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CENTER MANIFOLDS FOR ILL-POSED STOCHASTIC EVOLUTION EQUATIONS

Zonghao Li and Caibin Zeng

School of Mathematics, South China University of Technology
Guangzhou 510640, China

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Abstract. The aim of this paper is to develop a center manifold theory for a class of stochastic partial differential equations with a non-dense domain through the Lyapunov-Perron method. We construct a novel variation of constants formula by the resolvent operator to formulate the integrated solutions. Moreover, we impose an additional condition involving a non-decreasing map to deduce the required estimate since Young's convolution inequality is not applicable. As an application, we present a stochastic parabolic equation to illustrate the obtained results.

1. Introduction. In this paper, we shall study the existence of center invariant manifolds of the following Stratonovich stochastic evolution equation

$$\begin{cases} du = Audt + F(u) + u \circ dW, & t \in [0;T] \\ u(0) = u_0 \in \overline{D(A)}. \end{cases} \tag{1.1}$$

where $A : D(A) \subset X \to X$ is a linear operator and $\overline{D(A)} \neq X$, namely, its domain is non-densely defined. Indeed, W is a one-dimensional Brownian motion and $F : \overline{D(A)} \to X$ is globally Lipschitz continuous. Note that the Cauchy problem to (1.1) is ill-posed if the Hille-Yosida theory for C_0-semigroup generated by A breaks down. Such Cauchy problems with a non-dense domain cover several types of differential equations, such as delay differential equations, age-structured models and some partial differential equations, evolution equations with nonlinear boundary conditions. To overcome the embarrassment, the integrated semigroup theory and established the existence of random stable/unstable manifolds based on the Lyapunov-Perron method. A similar idea was extended to study the invariant foliations of (1.1) by Shen and Zeng [21]. The mentioned work brings much interest to discern the long-time dynamics of (1.1).

There are several results regarding invariant manifolds in the framework of random dynamical systems. Mohammed and Scheutzow [18] studied the existence of local stable and unstable manifolds of stochastic di erential equations driven by semimartingales. Results regarding stable and unstable manifolds for stochastic partial di erential equations can be looked up to the work of Duan et al. [11], Lu and Schmalfuss [13] and Caraballo et al. [5]. As for center manifolds, Chen et al. [8] studied center manifolds for stochastic partial di erential equations under an assumption of exponential trichotomy. Boxler [4] used the multiplicative erwhich is $(\mathcal{B}(\mathbb{R}) \otimes \mathcal{F} \otimes \mathcal{B}(X); \mathcal{B}(X))$-measurable and satisfies:

(i) $\varphi(0, \omega, x) = x$, for $x \in X$ and $\omega \in \Omega$.
(ii) $\varphi(t+s, \omega, x) = \varphi(t, \theta_s\omega, \varphi(s, \omega, x))$, for all $t, s \in \mathbb{R}, x \in X$ and all $\omega \in \Omega$.

We will use a coordinate transform to convert the stochastic partial differential equation (1.1) into a random evolution equation [6, 10, 11]. To do this, we consider the linear stochastic differential equation

$$dz + \mu z\,dt = dW, \tag{2.1}$$

where μ is a positive parameter. We extract the important results on the stationary Ornstein-Uhlenbeck process from [6, 11] in the following.

Lemma 2.1. (i) There exists a $\{\theta_t\}_{t\in\mathbb{R}}$-invariant set $\tilde\Omega \subset C_0(\mathbb{R};\mathbb{R})$ of full measure satisfying the sublinear growth condition

$$\lim_{t\to\pm\infty} \frac{|\omega(t)|}{|t|} = 0, \quad \omega \in \tilde\Omega.$$

(ii) The random variable $z(\omega) = -\mu\int_{-\infty}^0 e^{\mu s}\omega(s)\,ds$ is well defined for all $\omega \in \tilde\Omega$ and generates a unique stationary solution of (2.1) given by

$$(t, \omega) \mapsto z(\theta_t\omega) = -\mu\int_{-\infty}^0 e^{\mu s}\theta_t\omega(s)\,ds = -\mu\int_{-\infty}^0 e^{\mu s}\omega(s+t)\,ds + \omega(t).$$

The mapping $t \mapsto z(\theta_t\omega)$ is continuous.

(iii) Moreover

$$\lim_{t\to\pm\infty} \frac{|z(\theta_t\omega)|}{|t|} = 0,$$

for all $\omega \in \tilde\Omega$.

(iv) In addition

$$\lim_{t\to\pm\infty} \frac{1}{t}\int_0^t z(\theta_s\omega)\,ds = 0,$$

for all $\omega \in \tilde\Omega$.

From now on, we will work on $(\tilde\Omega, \tilde{\mathcal{F}},$

Therefore, it follows from [16, Lemma 2.1] that $\rho(A) = \rho(A_0)$, and thus $\sigma(A) = \sigma(A_0)$. Indeed, we need the following result.

Lemma 2.2. [14, Lemma 2.1] Let $(X, \|\cdot\|)$ be a Banach space and $A : D(A) \subset X \to X$ be a linear operator. Assume that there exists $\omega \in \mathbb{R}$, such that $(\omega, +\infty) \subset \rho(A)$ and

$$\limsup_{\lambda \to +\infty} \left\| \lambda (\lambda I - A)^{-1} \right\|_{L(X_0)} < +\infty.$$

Then the following assertions are equivalent:

(i) $\lim_{\lambda \to +\infty} \lambda (\lambda I - A)^{-1} x = x, \forall x \in X_0$.
(ii) $\lim_{\lambda \to +\infty} (\lambda I - A)^{-1} x = 0, \forall x \in X$.
(iii) $\overline{D(A_0)} = X_0$.

Now we are ready to present the definition of integrated semigroups.

Definition 2.2. Let $(X, \|\cdot\|)$ be a Banach space. A family of bounded linear operators $\{S(t)\}_{t \geq 0}$ on X is called an integrated semigroup if

(i) $S(0) = 0$.
(ii) The map $t \to S(t)x$ is continuous on $[0, +\infty)$ for each $x \in X$.
(iii) $S(t)$ satisfies

$$S(s)S(t) = \int_0^s (S(r+t) - S(r))\, dr, \forall s, t \geq 0.$$

We next discuss some basic assumptions about operator A and show that it can generate an integrated semigroup. Note that A is said to be a Hille-Yosida operator if there exist two constants, $\omega \in \mathbb{R}$ and $M \geq 1$, such that $(\omega, +\infty) \subset \rho(A)$ and

$$\left\| (\lambda I - A)^{-k} \right\|_{L(X)} \leq \frac{M}{(\lambda - \omega)^k}, \forall \lambda > \omega, \forall k \geq 1.$$

Lemma 2.3. [20, Theorem 5.3] A linear operator $A : D(A) \subset X \to X$ is an infinitesimal generator of a C_0-semigroup $(T(t))_{t \geq 0}$ satisfying $\|T(t)\| \leq Me^{\omega t}$, if and only if

(i) A is densely defined in X;
(ii) A is a Hille-Yosida operator.

Assumption 2.1. Let $(X, \|\cdot\|)$ be a Banach space and let $A : D(A) \subset X \to X$ be a linear operator. Assume that

(a) A is a Hille-Yosida operator on X_0;
(b) $\lim_{\lambda \to +\infty} (\lambda I - A)^{-1} x = 0, \forall x \in X$.

According to Lemma 2.3 and Assumption 2.1, we know that A_0 generates a C_0-semigroup on X_0 denoted by $(T(t))_{t \geq 0}$. It can also be denoted by $\left(e^{A_0 t}\right)_{t \geq 0}$. Besides, as concluded in [15, Proposition 2.5], A generates an integrated semigroup on X denoted by $(S(t))_{t \geq 0}$. For each $x \in X, t \geq 0$, and $\lambda > \omega$, $(S(t))_{t \geq 0}$ is given by

$$S(t)x = \lambda \int_0^t T(s)(\lambda I - A)^{-1} x\, ds + (I - T(t))(\lambda I - A)^{-1} x. \tag{2.2}$$

Also, the map $t \to S(t)$ is continuously differentiable if and only if $x \in X_0$ and

$$\frac{dS(t)x}{dt} = T(t)x, \forall t \geq 0, \forall x \in X_0.$$

Lemma 2.4. [14, Lemma 2.6] For each $f \in C^1([0,T];X)$, define
$$(S * f)(t) = \int_0^t S(t-s) f(s) ds, \quad \forall t \in [0,T].$$
Then we have the following:

(i) The map $t \to (S * f)(t)$ is continuously differentiable on $[0,T]$.
(ii) $(S * f)(t) \in D(A), \ \forall t \in [0,T]$.
(iii) For each $\lambda > \omega$, and each $t \in [0,T]$, we have
$$(\lambda I - A)^{-1} \frac{d}{dt}(S * f)(t) = \int_0^t T(t-s)(\lambda I - A)^{-1} f(s) ds.$$

Assumption 2.2. Assume that there exists real number $T > 0$, and a non-decreasing map $\delta : [0,T] \to [0,+\infty)$ such that for each $f \in C^1([0,T];X)$,
$$\left\| \frac{d}{dt}(S * f)(t) \right\| \le \delta(t) \sup_{s \in [0,t]} \|f(s)\|, \quad \forall t \in [0,T].$$

In order to define the mild solution to (1.1), we recall the non-homogeneous Cauchy problem [14, 15, 16]
$$\frac{du}{dt} = Au(t) + f(t), \quad t \in [0,T], \quad u(0) = x \in X_0, \tag{2.3}$$
where the linear operator A is the same as in this article. For each $f \in C^1([0,T];X)$, we denote
$$(S \diamond f)(t) = \frac{d}{dt}(S * f)(t), \quad \forall t \in [0,T].$$
According to Lemma 2.4 and Lemma 2.2(i), we have
$$(S \diamond f)(t) = \lim_{\lambda \to +\infty} \int_0^t T(t-s)\lambda(\lambda I - A)^{-1} f(s) ds, \quad \forall t \in [0,T], \ \forall \lambda > \omega.$$
Under Assumptions 2.1 and 2.2, equation (2.3) has a unique integrated solution u in $C([0,T];X_0)$, given by
$$u(t) = T(t)x + (S \diamond f)(t).$$

Assumption 2.3. $F : X_0 \to X$ is globally Lipschitz continuous, i.e.
$$|F(u_1) - F(u_2)| \le L|u_1 - u_2|,$$
for any $u_1, u_2 \in X_0$, where L is the Lipschitz constant and $F(0) = 0$.

To our aim, we define the coordinate transform
$$v = \mathcal{T}(u, \omega) = u e^{-z(\theta_t \omega)},$$
and its reverse transform
$$u = \mathcal{T}^{-1}(v, \omega) = v e^{z(\theta_t \omega)},$$
for $t \ge 0$. Performing the transformation $\mathcal{T}(v, \omega)$ into (1.1), we obtain
$$\begin{cases} dv = Av dt + z(\theta_t \omega) v dt + G(\omega, v) dt, \quad t \in [0,T] \\ v(0) = v_0 \in X_0, \end{cases} \tag{2.4}$$
where $G(\omega, v) = e^{-z(\theta_t \omega)} F\left(e^{z(\theta_t \omega)} v\right)$. Obviously, G and F have the same Lipschitz constant.

Definition 2.3. A mapping $v \in C([0,T];X)$ is called an integrated solution of (2.4) if and only if for $\forall t \in [0,T]$,

(i) $\int_0^t v(s)\,ds \in D(A)$.
(ii) It holds

$$v(t) = v_0 + A\int_0^t v(s)\,ds + \int_0^t z(\theta_s\omega)v(s)\,ds + \int_0^t G(\omega; v(s))\,ds.$$

We claim that if A is a closed operator, then an integrated solution always belongs to X_0. In fact,

$$v'(t) = \lim_{h\to 0}\frac{v(t+h)-v(t)}{h} \Longrightarrow v(t) = \lim_{h\to 0}\frac{1}{h}\int_t^{t+h} v(s)\,ds.$$

It follows from Definition 2.3 that

$$\int_t^{t+h} v(s)\,ds \in D(A);$$

and thus the required claim is sure. For the linear part of (2.4), we define

$$\Phi_{A_0}(t;s) = e^{A_0(t-s)+\int_s^t z(\theta_r\omega)dr} = T(t-s)e^{\int_s^t z(\theta_r\omega)dr}.$$

As in (2.3), one could define the mile solution to (2.4) as follows.

$$v(t) = \Phi_{A_0}(t;0)v_0 + \lim_{\lambda\to+\infty}\int_0^t \Phi_{A_0}(t;s)\lambda(\lambda I - A)^{-1}G(\theta_s\omega; v(s))\,ds. \tag{2.5}$$

The following result was proved in [19, Theorems 4.1 & 4.2].

Lemma 2.5. Under Assumptions 2.1-2.3, equation (2.4) has a unique integrated solution given by (2.5), which generates a random dynamical system φ. Moreover,

$$(t;\omega;x) \mapsto T^{-1}(\theta_t\omega; \varphi(t;\omega; T(\omega;x))) =: \hat{\varphi}(t;\omega;x);$$

is a random dynamical system. Also, this process is a solution of (1.1) for $x \in X_0$.

2.3. Exponential Trichotomy.

Assumption 2.4. We assume $T(t)$ satisfies the exponential trichotomy with exponents $\beta > \alpha \geq 0 > -\gamma$ and bound K, thus there exist two continuous linear projection operators with finite-rank $\Pi_{0c} \in L(X_0)\setminus\{0\}$ and $\Pi_{0u} \in L(X_0)$ such that

$$\Pi_{0u}\Pi_{0c} = \Pi_{0c}\Pi_{0u} = 0;$$

and

$$\Pi_{0k}T(t) = T(t)\Pi_{0k};\ \forall t \geq 0;\ \forall k \in \{c; u\}.$$

Denote

$$\Pi_{0s} = I_{X_0} - (\Pi_{0c} + \Pi_{0u});\ X_{0c} = \Pi_{0c}X_0;\ X_{0u} = \Pi_{0u}X_0;\ X_{0s} = \Pi_{0s}X_0;$$

and for all $x \in X_0$, it holds

$$\|T_{A_{0c}}(t)\Pi_{0c}x\| \leq Ke^{\alpha|t|}\|x\|;\ \forall t \in \mathbb{R}; \tag{2.6}$$

$$\|T_{A_{0u}}(t)\Pi_{0u}x\| \leq Ke^{\beta t}\|x\|;\ \forall t \leq 0; \tag{2.7}$$

$$\|T_{A_{0s}}(t)\Pi_{0s}x\| \leq Ke^{-\gamma t}\|x\|;\ \forall t \geq 0; \tag{2.8}$$

where $A_{0p} = A_0|_{X_{0p}}$ and $T_{A_{0p}}(t)$ is the C_0-semigroup generated by A_{0p}, $\forall p \in \{c; u; s\}$.

Remark 2.1. Since for each $k \in \{c; u\}$, Π_{0k} is finite-rank projection, then X_{0k} is a finite dimensional subspace of X_0. Then $T_{A_{0k}}(t)$ which acts on $x \in X_{0k}$ can be extended to a C_0-semigroup, $k \in \{c; u\}$.

Remark 2.2. Consulting [8], we have that if X_0 is a finite dimensional space and there exist eigenvalues with negative, zero, positive real parts, then the exponential trichotomy holds, and if X_0 is an infinite dimensional space and the spectrum of A_0 satisfies $\sigma(A_0) = \sigma_{0s} \cup \sigma_{0c} \cup \sigma_{0u}$, where $\sigma_{0s} = \{\lambda \in \sigma(A_0) : \mathrm{Re}(\lambda) < -\beta\}$, $\sigma_{0c} = \{\lambda \in \sigma(A_0) : |\mathrm{Re}(\lambda)| \le \beta\}$, $\sigma_{0u} = \{\lambda \in \sigma(A_0) : \mathrm{Re}(\lambda) > \beta\}$, and A_0 generates a strong continuous semigroup, then the exponential trichotomy holds, too (see [12, page 267]. Since $\sigma(A) = \sigma(A_0)$, the spectrum of A can be split as A_0.

Since we only require the Hille-Yosida condition on X_0, one needs to extend the mentioned projections from X_0 to X and outline the associated statement below.

Lemma 2.6. [16, Proposition 3.5] Let $\Pi_0 : X_0 \to X_0$ be a bounded linear projection satisfying

$$\Pi_0(\lambda I - A_0)^{-1} = (\lambda I - A_0)^{-1}\Pi_0, \ \forall \lambda > \omega.$$

and

$$\Pi_0 X_0 \subset D(A_0); \text{ and } A_0|_{\Pi_0 X_0} \text{ is bounded.}$$

Then there exists a unique bounded projection $\Pi : X \to X$ such that

(i) $\Pi|_{X_0} = \Pi_0$;
(ii) $\Pi(X) \subset X_0$;
(iii) $\Pi(\lambda I - A)^{-1} = (\lambda I - A)^{-1}\Pi$, $\forall \lambda > \omega$.

Furthermore, for each $x \in X$, we have

$$\Pi x = \lim_{\lambda \to +\infty} \Pi_0 \lambda(\lambda I - A)^{-1} x = \lim_{h \to 0^+} \frac{1}{h} \Pi_0 S(h) x,$$

where S is the integrated semigroup generated by A.

With the mentioned information at hand, we are ready to state the decomposition on X. For each $k \in \{c, u\}$, denote by $\Pi_k : X \to X$ the unique extension of Π_{0k} and

$$\Pi_s = I_X - (\Pi_c + \Pi_u).$$

Then we have for each $p \in \{c, u, s\}$,

$$\Pi_p(\lambda I - A)^{-1} = (\lambda I - A)^{-1}\Pi_p, \ \forall \lambda > \omega;$$

and

$$\Pi_p(X_0) \subset X_0.$$

Then for each $p \in \{c, u, s\}$, set

$$X_{0p} = \Pi_p(X_0); \ X_p = \Pi_p(X); \text{ and } A_p = A|_{X_p}.$$

Then for each $k \in \{c, u\}$,

$$X_{0k} = X_k; \ X_0 = X_{0s} \oplus X_{0c} \oplus X_{0u}; \text{ and } X = X_s \oplus X_c \oplus X_u.$$

Herein, X_{0c}, X_{0u} and X_{0s} are called centre subspace, unstable subspace and stable subspace of X_0 respectively. For more details, see [16, page 22].

2.4. **Estimate of Random Wick Convolution.** For the sake of notational simplicity, we introduce the notation of random Wick convolution to rewrite (2.5). In fact, it follows that

$$v(t) = \Phi_{A_0}(t;0)v_0 + (S \diamond G(v))(t);$$

where

$$\begin{aligned}(S \diamond G(v))(t) &= \lim_{\lambda\to+\infty}\int_0^t T(t-s)e^{\int_s^t z(\theta_r\omega)dr}\lambda(\lambda I - A)^{-1}G(\theta_s\omega; v(s))\,ds\\ &= \lim_{\lambda\to+\infty}\int_0^t \Phi_{A_0}(t;s)\lambda(\lambda I - A)^{-1}G(\theta_s\omega; v(s))\,ds.\end{aligned}$$

Before presenting the useful estimate, we recall a result on $(S \diamond f)(t)$.

Lemma 2.7. [16, Proposition 2.13] Let Assumptions 2.1 and 2.2 be satisfied. Then for $\gamma > \vartheta$, there exists $C_\gamma > 0$ such that $f \in C([0;T];X)$ and

$$\|(S \diamond f)(t)\| \le C_\gamma \sup_{s\in[0;t]} e^{\gamma(t-s)}\|f(t)\|; \ \forall t \in [0;T].$$

Moreover, for each $\varepsilon > 0$, if $\tau_\varepsilon > 0$ satisfying $M(\tau_\varepsilon) \le \varepsilon$, it holds

$$C_\gamma = \frac{2\varepsilon\max(1; e^{-\gamma\tau_\varepsilon})}{1 - e^{(\vartheta-\gamma)\tau_\varepsilon}}.$$

Proposition 2.1. Let Assumptions 2.1 and 2.4 be satisfied, then we have

$$\Pi_{0s}(S \diamond G(v))(t) = \lim_{\lambda\to+\infty}\int_0^t \Phi_{A_{0s}}(t;s)\lambda(\lambda I - A_s)^{-1}\Pi_s G(\theta_s\omega; v(s))\,ds; \tag{2.9}$$

while for each $k \in \{c; u\}$,

$$\Pi_{0k}(S \diamond G(v))(t) = \int_0^t \Phi_{A_{0k}}(t;s)\Pi_k G(\theta_s\omega; v(s))\,ds. \tag{2.10}$$

Moreover, for each $\gamma > \omega$, there exists $C_\gamma > 0$, such that for $\forall t \in [0;T]$, we obtain

$$\|\Pi_{0s}(S \diamond G(v))(t)\| \le C_\gamma \sup_{s\in[0;t]} e^{\gamma(t-s)}e^{\int_s^t z(\theta_r\omega)dr}\|G(\theta_s\omega; v(s))\|. \tag{2.11}$$

Proof. We first extract some facts from [16, Proposition 3.7] based on Assumption 2.1. To this end, denote $\Pi : X \to X$ by a bounded linear projection and assume

$$\Pi(\lambda I - A)^{-1} = (\lambda I - A)^{-1}\Pi; \ \forall \lambda > \vartheta.$$

Then the linear operator $A|_{\Pi(X)} = A_{\Pi(X)}$ satisfies Assumption 2.1 in $\Pi(X)$. Moreover, if Π has a finite rank and $\Pi(X) \subset X_0$, then $\Pi(X) = \Pi(X_0) \subset \Pi(D(A_0)) \subset D(A_0)$, $A|_{\Pi(X)}$ is a bounded linear operator from $\Pi(X_0)$ to itself. In particular, $A|_{\Pi(X)} = A_0|_{\Pi(X_0)}$. For each $x \in X_0$, if we set for each $t \in [0;T]$, that

$$v(t) = \Phi_{A_0}(t;0)x + (S \diamond G(v))(t);$$

then the map $t \to \Pi v(t)$ solves

$$\begin{cases} d\Pi v(t) = A_0|_{\Pi(X_0)}\Pi v(t)dt + z(\theta_t\omega)\Pi v(t)\,dt + \Pi G(\theta_t\omega; v(t))dt;\\ \Pi v(0) = \Pi x; \end{cases}$$

in $\Pi(X_0)$. This further implies that

$$\Pi v(t) = \Phi_{A_0|_{\Pi(X_0)}}(t;0)\Pi x + \left(S_{A|_{\Pi(X)}} \diamond \Pi G(v)\right)(t);$$

and

$$v(t) = x + A|_{(X)}\int_0^t v(s)ds + \int_0^t z(\theta_s\omega)v(s) + G(\theta_s\omega; v(s))ds.$$

We now turn to the associated projections and the required estimate. Note that $\Pi_s : X \to X$ is a bounded linear projection satisfying

$$\Pi_s(\lambda I - A)^{-1} = (\lambda I - A)^{-1}\Pi_s; \ \forall \lambda > \#.$$

Also, $A|_{\Pi_s(X)} = A|_{\Pi_s(X)} = A|_{X_s} = A_s$ satisfies Assumption 2.1 in $\Pi_s(X) = X_s$. Moreover, we know that Π_s has a finite rank and $\Pi_s(X) \subset X_0$, which further indicates that $A_0|_{\Pi_s(X_0)} = A_0|_{X_{0s}} = A_{0s}$ and $A|_{\Pi_s(X)} = A|_{X_s} = A_s$. Thus if for each $x \in X_0$ and each $t \in [0; T]$, we still set

$$v(t) = T_{A_0}(t; 0)x + (S \diamond G(v))(t);$$

then the map $t \to \Pi_s v(t)$ solves

$$\begin{aligned} d\Pi_s v(t) &= A_{0s}\Pi_s v(t)dt + \Pi_s z(\theta_t\omega)v(t)dt + \Pi_s G(\theta_t\omega; v(t))dt; \\ \Pi_s v(0) &= \Pi_s x; \end{aligned}$$

in $\Pi_s(X_0) = X_{0s}$. Then

$$\begin{aligned} \Pi_s v(t) &= T_{A_0|_{\Pi_s(X_0)}}(t; 0)\Pi_s x + \left(S_{A|_{\Pi_s(X)}} \diamond \Pi_s G(v)\right)(t) \\ &= T_{A_{0s}}(t; 0)\Pi_s x + (S_{A_s} \diamond \Pi_s G(v))(t) \\ &= T_{A_{0s}}(t; 0)\Pi_s x + \lim_{\lambda \to +\infty}\int_0^t T_{A_{0s}}(t; s)\lambda(\lambda I - A_s)^{-1}\Pi_s G(\theta_s\omega; v(s))ds. \end{aligned} \tag{2.12}$$

It is easy to prove that Π_s is a extension of Π_{0s} from X_0 to X, then we apply Π_{0s} to $v(t)$. In fact, we have

$$\Pi_{0s}v(t) = \Pi_s v(t) = T_{A_{0s}}(t; 0)\Pi_s x + \Pi_{0s}(S \diamond G(v))(t). \tag{2.13}$$

Comparing (2.12) and (2.13), we obtain the required expression (2.9). Similarly, we can prove that for each $k \in \{c; u\}$,

$$\Pi_{0k}(S \diamond G(v))(t) = \lim_{\lambda \to +\infty}\int_0^t T_{A_{0k}}(t; s)\lambda(\lambda I - A_k)^{-1}\Pi_k G(\theta_s\omega; v(s))ds.$$

Since $\Pi_k(X) \subset X_0$, (2.10) is sure by Lemma 2.2(i). Finally, it is straightforward to observe that the required estimate (2.11) is accessible by applying Lemma 2.7 to $\Pi_{0s}(S \diamond G(v))(t) = (S_{A_s} \diamond \Pi_s G(v))(t)$ and (2.9). □

3. **Center Manifolds.** This section is dedicated to the existence of invariant manifolds of (1.1) through the Lyapunov-Perron method. Precisely, we shall prove the existence of the invariant center manifold for (2.4), then use the inverse transformation T^{-1} to get a center manifold of (1.1).

Definition 3.1. A random set $M(\omega)$ is called invariant for a random dynamical system $\varphi(t; \omega; x)$ if $\varphi(t; \omega; M(\omega)) \subset M(\theta_t\omega)$ for $t \geq 0$.

For each $\eta > 0$ and a Banach space $(X; \|\cdot\|_X)$, we define the Banach space

$$C_\eta(\mathbb{R}; X) = \left\{ f \in C(\mathbb{R}; X) : \sup_{t \in \mathbb{R}} e^{-\eta|t| - \int_0^t z(\theta_s\omega)ds}\|f(t)\|_X < \infty \right\};$$

with the norm

$$\|f(t)\|_{C(\mathbb{R};X)} = \sup_{t\in\mathbb{R}} e^{\ |t|\ \int_0^t z(\ _s\omega)ds}\|f(t)\|_X.$$

Set

$$M^c(\omega) := \{v_0 \in X_0 : v(t;\omega;v_0) \in C\ (R;X_0)\};$$

where $v(t;\omega;v_0)$ is the solution of (2.4) with the initial data $v(0) = v_0$. We below prove that $M^c(\omega)$ is invariant and it is given by a graph of a Lipschitz function.

Theorem 3.1. Assume Assumptions 2.1-2.4 hold. If $\ <\ <\min\{\ ;\ \}$ and $\ <\ <\ $ with

$$KL\ \ C\ + \frac{1}{\ } + \frac{1}{\ }\ < 1; \tag{3.1}$$

then there exists a center invariant manifold for (2.4), given by

$$M^c(\omega) = \{\ + h^c(\ ;\omega):\ \in X_{0c}\};$$

where $h^c(\ ;\omega) : X_{0c} \to X_{0u} \bigoplus X_{0s}$ is a Lipschitz continuous map and $h^c(0;\omega) = 0$.

Proof. We proceed it in four steps.

Step 1. We prove that $v_0 \in M^c(\omega)$ if and only if there exists a function $v(\ ;\omega;v_0) \in C\ (R;X_0)$ with $v(0) = v_0$ which satisfies

$$\begin{aligned} v(t;\omega;v_0) = {}& _{A_{0c}}(t;0)\ \ + \int_0^t\ _{A_{0c}}(t;s)\ _cG(\ _s\omega;v(s))\,ds \\ & \int_t^{+\infty}\ _{A_{0u}}(t;s)\ _uG(\ _s\omega;v(s))\,ds \\ & + \lim_{\to+\infty}\int_{-\infty}^t\ _{A_{0s}}(t;s)\ (\ I\ \ A_s)^{-1}\ _sG(\ _s\omega;v(s))\,ds; \end{aligned} \tag{3.2}$$

where $\ = \ _{0c}v_0$.

To this aim, we let $v_0 \in M^c(\omega)$ and assume that $v \in C\ (R;X_0)$ is a solution of (2.4). Then by the procedure of proof to the Lemma 2.1, for $k \in \{c;u\}$, we have

$$\begin{aligned} \frac{d\ _kv(t)}{dt} = \frac{d\ _{0k}v(t)}{dt} &= A_0|_{\ _k(X_0)}\ _kv(t) + \ _kz(\ _t\omega)v(t) + \ _kG(\ _t\omega;v(t)) \\ &= A_{0k}\ _{0k}v(t) + \ _kz(\ _t\omega)v(t) + \ _kG(\ _t\omega;v(t)). \end{aligned}$$

By the variation of constants formula, one further gets

$$\ _{0c}v(t;\omega;v_0) = \ _{A_{0c}}(t;0)\ \ + \int_0^t\ _{A_{0c}}(t;s)\ _cG(\ _s\omega;v(s))\,ds; \tag{3.3}$$

for all $t;l \in R$ with $t < l$,

$$\ _{0u}v(t;\omega;v_0) = \ _{A_{0u}}(t;l)\ _{0u}v(l) + \int_l^t\ _{A_{0u}}(t;s)\ _uG(\ _s\omega;v(s))\,ds. \tag{3.4}$$

According to (2.7), for $l\ \ 0$,

$$\begin{aligned} \|\ _{A_{0u}}(t;l)\ _{0u}v(l)\| &\ Ke^{\ (t\ l)+\int_l^t z(\ _r\omega)dr}\|v(l)\| \\ & leq Ke^{\ (t\ l)+\ l+\int_0^t z(\ _r\omega)dr}e^{\ \ l\ \int_0^l z(\ _r\omega)dr}\|v(l)\| \\ &\ Ke^{(\ \ \ )l+\ t+\int_0^t z(\ _r\omega)dr}\|v\|_{C(\mathbb{R};X_0)}. \end{aligned}$$

Let $l \to +\infty$, we have

$$ {}_{0u} v(t;\omega;v_0) = \int_t^{+\infty} {}_{A_{0u}}(t;s) \quad {}_u G({}_s \omega; v(s)) ds. \tag{3.5}$$

Moreover, for all $t, l \in \mathbb{R}$ with $t \quad l$,

$$\begin{aligned} {}_{0s} v(t) &= {}_{A_{0s}}(t;l) \quad {}_{0s} v(l) + {}_{0s}(S \quad G(v(l+\cdot)))(t \quad l) \\ &= {}_{A_{0s}}(t;l) \quad {}_{0s} v(l) + \lim_{\to +\infty} \int_l^t {}_{A_{0s}}(t;s) \quad (\quad I \quad A_s)^{\quad 1} \\ &\qquad {}_s G({}_s \omega; v(s)) ds. \end{aligned} \tag{3.6}$$

According to (2.7), for $l \quad 0$,

$$\begin{aligned} \| {}_{A_{0u}}(t;l) \quad {}_{0u} v(l) \| &\quad K e^{(t \quad l) + \int_l^t z(\,{}_r\omega) dr} \| v(l) \| \\ &\quad K e^{(t \quad l) + \quad l + \int_0^t z(\,{}_r\omega) dr} e^{\quad l \quad \int_0^l z(\,{}_r\omega) dr} \| v(l) \| \\ &\quad K e^{(\quad) l + \quad t + \int_0^t z(\,{}_r\omega) dr} \| v \|_{C \quad (\mathbb{R}; X_0} \end{aligned}$$

To show this claim, we denote $J(v;\xi)$ the right side of (3.2), i.e.

$$
\begin{aligned}
J(v;\xi) = {} & \Phi_{A_{0c}}(t;0)\xi + \int_0^t \Phi_{A_{0c}}(t;s)\Pi_c G(\theta_s\omega; v(s))\,ds \\
& - \int_t^{+\infty} \Phi_{A_{0u}}(t;s)\Pi_u G(\theta_s\omega; v(s))\,ds \\
& + \lim_{\lambda\to+\infty}\int_{-\infty}^t \Phi_{A_{0s}}(t;s)\lambda(\lambda I - A_s)^{-1}\Pi_s G(\theta_s\omega; v(s))\,ds.
\end{aligned} \tag{3.10}
$$

So it needs to show that J maps from $C_\eta(\mathbb{R};X_0)\times X_{0c}$ to $C_\eta(\mathbb{R};X_0)$ and it is a uniform contraction. For $v,\bar v\in C_\eta(\mathbb{R};X_0)$

$$
\begin{aligned}
\|J(v;\xi) - J(\bar v;\xi)\|_{C_\eta(\mathbb{R};X_0)} &= \sup_{t\in\mathbb{R}} e^{-\eta|t| - \int_0^t z(\theta_s\omega)ds}\|J(v;\xi) - J(\bar v;\xi)\| \\
&:= \sup_{t\in\mathbb{R}} e^{-\eta|t| - \int_0^t z(\theta_s\omega)ds}(J_1 + J_2 + J_3);
\end{aligned}
$$

where

$$
\begin{aligned}
J_1 &= \left\|\int_0^t T_{A_{0c}}(t-s)e^{\int_s^t z(\theta_r\omega)dr}\Pi_c\big(G(\theta_s\omega; v(s)) - G(\theta_s\omega;\bar v(s))\big)ds\right\|;\\
J_2 &= \left\|\int_t^{+\infty} T_{A_{0u}}(t-s)e^{\int_s^t z(\theta_r\omega)dr}\Pi_u\big(G(\theta_s\omega;\bar v(s)) - G(\theta_s\omega; v(s))\big)ds\right\|;\\
J_3 &= \left\|\lim_{\lambda\to+\infty}\int_{-\infty}^t \Phi_{A_{0s}}(t;s)\lambda(\lambda I - A_s)^{-1}\Pi_s\big(G(\theta_s\omega; v(s)) - G(\theta_s\omega;\bar v(s))\big)ds\right\|.
\end{aligned}
$$

For the center part, it yields

$$
\begin{aligned}
\sup_{t\in\mathbb{R}} e^{-\eta|t| - \int_0^t z(\theta_s\omega)ds}J_1 &\le KL\sup_{t\in\mathbb{R}}\left|\int_0^t e^{(\gamma-\eta)|t-s| - \eta|s| - \int_0^s z(\theta_r\omega)dr}\|v-\bar v\|ds\right| \\
&\le KL\sup_{t\in\mathbb{R}}\left|\int_0^t e^{(\gamma-\eta)|t-s|}ds\right|\|v-\bar v\|_{C_\eta(\mathbb{R};X_0)} \\
&\le KL\frac{1}{\eta-\gamma}\|v-\bar v\|_{C_\eta(\mathbb{R};X_0)}.
\end{aligned}
$$

For the unstable part, it follows that

$$
\begin{aligned}
\sup_{t\in\mathbb{R}} e^{-\eta|t| - \int_0^t z(\theta_s\omega)ds}J_2 &\le KL\sup_{t\in\mathbb{R}}\int_t^{+\infty} e^{(\beta-\eta)(t-s) - \eta|s| - \int_0^s z(\theta_r\omega)dr}\|v-\bar v\|ds \\
&\le KL\sup_{t\in\mathbb{R}}\int_0^t e^{(\beta-\eta)(t-s)}ds\|v-\bar v\|_{C_\eta(\mathbb{R};X_0)} \\
&\le KL\frac{1}{\beta-\eta}\|v-\bar v\|_{C_\eta(\mathbb{R};X_0)}.
\end{aligned} \tag{3.11}
$$

For the stable part, according to (2.11), for with < < , we have

$$J_3 = \lim_{\to+\infty}\lim_{r\to-\infty}\int_r^t T_{A_{0s}}(t;s)\,(\ I - A_s)^{-1}\,\pi_s(G(\theta_s\omega;v(s)) - G(\theta_s\omega;\bar v(s)))\,ds$$

$$= \lim_{\to+\infty}\lim_{r\to-\infty}\int_0^{t-r} T_{A_{0s}}(t;l+r)\,(\ I - A_s)^{-1}$$

$$\pi_s(G(\theta_{l+r}\omega;v(l+r)) - G(\theta_{l+r}\omega;\bar v(l+r)))\,dl$$

$$\le \lim_{r\to-\infty} LC\sup_{l\in[0;t-r]} e^{(t-r-l)}e^{\int_{l+r}^{t} z(\ \omega)d}\,\|v(l+r) - \bar v(l+r)\|$$

$$= LC\sup_{\in(-\infty;t]} e^{(t-\ )}e^{\int^{t} z(\ \omega)d}\,\|v(\ ) - \bar v(\ )\|;$$

implying that

$$\sup_{t\in\mathbb{R}} e^{-|t|-\int_0^t z(\theta_s\omega)ds}J_3 \le LC\sup_{t\in\mathbb{R}} e^{-|t|}\sup_{\in(-\infty;t]} e^{|\ |+(t-\ )}$$

$$e^{-|\ |-\int_0 z(\ \omega)d}\,\|v(\ ) - \bar v(\ )\|$$

$$\le LC\sup_{t\in\mathbb{R}} e^{-|t|}\sup_{\in(-\infty;t]} e^{|\ |+(t-\ )}\|v - \bar v\|_{C\ (\mathbb{R};X_0)}.$$

Since $+ < 0$, if $t \le 0$,

$$LC\sup_{t\le 0} e^{-|t|}\sup_{\in(-\infty;t]} e^{|\ |+(t-\ )}\|v - \bar v\|_{C\ (\mathbb{R};X_0)}$$

$$= LC\sup_{t\le 0} e^{(\ +\ )t}\sup_{\in(-\infty;t]} e^{-(\ +\ )}\|v - \bar v\|_{C\ (\mathbb{R};X_0)}$$

$$= LC\sup_{t\le 0} e^{(\ +\ )t}e^{-(\ +\ )t}\|v - \bar v\|_{C\ (\mathbb{R};X_0)}$$

$$= LC\,\|v - \bar v\|_{C\ (\mathbb{R};X_0)}.$$

Since $+ < 0$ and > 0, we have > 0, if $t \ge 0$,

$$LC\sup_{t\ge 0} e^{-|t|}\sup_{\in(-\infty;t]} e^{|\ |+(t-\ )}\|v - \bar v\|_{C\ (\mathbb{R};X_0)}$$

$$= LC\sup_{t\ge 0} e^{(\ -\ )t}\sup_{\in(-\infty;t]} e^{|\ |-}\|v - \bar v\|_{C\ (\mathbb{R};X_0)}$$

$$\le LC\sup_{t\ge 0} e^{(\ -\ )t}\max\left(\sup_{\in(-\infty;0]} e^{-(\ +\ )};\ \sup_{\in[0;t]} e^{(\ -\ )}\right)\|v - \bar v\|_{C\ (\mathbb{R};X_0)}$$

$$\le LC\sup_{t\ge 0} e^{(\ -\ )t}e^{(\ -\ )t}\|v - \bar v\|_{C\ (\mathbb{R};X_0)}$$

$$= LC\,\|v - \bar v\|_{C\ (\mathbb{R};X_0)}.$$

Thus

$$\sup_{t\in\mathbb{R}} e^{-|t|-\int_0^t z(\theta_s\omega)ds}J_3 \le LC\,\|v - \bar v\|_{C\ (\mathbb{R};X_0)}. \tag{3.12}$$

By combining the above estimates, we have

$$\|J(v;\ ) - J(\bar v;\ )\|_{C\ (\mathbb{R};X_0)} \le KL\left(C + \frac{1}{\ } + \frac{1}{\ }\right)\|v - \bar v\|_{C\ (\mathbb{R};X_0)};$$

which implies that J is a uniform contraction with respect to v. Furthermore, it is straightforward that J is well defined from $C\ (\mathbb{R};X_0) \times X_{0c}$ to $C\ (\mathbb{R};X_0)$ by setting $\bar v = 0$. From the contraction mapping principle, for any given $\in X_{0c}$, $J(\ ;\ )$ has a unique fixed point $\tilde{v} \in C(\mathbb{R}; X_0)$. That is, for each $t \in \mathbb{R}$, $\tilde{v}(t; \omega; \ )$ is a unique solution to (3.2) and

$$J(\tilde{v}; \ ) = \tilde{v}(t; \omega; \ ):$$

Also, for $\forall \ _1; \ _2 \in X_{0c}$,

$$\begin{aligned} &\|J(\tilde{v}; \ _1) \quad J(\tilde{v}; \ _2)\|_{C \ (\mathbb{R};X_0)} \\ &\quad = \|\tilde{v}(t; \omega; \ _1) \quad \tilde{v}(t; \omega; \ _2)\|_{C \ (\mathbb{R};X_0)} \\ &\qquad K \, | \ _1 \quad _2| + K L \quad C \ + \frac{1}{\quad} + \frac{1}{\quad} \quad \|\tilde{v}(t; \omega; \ _1) \quad \tilde{v}(t; \omega; \ _2)\|_{C \ (\mathbb{R};X_0)}: \end{aligned}$$

Thus

$$\|\tilde{v}(t; \omega; \ _1) \quad \tilde{v}(t; \omega; \ _2)\|_{C \ (\mathbb{R};X_0)} \quad \frac{K}{1 \quad K L \quad C \ + \frac{1}{\ } + \frac{1}{\ }} | \ _1 \quad _2|: \tag{3.13}$$

So $\tilde{v}(t; \omega; \ )$ is Lipschitz continuous from X_{0c} to $C \ (\mathbb{R}; X_0)$. Since $\tilde{v}(\ ; \omega; \ )$ can be an ω-wise limit of the iteration of contraction mapping J starting at 0 and J maps a measurable function to a measurable function, $\tilde{v}(\ ; \omega; \ )$ is measurable. By [7, Lemma III.14], the mapping $\tilde{v}(t; \omega; \ )$ is measurable with respect to $(\ ; \omega; \ )$.

Step 3. We will prove that the center manifold is given by a graph of a Lipschitz continuous map.

Let $h^c(\ ; \omega) = \ _s\tilde{v}(0; \omega; \ ) + \ _u\tilde{v}(0; \omega; \ )$, then by (3.7) and (3.9),

$$\begin{aligned} h^c(\ ; \omega) = \lim_{\omega \to +1} &\int_{1}^{0} \ A_{0s}(0; s) \ (\ I \quad A_s) \ ^{1} \ _s G(\ _s\omega; \tilde{v}(s; \omega; \ ))\, ds \\ &\int_{0}^{+1} \ A_{0u}(0; s) \ \ _u G(\ _s\omega; \tilde{v}(s; \omega; \ ))\, ds: \end{aligned}$$

Since $\tilde{v}(0; \omega; v_0) = v_0$, we have $h^c(0; \omega) = 0$ and thus h^c is F-measurable. Indeed, by (3.11)-(3.13), it follows that

$$|h^c(\ _1; \omega) \quad h^c(\ _2; \omega)| \quad \frac{K(K L + C \ L(\qquad))}{(\qquad) \ 1 \quad K L \quad C \ + \frac{1}{\ } + \frac{1}{\ }} | \ _1 \quad _2|:$$

From the definition of $h^c(\ ; \omega)$ and the claim in **Step 1**, it follows that $v_0 \in \mathcal{M}^c(\omega)$ if and only if there exists $\ \in X_{0c}$ such that $v_0 = \ + h^c(\ ; \omega)$. Therefore, we have

$$\mathcal{M}^c(\omega) = \{ \ + h^c(\ ; \omega) : \ \in X_{0c}\}:$$

Step 4. Finally, we prove $\mathcal{M}^c(\omega)$ is invariant.

Following Definition 3.1, we need to show that for each $v_0 \in \mathcal{M}^c(\omega)$, $v(r; \omega; v_0) \in \mathcal{M}^c(\ r\omega)$ for all r

So $\mathcal{M}^c(\theta_t\omega)$ is invariant. Moreover, for $t \geq 0$,

$$\begin{aligned}\mathcal{M}^c(\theta_t\omega) &= \Phi^{-1}(\omega;\mathcal{M}^c(\omega))\\ &= \left\{u_0 = \Phi^{-1}(\omega;\xi + h^c(\xi;\omega)) : \xi \in X_{0c}\right\}\\ &= \left\{u_0 = e^{z(\theta_t\omega)}(\xi + h^c(\xi;\omega)) : \xi \in X_{0c}\right\}\\ &= \left\{u_0 = \xi + e^{z(\theta_t\omega)}h^c\left(e^{-z(\theta_t\omega)}\xi;\omega\right) : \xi \in X_{0c}\right\}.\end{aligned}$$

Therefore, $\mathcal{M}^c(\theta_t\omega)$ is a Lipschitz center invariant manifold given by the graph of a Lipschitz continuous function $h^c(\xi;\theta_t\omega) = e^{z(\theta_t\omega)}h^c\left(e^{-z(\theta_t\omega)}\xi;\omega\right)$ over X_{0c}. □

4. Illustrative example.

Consider the following stochastic Stratonovich parabolic partial equation

$$\begin{cases}\dfrac{\partial u(t,x)}{\partial t} = \left(\dfrac{\partial^2 u(t,x)}{\partial x^2} + \beta^2 u(t,x) + g_0(u(t,x))\right)dt + u(t,x)\circ dW(t)\\ \dfrac{\partial u(t,0)}{\partial x} = g_1(u(t,\cdot))\\ \dfrac{\partial u(t,1)}{\partial x} = g_2(u(t,\cdot))\\ u(0,\cdot) = u_0 \in L^2(0,1)\end{cases} \tag{4.1}$$

where $t > 0, x \in (0,1), g_0 : L^2(0,1) \to L^2(0,1)$, $g_1, g_2 : L^2(0,1) \to \mathbb{R}$ are arbitrary nonlinear functions, which are assumed to be globally Lipschitz continuous and $g_0(0) = g_1(0) = g_2(0) = 0$. Herein, W is a one-dimensional Brownian motion and $L^2(0,1)$ is the usual L^2-space on the interval $(0,1)$. In order to incorporate the boundary conditions, we define

$$Y = \mathbb{R}\times\mathbb{R}\times L^2(0,1) \text{ and } Y_0 = \{0\}\times\{0\}\times L^2(0,1),$$

which are Banach spaces equipped with the usual product norm. Let us denote

$$A\begin{pmatrix}0\\0\\\varphi\end{pmatrix} = \begin{pmatrix}\varphi'(0)\\-\varphi'(1)\\\varphi''\end{pmatrix}$$

with domain $D(A) = \{0\}\times\{0\}\times W^{2,2}(0,1)$ and its part

$$A_0\begin{pmatrix}0\\0\\\varphi\end{pmatrix} = \begin{pmatrix}0\\0\\\varphi''\end{pmatrix}$$

with domain $D(A_0) = \{0\}\times\{0\}\times\left\{\varphi \in W^{2,2}(0,1) : \varphi'(0) = \varphi'(1) = 0\right\}$, where $W^{2,2}(0,1)$ is the usual Sobolev space. Notice that

$$Y_0 = \overline{D(A)} = \{0\}\times\{0\}\times L^2(0,1) \neq Y$$

and take

$$F\begin{pmatrix}0\\0\\\varphi\end{pmatrix} = \begin{pmatrix}g_0(\varphi)\\g_1(\varphi)\\g_2(\varphi)\end{pmatrix}, \quad u(t,\cdot) = \begin{pmatrix}0\\0\\u(t,\cdot)\end{pmatrix}, \quad \omega(t) = \begin{pmatrix}0\\0\\W(t)\end{pmatrix}.$$

Then we can rewrite (4.1) as

$$\begin{cases}du(t) = \left(\left(A + \beta^2 I\right)u(t) + F(u(t))\right)dt + u(t)\circ d\omega(t),\\ u(0) = u_0 \in Y_0.\end{cases}$$

According to [17, Lemma 6.1 & Lemma 6.3] that the linear operator A_0 is the infinitesimal generator of $(T_{A_0}(t))_{t\ge 0}$ a C_0-semigroup on Y_0 and the non-densely defined operator A is 3/4-almost sectorial, namely, its resolvent satisfies

$$0 < \liminf_{\lambda\to\infty} \lambda^{\frac{3}{4}} \left\|(\lambda I - A)^{-1}\right\|_{L(X)} \le \limsup_{\lambda\to\infty} \lambda^{\frac{3}{4}} \left\|(\lambda I - A)^{-1}\right\|_{L(X)} < \infty.$$

Then [15, Lemma 8.5] guarantees Assumption 2.2. Moreover, the spectrum of A_0 is given by

$$\sigma(A_0) = \left\{ -(k\pi)^2 : k \in \mathbb{N} \right\}.$$

Moreover, $\left(A + \pi^2 I\right)_0$, the part of $\left(A + \pi^2 I\right)$, is the infinitesimal generator of a C_0-semigroup on Y_0 denoted by $\left\{T_{(A+\pi^2 I)_0}(t)\right\}_{t\ge 0}$. Also, according to [17, Lemma 6.4] and [14, 15, Proposition 2.5], we get that Assumption 2.1 is satisfied, which means A does not satisfy the Hille-Yosida condition but $\left(A + \pi^2 I\right)$ generates a integrated semigroup $\left\{S_{(A+\pi^2 I)}(t)\right\}_{t\ge 0}$. In addition, Assumption 2.3 is obvious by the Lipschitz continuity of $g_i;\ i = 0; 1; 2$. So we next check Assumption 2.4. In fact, by [17, Lemma 6.2] and [16, Lemma 2.1], we have $\sigma(A_0) = \sigma(A)$, then

$$\begin{aligned}
\sigma\left(A + \pi^2 I\right) = \sigma\left(A_0 + \pi^2 I\right) &= \left\{ -(k\pi)^2 + \pi^2 : k \in \mathbb{N} \right\} \\
&= \left\{ \pi^2; 0; -3\pi^2; -8\pi^2; -15\pi^2; \dots \right\};
\end{aligned}$$

and each enginvalue $\lambda_k = \left(1 - k^2\right)\pi^2$ corresponding to the enginfunction

$$e_k(x) = \sin(\pi k x).$$

By Remark 2.2, we could take $\gamma = 3\pi^2 - \varepsilon$, $\beta = \pi^2 + \varepsilon$ and $\eta \in \left(0; \min\left\{\pi^2 + \varepsilon; 3\pi^2 - \varepsilon\right\}\right)$ for any $\varepsilon > 0$. Thus the spectrum of $A_0 + \pi^2 I$ could be split into three parts $\sigma_{0s} = \left\{(1-k)^2\pi^2 : k = 2; 3; \dots\right\}$; $\sigma_{0c} = \{0\}$; $\sigma_{0u} = \left\{\pi^2\right\}$. So $A_0 + \pi^2 I$ satisfies the exponential trichotomy condition and the Assumption 2.4 is satisfied. The center subspace Y_{0c}, stable subspace Y_{0s} and unstable subspace Y_{0u} of Y_0 are span $\{\sin(\pi x)\}$, span $\{\sin(\pi n x); n = 2; 3; \dots\}$, and $\{0\}$, respectively.

Therefore, the results obtained in Theorems 3.1 and 3.2 are applicable and thus justify the existence of Lipschitz center manifolds for (4.1).

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E-mail address: 202020129528@mail.scut.edu.cn
E-mail address: macbzeng@scut.edu.cn