eta-2\nu}} \leq \varepsilon |\xi - \xi_0| \quad \text{for } |\xi - \xi_0| < \rho_0. \quad \square \quad \square$$

This implies $|I_c|_{C_{\eta-2\nu}} = o(|\xi - \xi_0|)$ as $\xi \rightarrow \xi_0$. Next, we prove $D_\xi \nabla(\bullet, \omega, \xi)$ is continuous with respect to ξ from X_{0c} to $L(X_{0c}, C_\eta)$. By [10, (3.2)], we have

$$\begin{aligned} D_\xi \nabla(t, \omega, \xi) &= \varphi_{A_{0c}}(t, 0) \Pi_{0c} \\ &\quad + \int_0^t \varphi_{A_{0c}}(t, r) \Pi_c D_v G(\theta_r \omega, \nabla(r, \omega, \xi)) D_\xi \nabla(r, \omega, \xi) dr \\ &\quad - \int_t^{+\infty} \varphi_{A_{0u}}(t, r) \Pi_u D_v G(\theta_r \omega, \nabla(r, \omega, \xi)) D_\xi \nabla(r, \omega, \xi) dr \\ &\quad + \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^t \varphi_{A_{0s}}(t, r) \lambda R_\lambda(A_s) \Pi_s D_v G(\theta_r \omega, \nabla(r, \omega, \xi)) D_\xi \nabla(r, \omega, \xi) dr. \end{aligned}$$

For $\xi \in X_{0c}$, $v \in C_\eta$, define the operator $H' : C_\eta \rightarrow C_\eta$ as follows:

$$\begin{aligned} H'v &= \int_0^t \varphi_{A_{0c}}(t, r) \Pi_c D_v G(\theta_s \omega, \nabla(r, \omega, \xi)) v(r) dr \\ &\quad - \int_t^{+\infty} \varphi_{A_{0u}}(t, r) \Pi_u D_v G(\theta_r \omega, \nabla(r, \omega, \xi)) v(r) dr \\ &\quad + \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^t \varphi_{A_{0s}}(t, r) \lambda R_\lambda(A_s) \Pi_s D_v G(\theta_r \omega, \nabla(r, \omega, \xi)) v(r) dr. \end{aligned}$$

Similarly, we have

$$1 - \kappa L \leq C_\zeta + \frac{1}{\eta - \gamma} + \frac{1}{\alpha - \eta} < 1. \quad \square \quad (3.13)$$

For $\xi, \xi_0 \in X_{0c}$,

$$\begin{aligned} & D_\xi \nabla(t, \omega, \xi) - D_\xi \nabla(t, \omega, \xi_0) \\ &= \int_0^t \varphi_{A_{0c}}(t, r) \Pi_c [D_v G(\theta_r \omega, \nabla(r, \omega, \xi)) D_\xi \nabla(r, \omega, \xi) \\ &\quad - D_v G(\theta_r \omega, \nabla(r, \omega, \xi_0)) D_\xi \nabla(r, \omega, \xi_0)] dr \\ &\quad + \int_{+\infty}^t \varphi_{A_{0u}}(t, r) \Pi_u [D_v G(\theta_r \omega, \nabla(r, \omega, \xi)) D_\xi \nabla(r, \omega, \xi) \\ &\quad - D_v G(\theta_r \omega, \nabla(r, \omega, \xi_0)) D_\xi \nabla(r, \omega, \xi_0)] dr \\ &\quad + \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^t \varphi_{A_{0s}}(t, r) \lambda R_\lambda(A_s) \Pi_s [D_v G(\theta_r \omega, \nabla(r, \omega, \xi)) D_\xi \nabla(r, \omega, \xi) \\ &\quad - D_v G(\theta_r \omega, \nabla(r, \omega, \xi_0)) D_\xi \nabla(r, \omega, \xi_0)] dr \\ &= H'(D_\xi \nabla(t, \omega, \xi) - D_\xi \nabla(t, \omega, \xi_0)) + I_c', \end{aligned} \quad (3.14)$$

where

$$\begin{aligned} I_c' &= \int_0^t \varphi_{A_{0c}}(t, r) \Pi_c [D_v G(\theta_r \omega, \nabla(r, \omega, \xi)) \\ &\quad - D_v G(\theta_r \omega, \nabla(r, \omega, \xi_0))] D_\xi \nabla(r, \omega, \xi_0) dr \\ &\quad + \int_{+\infty}^t \varphi_{A_{0u}}(t, r) \Pi_u [D_v G(\theta_s \omega, \nabla(r, \omega, \xi)) \\ &\quad - D_v G(\theta_r \omega, \nabla(r, \omega, \xi_0))] D_\xi \nabla(r, \omega, \xi_0) dr \\ &\quad + \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^t \varphi_{A_{0s}}(t, r) \lambda R_\lambda(A_s) \Pi_s [D_v G(\theta_r \omega, \nabla(r, \omega, \xi)) \\ &\quad - D_v G(\theta_r \omega, \nabla(r, \omega, \xi_0))] D_\xi \nabla(r, \omega, \xi_0) dr. \end{aligned}$$

By (3.13), $\text{id} + H'$ has a bounded inverse in $L(X_{0c}, C_\eta)$. From (3.14), we have

$$D_\xi \nabla(t, \omega, \xi) - D_\xi \nabla(t, \omega, \xi_0) = (\text{id} + H')^{-1} I_c'.$$

Then by (3.13),

$$D_\xi \nabla(t, \omega, \xi) - D_\xi \nabla(t, \omega, \xi_0) \leq \frac{\|I_c'\|_{L(X_{0c}, C_\eta)}}{1 - \kappa L \leq C_\zeta + \frac{1}{\eta - \gamma} + \frac{1}{\alpha - \eta}}. \quad (3.15)$$

Using the same procedure as for I_c , we have $\|I_c'\|_{L(X_{0c}, C_\eta)} = o(1)$ as $\xi \rightarrow \xi_0$. Then by (3.15), $D_\xi \nabla(\cdot, \omega, \xi)$ is continuous with respect to ξ . So we proved that $\nabla(t, \omega, \xi)$ is C^1 . $\square$

Step 2. We prove $\nabla(t, \omega, \xi)$ is $\mathbf{C}^k$.

Make the inductive assumption that $\nabla(\bullet, \omega, \xi)$ is $\mathbf{C}^j$ from X_{0c} to $\mathbf{C}_{j\eta}$ for all $1 \leq j \leq m-1$ and we prove it for $j = m$. By calculation, the $(m-1)$ -derivative $D_\xi^{m-1}\nabla(\bullet, \omega, \xi)$ satisfies the following equation:

$$\begin{aligned} D_\xi^{m-1}\nabla(t, \omega, \xi) = & \int_0^t \varphi_{A_{0c}}(t, r) \Pi_c D_v G(\theta_r \omega, \nabla(r, \omega, \xi)) D_\xi^{m-1}\nabla(r, \omega, \xi) dr \\ & - \int_t^{+\infty} \varphi_{A_{0u}}(t, r) \Pi_u D_v G(\theta_r \omega, \nabla(r, \omega, \xi)) D_\xi^{m-1}\nabla(r, \omega, \xi) dr \\ & + \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^t \varphi_{A_{0s}}(t, r) \lambda R_\lambda(A_s) \Pi_s D_v G(\theta_r \omega, \nabla(r, \omega, \xi)) \\ & \times D_\xi^{m-1}\nabla(r, \omega, \xi) dr + \int_0^t \varphi_{A_{0c}}(t, r) \Pi_c R_{m-1}(r, \omega, \xi) dr \\ & - \int_t^{+\infty} \varphi_{A_{0u}}(t, r) \Pi_u R_{m-1}(r, \omega, \xi) dr + \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^t \\ & \times \varphi_{A_{0s}}(t, r) \lambda R_\lambda(A_s) \Pi_s R_{m-1}(r, \omega, \xi) dr, \end{aligned}$$

where

$$R_{m-1}(r, \omega, \xi) = \sum_{i=1}^{m-3} \binom{m-2}{i} D_\xi^{m-2-i}(D_v G(\theta_r \omega, \nabla(r, \omega, \xi))) D_\xi^{i+1} \nabla(r, \omega, \xi).$$

Applying the chain rule to $D_\xi^{m-2-i}(D_v G(\theta_r \omega, \nabla(r, \omega, \xi)))$, we observe that each term in $R_{m-1}(r, \omega, \xi)$ contains factors: $D_v^{i_1} G(\theta_r \omega, \nabla(r, \omega, \xi))$ for some $2 \leq i_1 \leq m-1$ and at least two derivatives $D_\xi^{i_2} \nabla(r, \omega, \xi)$ and $D_\xi^{i_3} \nabla(r, \omega, \xi)$ for some $i_2, i_3 \in \{1, \dots, m-2\}$. Since $D_\xi^i \nabla(\bullet, \omega, \xi) \in \mathbf{C}_{i\eta}$ for $i = 1, \dots, m-1$ and F is $\mathbf{C}^k$, $R_{m-1}(\bullet, \omega, \xi) : X_{0c} \rightarrow L^{m-1}(X_{0c}, X_0)$ are $\mathbf{C}^1$ in ξ . Using the same argument we used in the case $k = 1$, we can show that $D_\xi^{m-1}\nabla(\bullet, \omega, \xi)$ is $\mathbf{C}^1$ from X_{0c} to $L^m(X_{0c}, \mathbf{C}_{m\eta})$. So $\nabla(t, \omega, \xi)$ is $\mathbf{C}^k$.

From [10], we know that $h^c(\xi, \omega) = \Pi_s \nabla(0, \omega, \xi) + \Pi_u \nabla(0, \omega, \xi)$ and

$$\begin{aligned} h^c(\xi, \omega) = & \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^0 \varphi_{A_{0s}}(0, r) \lambda (\lambda I - A_s)^{-1} \Pi_s G(\theta_r \omega, \nabla(r, \omega, \xi)) dr \\ & - \int_0^{+\infty} \varphi_{A_{0u}}(0, r) \Pi_u G(\theta_r \omega, \nabla(r, \omega, \xi)) dr. \end{aligned}$$

Thus, $h^c(\xi, \omega)$ is $\mathbf{C}^k$, which guarantees the $\mathbf{C}^k$ -smoothness of $M^c(\omega)$ and $M^{c*}(\omega)$. $\square$

4. Smoothness of Mean-Square Random Unstable Manifolds

This section is dedicated to the $\mathbf{C}^k$ -smoothness of mean-square random unstable invariant manifolds of (1.3). Given $\tau \in \mathbb{R}$, denote by $\mathbf{C}_\gamma^-$ the space of all X_0 -valued

F_t -progressively measurable processes $\xi(t) : t \in (\infty, \tau]$ such that $\xi : (\infty, \tau] \rightarrow L^2(\Omega, X_0)$ is continuous and

$$\sup_{t \leq \tau} (e^{-\gamma t} \xi(t))_{L^2(\cdot, X_0)} < \infty, \quad (4.1) \quad \square$$

with norm

$$\xi(t)_{C_\gamma} = \sup_{t \leq \tau} (e^{-\gamma t} \xi(t))_{L^2(\cdot, X_0)}, \quad \xi \in C_\gamma^-, \quad \square \quad \square \quad \square$$

where $\gamma \in (\beta, \alpha)$, we have the following result from [11, Theorem 3.1].

Lemma 4.1. *Let Assumptions 1, 2, 5 and 6 be satisfied. If $\beta < \zeta < \gamma < \alpha$ with*

$$K = \frac{L_1}{\alpha - \gamma} + L_1 C_\zeta + L_2 C_\zeta < 1, \quad (4.2)$$

then there exists a mean-square random unstable invariant manifold for (1.3) given by

$$M^u(\tau) = \{x + h^u(x, \tau) : x \in L^2(\Omega, F_\tau; X_{0u})\},$$

where $h^u(\cdot, \tau) : L^2(\Omega, F_\tau; X_{0u}) \rightarrow L^2(\Omega, F_\tau; X_{0s})$ is a Lipschitz continuous map.

We prove that $M^u(\tau)$ is a C^k mean-square unstable invariant manifold for (1.3) by induction.

Theorem 4.1. *Let Assumptions 1, 2, 5 and 6 be satisfied. Assume F and σ are C^k . If $\beta < \zeta < k\gamma < \alpha$ with*

$$K = \frac{L_1}{\alpha - i\gamma} + L_1 C_{\zeta+(i-1)\gamma} + L_2 C_{\zeta+(i-1)\gamma} < 1, \quad (4.3)$$

then $h^u(x, \tau)$ is C^k in x .

Proof. Step 1. We prove that the fixed point $\bar{\xi}(x) \in C_\gamma^-$ of the Lyapunov–Perron operator $J(\cdot, x, \tau)$ defined in [11, Theorem 3.1] is C^1 . Note that for $\beta < \gamma' < \gamma'' < \alpha$, $C_{\gamma'}^- \subset C_{\gamma''}^-$, thus a fixed point in $C_{\gamma'}^-$ must be in $C_{\gamma''}^-$. So $\bar{\xi}(\cdot, \tau, x)$ is independent of γ satisfying (4.2).

Clearly, there exists $v^* > 0$ such that for $0 < v \leq v^*$, we have

$$\beta < \zeta < i\gamma - 3v < i\gamma - 2v < i\gamma - v < \alpha, \quad (4.4)$$

$$K = \frac{L_1}{\alpha - i\gamma + jv} + L_1 C_{\zeta+jv} + L_2 C_{\zeta+jv} < 1, \quad \text{for all } 1 \leq i \leq k, j = 1, 2. \quad (4.5)$$

Then for given $\tau \in \mathbb{R}$, $J(\bullet, x, \tau)$ is a uniform contraction in $C_{\gamma-\nu}^- \subset C_\gamma^-$. By [11, (30)], for $t \leq \tau, x \in L^2(\Omega, F_\tau; X_{0u})$,

$$\begin{aligned} \bar{\xi}(t, \tau, x) = & \mathbb{E} \left[T_{A_{0u}}(t - \tau)x - \int_t^\tau T_{A_{0u}}(t - r) \Pi_u F(\bar{\xi}(r, \tau, x)) dr \mid F_t \right] \\ & + \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^t T_{A_{0s}}(t - r) \lambda R_\lambda(A_s) \Pi_s F(\bar{\xi}(r, \tau, x)) dr \\ & + \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^t T_{A_{0s}}(t - r) \lambda R_\lambda(A_s) \Pi_s \sigma(\bar{\xi}(r, \tau, x)) dW(r). \end{aligned}$$

By [11, (42)], for given $\tau \in \mathbb{R}, x, x_0 \in L^2(\Omega, F_\tau; X_{0u})$ and $i = 1, 2$, $\bar{\xi}$ satisfies

$$\begin{aligned} \bar{\xi}(t, \tau, x) - \bar{\xi}(t, \tau, x_0) & \in C_{\gamma-\nu} \quad \square \\ & \leq \frac{K}{1 - K \left(\frac{L_1}{\alpha - \gamma + i\nu} + L_1 C_{\zeta^+ i\nu} + L_2 C_{\zeta^+ i\nu} \right)} \|x - x_0\|_{L^2(\cdot, X_0)}. \quad (4.6) \end{aligned}$$

Fix $\nu \in (0, \nu^*]$, we first prove that $\bar{\xi}$ is differentiable from $L^2(\Omega, F_\tau; X_{0u})$ to $C_{\gamma-\nu}^-$. For $\tau \in \mathbb{R}, x_0 \in L^2(\Omega, F_\tau; X_{0u})$, $\xi \in C_{\gamma-\nu}^-$, define H as follows:

$$\begin{aligned} H\xi = & -\mathbb{E} \left[\int_t^\tau T_{A_{0u}}(t - r) \Pi_u D_\xi F(\bar{\xi}(r, \tau, x_0)) \xi(r) ds \mid F_t \right] \\ & + \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^t T_{A_{0s}}(t - r) \lambda R_\lambda(A_s) \Pi_s D_\xi F(\bar{\xi}(r, \tau, x_0)) \xi(r) dr \\ & + \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^t T_{A_{0s}}(t - r) \lambda R_\lambda(A_s) \Pi_s D_\xi \sigma(\bar{\xi}(r, \tau, x_0)) \xi(r) dW(r). \quad (4.7) \end{aligned}$$

Using the same arguments as we proved that J is a contraction and noticing that $D_\xi F(\xi) \in \text{Lip } F = L_1$, $D_\xi \sigma(\xi) \in \text{Lip } \sigma = L_2$, we have for $\xi \in C_{\gamma-\nu}^-$, $\square$

$$\sup_{t \leq \tau} e^{-(\gamma-\nu)t} \|\xi\|_{L^2(\cdot, X_0)} \leq K \left(\frac{L_1}{\alpha - \gamma + \nu} + L_1 C_{\zeta^+ \nu} + L_2 C_{\zeta^+ \nu} \right) \|\xi\|_{C_{\gamma-\nu}^-}.$$

Then H is a bounded linear operator from $C_{\gamma-\nu}^-$ into itself with norm

$$K \left(\frac{L_1}{\alpha - \gamma + \nu} + L_1 C_{\zeta^+ \nu} + L_2 C_{\zeta^+ \nu} \right) < 1$$

So $\text{id} + H$ has a bounded inverse in $C_{\gamma-\nu}^-$. For $x, x_0 \in L^2(\Omega, F_\tau; X_{0u})$, set

$$\begin{aligned} I_u(t) = & -\mathbb{E} \left[\int_t^\tau T_{A_{0u}}(t - r) \Pi_u [F(\bar{\xi}(r, \tau, x)) - F(\bar{\xi}(r, \tau, x_0))] \right. \\ & \left. - D_\xi F(\bar{\xi}(r, \tau, x_0))(\bar{\xi}(r, \tau, x) - \bar{\xi}(r, \tau, x_0)) \right] dr \mid F_t \end{aligned}$$

$$\begin{aligned}
& + \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^t T_{A_{0s}}(t-r) \lambda R_\lambda(A_s) \Pi_s [F(\bar{\xi}(r, \tau, x)) - F(\bar{\xi}(r, \tau, x_0))] \\
& - D_\xi F(\bar{\xi}(r, \tau, x_0))(\bar{\xi}(r, \tau, x) - \bar{\xi}(s, \tau, x_0))] dr \\
& + \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^t T_{A_{0s}}(t-r) \lambda R_\lambda(A_s) \Pi_s [\sigma(\bar{\xi}(r, \tau, x)) - \sigma(\bar{\xi}(r, \tau, x_0))] \\
& - D_\xi \sigma(\bar{\xi}(r, \tau, x_0))(\bar{\xi}(r, \tau, x) - \bar{\xi}(r, \tau, x_0))] dW(r).
\end{aligned}$$

We prove that $\|I_u\|_{C_{\gamma, \nu}} = o(|x - x_0|)$ as $x \rightarrow x_0$, then we have $\square \rightarrow$

$$\begin{aligned}
& \bar{\xi}(\bullet, \tau, x) - x \bar{\xi}(\bullet, \tau, x_0) + (\bar{\xi}(\bullet, \tau, x) - \bar{\xi}(\bullet, \tau, x_0)) \\
& = \mathbb{E}[T_{A_{0u}}(t - \tau)(x - x_0) | \mathcal{F}_t] + I_u(t).
\end{aligned}$$

Denote $Lx = \mathbb{E}[T_{A_{0u}}(t - \tau)x | \mathcal{F}_t]$, then we have

$$\bar{\xi}(\bullet, \tau, x) - \bar{\xi}(\bullet, \tau, x_0) = (\text{id} + H)^{-1} L(x - x_0) + o(|x - x_0|).$$

Denote $Lx = \mathbb{E}[T_{A_{0u}}(t - \tau)x | \mathcal{F}_t]$, then we have

$$\bar{\xi}(\bullet, \tau, x) - \bar{\xi}(\bullet, \tau, x_0) = (\text{id} + H)^{-1} L(x - x_0) + o(|x - x_0|).$$

Thus, $\bar{\xi}(\bullet, \tau, x)$ is differentiable in x , $D_x \bar{\xi}(\bullet, \tau, x) \in L(L^2(\Omega, \mathcal{F}_\tau; X_{0u}), \mathcal{C}_{\gamma-\nu}^-)$. Now, we prove $\|I_u\|_{C_{\gamma, \nu}} = o(|x - x_0|)$ as $x \rightarrow x_0$. We write $e^{\square(\gamma-\nu)t} \|I_u\|_{L^2(\cdot, X_0)}$ as a sum of six terms, i.e. $e^{-(\gamma-\nu)t} \|I_u\|_{L^2(\cdot, X_0)} = \sum_{i=1}^6 \|I_{ui}\|$, where for $t < N_1$, $\square \rightarrow$

$$\begin{aligned}
I_{u1} &= e^{-(\gamma-\nu)t} \mathbb{E} \int_{\tau}^{N_1} T_{A_{0u}}(t-r) \Pi_u [F(\bar{\xi}(r, \tau, x_0)) - F(\bar{\xi}(r, \tau, x))] \\
& - D_\xi F(\bar{\xi}(r, \tau, x_0))(\bar{\xi}(r, \tau, x_0) - \bar{\xi}(r, \tau, x))] dr | \mathcal{F}_t, \\
& L^2(\cdot, X_0)
\end{aligned}$$

and $I_{u1} = 0$ for $t \geq N_1$; For $t < N_1$,

$$\begin{aligned}
I_{u2} &= e^{-(\gamma-\nu)t} \mathbb{E} \int_{N_1}^t T_{A_{0u}}(t-r) \Pi_u [F(\bar{\xi}(r, \tau, x_0)) - F(\bar{\xi}(r, \tau, x))] \\
& - D_\xi F(\bar{\xi}(r, \tau, x_0))(\bar{\xi}(r, \tau, x_0) - \bar{\xi}(r, \tau, x))] dr | \mathcal{F}_t, \\
& L^2(\cdot, X_0)
\end{aligned}$$

and change N_1 to t if $t \geq N_1$. For $t > N_2$,

$$\begin{aligned}
I_{u3} &= e^{-(\gamma-\nu)t} \lim_{\lambda \rightarrow +\infty} \int_{N_2}^t T_{A_{0s}}(t-r) \lambda R_\lambda(A_s) \Pi_s [F(\bar{\xi}(r, \tau, x)) \\
& - F(\bar{\xi}(r, \tau, x_0)) - D_\xi F(\bar{\xi}(r, \tau, x_0))(\bar{\xi}(r, \tau, x) - \bar{\xi}(r, \tau, x_0))] dr, \\
& L^2(\cdot, X_0)
\end{aligned}$$

and $I_{u3} = 0$ if $t \leq N_2$; For $t > N_2$,

$$I_{u4} = e^{-(\gamma-\nu)t} \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^{N_2} T_{A_{0s}}(t-r) \lambda R_\lambda(A_s) \Pi_s [F(\bar{\xi}(r, \tau, x)) - F(\bar{\xi}(r, \tau, x_0)) - D_\xi F(\bar{\xi}(r, \tau, x_0))(\bar{\xi}(r, \tau, x) - \bar{\xi}(r, \tau, x_0))] dr, \quad L^2(-, X_0),$$

and change N_2 to t if $t \leq N_2$. For $t > N_3$,

$$I_{u5} = e^{-(\gamma-\nu)t} \lim_{\lambda \rightarrow +\infty} \int_{N_3}^t T_{A_{0s}}(t-r) \lambda R_\lambda(A_s) \Pi_s [\sigma(\bar{\xi}(r, \tau, x)) - \sigma(\bar{\xi}(r, \tau, x_0)) - D_\xi \sigma(\bar{\xi}(r, \tau, x_0))(\bar{\xi}(r, \tau, x) - \bar{\xi}(r, \tau, x_0))] dW(r), \quad L^2(-, X_0),$$

and $I_{u5} = 0$ if $t \leq N_3$; For $t > N_3$,

$$I_{u6} = e^{-(\gamma-\nu)t} \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^{N_3} T_{A_{0s}}(t-r) \lambda R_\lambda(A_s) \Pi_s [\sigma(\bar{\xi}(r, \tau, x)) - \sigma(\bar{\xi}(r, \tau, x_0)) - D_\xi \sigma(\bar{\xi}(r, \tau, x_0))(\bar{\xi}(r, \tau, x) - \bar{\xi}(r, \tau, x_0))] dW(r), \quad L^2(-, X_0),$$

and change N_3 to t if $t \leq N_3$. N_1, N_2, N_3 above are large positive numbers to be chosen later. For $t < N_1$,

$$|I_{u1}| \leq K \|\bar{\xi}(\cdot, \tau, x) - \bar{\xi}(\cdot, \tau, x_0)\|_{C_{\gamma-\nu}} e^{(\alpha-\gamma+\nu)t} \int_{\tau}^{N_1} e^{(\gamma-\nu-\alpha)r} \times \int_0^1 D_\xi F(b\bar{\xi}(r, \tau, x) + (1-b)\bar{\xi}(r, \tau, x_0)) - D_\xi F(\bar{\xi}(r, \tau, x_0)) db dr. \quad \square$$

Since $F(\xi) \in C^k$, $D_\xi F(\xi)$ is continuous. For any $\varepsilon > 0$, there exists $\rho_1 > 0$ such that if $\|x - x_0\|_{L^2(-, X_0)} < \rho_1$, $\square \quad \square$

$$D_\xi F(b\bar{\xi}(r, \tau, x) + (1-b)\bar{\xi}(r, \tau, x_0)) - D_\xi F(\bar{\xi}(r, \tau, x_0)) \leq \frac{\varepsilon(\alpha - \gamma + \nu) \{1 - K \frac{L_1}{\alpha - \gamma + \nu} + L_1 C_{\zeta+\nu} + L_2 C_{\zeta+\nu}\}}{6K^2 |e^{(\gamma-\nu-\alpha)(\tau-N_1)} - 1|}.$$

Then by (4.6),

$$\sup_{t \leq \tau} |I_{u1}| \leq \frac{\varepsilon}{6} \|x - x_0\|_{L^2(-, X_0)} \quad \text{for } \|x - x_0\|_{L^2(-, X_0)} < \rho_1. \quad \square \quad (4.8) \quad \square$$

And for $t < N_1$,

$$|I_{u2}| \leq K \|\bar{\xi}(\cdot, \tau, x) - \bar{\xi}(\cdot, \tau, x_0)\|_{C_{\gamma-\nu}} e^{(\alpha-\gamma+\nu)t} \int_{N_1}^t e^{(\gamma-\nu-\alpha)r} \times \int_0^1 D_\xi F(b\bar{\xi}(r, \tau, x) + (1-b)\bar{\xi}(r, \tau, x_0)) - D_\xi F(\bar{\xi}(r, \tau, x_0)) db dr. \quad \square$$

Similarly for any $\varepsilon > 0$, there exists $\rho_2 > 0$ such that if $\|x - x_0\|_L$

For $t > N_3$, by (4.6) and Lemma 2.6,

$$\begin{aligned}
|I_{u5}| &= e^{-(\gamma-\nu)t} \lim_{\lambda \rightarrow +\infty} \lim_{p \rightarrow -\infty} \int_0^{t-N_3} T_{A_{0s}}(t-l-N_3) \lambda R_\lambda(A_s) \\
&\quad \times \Pi_s[\sigma(\bar{\xi}(l+N_3, \tau, x)) - \sigma(\bar{\xi}(l+N_3, \tau, x_0)) - D_\xi \sigma(\bar{\xi}(l+N_3, \tau, x_0)) \\
&\quad \times (\bar{\xi}(l+N_3, \tau, x) - \bar{\xi}(l+N_3, \tau, x_0))] dW(l+N_3) \\
&\leq C_{\zeta+\nu} \sup_{l \in [0, t-N_3]} e^{(\zeta+\nu)(t-l-N_3)} e^{-(\gamma-\nu)t} (\mathbb{E} |\sigma(\bar{\xi}(l+N_3, \tau, x)) \\
&\quad - \sigma(\bar{\xi}(l+N_3, \tau, x_0)) - D_\xi \sigma(\bar{\xi}(l+N_3, \tau, x_0))(\bar{\xi}(l+N_3, \tau, x) \\
&\quad - \bar{\xi}(l+N_3, \tau, x_0))|^2_{\mathcal{L}(Y_0, X)})^{\frac{1}{2}} \\
&\leq 2L_2 C_{\zeta+\nu} \sup_{h \in [N_3, t]} e^{(\zeta+3\nu-\gamma)(t-h)} e^{-\nu t} |\bar{\xi}(h, \tau, x) - \bar{\xi}(h, \tau, x_0)|_{C_{\gamma, 2\nu}} \\
&\leq \frac{2KL_2 e^{-\nu N_3}}{1-K \frac{L_1}{\alpha-\gamma+2\nu} + L_1 C_{\zeta+2\nu} + L_2 C_{\zeta+2\nu}} \|x - x_0\|_{L^2(\cdot, X_0)}.
\end{aligned}$$

And finally for I_{u6} ,

$$\begin{aligned}
|I_{u6}| &= e^{-(\gamma-\nu)t} \lim_{\lambda \rightarrow +\infty} \lim_{p \rightarrow -\infty} \int_0^{N_3-p} T_{A_{0s}}(t-l-p) \lambda R_\lambda(A_s) \\
&\quad \times \Pi_s[\sigma(\bar{\xi}(l+p, \tau, x)) - \sigma(\bar{\xi}(l+p, \tau, x_0)) \\
&\quad - D_\xi \sigma(\bar{\xi}(l+p, \tau, x_0))(\bar{\xi}(l+p, \tau, x) - \bar{\xi}(l+p, \tau, x_0))] dW(l+p) \\
&\leq C_{\zeta+\nu} \sup_{h \in (-\infty, N_3]} e^{(\zeta+\nu)(t-h)} e^{-(\gamma-\nu)t} (\mathbb{E} |\sigma(\bar{\xi}(h, \tau, x)) - \sigma(\bar{\xi}(h, \tau, x_0)) \\
&\quad - D_\xi \sigma(\bar{\xi}(h, \tau, x_0))(\bar{\xi}(h, \tau, x) - \bar{\xi}(l+N_3, \tau, x_0))|^2_{\mathcal{L}(Y_0, X)})^{\frac{1}{2}} \\
&\leq 2L_2 C_{\zeta+\nu} \sup_{h \in (-\infty, N_3]} e^{(\zeta+3\nu-\gamma)(t-h)} e^{-\nu t} |\bar{\xi}(h, \tau, x) - \bar{\xi}(h, \tau, x_0)|_{C_{\gamma, 2\nu}} \\
&\leq \frac{2KL_2 e^{-\nu N_3}}{1-K \frac{L_1}{\alpha-\gamma+2\nu} + L_1 C_{\zeta+2\nu} + L_2 C_{\zeta+2\nu}} \|x - x_0\|_{L^2(\cdot, X_0)}.
\end{aligned}$$

Choose N_3 so large that

$$\sup_{t \leq \tau} |I_{u5}| \leq \frac{\varepsilon}{6} \|x - x_0\|_{L^2(\cdot, X_0)}. \quad (4.12)$$

$$\sup_{t \leq \tau} |I_{u6}| \leq \frac{\varepsilon}{6} \|x - x_0\|_{L^2(\cdot, X_0)}. \quad (4.13)$$

Take $\rho_0 = \min\{\rho_1, \rho_2\}$ and combining (4.8)–(4.13), we obtain

$$\|u\|_{C_{\gamma, \nu}} \leq \varepsilon \|x - x_0\|_{L^2(\cdot, X_0)} \quad \text{for } \|x - x_0\|_{L^2(\cdot, X_0)} < \rho_0. \quad \square \quad \square$$

So $\|u\|_{C_{\gamma, \nu}} = o(\|x - x_0\|)$ as $x \rightarrow x_0$. Next, we prove $D_{\xi} \bar{\xi}(\cdot, \tau, x)$ is continuous with respect to x from $L^2(\Omega, F_{\tau}; X_{0u})$ to $L(L^2(\Omega, F_{\tau}; X_{0u}), C_{\gamma}^-)$. By [11, (28)], we have

$$\begin{aligned} D_x \bar{\xi}(t, \tau, x) &= \mathbb{E} \left[T_{A_{0u}}(t - \tau) \Pi_{0u} \right. \\ &\quad - \int_t^{\tau} T_{A_{0u}}(t - r) \Pi_u D_{\xi} F(\bar{\xi}(r, \tau, x)) D_x \bar{\xi}(r, \tau, x) dr \mid F_t \\ &\quad + \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^t T_{A_{0s}}(t - r) \lambda R_{\lambda}(A_s) \Pi_s D_{\xi} F(\bar{\xi}(r, \tau, x)) D_x \bar{\xi}(r, \tau, x) dr \\ &\quad + \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^t T_{A_{0s}}(t - r) \lambda R_{\lambda}(A_s) \Pi_s D_{\xi} \sigma(\bar{\xi}(r, \tau, x)) \\ &\quad \times D_x \bar{\xi}(r, \tau, x) dW(r) \Big]. \end{aligned}$$

For $x \in L^2(\Omega, F_{\tau}; X_{0u})$, $\xi \in C_{\gamma}^-$, define the operator $H' : C_{\gamma}^- \rightarrow C_{\gamma}^-$ as follows:

$$\begin{aligned} H' \xi &= -\mathbb{E} \left[\int_t^{\tau} T_{A_{0u}}(t - r) \Pi_u D_{\xi} F(\bar{\xi}(r, \tau, x)) \xi(r) dr \mid F_t \right. \\ &\quad + \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^t T_{A_{0s}}(t - r) \lambda R_{\lambda}(A_s) \Pi_s D_{\xi} F(\bar{\xi}(r, \tau, x)) \xi(r) dr \\ &\quad + \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^t T_{A_{0s}}(t - r) \lambda R_{\lambda}(A_s) \Pi_s D_{\xi} \sigma(\bar{\xi}(r, \tau, x)) \xi(r) dW(r) \Big]. \quad (4.14) \end{aligned}$$

Similarly, we have

$$\|H'\xi\|_{C_{\gamma}^-} \leq K \left(\frac{L_1}{\alpha - \gamma} + L_1 C_{\zeta} + L_2 C_{\zeta} \right) \|\xi\|_{C_{\gamma}^-} < 1. \quad \square \quad \square \quad (4.15)$$

For $x, x_0 \in L^2(\Omega, F_{\tau}; X_{0u})$, $\tau \in \mathbb{R}$,

$$\begin{aligned} D_x \bar{\xi}(t, \tau, x) - D_x \bar{\xi}(t, \tau, x_0) &= -\mathbb{E} \left[\int_t^{\tau} T_{A_{0u}}(t - r) \Pi_u [D_{\xi} F(\bar{\xi}(r, \tau, x)) D_x \bar{\xi}(r, \tau, x) \right. \\ &\quad - D_{\xi} F(\bar{\xi}(r, \tau, x_0)) D_x \bar{\xi}(r, \tau, x_0)] dr \mid F_t \\ &\quad + \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^t T_{A_{0s}}(t - r) \lambda R_{\lambda}(A_s) \Pi_s [D_{\xi} F(\bar{\xi}(r, \tau, x)) D_x \bar{\xi}(r, \tau, x) \end{aligned}$$

$$\begin{aligned}
 & - D_\xi F(\bar{\xi}(r, \tau, x_0)) D_x \bar{\xi}(r, \tau, x_0)] dr \\
 & + \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^t T_{A_0 s}(t-r) \lambda R_\lambda(A_s) \Pi_s [D_\xi \sigma(\bar{\xi}(r, \tau, x)) D_x \bar{\xi}(r, \tau, x) \\
 & - D_\xi \sigma(\bar{\xi}(r, \tau, x_0)) D_x \bar{\xi}(r, \tau, x_0)] dW(r) \\
 & = H'(D_x \bar{\xi}(t, \tau, x) - D_x \bar{\xi}(t, \tau, x_0)) + I'_u,
 \end{aligned} \tag{4.16}$$

where

$$\begin{aligned}
 I'_u & = -\mathbb{E} \int_t^\tau T_{A_0 u}(t-r) \Pi_u [D_\xi F(\bar{\xi}(r, \tau, x)) \\
 & - D_\xi F(\bar{\xi}(r, \tau, x_0))] D_x \bar{\xi}(r, \tau, x_0) dr | \mathcal{F}_t \\
 & + \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^t T_{A_0 s}(t-r) \lambda R_\lambda(A_s) \Pi_s [D_\xi F(\bar{\xi}(r, \tau, x)) \\
 & - D_\xi F(\bar{\xi}(r, \tau, x_0))] D_x \bar{\xi}(r, \tau, x_0) dr \\
 & + \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^t T_{A_0 s}(t-r) \lambda R_\lambda(A_s) \Pi_s [D_\xi \sigma(\bar{\xi}(r, \tau, x)) \\
 & - D_\xi \sigma(\bar{\xi}(r, \tau, x_0))] D_x \bar{\xi}(r, \tau, x_0) dW(r).
 \end{aligned}$$

By (4.15), $\text{id} + H'$ has a bounded inverse in $L^2(\Omega, \mathcal{F}_\tau; X_{0u}, C_\gamma^-)$. From (4.16), we have

$$D_x \bar{\xi}(t, \tau, x) - D_x \bar{\xi}(t, \tau, x_0) = (\text{id} + H')^{-1} I'_u.$$

$(m-1)$ -derivative $D_x^{m-1}\bar{\xi}(\bullet, \tau, x)$ satisfies the following equation:

$$\begin{aligned}
D_\xi^{m-1}\bar{\xi}(t, \tau, x) = & -\mathbb{E} \int_t^\tau T_{A_{0u}}(t-r) \Pi_u D_\xi F(\bar{\xi}(r, \tau, x)) D_x^{m-1}\bar{\xi}(r, \tau, x) dr | \mathcal{F}_t \\
& + \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^t T_{A_{0u}}(t-r) \lambda R_\lambda(A_s) \Pi_s D_\xi F(\bar{\xi}(r, \tau, x)) \\
& \times D_x^{m-1}\bar{\xi}(r, \tau, x) dr \\
& + \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^t T_{A_{0s}}(t-r) \lambda R_\lambda(A_s) \Pi_s D_\xi \sigma(\bar{\xi}(r, \tau, x)) D_x^{m-1} \\
& \times \bar{\xi}(r, \tau, x) dW(r) - \mathbb{E} \int_t^\tau T_{A_{0u}}(t-r) \Pi_u R_{1,m-1}(r, \tau, x) dr | \mathcal{F}_t \\
& + \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^t T_{A_{0s}}(t-r) \lambda R_\lambda(A_s) \Pi_s R_{1,m-1}(r, \tau, x) ds \\
& + \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^t T_{A_{0s}}(t-r) \lambda R_\lambda(A_s) \Pi_s R_{2,m-1}(r, \tau, x) dW(r),
\end{aligned}$$

where

$$\begin{aligned}
R_{1,m-1}(r, \tau, x) &= \sum_{i=1}^{m-3} \binom{m-2}{i} D_x^{m-2-i} (D_\xi F(\bar{\xi}(r, \tau, x))) D_\xi^{i+1} \bar{\xi}(r, \tau, x), \\
R_{2,m-1}(r, \tau, x) &= \sum_{i=1}^{m-3} \binom{m-2}{i} D_x^{m-2-i} (D_\xi \sigma(\bar{\xi}(r, \tau, x))) D_\xi^{i+1} \bar{\xi}(r, \tau, x).
\end{aligned}$$

Applying the chain rule to $D_x^{m-2-i}(D_\xi F(\bar{\xi}(r, \tau, x)))$, we observe that each term in $R_{1,m-1}(r, \tau, x)$ contains factors: $D_\xi^{i_1} F(\bar{\xi}(r, \tau, x))$ for some $2 \leq i_1 \leq m-1$ and at least two derivatives $D_x^{i_2} \bar{\xi}(r, \tau, x)$ and $D_x^{i_3} \bar{\xi}(r, \tau, x)$ for some $i_2, i_3 \in \{1, \dots, m-2\}$. Since $D_x^i \bar{\xi}(\bullet, \tau, x) \in C_{i\gamma}^-$ for $i = 1, \dots, m-1$ and F is C^k , $R_{1,m-1}(\bullet, \tau, x) : L^2(\Omega, \mathcal{F}_\tau; X_{0u}) \rightarrow L^{m-1}(L^2(\Omega, \mathcal{F}_\tau; X_{0u}), L^2(\Omega, \mathcal{F}_\tau; X_0))$ are C^1 in x . We can obtain the same conclusion for $R_{2,m-1}(\bullet, \tau, x)$ by similar arguments. Using the same argument we used in the case $k = 1$, we can show that $D_x^{m-1}\bar{\xi}(\bullet, \tau, x)$ is C^1 from $L^2(\Omega, \mathcal{F}_\tau; X_{0u})$ to $L^m(L^2(\Omega, \mathcal{F}_\tau; X_{0u}), C_{m\gamma}^-)$. So $\bar{\xi}(t, \tau, x)$ is C^k . From [11], for given $\tau \in \mathbb{R}$, we know that $h^u(x, \tau) = \Pi_{0s} \bar{\xi}(x)(\tau)$ and

$$\begin{aligned}
h^u(x, \tau) &= \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^\tau T_{A_{0s}}(\tau-r) \lambda R_\lambda(A_s) \Pi_s F(\bar{\xi}(x)(r)) dr \\
&+ \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^\tau T_{A_{0s}}(\tau-r) \lambda R_\lambda(A_s) \Pi_s \sigma(\bar{\xi}(x)(r)) dW(r).
\end{aligned}$$

Thus $h^u(x, \tau)$ is C^k , which guarantees the C^k -smoothness of $M^u(\omega)$. $\square$

5. Smoothness of Mean-Square Random Stable Sets

This section is dedicated to the $\mathbf{C}^k$ -smoothness of mean-square random stable invariant sets of (1.3). Given $\tau \in \mathbb{R}$, denote by $\mathbf{C}_\gamma^+$ the space of all $\mathbf{X}_0$ -valued $\mathbf{F}_t$ -progressively measurable processes $\xi(t) : t \in [\tau, +\infty)$ such that $\xi : [\tau, +\infty) \rightarrow L^2(\Omega, \mathbf{X}_0)$ is continuous and

$$\sup_{t \geq \tau} (e^{-\gamma t} \|\xi(t)\|_{L^2(\cdot, \mathbf{X}_0)}) < \infty, \quad (5.1)$$

with norm

$$\|\xi\|_{\mathbf{C}_\gamma^+} = \sup_{t \geq \tau} (e^{-\gamma t} \|\xi(t)\|_{L^2(\cdot, \mathbf{X}_0)}), \quad \xi \in \mathbf{C}_\gamma^+, \quad \square \quad \square \quad \square$$

where $\gamma \in (\beta, \alpha)$. Let $\xi(t) \in \mathbf{C}_\gamma^+$ satisfy

$$\begin{aligned} \xi(t) = & \mathbf{T}_{A_{0s}}(t - \tau)\mathbf{x} - \int_t^\infty \mathbf{T}_{A_{0u}}(t - r)\Pi_u \mathbf{F}(\xi(r, \tau, \mathbf{x}))dr \\ & - \int_t^\infty \mathbf{T}_{A_{0u}}(t - r)\Pi_u \sigma(\xi(r, \tau, \mathbf{x}))d\mathbf{W}(r) \\ & + \lim_{\lambda \rightarrow +\infty} \int_\tau^t \mathbf{T}_{A_{0s}}(t - r)\lambda \mathbf{R}_\lambda(\mathbf{A}_s)\Pi_s \mathbf{F}(\xi(r, \tau, \mathbf{x}))dr \\ & + \lim_{\lambda \rightarrow +\infty} \int_\tau^t \mathbf{T}_{A_{0s}}(t - r)\lambda \mathbf{R}_\lambda(\mathbf{A}_s)\Pi_s \sigma(\xi(r, \tau, \mathbf{x}))d\mathbf{W}(r). \end{aligned} \quad (5.2)$$

Given $\tau \in \mathbb{R}$, define a subset of $L^2(\Omega, \mathbf{F}_\tau; \mathbf{X}_{0s})$ as $\mathbf{X}_{0s}^\tau$ which is given by

$$\mathbf{X}_{0s}^\tau = \{\mathbf{x} \in L^2(\Omega, \mathbf{F}_\tau; \mathbf{X}_{0s}) : \text{there exists } \xi \in \mathbf{C}_\gamma^+ \text{ satisfying (5.2)}\}.$$

$\mathbf{X}_{0s}^\tau$ is nonempty since $0 \in \mathbf{X}_{0s}^\tau$. We have the following result from [11, Theorem 4.1].

Lemma 5.1. *Let Assumptions 1, 2 and 6 be satisfied. If $\beta < \zeta < \gamma < \alpha$ with*

$$K = \frac{L_1}{\alpha - \gamma} + \sqrt{\frac{L_2}{2\alpha - 2\gamma}} + L_1 C_\zeta + L_2 C_\zeta < 1, \quad (5.3)$$

Proof. Step 1. By the definition of X_{0s}^τ , if $x \in X_{0s}^\tau$, then there exists at least one $\xi \in C_\tau^+$ which satisfies (5.2). We have proved that when (5.3) is satisfied, such ξ is unique (see [11, Theorem 4.1]), which is denoted by $\hat{\xi}$. We prove that $\hat{\xi}$ is C^1 .

Clearly, there exists $\nu^* > 0$ such that for $0 < \nu \leq \nu^*$, we have

$$\beta < \zeta < i\gamma - 3\nu < i\gamma - 2\nu < i\gamma - \nu < \alpha, \quad (5.5)$$

and for all $1 \leq i \leq k, j = 1, 2$

$$K \frac{L_1}{\alpha - i\gamma + j\nu} + \sqrt{\frac{L_2}{2\alpha - 2i\gamma + 2j\nu}} + L_1 C_{\zeta + j\nu} + L_2 C_{\zeta + j\nu} < 1. \quad (5.6)$$

Then $\hat{\xi}$ is a fixed point in $C_{\gamma-\nu}^+ \cap C_\gamma^+$. By [11, (55)], for $t \geq \tau, x \in X_{0s}^\tau$,

$$\begin{aligned} \hat{\xi}(t, \tau, x) = & T_{A_{0s}}(t - \tau)x - \int_t^{+\infty} T_{A_{0u}}(t - r)\Pi_u F(\hat{\xi}(r, \tau, x))dr \\ & - \int_t^{+\infty} T_{A_{0u}}(t - r)\Pi_u \sigma(\hat{\xi}(r, \tau, x))dW(r) \\ & + \lim_{\lambda \rightarrow +\infty} \int_\tau^t T_{A_{0s}}(t - r)\lambda R_\lambda(A_s)\Pi_s F(\hat{\xi}(r, \tau, x))ds \\ & + \lim_{\lambda \rightarrow +\infty} \int_\tau^t T_{A_{0s}}(t - r)\lambda R_\lambda(A_s)\Pi_s \sigma(\hat{\xi}(r, \tau, x))dW(r). \end{aligned}$$

By [11, (78)], for given $\tau \in \mathbb{R}, x, x_0 \in X_{0s}^\tau$ and $i = 1, 2$, $\hat{\xi}$ satisfies

$$\begin{aligned} & \hat{\xi}(t, \tau, x) - \hat{\xi}(t, \tau, x_0) \in C_{\gamma-i\nu} \\ & \leq \frac{K}{1 - K \frac{L_1}{\alpha - \gamma + i\nu} + \frac{L_2}{\sqrt{2\alpha - 2\gamma + 2i\nu}} + L_1 C_{\zeta + i\nu} + L_2 C_{\zeta + i\nu}} \|x - x_0\|_{L^2(\cdot, X_0)}. \end{aligned} \quad (5.7)$$

Fix $\nu \in (0, \nu^*]$, we first prove that $\hat{\xi}$ is differentiable from X_{0s}^τ to $C_{\gamma-\nu}^+$. For $\tau \in \mathbb{R}, x_0 \in X_{0s}^\tau, \xi \in C_{\gamma-\nu}^+$, define H as follows:

$$\begin{aligned} H\xi = & - \int_t^\infty T_{A_{0u}}(t - r)\Pi_u D_\xi F(\hat{\xi}(r, \tau, x_0))\xi(r)dr \\ & - \int_t^\infty T_{A_{0s}}(t - r)\Pi_u D_\xi \sigma(\hat{\xi}(r, \tau, x_0))\xi(r)dW(r) \\ & + \lim_{\lambda \rightarrow +\infty} \int_\tau^t T_{A_{0s}}(t - r)\lambda R_\lambda(A_s)\Pi_s D_\xi F(\hat{\xi}(r, \tau, x_0))\xi(r)dr \\ & + \lim_{\lambda \rightarrow +\infty} \int_\tau^t T_{A_{0s}}(t - r)\lambda R_\lambda(A_s)\Pi_s D_\xi \sigma(\hat{\xi}(r, \tau, x_0))\xi(r)dW(r). \end{aligned} \quad (5.8)$$

Noticing that $\|D_\xi F(\xi)\|_{\text{Lip}} = L_1$, $\|D_\xi \sigma(\xi)\|_{\text{Lip}} = L_2$, we can prove that for $\xi \in C_{\gamma-\nu}^+$, □ □

$$\sup_{t \geq \tau} e^{-(\gamma-\nu)t} \|\xi\|_{L^2(\cdot, X_0)} \leq K \left(\frac{L_1}{\alpha - \gamma + \nu} + \sqrt{\frac{L_2}{2\alpha - 2\gamma + 2\nu}} + L_1 C_{\zeta+j\nu} + L_2 C_{\zeta+\nu} \right) \|\xi\|_{C_{\gamma-\nu}}.$$

Then H is a bounded linear operator from $C_{\gamma-\nu}^+$ into itself with norm

$$K \left(\frac{L_1}{\alpha - \gamma + \nu} + \sqrt{\frac{L_2}{2\alpha - 2\gamma + 2\nu}} + L_1 C_{\zeta+j\nu} + L_2 C_{\zeta+\nu} \right) < 1.$$

So $\text{id} - H$ has a bounded inverse in $C_{\gamma-\nu}^+$. For $x, x_0 \in X_{0s}^\tau$, set

$$\begin{aligned} I_s(t) = & - \int_t^\infty T_{A_{0u}}(t-r) \Pi_u [F(\hat{\xi}(r, \tau, x)) - F(\hat{\xi}(r, \tau, x_0))] \\ & - D_\xi F(\hat{\xi}(r, \tau, x_0))(\hat{\xi}(r, \tau, x) - \hat{\xi}(r, \tau, x_0))] dr \\ & - \int_t^\infty T_{A_{0u}}(t-r) \Pi_u [\sigma(\hat{\xi}(r, \tau, x)) - \sigma(\hat{\xi}(r, \tau, x_0))] \\ & - D_\xi \sigma(\hat{\xi}(r, \tau, x_0))(\hat{\xi}(r, \tau, x) - \hat{\xi}(r, \tau, x_0))] dW(r) \end{aligned}$$

terms, i.e. $e^{-(\gamma-\nu)t} \mathbf{l}_s \in L^2(\cdot, X_0) = \sum_{i=1}^8 \mathbf{l}_{si}$, where for $t < N_1$, $\square \square$

$$\begin{aligned} \mathbf{l}_{s1} = & -e^{-(\gamma-\nu)t} \int_{N_1}^{+\infty} T_{A_{0u}}(t-r) \Pi_u [F(\hat{\xi}(r, \tau, x_0)) - F(\hat{\xi}(r, \tau, x))] \\ & - D_\xi F(\hat{\xi}(r, \tau, x_0))(\hat{\xi}(r, \tau, x_0) - \hat{\xi}(r, \tau, x))] dr \end{aligned} \quad ,$$

$L^2(\cdot, X_0)$

and $\mathbf{l}_{s1} = 0$ for $t \geq N_1$; For $t < N_1$,

$$\begin{aligned} \mathbf{l}_{s2} = & -e^{-(\gamma-\nu)t} \int_t^{N_1} T_{A_{0u}}(t-r) \Pi_u [F(\hat{\xi}(r, \tau, x_0)) - F(\hat{\xi}(r, \tau, x))] \\ & - D_\xi F(\hat{\xi}(r, \tau, x_0))(\hat{\xi}(r, \tau, x_0) - \hat{\xi}(r, \tau, x))] dr \end{aligned} \quad ,$$

$L^2(\cdot, X_0)$

and change N_1 to t if $t \geq N_1$. For $t < N_2$,

$$\begin{aligned} \mathbf{l}_{s3} = & -e^{-(\gamma-\nu)t} \int_{N_2}^{+\infty} T_{A_{0s}}(t-r) \Pi_u [\sigma(\hat{\xi}(r, \tau, x)) - \sigma(\hat{\xi}(r, \tau, x_0))] \\ & - D_\xi \sigma(\hat{\xi}(r, \tau, x_0))(\hat{\xi}(r, \tau, x) - \hat{\xi}(r, \tau, x_0))] dW(r) \end{aligned} \quad ,$$

$L^2(\cdot, X_0)$

and $\mathbf{l}_{s3} = 0$ if $t \geq N_2$; For $t < N_2$,

$$\begin{aligned} \mathbf{l}_{s4} = & -e^{-(\gamma-\nu)t} \int_t^{N_2} T_{A_{0s}}(t-r) \Pi_u [\sigma(\hat{\xi}(r, \tau, x)) - \sigma(\hat{\xi}(r, \tau, x_0))] \\ & - D_\xi \sigma(\hat{\xi}(r, \tau, x_0))(\hat{\xi}(r, \tau, x) - \hat{\xi}(r, \tau, x_0))] dW(r) \end{aligned} \quad ,$$

$L^2(\cdot, X_0)$

and change N_2 to t if $t \geq N_2$. For $t > N_3$,

$$\begin{aligned} \mathbf{l}_{s5} = & e^{-(\gamma-\nu)t} \lim_{\lambda \rightarrow +\infty} \int_{N_3}^t T_{A_{0s}}(t-r) \lambda R_\lambda(A_s) \Pi_s [F(\hat{\xi}(r, \tau, x)) \\ & - F(\hat{\xi}(r, \tau, x_0)) - D_\xi F(\hat{\xi}(r, \tau, x_0))(\hat{\xi}(r, \tau, x) - \hat{\xi}(r, \tau, x_0))] dr \end{aligned} \quad ,$$

$L^2(\cdot, X_0)$

and $\mathbf{l}_{s5} = 0$ if $t \leq N_3$; For $t > N_3$,

$$\begin{aligned} \mathbf{l}_{s6} = & e^{-(\gamma-\nu)t} \lim_{\lambda \rightarrow +\infty} \int_\tau^{N_3} T_{A_{0s}}(t-r) \lambda R_\lambda(A_s) \Pi_s [F(\hat{\xi}(r, \tau, x)) \\ & - F(\hat{\xi}(r, \tau, x_0)) - D_\xi F(\hat{\xi}(r, \tau, x_0))(\hat{\xi}(r, \tau, x) - \hat{\xi}(r, \tau, x_0))] dr \end{aligned} \quad ,$$

$L^2(\cdot, X_0)$

and change N_3 to t if $t \leq N_3$. For $t > N_4$,

$$\begin{aligned} \mathbf{l}_{s7} = & e^{-(\gamma-\nu)t} \lim_{\lambda \rightarrow +\infty} \int_{N_4}^t T_{A_{0s}}(t-r) \lambda R_\lambda(A_s) \Pi_s [\sigma(\hat{\xi}(r, \tau, x)) \\ & - \sigma(\hat{\xi}(r, \tau, x_0)) - D_\xi \sigma(\hat{\xi}(r, \tau, x_0))(\hat{\xi}(r, \tau, x) - \hat{\xi}(r, \tau, x_0))] dW(r) \end{aligned} \quad ,$$

$L^2(\cdot, X_0)$

and $I_{s7} = 0$ if $t \leq N_4$; For $t > N_4$,

$$I_{s8} = e^{-(\gamma-\nu)t} \lim_{\lambda \rightarrow +\infty} \int_{\tau}^{N_4} T_{A_{0s}}(t-r) \lambda R_{\lambda}(A_s) \Pi_s[\sigma(\hat{\xi}(r, \tau, x)) - \sigma(\hat{\xi}(r, \tau, x_0)) - D_{\xi} \sigma(\hat{\xi}(r, \tau, x_0))(\hat{\xi}(r, \tau, x) - \hat{\xi}(r, \tau, x_0))] dW(r),$$

$L^2(\cdot, X_0)$

and change N_4 to t if $t \leq N_4$. N_1, N_2, N_3, N_4 above are large positive numbers to be chosen later. By (2.9) and (5.7), for $t < N_1$,

$$|I_{s1}| \leq 2e^{-(\gamma-\nu)t} K L_1 \int_{N_1}^{+\infty} e^{\alpha(t-r)} \|\hat{\xi}(r, \tau, x) - \hat{\xi}(r, \tau, x_0)\|_{L^2(\cdot, X_0)} dr \quad \square$$

$$\leq 2K L_1 \|\hat{\xi}(\cdot, \tau, x) - \hat{\xi}(\cdot, \tau, x_0)\|_{C_{\gamma, 2\nu}^+} e^{(\alpha-\gamma+\nu)t} \int_{N_1}^{+\infty} e^{(\gamma-2\nu-\alpha)s} ds \quad \square$$

$$\leq \frac{2K^2 L_1 e^{-\nu N_1} \|x - x_0\|_{L^2(\cdot, X_0)}}{(\alpha - \gamma + 2\nu) \left(1 - K \left[\frac{L_1}{\alpha - \gamma + 2\nu} + \frac{L_2}{\sqrt{2\alpha - 2\gamma + 2\nu}} + L_1 C_{\zeta+2\nu} + L_2 C_{\zeta+2\nu}\right]\right)}. \quad \square$$

Choose N_1 so large that

$$\sup_{t \geq \tau} |I_{s1}| \leq \frac{\varepsilon}{8} \|x - x_0\|_{L^2(\cdot, X_0)}. \quad (5.9) \quad \square \quad \square$$

For $t < N_1$,

$$|I_{s2}| \leq K \|\hat{\xi}(\cdot, \tau, x) - \hat{\xi}(\cdot, \tau, x_0)\|_{C_{\gamma, \nu}^+} \int_t^{N_1} e^{(\alpha-\gamma+\nu)t} e^{(\gamma-\nu-\alpha)r} \times \int_0^1 D_{\xi} F(b\hat{\xi}(s, \tau, x) + (1-b)\hat{\xi}(r, \tau, x_0)) - D_{\xi} F(\hat{\xi}(r, \tau, x_0)) db dr. \quad \square$$

Since $F(\xi) \in C^k$, $D_{\xi} F(\xi)$ is continuous. For any $\varepsilon > 0$, there exists $\rho_1 > 0$ such that if $\|x - x_0\|_{L^2(\cdot, X_0)} < \rho_1$, $\square \quad \square$

$$D_{\xi} F(b\bar{\xi}(r, \tau, x) + (1-b)\bar{\xi}(r, \tau, x_0)) - D_{\xi} F(\bar{\xi}(r, \tau, x_0)) \leq \frac{\varepsilon(\alpha - \gamma + \nu) \left(1 - K \left[\frac{L_1}{\alpha - \gamma + \nu} + \frac{L_2}{\sqrt{2\alpha - 2\gamma + 2\nu}} + L_1 C_{\zeta+\nu} + L_2 C_{\zeta+\nu}\right]\right)}{8K^2}.$$

Then by (5.7),

$$\sup_{t \geq \tau} |I_{s2}| \leq \frac{\varepsilon}{8} \|x - x_0\|_{L^2(\cdot, X_0)} \quad \text{for } \|x - x_0\|_{L^2(\cdot, X_0)} < \rho_1. \quad \square \quad (5.10) \quad \square$$

Since $\sigma(\xi) \in C^k$, $D_\xi \sigma(\xi)$ is continuous. For any $\varepsilon > 0$, there exists $\rho_2 > 0$ such that if $\|x - x_0\|_{L^2(\cdot, X_0)} < \rho_2$, □ □

$$\begin{aligned} & D_\xi \sigma(b\hat{\xi}(r, \tau, x) + (1-b)\hat{\xi}(r, \tau, x_0)) - D_\xi \sigma(\hat{\xi}(r, \tau, x_0)) \\ & \leq \frac{\varepsilon(\alpha - \gamma + \nu) \|1 - K\| \left(\frac{L_1}{\alpha - \gamma + \nu} + \frac{L_2}{\sqrt{2\alpha - 2\gamma + 2\nu}} + L_1 C_{\zeta + \nu} + L_2 C_{\zeta + \nu} \right)^2}{16 K^4}. \end{aligned}$$

Then by (5.7),

$$\sup_{t \geq \tau} |I_{s4}| \leq \frac{\varepsilon}{6} \|x - x_0\|_{L^2(\cdot, X_0)} \quad \text{for } \|x - x_0\|_{L^2(\cdot, X_0)} < \rho_2. \quad \square \quad (5.12) \quad \square$$

For $t > N_3$, by (5.7) and Lemma 2.6

$$\begin{aligned} |I_{s5}| & \leq e^{-(\gamma - \nu)t} \lim_{\lambda \rightarrow +\infty} \int_0^{t - N_3} T_{A_0 s} (t - l - N_3) \lambda R_\lambda (A_s) \quad \square \\ & \quad \times \Pi_s [F(\hat{\xi}(l + N_3, \tau, x)) - F(\hat{\xi}(l + N_3, \tau, x_0))] \\ & \quad - D_\xi F(\hat{\xi}(l + N_3, \tau, x_0))(\hat{\xi}(l + N_3, \tau, x) - \hat{\xi}(l + N_3, \tau, x_0))] \, L^2(\cdot, X_0) dl \\ & \leq 2L_1 C_{\zeta + \nu} \sup_{h \in [N_3, t]} e^{(\zeta + 3\nu - \gamma)(t - h)} e^{-\nu t} \|\hat{\xi}(h, \tau, x) - \hat{\xi}(h, \tau, x_0)\|_{C_{\gamma, 2\nu}^+} \quad \square \\ & \leq \frac{2K L_1 e^{-\nu N_3}}{1 - K \left(\frac{L_1}{\alpha - \gamma + \nu} + \frac{L_2}{\sqrt{2\alpha - 2\gamma + 4\nu}} + L_1 C_{\zeta + 2\nu} + L_2 C_{\zeta + 2\nu} \right)} \|x - x_0\|_{L^2(\cdot, X_0)}. \end{aligned}$$

And for I_{s6} ,

$$\begin{aligned} |I_{s6}| & = e^{-(\gamma - \nu)t} \lim_{\lambda \rightarrow +\infty} \int_0^{N_3 - \tau} T_{A_0 s} (t - l - \tau) \lambda R_\lambda (A_s) \\ & \quad \times \Pi_s [F(\hat{\xi}(l + \tau, \tau, x)) - F(\hat{\xi}(l + \tau, \tau, x_0))] \\ & \quad - D_\xi F(\hat{\xi}(l + \tau, \tau, x_0))(\hat{\xi}(l + \tau, \tau, x) - \hat{\xi}(l + \tau, \tau, x_0))] dl \\ & \quad L^2(\cdot, X_0) \\ & \leq 2L_2 C_{\zeta + \nu} \sup_{h \in [\tau, N_3]} e^{(\zeta + 3\nu - \gamma)(t - h)} e^{-\nu t} \|\hat{\xi}(h, \tau, x) - \hat{\xi}(h, \tau, x_0)\|_{C_{\gamma, 2\nu}^+} \quad \square \\ & \leq \frac{2K L_1 e^{-\nu N_3}}{1 - K \left(\frac{L_1}{\alpha - \gamma + \nu} + \frac{L_2}{\sqrt{2\alpha - 2\gamma + 4\nu}} + L_1 C_{\zeta + 2\nu} + L_2 C_{\zeta + 2\nu} \right)} \|x - x_0\|_{L^2(\cdot, X_0)}. \end{aligned}$$

Choose N_3 so large that

$$\sup_{t \geq \tau} |I_{s5}| \leq \frac{\varepsilon}{8} \|x - x_0\|_{L^2(\cdot, X_0)}. \quad (5.13) \quad \square \quad \square$$

$$\sup_{t \geq \tau} |I_{s6}| \leq \frac{\varepsilon}{8} \|x - x_0\|_{L^2(\cdot, X_0)}. \quad (5.14) \quad \square \quad \square$$

For $t > N_4$, by (5.7) and Lemma 2.6

$$\begin{aligned}
 |I_{s7}| &= e^{-(\gamma-\nu)t} \lim_{\lambda \rightarrow +\infty} \int_0^{t-N_4} T_{A_{0s}}(t-l-N_4) \lambda R_\lambda(A_s) \\
 &\quad \times \Pi_s[\sigma(\hat{\xi}(l+N_4, \tau, x)) - \sigma(\hat{\xi}(l+N_4, \tau, x_0)) - D_\xi \sigma(\hat{\xi}(l+N_4, \tau, x_0)) \\
 &\quad \times (\hat{\xi}(l+N_4, \tau, x) - \hat{\xi}(l+N_4, \tau, x_0))] dW(l+N_4) \\
 &\leq C_{\zeta+\nu} \sup_{l \in [0, t-N_4]} e^{(\zeta+\nu)(t-l-N_4)} e^{-(\gamma-\nu)t} (\mathbb{E}[\|\sigma(\hat{\xi}(l+N_4, \tau, x)) - \sigma(\hat{\xi}(l \\
 &\quad + N_4, \tau, x_0)) - D_\xi \sigma(\hat{\xi}(l+N_4, \tau, x_0))(\hat{\xi}(l+N_4, \tau, x) - \hat{\xi}(l \\
 &\quad + N_4, \tau, x_0))\|_{\mathcal{L}(Y_0, X)}^2])^{\frac{1}{2}} \quad \square \\
 &\leq 2L_2 C_{\zeta+\nu} \sup_{h \in [N_4, t]} e^{(\zeta+3\nu-\gamma)(t-h)} e^{-\nu t} \|\hat{\xi}(h, \tau, x) - \hat{\xi}(h, \tau, x_0)\|_{C_{\gamma, 2\nu}^+} \quad \square \\
 &\leq 2KL_2 e
 \end{aligned}$$

Taking $\rho_0 = \min\{\rho_1, \rho_2\}$ and combining (5.9)–(5.16), we obtain

$$\| \mathbf{I}_s \mathbf{C}_{\gamma, \nu}^+ \| \leq \varepsilon \|\mathbf{x} - \mathbf{x}_0\|_{L^2(\cdot, X_0)} \quad \text{for } \|\mathbf{x} - \mathbf{x}_0\|_{L^2(\cdot, X_0)} < \rho_0. \quad \square \quad \square$$

So $\| \mathbf{I}_s \mathbf{C}_{\gamma, \nu}^+ \| = o(\|\mathbf{x} - \mathbf{x}_0\|)$ as $\mathbf{x} \rightarrow \mathbf{x}_0$. Next, we prove $\mathbf{D}_\xi \hat{\xi}(\cdot, \tau, \mathbf{x})$ is continuous with respect to $\mathbf{x}$ from X_{0s}^τ to $L(X_{0s}^\tau, \mathbf{C}_\gamma^+)$. By [11, (55)], we have

$$\begin{aligned} \mathbf{D}_\xi \hat{\xi}(\mathbf{t}, \tau, \mathbf{x}) &= \mathbf{T}_{A_{0s}}(\mathbf{t} - \tau) \Pi_{0s} - \int_t^{+\infty} \mathbf{T}_{A_{0u}}(\mathbf{t} - r) \Pi_u \mathbf{D}_\xi \mathbf{F}(\hat{\xi}(r, \tau, \mathbf{x})) \mathbf{D}_x \hat{\xi}(r, \tau, \mathbf{x}) dr \\ &\quad - \int_t^{+\infty} \mathbf{T}_{A_{0u}}(\mathbf{t} - r) \Pi_u \mathbf{D}_\xi \sigma(\hat{\xi}(r, \tau, \mathbf{x})) \mathbf{D}_x \hat{\xi}(r, \tau, \mathbf{x}) dW(r) \\ &\quad + \lim_{\lambda \rightarrow +\infty} \int_\tau^t \mathbf{T}_{A_{0s}}(\mathbf{t} - r) \lambda \mathbf{R}_\lambda(A_s) \Pi_s \mathbf{D}_\xi \mathbf{F}(\hat{\xi}(r, \tau, \mathbf{x})) \mathbf{D}_x \hat{\xi}(r, \tau, \mathbf{x}) dr \\ &\quad + \lim_{\lambda \rightarrow +\infty} \int_\tau^t \mathbf{T}_{A_{0s}}(\mathbf{t} - r) \lambda \mathbf{R}_\lambda(A_s) \Pi_s \mathbf{D}_\xi \sigma(\hat{\xi}(r, \tau, \mathbf{x})) \mathbf{D}_x \hat{\xi}(r, \tau, \mathbf{x}) dW(r). \end{aligned}$$

For $\mathbf{x} \in X_{0s}^\tau$, $\xi \in \mathbf{C}_\gamma^+$, define the operator $\mathbf{H}' : \mathbf{C}_\gamma^+ \rightarrow \mathbf{C}_\gamma^+$ as follows:

$$\begin{aligned} \mathbf{H}' \xi &= - \int_t^\infty \mathbf{T}_{A_{0u}}(\mathbf{t} - r) \Pi_u \mathbf{D}_\xi \mathbf{F}(\hat{\xi}(r, \tau, \mathbf{x})) \xi(r) dr \\ &\quad - \int_t^\infty \mathbf{T}_{A_{0s}}(\mathbf{t} - r) \Pi_u \mathbf{D}_\xi \sigma(\hat{\xi}(s, \tau, \mathbf{x})) \xi(r) dW(r) \\ &\quad + \lim_{\lambda \rightarrow +\infty} \int_\tau^t \mathbf{T}_{A_{0s}}(\mathbf{t} - r) \lambda \mathbf{R}_\lambda(A_s) \Pi_s \mathbf{D}_\xi \mathbf{F}(\hat{\xi}(r, \tau, \mathbf{x})) \xi(r) dr \\ &\quad + \lim_{\lambda \rightarrow +\infty} \int_\tau^t \mathbf{T}_{A_{0s}}(\mathbf{t} - r) \lambda \mathbf{R}_\lambda(A_s) \Pi_s \mathbf{D}_\xi \sigma(\hat{\xi}(r, \tau, \mathbf{x})) \xi(r) dW(r). \end{aligned} \quad (5.17)$$

Similarly, we have

$$\|\mathbf{H}'\| \leq K \left(\frac{L_1}{\alpha - \gamma} + \sqrt{\frac{L_2}{2\alpha - 2\gamma}} + L_1 C_\zeta + L_2 C_\zeta \right) < 1. \quad (5.18)$$

For $\mathbf{x}, \mathbf{x}_0 \in X_{0s}^\tau$, $\tau \in \mathbb{R}$,

$$\begin{aligned} &\mathbf{D}_x \hat{\xi}(\mathbf{t}, \tau, \mathbf{x}) - \mathbf{D}_x \hat{\xi}(\mathbf{t}, \tau, \mathbf{x}_0) \\ &= - \int_t^{+\infty} \mathbf{T}_{A_{0u}}(\mathbf{t} - r) \Pi_u [\mathbf{D}_\xi \mathbf{F}(\hat{\xi}(r, \tau, \mathbf{x})) \mathbf{D}_x \hat{\xi}(r, \tau, \mathbf{x}) \\ &\quad - \mathbf{D}_\xi \mathbf{F}(\hat{\xi}(r, \tau, \mathbf{x}_0)) \mathbf{D}_x \hat{\xi}(r, \tau, \mathbf{x}_0)] dr \\ &\quad - \int_t^{+\infty} \mathbf{T}_{A_{0u}}(\mathbf{t} - r) \Pi_u [\mathbf{D}_\xi \sigma(\hat{\xi}(r, \tau, \mathbf{x})) \mathbf{D}_x \hat{\xi}(r, \tau, \mathbf{x}) \\ &\quad - \mathbf{D}_\xi \sigma(\hat{\xi}(r, \tau, \mathbf{x}_0)) \mathbf{D}_x \hat{\xi}(r, \tau, \mathbf{x}_0)] dW(r) \end{aligned}$$

$$\begin{aligned}
& + \lim_{\lambda \rightarrow +\infty} \int_{\tau}^t T_{A_{0s}}(t-r) \lambda R_{\lambda}(A_s) \Pi_s [D_{\xi} F(\hat{\xi}(r, \tau, x)) D_x \hat{\xi}(r, \tau, x) \\
& - D_{\xi} F(\hat{\xi}(r, \tau, x_0)) D_x \hat{\xi}(r, \tau, x_0)] dr \\
& + \lim_{\lambda \rightarrow +\infty} \int_{\tau}^t T_{A_{0s}}(t-r) \lambda R_{\lambda}(A_s) \Pi_s [D_{\xi} \sigma(\hat{\xi}(r, \tau, x)) D_x \hat{\xi}(r, \tau, x) \\
& - D_{\xi} \sigma(\hat{\xi}(r, \tau, x_0)) D_x \hat{\xi}(r, \tau, x_0)] dW(r). \\
& = H'(D_x \hat{\xi}(t, \tau, x) - D_x \hat{\xi}(t, \tau, x_0)) + I'_s,
\end{aligned} \tag{5.19}$$

where

$$\begin{aligned}
I'_s = & - \int_t^{+\infty} T_{A_{0u}}(t-r) \Pi_u [D_{\xi} F(\hat{\xi}(r, \tau, x)) \\
& - D_{\xi} F(\hat{\xi}(r, \tau, x_0))] D_x \hat{\xi}(r, \tau, x_0) dr \\
& - \int_t^{+\infty} T_{A_{0u}}(t-r) \Pi_u [D_{\xi} \sigma(\hat{\xi}(r, \tau, x)) \\
& - D_{\xi} \sigma(\hat{\xi}(r, \tau, x_0))] D_x \hat{\xi}(r, \tau, x_0) dW(r) \\
& + \lim_{\lambda \rightarrow +\infty} \int_{\tau}^t T_{A_{0s}}(t-r) \lambda R_{\lambda}(A_s) \Pi_s [D_{\xi} F(\hat{\xi}(r, \tau, x)) \\
& - D_{\xi} F(\hat{\xi}(r, \tau, x_0))] D_x \hat{\xi}(r, \tau, x_0) dr \\
& + \lim_{\lambda \rightarrow +\infty} \int_{\tau}^t T_{A_{0s}}(t-r) \lambda R_{\lambda}(A_s) \Pi_s [D_{\xi} \sigma(\hat{\xi}(r, \tau, x)) \\
& - D_{\xi} \sigma(\hat{\xi}(r, \tau, x_0))] D_x \hat{\xi}(r, \tau, x_0) dW(r).
\end{aligned}$$

By (4.15), $(\text{id} + H')^{-1}$ has a bounded inverse in $L(X_{0s}^{\tau}, \mathbf{C}_{\gamma}^+)$. From (5.19), we have

$$D_x \hat{\xi}(t, \tau, x) - D_x \hat{\xi}(t, \tau, x_0) = (\text{id} + H')^{-1} I'_s.$$

Then by (5.18),

$$\|D_x \hat{\xi}(t, \tau, x) - D_x \hat{\xi}(t, \tau, x_0)\|_{L(X_{0s}^{\tau}, \mathbf{C}_{\gamma}^+)} \leq \frac{\|I'_s\|_{L(X_{0s}^{\tau}, \mathbf{C}_{\gamma}^+)}}{1 - K \left(\frac{L_1}{\alpha - \gamma} + \frac{L_2}{\sqrt{2\alpha - 2\gamma}} + L_1 C_{\zeta} + L_2 C_{\zeta} \right)}. \tag{5.20}$$

Using the same procedure as for I_s , we have $\|I'_s\|_{L(X_{0s}^{\tau}, \mathbf{C}_{\gamma}^+)} = o(1)$ as $x \rightarrow x_0$. Then by (5.20), $D_x \hat{\xi}(\bullet, \tau, x)$ is continuous with respect to x . So, we proved that $\hat{\xi}(t, \tau, x)$ is $\mathbf{C}^1$. □

Step 2. We prove $\hat{\xi}(t, \tau, x)$ is $\mathbb{Q}^k$.

Make the inductive assumption that $\hat{\xi}(\bullet, \tau, x)$ is $\mathbb{C}^j$ from X_{0s}^τ to $\mathbb{C}_{j\gamma}^+$ for all $1 \leq j \leq m-1$ and we prove it for $j = m$. By calculation, the $(m-1)$ -derivative $D_x^{m-1}\hat{\xi}(\bullet, \tau, x)$ satisfies the following equation:

$$\begin{aligned} D_\xi^{m-1}\hat{\xi}(t, \tau, x) &= (t-r)\Pi u \\ &= - \int_t^{+\infty} T_{A_{0u}}(t-r)\Pi_u D_\xi(\hat{\xi}(\tau, \tau, x)) D_x^{m-1}\hat{\xi}(r, \tau, x) dr \end{aligned}$$

is $\mathbf{C}^k$. From [11], for given $\tau \in \mathbb{R}$, we know that $h^s(x, \tau) = \Pi_{0u} \hat{\xi}(x)(\tau)$ and

$$\begin{aligned} h^s(x, \tau) = & - \int_{\tau}^{\infty} T_{A_{0u}}(\tau - r) \Pi_u F(\hat{\xi}(x)(r)) dr \\ & - \int_{\tau}^{\infty} T_{A_{0u}}(\tau - r) \Pi_u \sigma(\hat{\xi}(x)(r)) dW(r). \end{aligned}$$

Thus $h^s(x, \tau)$ is $\mathbf{C}^k$, which guarantees the $\mathbf{C}^k$ -smoothness of $M^s(\omega)$. $\square$

6. Illustrative Example

For simplicity, we will only provide one example for the case of linear noise. The case of nonlinear noise is quite similar. We consider the following stochastic parabolic equation with a nonlinear (and nonlocal) boundary condition:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} + \pi^2 u(t, x) + P(u(t, x)) + u(t, x) \circ dW(t), \quad t \geq 0, x > 0, \\ -\frac{u(t, 0)}{\partial x} &= Q(u(t, \bullet)), \end{aligned} \tag{6.1}$$

$$u(0, \bullet) = u_0 \in L^p((0, +\infty), \mathbb{R}).$$

where $P : L^p((0, +\infty), \mathbb{R}) \rightarrow L^p((0, +\infty), \mathbb{R})$ and $Q : L^p((0, +\infty), \mathbb{R}) \rightarrow \mathbb{R}$ are arbitrary nonlinear maps which are assumed to be global Lipschitz continuous and $P(0) = Q(0) = 0$. W is a one-dimensional Brownian motion. Make $X := \mathbb{R} \times L^p((0, +\infty), \mathbb{R})$. Consider a linear operator $A : D(A) \subset X \rightarrow X$ defined by

$$A \begin{pmatrix} 0 \\ \phi \end{pmatrix} = \begin{pmatrix} \phi'(0) \\ \phi'' \end{pmatrix}$$

with $D(A) = \{0_{\mathbb{R}}\} \times W^{2,p}((0, +\infty), \mathbb{R})$. Note that A_0 , the part of A in $\overline{D(A)} = \{0_{\mathbb{R}}\} \times L^p((0, +\infty), \mathbb{R})$, is the generator of the strongly continuous semigroup of bounded operators associated to

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2}, \quad t \geq 0, x > 0, \\ -\frac{u(t, 0)}{\partial x} &= 0, \end{aligned}$$

$$u(0, \bullet) = u_0 \in L^p((0, +\infty), \mathbb{R}).$$

That is

$$A_0 \begin{pmatrix} 0 \\ \phi \end{pmatrix} = \begin{pmatrix} 0 \\ \phi'' \end{pmatrix}$$

with

$$D(A_0) = \begin{pmatrix} 0 \\ \phi \end{pmatrix} \in \{0_{\mathbb{R}}\} \times W^{2,p}((0, +\infty), \mathbb{R}) : \phi'$$

Take

$$F \begin{pmatrix} 0 \\ \phi \end{pmatrix} = \begin{pmatrix} Q(\phi) \\ P(\phi) \end{pmatrix}, \quad u^*(t) = \begin{pmatrix} 0_{\mathbb{R}^2} \\ u(t, \bullet) \end{pmatrix}, \quad \omega(t) = \begin{pmatrix} 0 \\ W(t) \end{pmatrix}.$$

Then, we can rewrite (6.1) as

$$\begin{aligned} du^*(t) &= ((A + \pi^2 I)u^*(t) + F(u^*(t)))dt + u^*(t) \circ d\omega(t), \quad t \geq 0, \\ u^*(0, \bullet) &= u_0^* \square \overline{D(A)} := X_0. \end{aligned}$$

We will verify Assumptions 2-4 and 7 in turn. According to [15, Lemma 6.1], A_0 is the infinitesimal generator of a C_0 -semigroup $\{T_{A_0}(t)\}_{t \geq 0}$ on X_0 . Moreover, $(A + \pi^2 I)_0$, the part of $(A + \pi^2 I)$, is the infinitesimal generator of a C_0 -semigroup on X_0 denoted by

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