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Regular Article

Existence of smooth stable manifolds for a class of parabolic SPDEs with fractional noise ☆

Xiaofang Lin [a], Alexandra Neamţu [b], Caibin Zeng [a,*]

[a] School of Mathematics, South China University of Technology, Guangzhou 510640, PR China

[b] University of Konstanz, Department of Mathematics and Statistics, Universitätsstraße 10, 78464 Konstanz, Germany

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A B S T R A C T

Little seems to be known about the invariant manifolds for stochastic partial differential equations (SPDEs) driven by nonlinear multiplicative noise. Here we contribute to this aspect and analyze the Lu-Schmalfuß conjecture [Garrido-Atienza, et al., (2010) [14]] on the existence of stable manifolds for a class of parabolic SPDEs driven by nonlinear multiplicative fractional noise. We emphasize that stable manifolds for SPDEs are infinite-dimensional objects, and the classical Lyapunov-Perron method cannot be applied, since the Lyapunov-Perron operator does not give any information about the backward orbit. However, by means of interpolation theory, we construct a suitable function space in which the discretized Lyapunov-Perron-type operator has a unique fixed point. Based on this we further prove the existence and smoothness of local stable manifolds for such SPDEs.

☆ This

1. Introduction

Invariant manifolds including center, stable and unstable manifolds are powerful tools in the study of properties and structures for deterministic and random dynamical systems, such as the derivation of normal forms and understanding the nature of bifurcations [21,7,6,41]. Recently, a random version of the C^1 Hartamn theorem was proved in [29] by constructing a smooth weak-stable invariant manifold which provides a way to overcome di culties from nonuniformity and measurability. In general, there are two approaches to construct invariant manifolds: Hadamard's graph transform method [19] and the Lyapunov-Perron method [35,30]. The contribution list in this respect is too extensive to be completely written down herein. We refer to [8,9,27,32,3] and the references specified therein.

In this work we investigate stable manifolds for SPDEs given by

$$du(t) = [Au(t) + F(u(t))]dt + G(u(t))d\omega(t), \tag{1.1}$$

where ω is a Hilbert space-valued fractional Brownian motion (fBm) with Hurst index $H > \frac{1}{2}$. The assumptions on the linear operator A and on the nonlinear terms F and G

[14], in which the Lyapunov-Perron method is applied, the associated Lyapunov-Perron operator is related to the backward orbit. The existence of a fixed point for this operator implies the existence of a backward solution. It has been proved in [13] that the forward solution belongs to a subspace of $D((-A)^{\alpha})$ for $\alpha > 0$ (see Subsection 2.3 for a precise statement) and it is also Hölder continuous on the state space. For stable manifolds, this construction suggests that the backward solution is Hölder regular on the state space and should also belong to this domain for a certain $\alpha > 0$.

However, one cannot seek stable manifolds in such a space, because the Lyapunov-Perron operator does not give any information on the backward solution, see Subsection 2.3 for more details. Therefore one cannot a-priori make a similar ansatz as for unstable manifolds.

To overcome this difficulty, inspired by [16,17], one of the main novel contributions of this work is to introduce the function space $C^{\beta}_{-\beta}([0,T];\mathcal{B})$. This contains continuous mappings with values in the state space, which we denote by $\mathcal{B}$, and β-Hölder continuous mappings with values in $\mathcal{B}_{-\beta}$, where $\beta > \frac{1}{2}$. Here $\{\mathcal{B}_{\gamma}\}_{\gamma\in\mathbb{R}}$ is a monotone family of interpolation spaces associated to the linear part of the equation, as discussed in [17]. A similar approach combined with rough paths tools has been followed in [24] to construct center manifolds for (1.1), where the Hurst index of the fBm $H \in (\frac{1}{3}, \frac{1}{2}]$. Even though we deal with the case $H > \frac{1}{2}$, the fBm considered in this manuscript is infinite-dimensional, in contrast to [24]. Moreover, we only assume that the covariance operator Q of the fBm is of trace-class. This assumption is less restrictive than the condition imposed in [14] ($\mathrm{tr}Q^{1/2} < \infty$), which constructs unstable manifolds for (1.1). In order to establish the existence of stable manifolds, we first show the global existence of a solution and therefore obtain by common methods an RDS. Putting these deliberations together we construct a Lyapunov-Perron type operator [14,24]. Under a certain spectral gap condition, this operator is proved to have a unique fixed point in the function space $C^{\beta}_{-\beta}([0,T];\mathcal{B})$. This fixed point is further used to characterize the stable manifold. Due to the nonlinear diffusion coefficient G and in order to control the growth of the noise, a truncation argument is additionally required for the fixed point argument. In conclusion our results provide a positive answer to the Lu-Schmalfuß conjecture [14,28]. The pathwise integrals occurring in the Lyapunov-Perron method are constructed using fractional calculus as in [5]. Nevertheless one should obtain similar results using Young's integral, see Remark 2.7. Moreover we believe that these results can be generalized to the rough case, i.e. $H \in (\frac{1}{3}, \frac{1}{2}]$ using rough paths theory. The methods presented in this work combined with the tools developed in [24] are expected to entail the existence of stable manifolds provided that the driving noise is finite-dimensional. For infinite-dimensional trace-class noise, as considered here, similar arguments could be conducted based on the results obtained in [20]. This case is technically more involved and therefore postponed to another work.

This work is structured as follows. In Section 2, we introduce basic concepts on RDS, Hilbert space-valued fBm and a pathwise integral with respect to this process via fractional calculus. Furthermore, we formulate the equation we consider and prove the global well-posedness and the generation of an RDS (Theorem 2.12). Section 3 contains our main results. In Theorem 3.7 we prove the existence of a local stable manifold for (1.1) in $C^{\beta}_{-\beta}([0,T];\mathcal{B})$. Under additional assumptions on the coefficients, the smoothness of this manifold is established in Subsection 3.3. Assertions regarding the smoothness of invariant manifolds have been derived in [9,27] for linear multiplicative (Brownian) noise. To our best knowledge, this is the first work that provides a complete proof of the smoothness of random invariant manifolds for SPDEs with nonlinear multiplicative fractional noise.

2. Preliminaries

In this section, we present some basic concepts on RDS, fBm and the construction of the integral concerning β-Hölder continuous integrator from [1,4,31,42,14] and references therein. For the sake of completeness, we also establish the well-posedness of SPDEs with fractional noise in a different function space as in [33,5,13]. Some key estimates for the computations of stable manifolds will be provided in this setting.

2.1. Random dynamical systems

Given a metric dynamical system $(\Omega,\mathcal{F},\mathbb{P},\{\theta_t\}_{t\in\mathbb{R}})$, a function $X:\Omega\to\mathbb{R}$ is called a random variable if it is $(\mathcal{F};\mathscr{B}(\mathbb{R}))$-measurable. It is tempered if

$$\lim_{t\to\pm\infty}\frac{\log X(\theta_t\omega)}{t}=0,\quad \mathbb{P}\text{-almost surely.}$$

In particular, it is tempered from above if

$$\lim_{t\to\pm\infty}\frac{\log^+ X(\theta_t\omega)}{t}=0,\quad \mathbb{P}\text{-almost surely,}$$

and tempered from below if $1/X$ is tempered from above. Here we recall that $\log^+ x:=\max\{\log x,0\}$. Let $(\mathcal{B},|\cdot|,\langle\cdot,\cdot\rangle)$ be a separable Hilbert space. A set-valued mapping $M:\Omega\to 2^{\mathcal{B}}\setminus\emptyset$, $\omega\mapsto M(\omega)$ is said to be a random set if $M(\omega)$ is a closed set and $\omega\mapsto\inf_{y\in M(\omega)}|x-y|$ is a random variable for each $x\in\mathcal{B}$.

Definition 2.1. A mapping $\varphi:\mathbb{R}^+\times\Omega\times\mathcal{B}\to\mathcal{B}$ defines a random dynamical system if

(i) φ is $(\mathscr{B}(\mathbb{R}^+)\otimes\mathcal{F}\otimes\mathscr{B}(\mathcal{B}),\mathscr{B}(\mathcal{B}))$-measurable;
(ii) the mapping $\varphi(t,\omega):=\varphi(t,\omega,\cdot):\mathcal{B}\to\mathcal{B}$ forms a cocycle over $\{\theta_t\}_{t\in\mathbb{R}}$ meaning that

$$\begin{aligned}\varphi(0,\omega)&=\mathrm{id}_{\mathcal{B}},\quad \text{for all } \omega\in\Omega,\\ \varphi(t+s,\omega)&=\varphi(t,\theta_s\omega)\circ\varphi(s,\omega),\quad \text{for all } t,s\in\mathbb{R}^+ \text{ and all } \omega\in\Omega.\end{aligned}$$

Definition 2.2. A random set M is called an invariant set for φ if

$$\varphi(t,\omega,M(\omega)) \subset M(\theta_t\omega) \text{ for } t \geq 0.$$

It is called a random Lipschitz manifold if it can be characterized by the graph of a Lipschitz mapping, meaning that there exists a function m such that

$$M(\omega) = \{x \in \mathcal{B} : x = \xi + m(\xi,\omega)\}. \tag{2.1}$$

Here $m(\cdot,\omega) : \mathcal{B}^- \to \mathcal{B}^+$ is Lipschitz continuous and $\mathcal{B}^-$ and $\mathcal{B}^+$ are unstable and stable subspaces as defined in Section 2.3.

Definition 2.3. A random manifold $M(\omega)$ is called a local stable manifold at zero for φ if m is defined in a neighborhood $U \subset \mathcal{B}^-$ of zero and there exists a neighborhood $W(\omega) \subset \mathcal{B}$ of zero such that

(1) for $x \in M(\omega) \cap W(\omega)$,

$$\lim_{t \to +\infty} \varphi(t,\omega,x) = 0 \quad \text{exponentially},$$

(2)

$$\lim_{|x| \to 0} t_0(\omega,x) = \infty,$$

where $x \in M(\omega)$ and

$$t_0(\omega,x) = \inf\{t \in \mathbb{R}^+ : \varphi(t,\omega,x) \notin M(\theta_t\omega)\}.$$

2.2. Hilbert space-valued fractional Brownian motion and stochastic integral

The two-sided one-dimensional fBm $(\beta^H(t))_{t\in\mathbb{R}}$, where $\{\beta_i^H(t)\}_{i\in\mathbb{N}}$ is a sequence of stochastically independent one-dimensional fBms. We next construct a canonical probability space associated to B^H. Denote by $\Omega = C_0(\mathbb{R};\mathcal{B})$ the space of continuous functions $\omega : \mathbb{R} \to \mathcal{B}$ such that $\omega(0) = 0$, equipped with the compact open topology, $\mathcal{F}$ represents the associated Borel-σ-algebra of Ω and $\mathbb{P}$is

$$\int_0^T Z(t)d\omega(t) = (-1)^\alpha \sum_{j\in\mathbb{N}} \left(\sum_{i\in\mathbb{N}} \int_0^T D_{0+}^\alpha \langle e_j, Z(\cdot)e_i\rangle[t] D_{T-}^{1-\alpha}\langle e_i, \omega(\cdot)\rangle[t]dt \right) e_j \tag{2.2}$$

is well-defined. Also, the estimate of its norm follows

$$\left| \int_0^T Z(t)d\omega(t) \right|_{\mathcal{B}} \leq \int_0^T \left\| D_{0+}^\alpha Z[t] \right\|_{L_2(\mathcal{B};\mathcal{B})} \left| D_{T-}^{1-\alpha}\omega_{T-}[t] \right| dt, \tag{2.3}$$

and the additivity of such integral yields

$$\int_0^T Z(t)d\omega(t) = \int_{-\tau}^{T-\tau} Z(t+\tau)d\theta_\tau\omega(t), \qquad \forall\, \tau \in \mathbb{R}. \tag{2.4}$$

Remark 2.4. Note that the trace-class assumption on the noise is less restrictive than the condition $\mathrm{tr}Q^{1/2} < \infty$ imposed in [14]. Alternatively, one can use Young's integral or rough paths theory as in [20] to define (2.2). Here we use for simplicity fractional calculus relying on the definition of the stochastic integral introduced in [5].

2.3. Equation of study

We consider the following stochastic evolution equation on the separable Hilbert space $\mathcal{B}$

$$du(t) = [Au(t) + F(u(t))]dt + G(u(t))d\omega(t), \tag{2.5}$$

where ω denotes the infinite dimensional fBm with Hurst index $H > \frac{1}{2}$ and the stochastic integral with respect to ω is interpreted in the sense of (2.2). We impose the following assumptions:

($\mathbf{A_1}$) A is the generator of an analytic semigroup $S(\cdot)$ on a monotone family of interpolation spaces $\{\mathcal{B}_\theta\}_{\theta\in\mathbb{R}}$. We recall that a family of separable Banach spaces $(\mathcal{B}_\theta, |\cdot|_\theta)_{\theta\in\mathbb{R}}$ is called a monotone family of interpolation spaces if for $\beta_1 \leq \beta_2$, the space $\mathcal{B}_{\beta_2} \subset \mathcal{B}_{\beta_1}$ with dense and continuous embedding and the following interpolation inequality holds for $\theta \leq \beta \leq \gamma$ and $x \in \mathcal{B}_\gamma$:

$$|x|_\beta^{\gamma-\theta} \lesssim |x|_\theta^{\gamma-\beta}|x|_\gamma^{\beta-\theta}. \tag{2.6}$$

We set $\mathcal{B}_0 := \mathcal{B}$. Furthermore the spectrum of A is discrete and, in particular, does not contain 0. The eigenvalues of A are real and can be ordered as $\lambda_1 > \lambda_2 > \cdots > \lambda_n \cdots$, and $\lim_{n\to\infty} \lambda_n = -\infty$.

(A₂) The mapping $F \in C_b^1(\mathcal{B};\mathcal{B})$ with $F(0) = DF(0) = 0$ and there exists a constant $L_F > 0$ such that

$$|F(u) - F(v)| \leq L_F|u - v| \quad \text{for all } u, v \in \mathcal{B}. \tag{2.7}$$

(A₃) The mapping $G \in C_b^2(\mathcal{B}; L_2(\mathcal{B};\mathcal{B}))$ with $G(0) = DG(0) = 0$, i.e. is a twice continuously Fréchet-di erentiable operator with bounded first derivative DG and second derivative D^2G. There also exists a constant $L_G > 0$ such that

$$\|G(u) - G(v)\|_{L_2(\mathcal{B};\mathcal{B})} \leq L_G|u - v|_{-\beta} \quad \text{for all } u, v \in \mathcal{B} \tag{2.8}$$

$$\|DG(u)h - DG(v)h\|_{L_2(\mathcal{B};\mathcal{B}_{-\beta})} \leq L_G|u - v|_{-\beta}|h|_{-\beta}, \quad \text{for all } u, v, h \in \mathcal{B}. \tag{2.9}$$

(A₄) $\frac{1}{2} < \beta < \beta\ < H < 1$ and $1 - \beta\ < \alpha < \beta$.

Note that due to (2.9) we have for $u_1, v_1, u_2, v_2 \in \mathcal{B}$ that

$$\begin{aligned}&\|G(u_1) - G(v_1) - G(u_2) + G(v_2)\|_{L_2(\mathcal{B};\mathcal{B}_{-\beta})}\\ &\quad \leq L_G(|u_1 - u_2|_{-\beta} + |v_1 - v_2|_{-\beta})|u_1 - v_1|_{-\beta} + (|u_2|_{-\beta} + |v_2|_{-\beta})|u_1 - v_1 - u_2 + v_2|_{-\beta}.\end{aligned}$$

Remark 2.5. Note that $\mathcal{B}$ is continuously embedded into $\mathcal{B}_{-\beta}$, therefore we can always find a constant $C > 0$ such that $|u|_{-\beta} \leq C|u|$. Hence, there exist constants, still denoted by L_F and L_G for the simplicity of notation, such that $|F(u) - F(v)|_{-\beta} \leq L_F|u - v|$. Similarly for G we have

$$\|G(u) - G(v)\|_{L_2(\mathcal{B};\mathcal{B})} \leq L_G|u - v|, \tag{2.10}$$

$$\|G(u) - G(v)\|_{L_2(\mathcal{B};\mathcal{B}_{-\beta})} \leq L_G|u - v|, \tag{2.11}$$

$$\|G(u) - G(v)\|_{L_2(\mathcal{B};\mathcal{B}_{-\beta})} \leq L_G|u - v|_{-\beta}. \tag{2.12}$$

Remark 2.6. Under weaker assumptions on G, the global existence (and not uniqueness) of solutions for (2.5) was proved in [38, Theorem 8, Lemma 10].

For a better comprehension, we discuss some important consequences of the imposed assumptions. First, based on **(A₁)** we can decompose the space $\mathcal{B}$ into a direct sum of closed invariant subspaces:

$$\mathcal{B} = \mathcal{B}^+ \oplus \mathcal{B}^-,$$

where the finite dimensional linear subspace $\mathcal{B}^+$ is spanned by the eigenvectors with eigenvalues larger than 0, and $\mathcal{B}^-$ is the linear subspace spanned by the eigenvectors with eigenvalues less than 0. Let us denote by $\pi^\pm$ the orthogonal projections associated with this splitting, and $S^\pm$ the restriction of the semigroup S to $\mathcal{B}^\pm$. In other words, S^+ is generated by the bounded operator π^+A on $\mathcal{B}^+$ and S^- is generated by the strictly negative operator $\pi^- A$ on $\mathcal{B}^-$. The projections commute with the semigroup, namely it holds that $\pi^\pm S = S\pi^\pm$ and there exist positive constants $\hat{\mu}$ and c_S and a negative constant $\check{\mu}$ such that

$$\|S^+(t)x\| \le c_S e^{\hat{\mu}t}\|x\|, \quad t \le 0,\ x \in \mathcal{B}, \tag{2.13a}$$

$$\|S^-(t)x\| \le c_S e^{\check{\mu}t}\|x\|, \quad t \ge 0,\ x \in \mathcal{B}. \tag{2.13b}$$

Here $\hat{\mu}$ can be any positive number less than the smallest positive eigenvalue of A and $\check{\mu}$ can be any negative number larger than the largest negative eigenvalue of A. Moreover it follows for e.g. from [37, p. 118] that for any $\sigma \in \mathbb{R}$, the interpolation space $\mathcal{B}_\sigma$ admits a decomposition given by $\mathcal{B}_\sigma = \mathcal{B}^+ \oplus \mathcal{B}_\sigma^-$, where $\mathcal{B}_\sigma^- = \mathcal{B}_\sigma \cap \mathcal{B}^-$. We refrain from giving an additional index to the finite dimensional space $\mathcal{B}^+$. We recall that $\mathcal{B}_0 = \mathcal{B}$ and $\mathcal{B}_{\sigma_2} \subset \mathcal{B}_{\sigma_1}$ for $\sigma_1 \le \sigma_2$ with dense and continuous embeddings. Let $L(\mathcal{B}_\eta; \mathcal{B}_\gamma)$ be the space of continuous linear operators from $\mathcal{B}_\eta$ to $\mathcal{B}_\gamma$. Thanks to the analyticity of the semigroup, it follows from [34] that for $0 \le \sigma \le 1$, $x \in \mathcal{B}_{\gamma+\sigma}$

$$|S(t)x|_{\gamma+\sigma} \le c_S t^{-\sigma}|x|_\gamma, \tag{2.14}$$

and

$$|(S(t) - \mathrm{id})x|_\gamma \le c_S t^\sigma |x|_{\gamma+\sigma}. \tag{2.15}$$

It is straightforward from the above inequalities that

$$\|S(t-r) - S(t-\tau)\|_{L(\mathcal{B}_\eta,\mathcal{B}_\gamma)} \le c_S(r-\tau)^\nu (t-r)^{-\nu-\gamma+\eta}, \quad \text{for } 0 \le \eta - \nu \le \gamma, \tag{2.16}$$

and

$$\|S(t-r) - S(s-r) - S(t-\tau) + S(s-\tau)\|_{L(\mathcal{B};\mathcal{B})} \le c_S(t-s)^\kappa (r-\tau)^\iota (s-r)^{-(\kappa+\iota)}, \tag{2.17}$$

for $\kappa, \iota \ge 0$, $\tau \le r \le s \le t$.

We interpret (2.5) in the mild form

$$u(t) = S(t)u_0 + \int_0^t S(t-\tau)F(u(\tau))d\tau + \int_0^t S(t-\tau)G(u(\tau))d\omega(\tau), \tag{2.18}$$

where the initial condition $u(0) = u_0 \in \mathcal{B}$. Existence of solutions to (2.5) was first studied in [5] (see also [12, Theorem 3.2]) in $C^{\beta,\sim}([0,T];\mathcal{B})$, the space of continuous functions $u : [0,T] \to \mathcal{B}$ such that

$$\|u\|_{\beta,\sim,[0,T]} = \sup_{t\in[0,T]} |u(t)| + \sup_{0\le s<t\le T} s^\beta \frac{|u(t) - u(s)|}{(t-s)^\beta}.$$

If the initial value $u_0 \in \mathcal{B}_\beta$, the solution $u \in C^\beta([0,T];\mathcal{B})$.

In [14], working with the scale $\mathcal{B}_\beta := D(\pi^+ id_{\mathcal{B}} + \pi^-(-A\pi^-)^\beta)$ (equipped with the graph norm $|x|_{\mathcal{B}_\beta} := |\pi^+ x| + |\pi^-(-A\pi^-)^\beta x|$), the unstable invariant manifold is obtained in $\mathcal{B}_\delta$ for $\delta \in [0, 1-\xi),\ \xi \in [\alpha, 1-\alpha)$. As already stated, the operator $A\pi^-$ is strictly negative definite on $\mathcal{B}^-$ (A itself is not, due to the assumed spectral decomposition). Therefore it makes sense to construct its fractional powers $(-A\pi^-)^\beta$ for $\beta > 0$. Further, we remark that it is reasonable to have this prediction on the manifold (i.e. that it should belong to $\mathcal{B}_\delta$) before even obtaining it, caus946 Tm .0007 Tc [(ha)27.3(v)28.1(e84Tj /F4 1 T

2.4. Existence and uniqueness of a global solution

We prove that the SPDE (2.5) has a unique global-in-time solution in the function space $C^{\beta}_{-\beta}([0,T];\mathcal{B})$, which generates a random dynamical system. For our aims, we begin with some preliminary considerations for the fractional integral and present several estimates of the stochastic convolution required in (2.18). Based on the definition of the fractional derivative, on the assumptions on the nonlinear term G and regarding (2.19) we obtain the following preliminary lemmas.

Lemma 2.8. *For $\beta > \alpha$ we have*

$$\|D^{\alpha}_{s+}S(t-\cdot)G(u(\cdot))[r]\|_{L_2(\mathcal{B};\mathcal{B}_{-\beta})} \le c_S C_{\alpha,\beta} L_G \|u\|_{\beta,\rho,-\beta}[(r-s)^{-\alpha} + (r-s)^{\beta-\alpha}]e^{\rho r}$$

as well as

$$\|D^{\alpha}_{s+}S(t-\cdot)G(u(\cdot))[r]\|_{L_2(\mathcal{B};\mathcal{B})}$$

$$
\begin{aligned}
&\leq \frac{1}{\Gamma(1-\alpha)}\Bigg(\frac{\|S(t-r)G(u(r))\|_{L_2(\mathcal{B};\mathcal{B})}}{(r-s)^{\alpha}}\\
&\qquad + \alpha\int_s^r \frac{\|S(t-r)G(u(r))-S(t-q)G(u(q))\|_{L_2(\mathcal{B};\mathcal{B})}}{(r-q)^{\alpha+1}}dq\Bigg)\\
&\leq c_S C_\alpha |u(r)|(r-s)^{-\alpha} + C_\alpha\int_s^r \frac{\|S(t-r)-S(t-q)\|_{L(\mathcal{B};\mathcal{B})}\|G(u(r))\|_{L_2(\mathcal{B};\mathcal{B})}}{(r-q)^{1+\alpha}}dq\\
&\quad + C_\alpha\int_s^r \frac{\|S(t-q)\|_{L(\mathcal{B}_{-\beta};\mathcal{B})}\|G(u(r))-G(u(q))\|_{L_2(\mathcal{B};\mathcal{B}_{-\beta})}}{(r-q)^{1+\alpha}}dq\\
&\leq c_S C_\alpha L_G (r-s)^{-\alpha}e^{\rho r}\|u\|_{\beta,\rho,-\beta} + c_S C_\alpha L_G\|u\|_{\beta,\rho,-\beta}(t-r)^{-\beta}\int_s^r (r-q)^{\beta-\alpha-1}e^{\rho r}dq\\
&\quad + c_S C_\alpha L_G\|u\|_{\beta,\rho,-\beta}\int_s^r e^{\rho r}(t-q)^{-\beta}(r-q)^{\beta-\alpha-1}dq\\
&\leq c_S C_{\alpha,\beta} L_G\|u\|_{\beta,\rho,-\beta}[(r-s)^{-\alpha} + (t-r)^{-\beta}(r-s)^{\beta-\alpha}]e^{\rho r}. \qquad \square
\end{aligned}
$$

Taking additionally (2.9) into account, we analogously obtain for the fractional derivative of the convolution of the di erence of two terms the following estimates.

Lemma 2.9. *Let $\beta > \alpha$. The following estimates hold true*

$$
\begin{aligned}
&\|D^\alpha_{s+}S(t-\cdot)[G(u(\cdot))-G(v(\cdot))][r]\|_{L_2(\mathcal{B};\mathcal{B}_{-\beta})}\\
&\leq c_S C_{\alpha,\beta} L_G(1+\|u\|_{\beta,-\beta}+\|v\|_{\beta,-\beta})\|u-v\|_{\beta,\rho,-\beta}[(r-s)^{-\alpha}+(r-s)^{\beta-\alpha}]e^{\rho r}
\end{aligned}
$$

and

$$
\begin{aligned}
&\|D^\alpha_{s+}S(t-\cdot)[G(u(\cdot))-G(v(\cdot))][r]\|_{L_2(\mathcal{B};\mathcal{B})}\\
&\leq c_S C_{\alpha,\beta} L_G(1+\|u\|_{\beta,-\beta}+\|v\|_{\beta,-\beta})\|u\|_{\beta,\rho,-\beta}[1+(t-r)^{-\beta}(r-s)^{\beta}](r-s)^{-\alpha}e^{\rho r}.
\end{aligned}
$$

Proof. Using (2.9) we compute

$$
\begin{aligned}
&\|D^\alpha_{s+}S(t-\cdot)[G(u(\cdot))-G(v(\cdot))][r]\|_{L_2(\mathcal{B};\mathcal{B}_{-\beta})}\\
&\quad\leq \frac{1}{\Gamma(1-\alpha)}\Bigg(\frac{\|S(t-r)[G(u(r))-G(v(r))]\|_{L_2(\mathcal{B};\mathcal{B}_{-\beta})}}{(r-s)^{\alpha}}\\
&\qquad + \alpha\int_s^r \frac{\|S(t-r)[G(u(r))-G(v(r))]-S(t-q)[G(u(q))-G(v(q))]\|_{L_2(\mathcal{B};\mathcal{B}_{-\beta})}}{(r-q)^{\alpha+1}}dq\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&\leq c_S C_\alpha |u(r)-v(r)|(r-s)^{-\alpha} \\
&+ C_\alpha \int_s^r \frac{\|S(t-r)-S(t-q)\|_{L(\mathcal{B};\mathcal{B}_{-\beta})}\|G(u(r))-G(v(r))\|_{L_2(\mathcal{B};\mathcal{B})}}{(r-q)^{1+\alpha}}dq \\
&+ C_\alpha \int_s^r \frac{\|S(t-q)\|_{L(\mathcal{B}_{-\beta};\mathcal{B}_{-\beta})}\|G(u(r))-G(v(r))-G(u(q))+G(v(q))\|_{L_2(\mathcal{B};\mathcal{B}_{-\beta}))}}{(r-q)^{1+\alpha}}dq \\
&\leq c_S C_\alpha L_G (r-s)^{-\alpha} e^{\rho r}\|u-v\|_{\beta,\rho,-\beta} + c_S C_\alpha L_G \|u-v\|_{\beta,\rho,-\beta}\int_s^r (r-q)^{\beta-\alpha-1}e^{\rho r}dq \\
&\quad + c_S C_\alpha L_G(\|u\|_{\beta,-\beta}+\|v\|_{\beta,-\beta})\|u-v\|_{\beta,\rho,-\beta}\int_s^r e^{\rho r}(r-q)^{\beta-\alpha-1}dq \\
&\leq c_S C_{\alpha,\beta} L_G(1+\|u\|_{\beta,-\beta}+\|v\|_{\beta,-\beta})\|u-v\|_{\beta,\rho,-\beta}[(r-s)^{-\alpha}+(r-s)^{\beta-\alpha}]e^{\rho r},
\end{aligned}
$$

entailing the first statement. Furthermore, due to (2.9) and (2.10) we have

$$
\begin{aligned}
&\|D^\alpha_{s+}S(t-\cdot)[G(u(\cdot))-G(v(\cdot))][r]\|_{L_2(\mathcal{B};\mathcal{B})} \\
&\quad \leq \frac{1}{\Gamma(1-\alpha)}\Bigg(\frac{\|S(t-r)[G(u(r))-G(v(r))]\|_{L_2(\mathcal{B};\mathcal{B})}}{(r-s)^\alpha} \\
&\qquad + \alpha\int_s^r \frac{\|S(t-r)[G(u(r))-G(v(r))]-S(t-q)[G(u(q))-G(v(q))]\|_{L_2(\mathcal{B};\mathcal{B})}}{(r-q)^{\alpha+1}}dq\Bigg) \\
&\leq c_S C_\alpha |u(r)-v(r)|(r-s)^{-\alpha} \\
&\quad + C_\alpha \int_s^r \frac{\|S(t-r)-S(t-q)\|_{L(\mathcal{B};\mathcal{B})}\|G(u(r))-G(v(r))\|_{L_2(\mathcal{B};\mathcal{B})}}{(r-q)^{1+\alpha}}dq \\
&\quad + C_\alpha \int_s^r \frac{\|S(t-q)\|_{L(\mathcal{B}_{-\beta};\mathcal{B})}\|G(u(r))-G(v(r))-G(u(q))+G(q(r))\|_{L_2(\mathcal{B};\mathcal{B}_{-\beta})}}{(r-q)^{1+\alpha}}dq \\
&\leq c_S C_\alpha L_G (r-s)^{-\alpha}e^{\rho r}\|u-v\|_{\beta,\rho,-\beta} \\
&\quad + c_S C_\alpha L_G\|u-v\|_{\beta,\rho,-\beta}(t-r)^{-\beta}\int_s^r (r-q)^{\beta-\alpha-1}e^{\rho r}dq \\
&\quad + c_S C_\alpha L_G(\|u\|_{\beta,-\beta}+\|v\|_{\beta,-\beta})\|u-v\|_{\beta,\rho,-\beta}\int_s^r e^{\rho r}(t-q)^{-\beta}(r-q)^{\beta-\alpha-1}dq \\
&\leq c_S C_{\alpha,\beta}L_G(1+\|u\|_{\beta,-\beta}+\|v\|_{\beta,-\beta})\|u\|_{\beta,\rho,-\beta}[1+(t-r)^{-\beta}(r-s)^\beta](r-s)^{-\alpha}e^{\rho r}. \quad \square
\end{aligned}
$$

Lemma 2.10. *Let* $\alpha \in (0,1)$ *such that* $\alpha - \alpha - 1 > -1$. *Then*

$$\begin{aligned}&\|D^{\alpha}_{0+}[S(t-\cdot)-S(s-\cdot)]G(u(\cdot))[r]\|_{L_2(\mathcal{B};\mathcal{B}_{-\beta})}\\&\quad\le C_{\alpha,\beta}c_S L_G\|u\|_{\beta,\rho,-\beta}e^{\rho r}(t-s)^{\beta}[(s-r)^{-\alpha'}r^{\alpha'-\alpha}+(s-r)^{-\beta}r^{\beta-\alpha}]\end{aligned}$$

and

$$\begin{aligned}&\|D^{\alpha}_{0+}[S(t-\cdot)-S(s-\cdot)][G(u(\cdot))-G(v(\cdot))][r]\|_{L_2(\mathcal{B};\mathcal{B}_{-\beta})}\\&\quad\le c_S C_{\alpha,\beta}L_G(1+\|u\|_{\beta,-\beta}+\|v\|_{\beta,-\beta})\|u-v\|_{\beta,\rho,-\beta}e^{\rho r}(t-s)^{\beta}\\&\qquad\times[(s-r)^{-\alpha'}r^{\alpha'-\alpha}+(s-r)^{-\beta}r^{\beta-\alpha}].\end{aligned}$$

Proof. By similar computations we derive

$$\begin{aligned}&\|D^{\alpha}_{0+}[S(t-\cdot)-S(s-\cdot)]G(u(\cdot))[r]\|_{L_2(\mathcal{B};\mathcal{B}_{-\beta})}\\
&\quad\le\frac{C_\alpha\|[S(t-r)-S(s-r)]G(u(r))\|_{L_2(\mathcal{B};\mathcal{B}_{-\beta})}}{r^\alpha}\\
&\qquad+C_\alpha\int_0^r\frac{\|[S(t-r)-S(s-r)]G(u(r))-[S(t-q)-S(s-q)]G(u(q))\|_{L_2(\mathcal{B};\mathcal{B}_{-\beta})}}{(r-q)^{\alpha+1}}dq\\
&\quad\le\frac{C_\alpha\|S(t-s)-\mathrm{id}\|_{L(\mathcal{B};\mathcal{B}_{-\beta})}\|S(s-r)\|_{L(\mathcal{B};\mathcal{B})}\|G(u(r))\|_{L_2(\mathcal{B};\mathcal{B})}}{r^\alpha}\\
&\qquad+C_\alpha\int_0^r\frac{\|[S(t-r)-S(s-r)-S(t-q)+S(s-q)]G(u(r))\|_{L_2(\mathcal{B};\mathcal{B}_{-\beta})}}{(r-q)^{\alpha+1}}dq\\
&\qquad+C_\alpha\int_0^r\frac{\|[S(t-q)-S(s-q)][G(u(r))-G(u(q))]\|_{L_2(\mathcal{B};\mathcal{B}_{-\beta})}}{(r-q)^{\alpha+1}}dq\\
&\quad\le\frac{C_\alpha\|S(t-s)-\mathrm{id}\|_{L(\mathcal{B};\mathcal{B}_{-\beta})}\|S(s-r)\|_{L(\mathcal{B};\mathcal{B})}\|G(u(r))\|_{L_2(\mathcal{B};\mathcal{B})}}{r^\alpha}\\
&\qquad+C_\alpha\int_0^r\frac{1}{(r-q)^{\alpha+1}}\|S(t-s)-\mathrm{id}\|_{L(\mathcal{B};\mathcal{B}_{-\beta})}\|S(s-r)\|_{L(\mathcal{B}_{-\alpha'};\mathcal{B})}\\
&\qquad\times\|\mathrm{id}-S(r-q)\|_{L(\mathcal{B};\mathcal{B}_{-\alpha'})}\|G(u(r))\|_{L_2(\mathcal{B};\mathcal{B})}dq\\
&\qquad+C_\alpha\int_0^r\frac{\|S(t-s)-\mathrm{id}\|_{L(\mathcal{B};\mathcal{B}_{-\beta})}\|S(s-q)\|_{L(\mathcal{B}_{-\beta};\mathcal{B})}\|G(u(r))-G(u(q))\|_{L_2(\mathcal{B};\mathcal{B}_{-\beta}))}}{(r-q)^{\alpha+1}}dq\\
&\quad\le C_\alpha c_S L_G\|u\|_{\beta,\rho,-\beta}e^{\rho r}(t-s)^{\beta}\Big[r^{-\alpha}+\int_0^r(s-r)^{-\alpha'}(r-q)^{\alpha'-\alpha-1}dq\end{aligned}$$

$$+ \int_0^r (s-q)^{-\beta}(r-q)^{\beta-\alpha-1}dq\Big]$$

$$\leq C_{\alpha,\beta}c_S L_G\|u\|_{\beta,\rho,-\beta}e^{\rho r}(t-s)^{\beta}[(s-r)^{-\alpha'}r^{\alpha'-\alpha} + (s-r)^{-\beta}r^{\beta-\alpha}].$$

Finally according to (2.17) and assumption (A_4), it holds

$$\begin{aligned}
&\|D_{0+}^{\alpha}[S(t-\cdot)-S(s-\cdot)][G(u(\cdot))-G(v(\cdot))][r]\|_{L_2(\mathcal{B};\mathcal{B}_{-\beta})}\\
&\leq \frac{C_\alpha\|[S(t-r)-S(s-r)][G(u(r))-G(v(r))]\|_{L_2(\mathcal{B};\mathcal{B}_{-\beta})}}{r^\alpha}\\
&\quad + C_\alpha\int_0^r \frac{1}{(r-q)^{\alpha+1}}\Big\|[S(t-r)-S(s-r)][G(u(r))-G(v(r))]\\
&\quad - [S(t-q)-S(s-q)][G(u(q))-G(v(q))]\Big\|_{L_2(\mathcal{B};\mathcal{B}_{-\beta})}dq\\
&\leq \frac{C_\alpha\|S(t-s)-\mathrm{id}\|_{L(\mathcal{B};\mathcal{B}_{-\beta})}\|S(s-r)\|_{L(\mathcal{B};\mathcal{B})}\|G(u(r))-G(v(r))\|_{L_2(\mathcal{B};\mathcal{B})}}{r^\alpha}\\
&+ C_\alpha\int_0^r \frac{\|[S(t-r)-S(s-r)-S(t-q)+S(s-q)][G(u(r))-G(v(r))]\|_{L_2(\mathcal{B};\mathcal{B}_{-\beta}))}}{(r-q)^{\alpha+1}}dq\\
&+ C_\alpha\int_0^r \frac{\|[S(t-q)-S(s-q)][G(u(r))-G(v(r))-G(u(q))+G(v(q))]\|_{L_2(\mathcal{B};\mathcal{B}_{-\beta})}}{(r-q)^{\alpha+1}}dq\\
&\leq \frac{C_\alpha\|S(t-s)-\mathrm{id}\|_{L(\mathcal{B};\mathcal{B}_{-\beta})}\|S(s-r)\|_{L(\mathcal{B};\mathcal{B})}\|G(u(r))-G(v(r))\|_{L_2(\mathcal{B};\mathcal{B})}}{r^\alpha}\\
&\quad + C_\alpha\int_0^r \frac{1}{(r-q)^{\alpha+1}}\|S(t-s)-\mathrm{id}\|_{L(\mathcal{B};\mathcal{B}_{-\beta})}\|S(s-r)\|_{L(\mathcal{B}_{-\alpha'};\mathcal{B})}\\
&\quad \times\|\mathrm{id}-S(r-q)\|_{L(\mathcal{B};\mathcal{B}_{-\alpha'})}\|G(u(r))-G(v(r))\|_{L_2(\mathcal{B};\mathcal{B})}dq\\
&\quad + C_\alpha\int_0^r \frac{1}{(r-q)^{\alpha+1}}\|S(t-s)-\mathrm{id}\|_{L(\mathcal{B};\mathcal{B}_{-\beta})}\|S(s-q)\|_{L(\mathcal{B}_{-\beta};\mathcal{B})}\\
&\quad \times\|G(u(r))-G(v(r))-G(u(q))+G(v(q))\|_{L_2(\mathcal{B};\mathcal{B}_{-\beta})}dq\\
&\leq c_S C_{\alpha,\beta}L_G\|u-v\|_{\beta,\rho,-\beta}e^{\rho r}(t-s)^\beta\Big[r^{-\alpha} + \int_0^r (s-r)^{-\alpha'}(r-q)^{\alpha'-\alpha-1}dq\\
&\quad + (\|u\|_{\beta,-\beta} + \|v\|_{\beta,-\beta})\int_0^r (s-q)^{-\beta}(r-q)^{\beta-\alpha-1}dq\Big]\\
&\leq c_S C_{\alpha,\beta}L_G(1+\|u\|_{\beta,-\beta}+\|v\|_{\beta,-\beta})\|u-v\|_{\beta,\rho,-\beta}e^{\rho r}\\
&\quad \times[(s-r)^{-\alpha'}r^{\alpha'-\alpha} + (s-r)^{-\beta}r^{\beta-\alpha}]. \quad \square
\end{aligned}$$

In the following computations we will apply the following useful result. For any $s, t \in \mathbb{R}$, $a, b > -1$, due to the definition of Euler's Gamma and Beta-functions, it is straightforward that

$$\int_s^t (t-r)^a (r-s)^b dr = \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+2)} (t-s)^{a+b+1}. \tag{2.20}$$

We further set

$$K_1(\rho) := \sup_{0 \le s < t \le T} \int_s^t e^{-\rho(t-r)} (r-s)^a (t-r)^b dr. \tag{2.21}$$

Combining the fact that

$$\int_s^t e^{-\rho(t-r)} (r-s)^a (t-r)^b dr = (t-s)^{a+b+1} \int_0^1 e^{-\rho(t-s)(1-r)} r^a (1-r)^b dr$$

and [2, Lemma 8], we know that $K_1(\rho) \to 0$ as $\rho \to \infty$ if $a, b > -1$ and $a + b + 1 > 0$.

Lemma 2.11. *Under the assumptions* $(\mathbf{A_1})$*,* $(\mathbf{A_3})$ *and* $(\mathbf{A_4})$*, there exist constants* $c_S, C_{\alpha,\beta}$ *such that for any* $u, v \in C^{\beta}_{-\beta}([0,T]; \mathcal{B})$*,*

$$\left\| \int_0^{\cdot} S(\cdot - r) G(u(r)) d\omega(r) \right\|_{\beta,\rho,-\beta} \le c_S C_{\alpha,\beta} K_1(\rho) L_G \|u\|_{\beta,\rho,-\beta} |||\omega|||_{\beta',0,T},$$

and

$$\left\| \int_0^{\cdot} S(\cdot - r)[G(u(r)) - G(v(r))] d\omega(r) \right\|_{\beta,\rho,-\beta}$$
$$\le C_{\alpha,\beta} c_S K_1(\rho) L_G (1 + \|u\|_{\beta,-\beta} + \|v\|_{\beta,-\beta}) \|u - v\|_{\beta,\rho,-\beta} |||\omega|||_{\beta',0,T}.$$

Proof. First, it is easy to check that

$$|D^{1-\alpha}_{t-} \omega[r]| \le |||\omega|||_{\beta',0,T} (t-r)^{\alpha+\beta'-1}.$$

Applying the estimates above for each D^{α}_{s+}, we have

$$\left| \int_s^t S(t-r)[G(u(r)) - G(v(r))] d\omega(r) \right|_{-\beta}$$

$$\leq \int_s^t \|D_{s+}^\alpha S(t-\cdot)[G(u(\cdot)) - G(v(\cdot))][r]\|_{L_2(\mathcal{B};\mathcal{B}_{-\beta}))} |D_{t-}^{1-\alpha}\omega[r]| dr$$

$$\leq c_S C_{\alpha,\beta} L_G (1 + \|u\|_{\beta,-\beta} + \|v\|_{\beta,-\beta}) \|u\|_{\beta,\rho,-\beta} |||\omega|||_{\beta',0,T}$$

$$\times \int_s^t (t-r)^{\alpha+\beta'-1}(r-s)^{-\alpha}[1 + (r-s)^\beta] e^{\rho r} dr,$$

$$\left| \int_s^t S(t-r)G(u(r))d\omega(r) \right|$$

$$\leq \int_s^t \|D_{s+}^\alpha S(t-\cdot)G(u(\cdot))[r]\|_{L_2(\mathcal{B};\mathcal{B})} |D_{t-}^{1-\alpha}\omega[r]| dr$$

$$\leq c_S C_{\alpha,\beta} L_G \|u\|_{\beta,\rho,-\beta} |||\omega|||_{\beta',0,T} \int_s^t (t-r)^{\alpha+\beta'-1}(r-s)^{-\alpha}[1 + (t-r)^{-\beta}(r-s)^\beta] e^{\rho r} dr,$$

and

$$\begin{aligned} &\left| \int_s^t S(t-r)[G(u(r)) - G(v(r))]d\omega(r) \right| \\ &\leq \int_s^t \|D_{s+}^\alpha S(t-\cdot)[G(u(\cdot)) - G(v(\cdot))][r]\|_{L_2(\mathcal{B};\mathcal{B})} |D_{t-}^{1-\alpha}\omega[r]| dr \\ &\leq c_S C_{\alpha,\beta} L_G (1 + \|u\|_{\beta,-\beta} + \|v\|_{\beta,-\beta}) \|u\|_{\beta,\rho,-\beta} |||\omega|||_{\beta',0,T} \\ &\quad \times \int_s^t (t-r)^{\alpha+\beta'-1}[(r-s)^{-\alpha} + (t-r)^{-\beta}(r-s)^{\beta-\alpha}] e^{\rho r} dr. \end{aligned} \tag{2.22}$$

Taking the $\mathcal{B}_{-\beta}$-norm, we have

$$\left| \int_s^t S(t-r)G(u(r))d\omega(r) \right|_{-\beta}$$

$$\leq \int_s^t \|D_{s+}^\alpha S(t-\cdot)G(u(\cdot))[r]\|_{L_2(\mathcal{B};\mathcal{B}_{-\beta}))} |D_{t-}^{1-\alpha}\omega[r]| dr$$

$$\leq c_S C_{\alpha,\beta} L_G \|u\|_{\beta,\rho,-\beta} |||\omega|||_{\beta',0,T} \int_s^t (t-r)^{\alpha+\beta'-1}[(r-s)^{-\alpha} + (r-s)^{\beta-\alpha}] e^{\rho r} dr.$$

$$\begin{aligned}
&\left|\int_0^s [S(t-r)-S(s-r)]G(u(r))d\omega(r)\right|_{-\beta} \\
&\quad \le c_S C_{\alpha,\beta} L_G \|u\|_{\beta,\rho,-\beta} |||\omega|||_{\beta',0,T}(t-s)^{\beta} \\
&\qquad \times \int_0^s e^{\rho r}(s-r)^{\alpha+\beta'-1}[(s-r)^{-\alpha'}r^{\alpha'-\alpha} + (s-r)^{-\beta}r^{\beta-\alpha}]dr \\
&\quad \le c_S C_{\alpha,\beta} L_G \|u\|_{\beta,\rho,-\beta} e^{\rho s} |||\omega|||_{\beta',0,T}(t-s)^{\beta} s^{\beta'},
\end{aligned}$$

and

$$\begin{aligned}
&\left|\int_0^s [S(t-r)-S(s-r)][G(u(r))-G(v(r))]d\omega(r)\right|_{-\beta} \\
&\quad \le c_S C_{\alpha,\beta} L_G (1+\|u\|_{\beta,-\beta}+\|v\|_{\beta,-\beta})\|u-v\|_{\beta,\rho,-\beta} |||\omega|||_{\beta',0,T}(t-s)^{\beta} \\
&\qquad \times \int_0^s e^{\rho r}(s-r)^{\alpha+\beta'-1}[(s-r)^{-\alpha'}r^{\alpha'-\alpha} + (s-r)^{-\beta}r^{\beta-\alpha}]dr \\
&\quad \le c_S C_{\alpha,\beta} L_G (1+\|u\|_{\beta,-\beta}+\|v\|_{\beta,-\beta})\|u-v\|_{\beta,\rho,-\beta} e^{\rho s} |||\omega|||_{\beta',0,T}(t-s)^{\beta} s^{\beta'}.
\end{aligned}$$

Hence for $u \in C^{\beta}_{-\beta}([0,T];\mathcal{B})$ we compute regarding (2.19)

$$\begin{aligned}
&\left\|\int_0^{\cdot} S(\cdot-r)G(u(r))d\omega(r)\right\|_{\beta,\rho,-\beta} \\
&\quad \le \sup_{t\in[0,T]} e^{-\rho t}\left|\int_0^t S(t-r)G(u(r))d\omega(r)\right| + \sup_{s<t\in[0,T]} \frac{e^{-\rho t}\left|\int_s^t S(t-r)G(u(r))d\omega(r)\right|_{-\beta}}{(t-s)^{\beta}} \\
&\qquad + \sup_{s<t\in[0,T]} \frac{e^{-\rho t}\left|\int_0^s [S(t-r)-S(s-r)]G(u(r))d\omega(r)\right|_{-\beta}}{(t-s)^{\beta}} \\
&\quad \le \sup_{t\in[0,T]} c_S C_{\alpha,\beta} L_G \|u\|_{\beta,\rho,-\beta} |||\omega|||_{\beta',0,T} \\
&\qquad \times \int_s^t (t-r)^{\alpha+\beta'-1}[(r-s)^{-\alpha} + (t-r)^{-\beta}(r-s)^{\beta-\alpha}]e^{\rho-(t-r)}dr \\
&\qquad + \sup_{s<t\in[0,T]} c_S C_{\alpha,\beta} L_G \|u\|_{\beta,\rho,-\beta} |||\omega|||_{\beta',0,T} \\
&\qquad \times \frac{\int_s^t (t-r)^{\alpha+\beta'-1}[(r-s)^{-\alpha} + (r-s)^{\beta-\alpha}]e^{-\rho(t-r)}dr}{(t-s)^{\beta}}
\end{aligned}$$

$$
\begin{aligned}
&+ \sup_{s<t\in[0,T]} c_S C_{\alpha,\beta} L_G \|u\|_{\beta,\rho,-\beta} |||\omega|||_{\beta',0,T} (t-s)^\beta \\
&\times \frac{\int_0^s e^{-\rho(t-r))}(s-r)^{\alpha+\beta'-1}[(s-r)^{-\alpha'} r^{\alpha'-\alpha} + (s-r)^{-\beta} r^{\beta-\alpha}]dr}{(t-s)^\beta} \\
&\leq c_S C_{\alpha,\beta} K_1(\rho) L_G \|u\|_{\beta,\rho,-\beta} |||\omega|||_{\beta',0,T}.
\end{aligned}
$$

For $u, v \in C^\beta_{-\beta}([0,T];\mathcal{B})$ we further have that

$$
\begin{aligned}
&\left\| \int_0^{\cdot} S(\cdot - r)[G(u(r)) - G(v(r))]d\omega(r) \right\|_{\beta,\rho,-\beta} \\
&= \sup_{t\in[0,T]} e^{-\rho t} \left| \int_0^t S(t-r)[G(u(r)) - G(v(r))]d\omega(r) \right| \\
&+ \sup_{s<t\in[0,T]} \frac{e^{-\rho t} \left| \int_0^t S(t-r)[G(u(r)) - G(v(r))]d\omega(r) - \int_0^s S(s-r)[G(u(r)) - G(v(r))]d\omega(r) \right|_{-\beta}}{(t-s)^\beta} \\
&\leq \sup_{t\in[0,T]} e^{-\rho t} \left| \int_0^s S(t-r)[G(u(r)) - G(v(r))]d\omega(r) \right| \\
&+ \sup_{s<t\in[0,T]} \frac{e^{-\rho t} \left| \int_0^s [S(t-r) - S(s-r)][G(u(r)) - G(v(r))]d\omega(r)d\omega(r) \right|_{-\beta}}{(t-s)^\beta} \\
&+ \sup_{s<t\in[0,T]} \frac{e^{-\rho t} \left| \int_s^t S(t-r)[G(u(r)) - G(v(r))]d\omega(r) \right|_{-\beta}}{(t-s)^\beta} \\
&\leq C_{\alpha,\beta} c_S K_1(\rho) L_G (1 + \|u\|_{\beta,-\beta} + \|v\|_{\beta,-\beta}) \|u-v\|_{\beta,\rho,-\beta} |||\omega|||_{\beta',0,T}. \quad \square
\end{aligned}
$$

Putting all these deliberations together we infer.

Theorem 2.12. *Assume that* $(\mathbf{A_1})$*-*$(\mathbf{A_4})$ *are satisfied. Then the SPDE* (2.5) *has a unique solution* $u \in C^\beta_{-\beta}([0,T];\mathcal{B})$ *with* $u(0) = u_0 \in \mathcal{B}$*. Moreover, its solution operator generates a random dynamical system* $\varphi : \mathbb{R}^+ \times \Omega \times \mathcal{B} \to \mathcal{B}$*.*

Proof. We apply the Banach fixed point theorem in order to obtain the existence and uniqueness of a solution to (2.5). By Lemma 2.11, we need to focus on the estimates of $S(\cdot)u_0$ and the term containing the drift F. In fact, given $u_0 \in \mathcal{B}$ and $\beta \geq 0$, it is straightforward to check that

$$
\begin{aligned}
\|S(\cdot)u_0\|_{\beta,\rho,-\beta} &\leq \sup_{t\in 0,T} e^{-\rho t}|S(t)u_0| + \sup_{0\leq s<t\leq T} \frac{e^{-\rho t}\|S(t) - S(s)\|_{L(\mathcal{B},\mathcal{B}_{-\beta})}|u_0|}{(t-s)^\beta} \\
&\leq c_S|u_0|. \qquad (2.23)
\end{aligned}
$$

As for the drift term, regarding again (2.19) and (2.7), we directly obtain that

$$
\begin{aligned}
&\left\| \int_0^{\cdot} S(\cdot - r)F(u(r))dr \right\|_{\beta,\rho,-\beta} \\
&= \sup_{t\in[0,T]} e^{-\rho t} \left| \int_0^t S(t-r)F(u(r))dr \right| \\
&\quad + \sup_{s<t\in[0,T]} \frac{e^{-\rho t} \left| \int_0^t S(t-r)F(u(r))dr - \int_0^s S(s-r)F(u(r))dr \right|_{-\beta}}{(t-s)^\beta} \\
&\leq \sup_{t\in[0,T]} e^{-\rho t} \left| \int_0^t S(t-r)F(u(r))dr \right| + \sup_{s<t\in[0,T]} \frac{e^{-\rho t} \left| \int_s^t S(t-r)F(u(r))dr \right|_{-\beta}}{(t-s)^\beta} \\
&\quad + \sup_{s<t\in[0,T]} \frac{e^{-\rho t} \left| \int_0^s [S(t-r) - S(s-r)]F(u(r))dr \right|_{-\beta}}{(t-s)^\beta} \\
&\leq c_S L_F \frac{1-e^{-\rho T}}{\rho} \|u\|_{\beta,\rho,-\beta} (1 + T^{1-\beta}) \\
&\quad + \sup_{s<t\in[0,T]} \frac{e^{-\rho t} \int_0^s \|S(t-r) - S(s-r)\|_{L(\mathcal{B};\mathcal{B}_{-\beta})} |F(u(r))| dr}{(t-s)^\beta} \\
&\leq c_S L_F \frac{1-e^{-\rho T}}{\rho} \|u\|_{\beta,\rho,-\beta} (1 + T^{1-\beta} + T) \\
&= c_S L_F K_2(\rho) \|u\|_{\beta,\rho,-\beta} (1 + T^{1-\beta} + T),
\end{aligned}
$$

and

$$
\begin{aligned}
&\left\| \int_0^{\cdot} S(\cdot - r)[F(u(r)) - F(v(r))]dr \right\|_{\beta,\rho,-\beta} \\
&= \sup_{t\in[0,T]} e^{-\rho t} \left| \int_0^t S(t-r)[F(u(r)) - F(v(r))]dr \right| \\
&\quad + \sup_{s<t\in[0,T]} \frac{e^{-\rho t} \left| \int_0^t S(t-r)[F(u(r)) - F(v(r))]dr - \int_0^s S(s-r)[F(u(r)) - F(v(r))]dr \right|_{-\beta}}{(t-s)^\beta} \\
&\leq \sup_{t\in[0,T]} e^{-\rho t} \left| \int_0^t S(t-r)[F(u(r)) - F(v(r))]dr \right|
\end{aligned}
$$

$$
\begin{aligned}
&+ \sup_{s<t\in[0,T]} \frac{e^{-\rho t}\left|\int_s^t S(t-r)[F(u(r))-F(v(r))]dr\right|_{-\beta}}{(t-s)^{\beta}} \\
&+ \sup_{s<t\in[0,T]} \frac{e^{-\rho t}\left|\int_0^s [S(t-r)-S(s-r)][F(u(r))-F(v(r))]dr\right|_{-\beta}}{(t-s)^{\beta}} \\
&\le c_S L_F \frac{1-e^{-\rho T}}{\rho}\|u-v\|_{\beta,\rho,-\beta}(1+T^{1-\beta}) \\
&+ \sup_{s<t\in[0,T]} \frac{e^{-\rho t}\int_0^s \|S(t-r)-S(s-r)\|_{L(\mathcal{B};\mathcal{B}_{-\beta})}|F(u(r))-F(v(r))|dr}{(t-s)^{\beta}} \\
\le\, & c_S L_F K_2(\rho)\|u-v\|_{\beta,\rho,-\beta}(1+T^{1-\beta}+T),
\end{aligned}
$$

where $K_2(\rho) = \frac{1-e^{-\rho T}}{\rho}$ which converges to 0 as $\rho \to \infty$.

For simplicity, we set $T := 1$. Let $\mathcal{T}$ be the mapping from $C^{\beta}_{-\beta}([0,1];\mathcal{B})$ into itself given by

$$
\mathcal{T}(u)(t) := S(t)u_0 + \int_0^t S(t-\tau)F(u(\tau))d\tau + \int_0^t S(t-\tau)G(u(\tau))d\omega(\tau).
$$

Taking Lemma 2.11

Therefore, considering $u_0 = v_0$, it is possible to find a ρ_1 such that for $u, v \in B$,

$$\begin{aligned}\|\mathcal{T}(u) - \mathcal{T}(v)\|_{\beta,\rho_1,-\beta} &\leq c_S L_F K_2(\rho_1)\|u - v\|_{\beta,\rho_1,-\beta} \\ &\quad + c_S C_{\alpha,\beta} L_G K_1(\rho_1)(1 + 2R_1)\|u - v\|_{\beta,\rho_1,-\beta}|||\omega|||_{\beta',0,1} \\ &\leq \frac{1}{2}\|u - v\|_{\beta,\rho_1,-\beta}.\end{aligned}$$

Hence by the Banach fixed point theorem, there is a unique fixed point in B such that $\mathcal{T}(u) = u$ for given $u_0 \in \mathcal{B}$. In conclusion, following a similar argument to [5, Theorem 16], we conclude that the solution operator of (2.5) generates a random dynamical system. $\square$

3. Main results

The aim of this section is to prove the existence of a C^k-smooth stable manifold for the dynamical system φ generated by the unique global solution of (2.18). Let $\mathcal{H}_\kappa, \check{\mu} < 0 < \kappa < \min\{-\check{\mu}, \hat{\mu}\}$ be a space with elements $U = \{u_i\}_{i\in\mathbb{Z}^+}$, where $u_i \in C^\beta_{-\beta}([0,1]; \mathcal{B})$, with the norm,

$$\|U\|_{\mathcal{H}_\kappa} := \sup_{i\in\mathbb{Z}^+} e^{\kappa i}\|u_i\|_{\beta,-\beta} < \infty, \quad u_i(1) = u_{i+1}(0).$$

The space $(\mathcal{H}_\kappa, \|\cdot\|_{\mathcal{H}_\kappa})$ is a Banach space. Let also $\mathcal{H}^\xi_\kappa$ be the subset of $\mathcal{H}_\kappa$ such that $u^-_0(0) = \xi \in \mathcal{B}^-$. Since $\mathcal{H}^\xi_\kappa$ is a closed subset of $\mathcal{H}_\kappa$, it is complete.

3.1. Derivation of the Lyapunov-Perron operator

Similar to [14,24] we introduce now a discrete Lyapunov-Perron operator associated with (2.18).

Lemma 3.1. *Given $\xi \in \mathcal{B}^-, U = \{u_i\}_{i\in\mathbb{Z}^+} \in \mathcal{H}^\xi_\kappa$, $t \in [0,1]$, let $u(t+m) = u_m(t), t \in [0,1]$. Then $u(\cdot)$ is the solution of (2.18) with $u^-(0) = \xi$ if and only if U satisfies the following equality:*

$$\begin{aligned}[J(U)]_m(t) &= S^-(t+m)\xi \\ &\quad + \sum_{i=1}^{m} S^-(t+m-i)\int_0^1 S^-(1-\tau)F(u_{i-1}(\tau))d\tau \\ &\quad + \sum_{i=1}^{m} S^-(t+m-i)\int_0^1 S^-(1-\tau)G(u_{i-1}(\tau))d\theta_{i-1}\omega(\tau) \\ &\quad + \int_0^t S^-(t-\tau)F(u_m(\tau))d\tau + \int_0^t S^-(t-\tau)G(u_m(\tau))d\theta_m\omega(\tau) \quad (3.1)\end{aligned}$$

$$- \int_t^1 S^+(t-\tau)F(u_m(\tau))d\tau - \int_t^1 S^+(t-\tau)G(u_m(\tau))d\theta_m\omega(\tau)$$
$$- \sum_{i=m+2} S^+(t+m-i)\int_0^1 S^+(1-\tau)F(u_{i-1}(\tau))d\tau$$
$$- \sum_{i=m+2} S^+(t+m-i)\int_0^1 S^+(1-\tau)G(u_{i-1}(\tau))d\theta_{i-1}\omega(\tau),$$

with the convention that $\sum_{i=k}^n f_i = 0$ *for any* $n < k$.

Proof. If u is the solution of (2.18) with initial value $u(0)$ with $u^-(0) = \xi$, denoting $u(t+m) = u_m(t), t \in [0,1]$ and $U = (u_i)_{i\in\mathbb{Z}^+}$, then by the variation of constants formula, we obtain that

$$\begin{aligned}
u_m^-(t) &= \varphi^-(t+m,\omega,u(0)) \\
&= S^-(t+m)\xi + \sum_{i=1}^m \int_{i-1}^i S^-(t+m-\tau)F(u(\tau))d\tau + \int_m^{t+m} S^-(t+m-\tau)F(u(\tau))d\tau \\
&\quad + \sum_{i=1}^m \int_{i-1}^i S^-(t+m-\tau)G(u(\tau))d\omega(\tau) + \int_m^{t+m} S^-(t+m-\tau)G(u(\tau))d\omega(\tau) \\
&= S^-(t+m)\xi + \sum_{i=1}^m S^-(t+m-i)\int_0^1 S^-(1-\tau)F(u_{i-1}(\tau))d\tau \\
&\quad + \sum_{i=1}^m S^-(t+m-i)\int_0^1 S^-(1-\tau)G(u_{i-1}(\tau))d\theta_{i-1}\omega(\tau) \\
&\quad + \int_0^t S^-(t-\tau)F(u_m(\tau))d\tau + \int_0^t S^-(t-\tau)G(u_m(\tau))d\theta_m\omega(\tau).
\end{aligned} \tag{3.2}$$

Regarding the part in $\mathcal{B}^+$, by the definition of the solution of (2.18), we have

$$u^+(s) = S^+(s)u^+(0) + \int_0^s S^+(s-r)F(u(r))dr + \int_0^s S^+(s-r)G(u(r))d\omega(r)$$

and for $n \geq s$, by the cocycle property, it holds that

$$u^+(n) = S^+(n-s)u^+(s) + \int_s^n S^+(n-r)F(u(r))dr + \int_s^n S^+(n-r)G(u(r))d\omega(r).$$

Since $\mathcal{B}^+$ is finite-dimensional, the semigroup $S^+(\cdot$

which vanishes as $n \to +\infty$ since $\kappa + \hat{\mu} > 0$. So taking the limit $n \to +\infty$ in (3.3) results in

$$
\begin{aligned}
u_m^+(t) = &- \int_t^1 S^+(t-\tau)F(u_m(\tau))d\tau - \int_t^1 S^+(t-\tau)G(u_m(\tau))d\theta_m\omega(\tau) \\
&- \sum_{i=m+2} S^+(t+m-i)\int_0^1 S^+(1-\tau)F(u_{i-1}(\tau))d\tau \\
&- \sum_{i=m+2} S^+(t+m-i)\int_0^1 S^+(1-\tau)G(u_{i-1}(\tau))d\theta_{i-1}\omega(\tau).
\end{aligned}
\tag{3.4}
$$

Then U satisfies (3.1) by combining (3.2) with (3.4). Conversely, if $U \in \mathcal{H}_\kappa^\xi$ satisfies (3.1), then denoting by $u(t) = u_i(s)$ with $i + s = t$ for any $t \geq 0$, it follows from a straightforward calculation that $u(t)$ is the solution of (2.18) with $u^-(0) = \xi$. $\square$

As commonly met in the theory of invariant manifolds [14,24], we have to truncate the nonlinear terms F and G. To this aim let χ be a smooth function on $C^\beta_{-\beta}([0,1];\mathcal{B})$ given by

$$
\chi(u) := \begin{cases} u, \|u\|_{\beta,-\beta} \leq \dfrac{1}{2} \\ 0, \|u\|_{\beta,-\beta} > 1, \end{cases}
$$

with continuous first and second derivative bounded by the constants D_χ, D_χ^2. For $R > 0$ (which will be determined in Subsection 3.2), let

$$
\chi_R(u) := R\chi\left(\frac{u}{R}\right) = \begin{cases} u, \|u\|_{\beta,-\beta} \leq \dfrac{R}{2} \\ 0, \|u\|_{\beta,-\beta} > R. \end{cases}
$$

We further define $F_R(u) := F \circ \chi_R(u)$ and $G_R(u) := G \circ \chi_R(u)$ for $u \in C^\beta_{-\beta}([0,1];\mathcal{B})$. Denote by

$$
\begin{aligned}
L_F(R) = \max\{&D_\chi \sup_{u,v\in B_{\mathcal{B}}(0,R),\eta\in[0,1]} \|DF(\eta\chi_R(u) + (1-\eta)\chi_R(v))\|_{L(\mathcal{B};\mathcal{B})}, \\
&D_\chi \sup_{u,v\in B_{\mathcal{B}}(0,R),\eta\in[0,1]} \|DF(\eta\chi_R(u) + (1-\eta)\chi_R(v))\|_{L(\mathcal{B};\mathcal{B}_{-\beta})}\},
\end{aligned}
$$

and

$$
\begin{aligned}
&L_G(R) \\
&\quad = \max\{D_\chi \sup_{w\in B_{\mathcal{B}}(0,R)} \|DG(w)\|_{L(\mathcal{B};L_2(\mathcal{B};\mathcal{B}))}, D_\chi \sup_{w\in B_{\mathcal{B}}(0,R)} \|DG(w)\|_{L(\mathcal{B}_{-\beta};L_2(\mathcal{B};\mathcal{B}_{-\beta}))}\}.
\end{aligned}
$$

Note that due to the boundedness of DF, DG and due to the conditions $DF(0) = 0, DG(0) = 0$, $L_F(R)$ and $L_G(R)$ converge to 0 as $R \to 0$. Moreover we have that $L_F(R) = C[D_\chi, C_{DF}]R$ and $L_G(R) = C[D_\chi, D_\chi^2, C_{DG}, C_{D^2G}]R$, where C is a constant which depends only on D_χ, D_χ^2 and on the bounds of the derivatives of F and G denoted by C_{DF} respectively C_{DG}, C_{D^2G}. Here we disregard the embedding constants since these do not change the argument. Due to assumption $(\mathbf{A_2})$ and $(\mathbf{A_3})$, it holds that

$$\begin{aligned}
|F_R(u) - F_R(v)| &= |F(\chi_R(u)) - F(\chi_R(v))| \\
&= \int_0^1 DF(\eta\chi_R(u) + (1-\eta)\chi_R(v))(\chi_R(u) - \chi_R(v))d\eta \\
&\leq D_\chi \sup_{u,v\in B_{\mathcal{B}}(0,R),\eta\in[0,1]} \|DF(\eta\chi_R(u) + (1-\eta)\chi_R(v))\||u-v| \\
&\leq L_F(R)|u-v|
\end{aligned}$$

and

$$\begin{aligned}
|F_R(u) - F_R(v)|_{-\beta} &= |F(\chi_R(u)) - F(\chi_R(v))|_{-\beta} \\
&= \int_0^1 DF(\eta\chi_R(u) + (1-\eta)\chi_R(v))(\chi_R(u) - \chi_R(v))d\eta \\
&\leq D_\chi \sup_{u,v\in B_{\mathcal{B}}(0,R),\eta\in[0,1]} \|DF(\eta\chi_R(u) + (1-\eta)\chi_R(v))\|_{L(\mathcal{B};\mathcal{B}_{-\beta})}|u-v| \\
&\leq L_F(R)|u-v|.
\end{aligned}$$

Similarly, for G, we have

$$\begin{aligned}
\|G_R(u) - G_R(v)\|_{L_2(\mathcal{B};\mathcal{B})} &= \|G(\chi_R(u)) - G(\chi_R(u))\|_{L_2(\mathcal{B};\mathcal{B})} \\
&= \left\| \int_0^1 DG(\eta\chi_R(u) + (1-\eta)\chi_R(v))(\chi_R(u) - \chi_R(v))d\eta \right\| \\
&\leq D_\chi \sup_{u,v\in B_{\mathcal{B}}(0,R)} \|DG(\eta\chi_R(u) + (1-\eta)\chi_R(v))\||u-v| \\
&\leq D_\chi \sup_{w\in B_{\mathcal{B}}(0,R)} \|DG(w)(u-v)\|_{L_2(\mathcal{B};\mathcal{B})} \\
&\leq L_G(R)|u-v|,
\end{aligned}$$

and

$$\|G_R(u) - G_R(v)\|_{L_2(\mathcal{B};\mathcal{B}_{-\beta})}$$

$$
\begin{aligned}
&= \|G(\chi_R(u)) - G(\chi_R(u))\|_{L_2(\mathcal{B};\mathcal{B}_{-\beta})} \\
&= \left\| \int_0^1 DG(\eta\chi_R(u) + (1-\eta)\chi_R(v))(\chi_R(u) - \chi_R(v))d\eta \right\|_{L_2(\mathcal{B};\mathcal{B}_{-\beta})} \\
&\leq D_\chi \sup_{u,v \in B_{\mathcal{B}}(0,R)} \|DG(\eta\chi_R(u) + (1-\eta)\chi_R(v))(u-v)\|_{L_2(\mathcal{B};\mathcal{B}_{-\beta})} \\
&\leq D_\chi \sup_{w \in B_V(0,R)} \|DG(w)(u-v)\|_{L_2(\mathcal{B};\mathcal{B}_{-\beta})} \\
&\leq L_G(R)|u-v|_{-\beta}.
\end{aligned}
$$

Since $\mathcal{B}$ is densely embedded into $\mathcal{B}_{-\beta}$, we have, dropping the embedding constant

$$
\|G_R(u) - G_R(v)\|_{L_2(\mathcal{B};\mathcal{B}_{-\beta})} \leq L_G(R)|u-v|.
$$

Lastly,

$$
\begin{aligned}
&\|DG_R(u)h - DG_R(v)h\|_{L_2(\mathcal{B};\mathcal{B}_{-\beta}))} \\
&\quad = \left\| \int_0^1 D^2G(\eta\chi_R(u) + (1-\eta)\chi_R(v))d\eta(\chi_R(u) - \chi_R(v))h \right\|_{L_2(\mathcal{B};\mathcal{B}_{-\beta})} \\
&\quad \leq D_\chi \sup_{w \in B_{\mathcal{B}}(0,R)} \|D^2G(w)(u-v)h\|_{L_2(\mathcal{B};\mathcal{B}_{-\beta}))} \\
&\leq D_\chi L_G|u-v|_{-\beta}|h|_{-\beta}.
\end{aligned}
$$

Hence for $u_1, v_1, u_2, v_2 \in \mathcal{B}$, we have

$$
\begin{aligned}
&\|G_R(u_1) - G_R(v_1) - G_R(u_2) + G_R(v_2)\|_{L_2(\mathcal{B};\mathcal{B}_{-\beta})} \\
&\leq \Bigg\| \int_0^1 \int_0^1 D^2G(h(\xi\chi_R(u_1) + (1-\xi)\chi_R(v_1)) + (1-h)(\xi\chi_R(u_2) + (1-\xi)\chi_R(v_2)))dh \\
&\quad \times [\xi\chi_R(u_1) + (1-\xi)\chi_R(v_1) - \xi\chi_R(u_2) - (1-\xi)\chi_R(v_2)]d\xi(\chi_R(u_1) - \chi_R(v_1)) \Bigg\|_{L_2(\mathcal{B};\mathcal{B}_{-\beta})} \\
&\quad + \left\| \int_0^1 DG(\xi\chi_R(u_2) + (1-\xi)\chi_R(v_2))d\xi(\chi_R(u_1) - \chi_R(v_1) - \chi_R(u_2) + \chi_R(v_2)) \right\|_{L_2(\mathcal{B};\mathcal{B}_{-\beta})} \\
&\leq L_G(|\chi_R(u_1) - \chi_R(u_2)|_{-\beta} + |\chi_R(v_1) - \chi_R(v_2)|_{-\beta})|\chi_R(u_1) - \chi_R(v_1)|_{-\beta} \\
&\quad + L_G(R)|\chi_R(u_1) - \chi_R(v_1) - \chi_R(u_2) - \chi_R(v_2)|_{-\beta} \\
&\leq L_G(D_\chi)^2(|u_1 - u_2|_{-\beta} + |v_1 - v_2|_{-\beta})|u_1 - v_1|_{-\beta} + L_G(R)D_\chi^2|u_1 - v_1 - u_2 - v_2|_{-\beta}.
\end{aligned}
$$

Remark 3.2. Due to the previous estimates, one can show that the modified equation (2.5) (i.e. replacing F and G by F_R and G_R which are now path-dependent) has a unique global-in-time solution in $C^{\beta}_{-\beta}([0,T],\mathcal{B})$. This follows via a fixed point method as in Section 2.3, see [24, Theorem 4.3] for a similar argument.

For $K > 0$, let $R(\theta_i\omega)$ be the solution of

$$c_S L_F(R(\theta_i\omega)) + c_S C_{\alpha,\beta} L_G(R(\theta_i\omega)) |||\theta_i\omega|||_{\beta',0,1} = K. \tag{3.5}$$

The existence of such a random variable is ensured due to the structure of $L_F(R)$ and $L_G(R)$ specified above. The random variable $R(\omega) > 0$ is tempered from below since $|||\theta_i\omega|||_{\beta',0,1}$ is tempered from above. Next, we consider the following Lyapunov-Perron-type operator:

$$\begin{aligned}
[\mathcal{J}_R(U)]_m(t) = {} & S^-(t+m)\xi + \sum_{i=1}^{m} S^-(t+m-i)\int_0^1 S^-(1-\tau)F_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))d\tau \\
& + \sum_{i=1}^{m} S^-(t+m-i)\int_0^1 S^-(1-\tau)G_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))d\theta_{i-1}\omega(\tau) \\
& + \int_0^t S^-(t-\tau)F_{R(\theta_m\omega)}(u_m(\tau))d\tau + \int_0^t S^-(t-\tau)G_{R(\theta_m\omega)}(u_m(\tau))d\theta_m\omega(\tau) \\
& - \sum_{i=m+2} S^+(t+m-i)\int_0^1 S^+(1-\tau)F_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))d\tau \\
& - \sum_{i=m+2} S^+(t+m-i)\int_0^1 S^+(1-\tau)G_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))d\theta_{i-1}\omega(\tau) \\
& - \int_t^1 S^+(t-\tau)F_{R(\theta_m\omega)}(u_m(\tau))d\tau - \int_t^1 S^+(t-\tau)G_{R(\theta_m\omega)}(u_m(\tau))d\theta_m\omega(\tau).
\end{aligned} \tag{3.6}$$

This coincides with the Lyapunov-Perron operator of the original equation when $\|u_i\|_{\beta,-\beta} \leq \frac{R(\theta_i\omega)}{2}$, for all $i = 0, 1, \cdots$.

3.2. Existence of local stable manifolds

Theorem 3.3. *Assume that* $(\mathbf{A_1})$*-*$(\mathbf{A_4})$ *hold and that* K *satisfies the gap condition:*

$$K\left(\frac{-1}{1-e^{-(\check{\mu}+\kappa)}} + \frac{1}{1-e^{-(\hat{\mu}+\kappa)}}\right) \leq \frac{1}{2}.$$

Then for given $\xi \in \mathcal{B}^-$, there exists a unique element $\Gamma(\xi,\omega) = \{\Gamma(\xi,\omega)(i,\cdot)\}_{i\in\mathbb{Z}^+} \in \mathcal{H}_\kappa^\xi$ such that

$$[\mathcal{J}_R(\Gamma(\xi,\omega))](i,\cdot) = \Gamma(\xi,\omega)(i,\cdot), \qquad \Gamma^-(\xi,\omega)(0,0) = \xi.$$

Moreover, $\Gamma(\xi,\omega)(i,\cdot)$ is Lipschitz with respect to $\xi \in \mathcal{B}^-$.

Proof. We denote each part of $\mathcal{J}_R(U,\xi)$, given as (3.6), as follows,

$$\begin{aligned}[\mathcal{J}_R(U,\xi)]_m(t) := {} & S^-(t+m)\xi + I_1(U) + I_2(U) + I_3(U) + I_4(U) - I_5(U) - I_6(U) \\ & - I_7(U) - I_8(U),\end{aligned}$$

and estimate all of them individually under the norm $\|\cdot\|_{\beta,-\beta}$. Based on the previous computations, we just need to take into account the exponential contraction and expansion of S^- and S^+ given by (2.13b) and (2.13a). First, it follows from Lemma 2.11 and Theorem 2.12 that

$$\begin{aligned}
\|I_3(U)\|_{\beta,-\beta} &\le c_S L_F(R(\theta_m\omega))\|u_m\|_{\beta,-\beta}, \\
\|I_4(U)\|_{\beta,-\beta} &\le c_S C_{\alpha,\beta} L_G(R(\theta_m\omega))\|u_m\|_{\beta,-\beta}|||\theta_m\omega|||_{\beta'}, \\
\|I_7(U)\|_{\beta,-\beta} &\le c_S L_F(R(\theta_m\omega))\|u_m\|_{\beta,-\beta}, \\
\|I_8(U)\|_{\beta,-\beta} &\le c_S C_{\alpha,\beta} L_G(R(\theta_m\omega))\|u_m\|_{\beta,-\beta}|||\theta_m\omega|||_{\beta'}.
\end{aligned}$$

Afterwards, we need to treat the terms $S^-(\cdot)\xi$ and I_1, I_2, I_5, I_6. Following the steps of (2.23), we can directly obtain that

$$\begin{aligned}
\|S^-(\cdot+m)\xi\|_{\beta,-\beta} &\le c_S e^{\breve{\mu} m}|\xi| + \sup_{0\le s<t<1} \frac{\|S^-(s+m)\|_{L(\mathcal{B};\mathcal{B})}\|S^-(t-s) - id\|_{L(\mathcal{B};\mathcal{B}_{-\beta})}|\xi|}{(t-s)^\beta} \\
&\le c_S e^{\breve{\mu} m}|\xi|,
\end{aligned}$$

and

$$\begin{aligned}
&\|S^-(\cdot+m)(\xi-\xi_0)\|_{\beta,-\beta} \\
&\quad \le c_S e^{\breve{\mu} m}|\xi-\xi_0| + \sup_{0\le s<t<1} \frac{\|S^-(s+m)\|_{L(\mathcal{B};\mathcal{B})}\|S^-(t-s) - id\|_{L(\mathcal{B};\mathcal{B}_{-\beta})}|\xi-\xi_0|}{(t-s)^\beta} \\
&\quad \le c_S e^{\breve{\mu} m}|\xi-\xi_0|
\end{aligned}$$

for $\xi, \xi_0 \in \mathcal{B}^-$. Since for any $i = 1, \cdots, m$, $0 \le t < 1$,

$$\begin{aligned}
&\max\left\{\|S^-(t+m-i)\|_{L(\mathcal{B};\mathcal{B})}, \frac{1}{(t-s)^\beta}\|S^-(t+m-i) - S^-(s+m-i)\|_{L(\mathcal{B};\mathcal{B}_{-\beta})}\right\} \\
&\quad \le c_S e^{\breve{\mu}(m-i)},
\end{aligned}$$

we have

$$\begin{aligned}
\|I_1(U)\|_{\beta,-\beta} &= \left\| \sum_{i=1}^{m} S^-(\cdot + m - i) \int_0^1 S^-(1-\tau) F_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau)) d\tau \right\|_{\beta,-\beta} \\
&\le c_S \sum_{i=1}^{m} e^{\check{\mu}(m-i)} \left| \int_0^1 S^-(1-\tau) F_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau)) d\tau \right| \\
&\le c_S \sum_{i=1}^{m} L_F(R(\theta_{i-1}\omega)) e^{\check{\mu}(m-i)} \|u_{i-1}\|_{\beta,-\beta}, \\
\|I_2(U)\|_{\beta,-\beta} &= \left\| \sum_{i=1}^{m} S^-(\cdot + m - i) \int_0^1 S^-(1-\tau) G_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau)) d\theta_{i-1}\omega(\tau) \right\|_{\beta,-\beta} \\
&\le c_S \sum_{i=1}^{m} e^{\check{\mu}(m-i)} \left| \int_0^1 S^-(1-\tau) G_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau)) d\theta_{i-1}\omega(\tau) \right| \\
&\le c_S C_{\alpha,\beta} \sum_{i=1}^{m} L_G(R(\theta_{i-1}\omega)) e^{\check{\mu}(m-i)} \|u_{i-1}\|_{\beta,-\beta} |||\theta_{i-1}\omega|||_{\beta'}.
\end{aligned}$$

Similarly, for any $i \ge m+2, 0 \le t < 1$, it holds

$$\begin{aligned}
&\max\left\{ \|S^+(t+m-i)\|_{L(\mathcal{B};\mathcal{B})}, \frac{1}{(t-s)^\beta} \|S^+(t+m-i) - S^+(s+m-i)\|_{L(\mathcal{B};\mathcal{B}_{-\beta})} \right\} \\
&\quad \le c_S e^{\hat{\mu}(m-i+1)}.
\end{aligned}$$

Then we have

$$\begin{aligned}
\|I_5(U)\|_{\beta,-\beta} &= \left\| \sum_{i=m+2} S^+(\cdot + m - i) \int_0^1 S^+(1-\tau) F_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau)) d\tau \right\|_{\beta,-\beta} \\
&\le c_S \sum_{i=m+2} L_F(R(\theta_{i-1}\omega)) e^{\hat{\mu}(m-i+1)} \|u_{i-1}\|_{\beta,-\beta},
\end{aligned}$$

and

$$\begin{aligned}
\|I_6(U)\|_{\beta,-\beta} &= \left\| \sum_{i=m+2} S^+(\cdot + m - i) \int_0^1 S^+(1-\tau) G_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau)) d\theta_{i-1}\omega(\tau) \right\|_{\beta,-\beta} \\
&\le c_S C_{\alpha,\beta} \sum_{i=m+2} L_G(R(\theta_{i-1}\omega)) e^{\hat{\mu}(m-i+1)} \|u_{i-1}\|_{\beta,-\beta} |||\theta_{i-1}\omega|||_{\beta'}.
\end{aligned}$$

To sum up, we have

$$
\begin{aligned}
&\|[\mathcal{J}_R(U,\xi)]_m\|_{\beta,-\beta} \\
&\quad \le c_S e^{\check{\mu} m}|\xi| + c_S \sum_{i=0}^{m} e^{\check{\mu}(m-i)}\|u_i\|_{\beta,-\beta}\left[L_F(R(\theta_i\omega)) + C_{\alpha,\beta}L_G(R(\theta_i\omega))|||\theta_i\omega|||_{\beta'}\right] \\
&\qquad + c_S \sum_{i=m} e^{\hat{\mu}(m-i)}\|u_i\|_{\beta,-\beta}\left[L_F(R(\theta_i\omega)) + C_{\alpha,\beta}L_G(R(\theta_i\omega))|||\theta_i\omega|||_{\beta'}\right].
\end{aligned}
$$

Referring to the result of (2.22), we can similarly obtain that for $U, V \in \mathcal{H}_\kappa$ with $u^-(0) = \xi, v^-(0) = \xi_0$,

$$
\begin{aligned}
&\|[\mathcal{J}_R(U,\xi) - \mathcal{J}_R((V,\xi_0))]_m\|_{\beta,-\beta} \\
&\le c_S e^{\check{\mu} m}|\xi - \xi_0| + c_S \sum_{i=0}^{m} e^{\check{\mu}(m-i)}\|u_i - v_i\|_{\beta,-\beta}\left[L_F(R(\theta_i\omega)) + C_{\alpha,\beta}L_G(R(\theta_i\omega))|||\theta_i\omega|||_{\beta'}\right] \\
&\quad + c_S \sum_{i=m} e^{\hat{\mu}(m-i)}\|u_i - v_i\|_{\beta,-\beta}\left[L_F(R(\theta_i\omega)) + C_{\alpha,\beta}L_G(R(\theta_i\omega))|||\theta_i\omega|||_{\beta'}\right].
\end{aligned}
$$

Lastly, by (3.5), we can conclude that

$$
\begin{aligned}
e^{\kappa m}\|[\mathcal{J}_R(U,\xi)]_m\|_{\beta,-\beta} \le c_S e^{(\check{\mu}+\kappa)m}|\xi| &+ K\sum_{i=0}^{m} e^{\check{\mu}(m-i)+\kappa(m-i)}e^{\kappa i}\|u_i\|_{\beta,-\beta} \\
&+ K\sum_{i=m} e^{\hat{\mu}(m-i)+\kappa(m-i)}e^{\kappa i}\|u_i\|_{\beta,-\beta},
\end{aligned}
$$

and further

$$
\begin{aligned}
&e^{\kappa m}\|[\mathcal{J}_R(U,\xi) - \mathcal{J}_R((V,\xi_0))]_m\|_{\beta,-\beta} \\
&\quad \le c_S e^{(\check{\mu}+\kappa)m}|\xi - \xi_0| + K\sum_{i=0}^{m} e^{\check{\mu}(m-i)+\kappa(m-i)}e^{\kappa i}\|u_i - v_i\|_{\beta,-\beta} \\
&\qquad + K\sum_{i=m} e^{\hat{\mu}(m-i)+\kappa(m-i)}e^{\kappa i}\|u_i - v_i\|_{\beta,-\beta}.
\end{aligned}
$$

Moreover, since

$$
\sup_{m\in\mathbb{Z}^+}\sum_{i=0}^{m} e^{(\check{\mu}+\kappa)(m-i)} = \frac{-1}{1-e^{-(\check{\mu}+\kappa)}}, \quad \sum_{i=m} e^{(\hat{\mu}+\kappa)(m-i)} = \frac{1}{1-e^{-(\hat{\mu}+\kappa)}}, \quad 0 < \kappa < -\check{\mu},
$$

we have

$$\|\mathcal{J}_R(U,\xi)\|_{\mathcal{H}_\kappa} \le c_S|\xi| + K\left(\frac{-1}{1-e^{-(\check{\mu}+\kappa)}} + \frac{1}{1-e^{-(\hat{\mu}+\kappa)}}\right)\|U\|_{\mathcal{H}_\kappa}$$
$$\le c_S|\xi| + \frac{1}{2}\|U\|_{\mathcal{H}_\kappa},$$

and

$$\|\mathcal{J}_R(U,\xi) - \mathcal{J}_R(V,\xi)\|_{\mathcal{H}_\kappa} \le c_S|\xi-\xi_0| + K\left(\frac{-1}{1-e^{-(\check{\mu}+\kappa)}} + \frac{1}{1-e^{-(\hat{\mu}+\kappa)}}\right)\|U-V\|_{\mathcal{H}_\kappa}$$
$$\le c_S|\xi-\xi_0| + \frac{1}{2}\|U-V\|_{\mathcal{H}_\kappa}.$$

Hence by the Banach fixed point theorem, there is a fixed point $\Gamma(\xi,\omega) \in \mathcal{H}_\kappa^\xi$ of $\mathcal{J}_R$ for given $\xi \in \mathcal{B}^-$. Since $|||\omega|||_{\beta',0,1}$ is tempered from above, it is obvious that $R(\omega)$ is tempered from below.

Now, we consider the regularity of $\Gamma(\cdot,\omega)$. Let $\Gamma(\xi,\omega)$ and $\Gamma(\xi_0,\omega)$ be the fixed points for $\xi, \xi_0 \in \mathcal{B}^-$. Then we have t_0 7 230.018 478.506 Tm 0 Tc T160.0865 1 (0 10 85.4047 407.748 Tm 80098f16))T5 1 (J)]

$$\|\Gamma(\xi,\omega) - \Gamma(\xi_0,\omega)\|_{\mathcal{H}_\kappa} \le \|\mathcal{J}_R(\Gamma(\xi,\omega),\xi) - \mathcal{J}_R(\Gamma(\xi_0,\omega),\xi_0)\|_{\mathcal{H}_\kappa}$$
$$\le \|\mathcal{J}_R(\Gamma(\xi,\omega),\xi) - \mathcal{J}_R(\Gamma(\xi,\omega),\xi_0)\|_{\mathcal{H}_\kappa}$$
$$+ \|\mathcal{J}_R(\Gamma(\xi,\omega),\xi_0) - \mathcal{J}_R(\Gamma(\xi_0,\omega),\xi_0)\|_{\mathcal{H}_\kappa}$$
$$\le \sup_{m\in\mathbb{Z}^+} c_S e^{(\check{\mu}+\kappa)m}\|\xi_0-\xi_0\| + 1$$

Proof. By definition, we have for $t \in [0, 1]$

$$
\begin{aligned}
&\Gamma(\xi, \omega)(m+1, t) \\
&= S^{-}(t+m+1)\xi + \sum_{i=1}^{m+1} S^{-}(t+m+1-i) \int_0^1 S^{-}(1-\tau) F_{R(\theta_{i-1}\omega)}(\Gamma(\xi, \omega)(i-1, \tau)) d\tau \\
&\quad + \sum_{i=1}^{m+1} S^{-}(t+m+1-i) \int_0^1 S^{-}(1-\tau) G_{R(\theta_{i-1}\omega)}(\Gamma(\xi, \omega)(i-1, \tau)) d\theta_{i-1}\omega(\tau) \\
&\quad + \int_0^t S^{-}(t-\tau) F_{R(\theta_{m+1}\omega)}(\Gamma(\xi, \omega)(m+1, \tau)) d\tau \\
&\quad + \int_0^t S^{-}(t-\tau) G_{R(\theta_{m+1}\omega)}(\Gamma(\xi, \omega)(m+1, \tau)) d\theta_{m+1}\omega(\tau) \\
&\quad - \sum_{i=m+3} S^{+}(t+m+1-i) \int_0^1 S^{+}(1-\tau) F_{R(\theta_{i-1}\omega)}(\Gamma(\xi, \omega)(i-1, \tau)) d\tau \\
&\quad - \sum_{i=m+3} S^{+}(t+m+1-i) \int_0^1 S^{+}(1-\tau) G_{R(\theta_{i-1}\omega)}(\Gamma(\xi, \omega)(i-1, \tau)) d\theta_{i-1}\omega(\tau) \\
&\quad - \int_t^1 S^{+}(t-\tau) F_{R(\theta_{m+1}\omega)}(\Gamma(\xi, \omega)(m+1, \tau)) d\tau \\
&\quad - \int_t^1 S^{+}(t-\tau) G_{R(\theta_{m+1}\omega)}(\Gamma(\xi, \omega)(m+1, \tau)) d\theta_{m+1}\omega(\tau),
\end{aligned}
$$

and

$$
\begin{aligned}
&\Gamma(u^{-}(1), \theta_1\omega)(m, t) \\
&= S^{-}(t+m)u^{-}(1) + \sum_{i=1}^{m} S^{-}(t+m-i) \int_0^1 S^{-}(1-\tau) F_{R(\theta_i\omega)}(\Gamma(u^{-}(1), \theta_1\omega)(i-1, \tau)) d\tau \\
&\quad + \sum_{i=1}^{m} S^{-}(t+m-i) \int_0^1 S^{-}(1-\tau) G_{R(\theta_i\omega)}(\Gamma(u^{-}(1), \theta_1\omega)(i-1, \tau)) d\theta_i\omega(\tau) \\
&\quad + \int_0^t S^{-}(t-\tau) F_{R(\theta_{m+1}\omega)}(\Gamma(u^{-}(1), \theta_1\omega)(m, \tau)) d\tau
\end{aligned}
$$

$$
\begin{aligned}
&+ \int_0^t S^-(t-\tau) G_{R(\theta_{m+1}\omega)}(\Gamma(u^-(1), \theta_1\omega)(m, \tau)) d\theta_{m+1}\omega(\tau) \\
&- \sum_{i=m+2} S^+(t+m-i) \int_0^1 S^+(1-\tau) F_{R(\theta_i\omega)}(\Gamma(u^-(1), \theta_1\omega)(i-1, \tau)) d\tau \\
&- \sum_{i=m+2} S^+(t+m-i) \int_0^1 S^+(1-\tau) G_{R(\theta_i\omega)}(\Gamma(u^-(1), \theta_1\omega)(i-1, \tau)) d\theta_i\omega(\tau) \\
&- \int_t^1 S^+(t-\tau) F_{R(\theta_{m+1}\omega)}(\Gamma(u^-(1), \theta_1\omega)(m, \tau)) d\tau \\
&- \int_t^1 S^+(t-\tau) G_{R(\theta_{m+1}\omega)}(\Gamma(u^-(1), \theta_1\omega)(m, \tau)) d\theta_{m+1}\omega(\tau).
\end{aligned}
$$

For the part containing $S^-(\cdot)$, we have by the definition of mild solution

$$
\begin{aligned}
u^-(1) = S^-(1)\xi &+ \int_0^1 S^-(1-\tau) F_{R(\omega)}(\Gamma(\xi, \omega)(0, \tau)) d\tau \\
&+ \int_0^1 S^-(1-\tau) G_{R(\omega)}(\Gamma(\xi, \omega)(0, \tau)) d\omega(\tau).
\end{aligned}
$$

This further leads to

$$
\begin{aligned}
&\Gamma^-(u^-(1), \theta_1\omega)(m, t) \\
&= S^-(t+m+1)\xi + S^-(t+m) \int_0^1 S^-(1-\tau) F_{R(\omega)}(\Gamma(\xi, \omega)(0, \tau)) d\tau \\
&\quad + S^-(t+m) \int_0^1 S^-(1-\tau) G_{R(\omega)}(\Gamma(\xi, \omega)(0, \tau)) d\omega(\tau)\tau \\
&\quad + \sum_{i=1}^m S^-(t+m-i) \int_0^1 S^-(1-\tau) F_{R(\theta_i\omega)}(\Gamma(u^-(1), \theta_1\omega)(i-1, \tau)) d\tau \\
&\quad + \sum_{i=1}^m S^-(t+m-i) \int_0^1 S^-(1-\tau) G_{R(\theta_i\omega)}(\Gamma(u^-(1), \theta_1\omega)(i-1, \tau)) d\theta_i\omega(\tau)
\end{aligned}
\tag{3.8}
$$

$$+\int_0^t S^-(t-\tau)F_{R(\theta_{m+1}\omega)}(\Gamma(u^-(1),\theta_1\omega)(m,\tau))d\tau$$
$$+\int_0^t S^-(t-\tau)G_{R(\theta_{m+1}\omega)}(\Gamma(u^-(1),\theta_1\omega)(m,\tau))d\theta_{m+1}\omega(\tau),$$

and

$$\begin{aligned}\Gamma^-(\xi,\omega)(m+1,t) = S^-(t+m+1)\xi \\ &+\sum_{i=0}^m S^-(t+m-i)\int_0^1 S^-(1-\tau)F_{R(\theta_i\omega)}(\Gamma(\xi,\omega)(i,\tau))d\tau \\ &+\sum_{i=0}^m S^-(t+m-i)\int_0^1 S^-(1-\tau)G_{R(\theta_i\omega)}(\Gamma(\xi,\omega)(i,\tau))d\theta_i\omega(\tau) \\ &+\int_0^t S^-(t-\tau)F_{R(\theta_{m+1}\omega)}(\Gamma(\xi,\omega)(m+1,\tau))d\tau \\ &+\int_0^t S^-(t-\tau)G_{R(\theta_{m+1}\omega)}(\Gamma(\xi,\omega)(m+1,\tau))d\theta_{m+1}\omega(\tau).\end{aligned} \tag{3.9}$$

Then (3.9)-(3.8) gives that

$$\|\Gamma^-(\xi,\omega)(\cdot+1,\cdot)-\Gamma^-(u^-(1),\theta_1\omega)(\cdot,\cdot)\|_{\mathcal{H}_\kappa}\le\frac{1}{2}\|\Gamma(\xi,\omega)(\cdot+1,\cdot)-\Gamma(u^-(1),\theta_1\omega)(\cdot,\cdot)\|_{\mathcal{H}_\kappa},$$

which is only satisfied when $\Gamma^-(\xi,\omega)(m+1,\cdot)=\Gamma^-(u^-(1),\theta_1\omega)(m,\cdot)$. Similar produce can be conducted about the positive part and leads to $\Gamma^+(\xi,\omega)(m+1,\cdot)=\Gamma^+(u^-(1),\theta_1\omega)(m,\cdot)$. Hence we can conclude that $\Gamma(\xi,\omega)(m+1,\cdot)=\Gamma(u^-(1),\theta_1\omega)(m,\cdot)$ with $u(1)=\Gamma(\xi,$

$$\begin{aligned}
\hat{\rho}(\theta_i\omega)e^{\kappa i} &= \frac{1}{L_\Gamma}e^{i\kappa}\inf_{j\in\mathbb{Z}^+} e^{j\kappa}\hat{r}(\theta_{j+i}\omega) = \frac{1}{L_\Gamma}\inf_{j\in\mathbb{Z}^+} e^{(j+i)\kappa}\hat{r}(\theta_{j+i}\omega) \\
&= \frac{1}{L_\Gamma}\inf_{k\geq i} e^{k\kappa}
\end{aligned}$$

Theorem

$$\begin{aligned}\|\Gamma(\xi,\omega)(m,\cdot)\|_{\beta,-\beta} &= \|\Gamma(u^-(m),\theta_m\omega)(0,\cdot)\|_{\beta,-\beta} \le L_\Gamma|u^-(m)| \\ &\le L_\Gamma \hat{r}(\theta_m\omega) = \frac{R(\theta_m\omega)}{2}.\end{aligned} \tag{3.15}$$

As discussed in Subsection 3.1 and due to (3.15), for $x \in M(\omega)\cap W(\omega)$, $m = 0,1,\cdots$, $s \in [0,1]$, we conclude that $\Gamma(\xi,\omega)(m,s)$ is the solution of (2.18) at $m+s$ with $x = \Gamma(\xi,\omega)(0,0)$ as initial datum, i.e.

$$\Gamma(\xi,\omega)(m,s) = \varphi(m+s,\omega,x) = \varphi(m,\theta_s\omega,\varphi(s,\omega,x)).$$

In particular we have

$$\varphi(s,\omega,x) = \Gamma(\xi,\omega)(0,s) =: u(s).$$

By Lemma 3.6, we have $|\Gamma^-(\xi,\omega)(0,s)| \le \hat{\rho}(\theta_s\omega)$ and further

$$\varphi(m,\theta_s\omega,\Gamma(\xi,\omega)(0,s)) = \Gamma(\Gamma^-(\xi,\omega)(0,s),\theta_s\omega)(m,0). \tag{3.16}$$

Now, we can conclude that

$$\Gamma(\xi,\omega)(m,s) = \varphi(m+s,\omega,x) = \varphi(m,\theta_s\omega,u(s)) = \Gamma(u^-(s),\theta_s\omega)(m,0),$$

for $x \in M(\omega)\cap W(\omega)$, where the first equality implies

$$\lim_{t\to\infty} \varphi(t,\omega,x) = 0, \quad \text{exponentially}.$$

Furthermore, together with (3.13), the second equality entails

$$\varphi(m+s,\omega,x) = \Gamma(u^-(m+s),\theta_{m+s}\omega)(0,0) = u^-(m+s) + \Gamma^+(u^-(m+s),\theta_{m+s}\omega)(0,0).$$

In which $u^-(k+s) := \Gamma^-(u^-(k-1+s),\theta_{k-1}\omega)(1,0)$ satisfying $|u^-(k+s)| \le \hat{r}(\theta_{k+s}\omega)$, $k = 1,2\cdots,m$, due to (3.14). Hence $\varphi(t,\omega,M(\omega)\cap W(\omega)) \subset M(\theta_t\omega)$ for any $t \ge 0$, which naturally implies that

$$\lim_{|x|\to 0} t_0(\omega,x) = \infty, \text{ for } x \in M(\omega).$$

Hence by Definition 2.3, we conclude that M is a local stable manifold for φ. The same procedure can be conducted for $L_\Gamma \le 1$ using Lemma 3.6 and replacing W by

$$\tilde{W}(\omega) := \{x \in \mathcal{B} : \xi \in B_{\mathcal{B}^-}(0,\overline{\rho}(\omega)),\ \xi = x^-\}. \quad \square$$

3.3. Smoothness of local stable manifolds

Fixing $\xi_0 \in \mathcal{B}^-$, denoting $U = \{u_i\}_{i\in\mathbb{Z}^+}$ with $u_i(t) = \Gamma(\xi_0, \omega)(i, t)$, we now prove that U belongs to $C^k(\mathcal{B}^-; \mathcal{H}_\kappa)$ (with respect to ξ_0) for $k \geq 1$ under suitable assumptions.

($\mathbf{A_2^k}$) The mapping $F \in C_b^k(\mathcal{B}; \mathcal{B})$ with $F(0) = 0, DF(0) = 0, \cdots, D^kF(0) = 0$ and satisfies the following Lipschitz condition

$$\|D^lF(u) - D^lF(v)\|_{L^l(\mathcal{B};\mathcal{B})} \leq L_F|u - v|,$$

for some constant $L_F > 0, l = 0, 1, \cdots, k - 1$.

($\mathbf{A^k}$

$$[\mathcal{L}(V)]_m(t)$$

$$:= \sum_{i=1}^{m} S^-(t+m-i) \int_0^1 S^-(1$$

$$\le \sup_{|w|_{\beta,-\beta}\le R} \|DG(w)\|_{L(\mathcal{B}_{-\beta};L_2(\mathcal{B};\mathcal{B}_{-\beta}))}|v(r)-v(q)|_{-\beta} + L_G|u(r)-u(q)|_{-\beta}|v(q)|$$

$$\le L_G(R)\|v\|_{\beta,-\beta}(r-q)^{\beta}.$$

Based on the last two relations and (2.9), we infer

$$\begin{aligned}
&\|D^{\alpha}_{s+}S(t-\cdot)DG_R(u(\cdot))v(\cdot)[r]\|_{L_2(\mathcal{B};\mathcal{B})}\\
&\quad\le \frac{1}{\Gamma(1-\alpha)}\Bigg(\frac{\|S(t-r)DG_R(u(r))v(r)\|_{L_2(\mathcal{B};\mathcal{B})}}{(r-s)^{\alpha}}\\
&\qquad+\alpha\int_s^r \frac{\|S(t-r)DG_R(u(r))v(r)-S(t-q)DG_R(u(q))v(q)\|_{L_2(\mathcal{B};\mathcal{B})}}{(r-q)^{1+\alpha}}\,dq\Bigg)\\
&\quad\le c_S C_\alpha L_G(R)(r-s)^{-\alpha}|v(r)|\\
&\qquad+C_\alpha\int_s^r \frac{\|S(t-r)-S(t-q)\|_{L(\mathcal{B};\mathcal{B})}\|DG_R(u(r))v(r)\|_{L_2(\mathcal{B};\mathcal{B})}}{(r-q)^{1+\alpha}}\,dq
\end{aligned}$$

$$
\begin{aligned}
&+C_\alpha \int_s^r \frac{\|S(t-q)\|_{L(\mathcal{B}_{-\beta};\mathcal{B}_{-\beta})}\|DG_R(u(r))v(r) - DG_R(u(q))v(q)\|_{L_2(\mathcal{B};\mathcal{B}_{-\beta})}}{(r-q)^{1+\alpha}}dq \\
&\le c_S C_\alpha L_G(R)(r-s)^{-\alpha}\|v\|_{\beta,-\beta} + c_S C_\alpha L_G(R)\|v\|_{\beta,-\beta}\int_s^r (r-q)^{\beta-\alpha-1}dq \\
&+c_S C_\alpha L_G(R)\|v\|_{\beta,-\beta}\int_s^r (r-q)^{\beta-\alpha-1}dq \\
&\le c_S C_{\alpha,\beta} L_G(R)\|v\|_{\beta,-\beta}[(r-s)^{-\alpha} + (r-s)^{\beta-\alpha} + (r-s)^{\beta-\alpha}].
\end{aligned}
$$

Furthermore, for $\alpha \; - \alpha > 0$, we infer that

$$
\begin{aligned}
&\|D^\alpha_{0+}[S(t-\cdot) - S(s-\cdot)]DG_R(u(\cdot))v(\cdot)[r]\|_{L_2(\mathcal{B};\mathcal{B}_{-\beta})} \\
&\le \frac{C_\alpha\|[S(t-r) - S(s-r)]DG_R(u(r))v(r)\|_{L_2(\mathcal{B};\mathcal{B}_{-\beta})}}{r^\alpha} \\
&+C_\alpha\int_0^r \frac{1}{(r-q)^{\alpha+1}}\|[S(t-r) - S(s-r)]DG_R(u(r))v(r) \\
&-[S(t-q) - S(s-q)]DG_R(u(r))v(q)\|_{L_2(\mathcal{B};\mathcal{B}_{-\beta})}dq \\
&\le \frac{C_\alpha\|S(t-s) - \mathrm{id}\|_{L(\mathcal{B};\mathcal{B}_{-\beta})}\|S(s-r)\|_{L(\mathcal{B};\mathcal{B})}\|DG_R(u(r))v(r)\|_{L_2(\mathcal{B};\mathcal{B})}}{r^\alpha} \\
&+C_\alpha\int_0^r \frac{\|[S(t-r) - S(s-r) - S(t-q) + S(s-q)]DG_R(u(r))v(r)\|_{L_2(\mathcal{B};\mathcal{B}_{-\beta})}}{(r-q)^{\alpha+1}}dq \\
&+C_\alpha\int_0^r \frac{\|[S(t-q) - S(s-q)][DG_R(u(r))v(r) - DG_R(u(r))v(q)]\|_{L_2(\mathcal{B};\mathcal{B}_{-\beta})}}{(r-q)^{\alpha+1}}dq
\end{aligned}
$$

348.852 Tm (L)Tj /F10 1 Tf 5 0 0 5 247.458 34

$$
\begin{aligned}
&\le \frac{C_\alpha\|S(t-s) - \mathrm{id}\|_{L(\mathcal{B};\mathcal{B}_{-\beta})}\|S(s-r)\|_{L(\mathcal{B};\mathcal{B})}\|DG_R(u(r))v(r)\|_{L_2(\mathcal{B};\mathcal{B})}}{r^\alpha} \qquad (3.24) \\
&+C_\alpha\int_0^r \frac{1}{(r-q)^{\alpha+1}}\|S(t-s) - \mathrm{id}\|_{L(\mathcal{B};\mathcal{B}_{-\beta})}\|S(s-r)\|_{L(\mathcal{B}_{-\alpha'};\mathcal{B})} \\
&\times\|\mathrm{id} - S(r-q)\|_{L(\mathcal{B};\mathcal{B}_{-\alpha'})}\|DG_R(u(r))v(r)\|_{L_2(\mathcal{B};\mathcal{B})}dq \\
&+C_\alpha\int^r
\end{aligned}
$$

$B;\ \alpha'\mathcal{B}$ — DG

$mj\gamma/\alpha \quad TD\gamma\ C\ Tj\gamma/F \quad Tf\gamma \quad .$

$$+ \int_0^r (s-q)^{-\beta}(r-q)^{\beta-\alpha-1}\Big]$$

$$\leq c_S C_{\alpha,\beta} L_G(R)\|u\|_{\beta,-\beta}(t-s)^{\beta}[(s-r)^{-\alpha'}r^{\alpha'-\alpha} + (s-r)^{-\beta}r^{\beta-\alpha}].$$

Lemma 3.9. *The operator $\mathcal{L} : \mathcal{H}_{\kappa+\gamma} \to \mathcal{H}_{\kappa+\gamma}$ is bounded for any $\gamma \in [0, 2\gamma_0]$.*

Proof. The fact that $\mathcal{L}$ is well-defined on $\mathcal{H}_{\kappa+\gamma}$ can be obtained by a similar approach as for $\mathcal{J}$ using $(\mathbf{A_2})$, $(\mathbf{A_3})$ and (3.18). Therefore we omit the details. We mainly focus on its boundedness. Note that $F \in C_b^1(\mathcal{B};\mathcal{B})$ with $DF(0) = 0$, thus

$$|DF_{R(\theta_{i-1}\omega)}(u_{i-1})v_{i-1}| \leq L_F(R(\theta_{i-1}\omega))|v_{i-1}|.$$

Then by the similar arguments as for I_1, I_5, I_3 and I_7 in Theorem 3.3, we have

$$\begin{aligned}
\|\mathcal{L}_1(V)\|_{\beta,-\beta} &\leq \sum_{i=1}^{m} \left\| S^-(\cdot + m - i) \int_0^1 S^-(1-\tau) DF_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))v_{i-1}(\tau)d\tau \right\|_{\beta,-\beta} \\
&\leq c_S \sum_{i=1}^{m} L_F(R(\theta_{i-1}\omega))e^{\check{\mu}(m-i)}\|v_{i-1}\|_{\beta,-\beta}, \\
\|\mathcal{L}_3(V)\|_{\beta,-\beta} &\leq \left\| \int_0^{\cdot} S^-(\cdot - \tau) DF_{R(\theta_m\omega)}(u_m(\tau))v_m(\tau)d\tau \right\|_{\beta,-\beta} \\
&\leq c_S L_F(R(\theta_m\omega))\|v_m\|_{\beta,-\beta}, \\
\|\mathcal{L}_5(V)\|_{\beta,-\beta} &\leq \sum_{i=m+2} \left\| S^+(\cdot + m - i) \int_0^1 S^+(1-\tau) DF_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))v_{i-1}(\tau)d\tau \right\|_{\beta,-\beta} \\
&\leq c_S \sum_{i=m+2} L_F(R(\theta_{i-1}\omega))e^{\hat{\mu}(m-i)}\|v_{i-1}\|_{\beta,-\beta}, \\
\|\mathcal{L}_7(V)\|_{\beta,-\beta} &\leq \left\| \int_{\cdot}^0 S^+(\cdot - \tau) DF_{R(\theta_m\omega)}(u_m(\tau))v_m(\tau)d\tau \right\|_{\beta,-\beta} \\
&\leq c_S L_F(R(\theta_m\omega))\|v_m\|_{\beta,-\beta}.
\end{aligned}$$

On the other hand, (3.22), (3.23), (3.24) entail that

$$\left| \int_s^t S(t-r) DG_R(u(r))v(r)d\omega(r) \right|$$

$$\leq c_S C_{\alpha,\beta} L_G(R) \|v\|_{\beta,-\beta} |||\omega|||_{\beta',0,1} \int_s^t (t-r)^{\alpha+\beta'-1}(r-s)^{-\alpha} dr$$

$$+ c_S C_{\alpha,\beta} L_G(R) \|v\|_{\beta,-\beta} |||\omega|||_{\beta',0,1} \int_s^t (t-r)^{\alpha+\beta'-1}(t-r)^{-\beta}(r-s)^{\beta-\alpha} dr$$

$$\leq c_S C_{\alpha,\beta} L_G(R) \|v\|_{\beta,-\beta} |||\omega|||_{\beta',0,1} (t-s)^{\beta'},$$

and

$$\left| \int_s^t S(t-r) DG_R(u(r)) v(r) d\omega(r) \right|_{-\beta}$$

$$\leq c_S C_{\alpha,\beta} L_G(R) \|v\|_{\beta,-\beta} |||\omega|||_{\beta',0,1} \int_s^t (t-r)^{\alpha+\beta'-1}[(r-s)^{-\alpha} + (r-s)^{\beta-\alpha}] dr$$

$$\leq c_S C_{\alpha,\beta} L_G(R) \|v\|_{\beta,-\beta} |||\omega|||_{\beta',0,1} \left[(t-s)^{\beta'} + (t-s)^{\beta+\beta'} \right],$$

and

$$\left| \int_0^s [S(t-r) - S(s-r)] DG_R(u(r)) v(r) d\omega(r) \right|_{-\beta}$$

$$\leq C_{\alpha,\beta} c_S L_G(R) \|u\|_{\beta,-\beta} (t-s)^{\beta} |||\omega|||_{\beta',0,1}$$

$$\times \int_0^s (s-r)^{\alpha+\beta'-1}[(s-r)^{-\alpha'} r^{\alpha'-\alpha} + (s-r)^{-\beta} r^{\beta-\alpha}] dr$$

$$\leq C_{\alpha,\beta} c_S L_G(R) \|u\|_{\beta,-\beta} (t-s)^{\beta} |||\omega|||_{\beta',0,1} (t-s)^{\beta} s^{\beta'}.$$

Hence, by the definition of the norm on $C^{\beta}_{-\beta}$, we have

$$\left\| \int_0 S(\cdot - r) DG_R(u(r)) v(r) d\omega(r) \right\|_{\beta,-\beta}$$

$$= \sup_{t \in [0,1]} \left| \int_0^t S(t-r) DG_R(u(r)) v(r) d\omega(r) \right|$$

$$+ \sup_{s<t \in [0,1]} \frac{\left| \int_0^t S(t-r) DG_R(u(r)) v(r) d\omega(r) - \int_0^s S(s-r) DG_R(u(r)) v(r) d\omega(r) \right|_{-\beta}}{(t-s)^{\beta}}$$

$$\le \sup_{t\in[0,1]} \left| \int_0^t S(t-r)DG_R(u(r))v(r)d\omega(r) \right|$$

$$+ \sup_{s<t\in[0,1]} \frac{\left|\int_s^t S(t-r)DG_R(u(r))v(r)d\omega(r)\right|_{-\beta}}{(t-s)^\beta}$$

$$+ \sup_{s<t\in[0,1]} \frac{\left|\int_0^s [S(t-r)-S(s-r)]DG_R(u(r))v(r)d\omega(r)\right|_{-\beta}}{(t-s)^\beta}$$

$$\le c_S C_{\alpha,\beta} L_G(R)\|v\|_{\beta,-\beta} |||\omega|||_{\beta',0,1}.$$

Then we further obtain

$$\|\mathcal{L}_2(V)\|_{\beta,-\beta}$$

$$\le \sum_{i=1}^m \left\| S^-(\cdot + m - i) \int_0^1 S^-(1-\tau)DG_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))v_{i-1}(\tau)d\theta_{i-1}\omega\tau \right\|_{\beta,-\beta}$$

$$\le c_S C_{\alpha,\beta} \sum_{i=1}^m L_G(R(\theta_{i-1}\omega))|||\theta_{i-1}\omega|||_{\beta',0,1} e^{\check{\mu}(m-i)}\|v_{i-1}\|_{\beta,-\beta},$$

$$\|\mathcal{L}_4(V)\|_{\beta,-\beta} \le \left\| \int_0 S^-(\cdot - \tau)DG_{R(\theta_m\omega)}(u_m(\tau))v_m(\tau)d\theta_m\omega(\tau) \right\|_{\beta,-\beta}$$

$$\le c_S C_{\alpha,\beta} L_G(R(\theta_m\omega))|||\theta_m\omega|||_{\beta',0,1}\|v_m\|_{\beta,-\beta},$$

$$\|\mathcal{L}_6(V)\|_{\beta,-\beta}$$

$$\le \sum_{i=m+2} \left\| S^+(\cdot + m - i) \int_0^1 S^+(1-\tau)DG_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))v_{i-1}(\tau)d\theta_{i-1}\omega(\tau) \right\|_{\beta,-\beta}$$

$$\le c_S C_{\alpha,\beta} \sum_{i=m+2} L_G(R(\theta_{i-1}\omega))e^{\hat{\mu}(m-i)}\|v_{i-1}\|_{\beta,-\beta}|||\theta_{i-1}\omega|||_{\beta',0,1},$$

$$\|\mathcal{L}_8(V)\|_{\beta,-\beta} \le \left\| \int^0 S^+(\cdot - \tau)DG_{R(\theta_m\omega)}(u_m(\tau))v_m(\tau)d\theta_m\omega(\tau) \right\|_{\beta,-\beta}$$

$$\le c_S C_{\alpha,\beta} L_G(R(\theta_m\omega))|||\theta_m\omega|||_{\beta',0,1}\|v_m\|_{\beta,-\beta}.$$

In conclusion, we have

$$\|[\mathcal{L}(V)]_m\|_{\beta,-\beta} \le c_S \sum_{i=0}^m e^{\check{\mu}(m-i)}\|v_i\|_{\beta,-\beta} \left[L_F(R(\theta_i\omega)) + C_{S,\alpha,\beta}L_G(R(\theta_i\omega))|||\theta_i\omega|||_{\beta'} \right]$$

$$+ c_S \sum_{i=m} e^{\hat{\mu}(m-i)} \|v_i\|_{\beta,-\beta} \left[L_F(R(\theta_i\omega)) + C_{\alpha,\beta} L_G(R(\theta_i\omega)) |||\theta_i\omega|||_{\beta'} \right].$$

At last, combining the gap condition (3.18), we conclude that

$$\begin{aligned} \|[\mathcal{L}(V)]\|_{\mathcal{H}_{\kappa+\gamma}} &\leq K \sup_{m\in\mathbb{Z}^+} \left[\sum_{i=0}^{m} e^{(\check{\mu}+\kappa+\gamma)(m-i)} + \sum_{i=m} e^{(\hat{\mu}+\kappa+\gamma)(m-i)} \right] \|V\|_{\mathcal{H}_{\kappa+\gamma}} \\ &\leq \frac{1}{2} \|V\|_{\mathcal{H}_{\kappa+\gamma}}. \end{aligned} \tag{3.25}$$

Hence $\mathcal{L}$ is a bounded linear operator and also a uniform contraction on $\mathcal{H}_{\kappa+\gamma}$. □

Remark 3.10. Naturally the operator $id - \mathcal{L}$ is injective on $\mathcal{H}_{\kappa+\gamma}$. It remains to show that this is also surjective, meaning that for any $V \in \mathcal{H}_{\kappa+\gamma}$, one can find a $V \ \in \mathcal{H}_{\kappa+\gamma}$ such that

$$(\mathcal{L} - id)V \ = V.$$

To this aim, we set $\mathcal{L}_V x := \mathcal{L}x - V$ for $x \in \mathcal{H}_{\kappa+\gamma}$. This is a contraction and has therefore a fixed point V such that

$$\mathcal{L}V \ - V = V \ ,$$

that is

$$\mathcal{L}V \ - V \ = V.$$

In conclusion $id - \mathcal{L}$ is surjective. Therefore it has an inverse which is a bounded linear operator on $\mathcal{H}_{\kappa+\gamma}$.

With $\mathcal{L}$ at hand, we define another operator $\mathcal{I}$ on $\mathcal{H}_{\kappa+\gamma}$ with $\gamma \in [0, \gamma_0]$, which is given by

$$[\mathcal{I}(V - U)]_m(t) = [V - U]_m(t) - [\mathcal{L}(V - U)]_m(t) - S^-(t + m)(\xi - \xi_0). \tag{3.26}$$

We claim that $\|\mathcal{I}(V - U)\|_{\mathcal{H}_{\kappa+\gamma}} = o(|\xi - \xi_0|)$ as $\xi \to \xi_0$. Then it yields

$$v_m(t) - u_m(t) - [\mathcal{L}(V - U)]_m(t) = S(t + m)(\xi - \xi_0) + o(|\xi - \xi_0|)$$

as $\xi \to \xi_0$ for any $t \in [0, 1], m \in \mathbb{Z}^+$, which implies

$$v_m(t) - u_m(t) = (\mathrm{id} - \mathcal{L})^{-1} S(t + m)(\xi - \xi_0) + o(|\xi_0 - \xi|).$$

Hence $u(\cdot, \xi, \omega)$ is di erentiable with respect to $\xi \in \mathcal{B}^-$ and its derivative satisfies $D_\xi u(\cdot, \xi, \omega) \in L(\mathcal{B}^-; \mathcal{H}_{\kappa+\gamma})$.

Lemma 3.11. *The operator defined in* (3.26) *satisfies* $\|\mathcal{I}(V-U)\|_{\mathcal{H}_{\kappa+\gamma}} = o(|\xi-\xi_0|)$ *as* $\xi \to \xi_0$.

Proof. For an $m_1 > 0$ to be determined, we decompose $\mathcal{I}$ into a sum as follows:

$$[\mathcal{I}(V-U)]_m(t) := \mathcal{I}_1^-(t) + \mathcal{I}_2^-(t) + \mathcal{I}_1^+(t) + \mathcal{I}_2^+(t),$$

where

$$\mathcal{I}_1^-(t) = \begin{cases} \sum_{i=1}^m S^-(t+m-i)\int_0^1 S^-(1-\tau)[F_{R(\theta_{i-1}\omega)}(v_{i-1}(\tau)) \\ \quad -F_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau)) \\ \quad -DF_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))(v_{i-1}(\tau)-u_{i-1}(\tau))]d\tau \\ \quad +\sum_{i=1}^m S^-(t+m-i)\int_0^1 S^-(1-\tau)[G_{R(\theta_{i-1}\omega)}(v_{i-1}(\tau)) \\ \quad -G_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau)) \\ \quad -DG_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))(v_{i-1}(\tau)-u_{i-1}(\tau))]d\theta_{i-1}\omega(\tau) \\ \quad +\int_0^t S^-(t-\tau)[F_{R(\theta_m\omega)}(v_m(\tau))-F_{R(\theta_m\omega)}(u_m(\tau)) \\ \quad -DF_{R(\theta_m\omega)}(u_m(\tau))(v_m(\tau)-u_m(\tau))]d\tau \\ \quad +\int_0^t S^-(t-\tau)[G_{R(\theta_m\omega)}(v_m(\tau))-G_{R(\theta_m\omega)}(u_m(\tau)) \\ \quad -DG_{R(\theta_m\omega)}(u_m(\tau))(v_m(\tau)-u_m(\tau))]d\theta_m\omega(\tau), & 1\le m<m_1, \\ 0, & m\ge m_1, \end{cases}$$

$$\mathcal{I}_2^-(t) = \begin{cases} 0, & 1\le m<m_1, \\ \sum_{i=1}^m S^-(t+m-i)\int_0^1 S^-(1-\tau)[F_{R(\theta_{i-1}\omega)}(v_{i-1}(\tau)) \\ \quad -F_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau)) \\ \quad -DF_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))(v_{i-1}(\tau)-u_{i-1}(\tau))]d\tau \\ \quad +\sum_{i=1}^m S^-(t+m-i)\int_0^1 S^-(1-\tau)[G_{R(\theta_{i-1}\omega)}(v_{i-1}(\tau)) \\ \quad -G_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau)) \\ \quad -DG_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))(v_{i-1}(\tau)-u_{i-1}(\tau))]d\theta_{i-1}\omega(\tau) \\ \quad +\int_0^t S^-(t-\tau)[F_{R(\theta_m\omega)}(v_m(\tau))-F_{R(\theta_m\omega)}(u_m(\tau)) \\ \quad -DF_{R(\theta_m\omega)}(u_m(\tau))(v_m(\tau)-u_m(\tau))]d\tau \\ \quad +\int_0^t S^-(t-\tau)[G_{R(\theta_m\omega)}(v_m(\tau))-G_{R(\theta_m\omega)}(u_m(\tau)) \\ \quad -DG_{R(\theta_m\omega)}(u_m(\tau))(v_m(\tau)-u_m(\tau))]d\theta_m\omega(\tau), & m\ge m_1, \end{cases}$$

$$\mathcal{I}_1^+(t) = \begin{cases} -\sum_{i=m+2}^{m_1} S^+(t+m-i)\int_0^1 S^+(1-\tau)[F_{R(\theta_{i-1}\omega)}(v_{i-1}(\tau)) \\ \quad -F_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau)) \\ \quad -DF_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))(v_{i-1}(\tau)-u_{i-1}(\tau))]d\tau \\ \quad -\sum_{i=m+2}^{m_1} S^+(t+m-i)\int_0^1 S^+(1-\tau)[G_{R(\theta_{i-1}\omega)}(v_{i-1}(\tau)) \\ \quad -G_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau)) \\ \quad -DG_{R(\theta_{i-1}\omega)}(u_m(\tau))(v_{i-1}(\tau)-u_{i-1}(\tau))]d\theta_{i-1}\omega(\tau) \\ \quad -\int_t^1 S^+(t-\tau)[F_{R(\theta_m\omega)}(v_m(\tau))-F_{R(\theta_m\omega)}(u_m(\tau)) \\ \quad -DF_{R(\theta_m\omega)}(u_m(\tau))(v_m(\tau)-u_m(\tau))]d\tau \\ \quad -\int_t^1 S^+(t-\tau)[G_{R(\theta_m\omega)}(v_m(\tau))-G_{R(\theta_m\omega)}(u_m(\tau)) \\ \quad -DG_{R(\theta_m\omega)}(u_m(\tau))(v_m(\tau)-u_m(\tau))]d\theta_m\omega(\tau), & 1\le m<m_1, \\ 0, & m\ge m_1, \end{cases}$$

$$\mathcal{I}_2^+(t)=\begin{cases}-\sum_{i=m_1+2}^{\infty}S^+(t+m-i)\int_0^1 S^+(1-\tau)[F_{R(\theta_{i-1}\omega)}(v_{i-1}(\tau))\\ \quad -F_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))\\ \quad -DF_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))(v_{i-1}(\tau)-u_{i-1}(\tau))]d\tau\\ \quad -\sum_{i=m_1+2}^{\infty}S^+(t+m-i)\int_0^1 S^+(1-\tau)[G_{R(\theta_{i-1}\omega)}(v_{i-1}(\tau))\\ \quad -G_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))\\ \quad -DG_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))(v_{i-1}(\tau)-u_{i-1}(\tau))]d\theta_{i-1}\omega(\tau)\\ \quad -\int_0^1 S^+(t+m-m_1-1)[F_{R(\theta_{m_1}\omega)}(v_{m_1}(\tau))-F_{R(\theta_{m_1}\omega)}(u_{m_1}(\tau))\\ \quad -DF_{R(\theta_{m_1}\omega)}(u_{m_1}(\tau))(v_{m_1}(\tau)-u_{m_1}(\tau))]d\tau\\ \quad -\int_0^1 S^+(t+m-m_1-1)[G_{R(\theta_{m_1}\omega)}(v_{m_1}(\tau))-G_{R(\theta_{m_1}\omega)}(u_{m_1}(\tau))\\ \quad -DG_{R(\theta_{m_1}\omega)}(u_{m_1}(\tau))(v_{m_1}(\tau)-u_{m_1}(\tau))]d\theta_{m_1}\omega(\tau), & 1\le m<m_1,\\ -\sum_{i=m+2}^{\infty}S^+(t+m-i)\int_0^1 S^+(1-\tau)[F_{R(\theta_{i-1}\omega)}(v_{i-1}(\tau))\\ \quad -F_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))\\ \quad -DF_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))(v_{i-1}(\tau)-u_{i-1}(\tau))]d\tau\\ \quad -\sum_{i=m+2}^{\infty}S^+(t+m-i)\int_0^1 S^+(1-\tau)[G_{R(\theta_{i-1}\omega)}(v_{i-1}(\tau))\\ \quad -G_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))\\ \quad -DG_{R(\theta_{i-1}\omega)}(u_m(\tau))(v_{i-1}(\tau)-u_{i-1}(\tau))]d\theta_{i-1}\omega(\tau)\\ \quad -\int_t^1 S^+(t-\tau)[F_{R(\theta_m\omega)}(v_m(\tau))-F_{R(\theta_m\omega)}(u_m(\tau))\\ \quad -DF_{R(\theta_m\omega)}(u_m(\tau))(v_m(\tau)-u_m(\tau))]d\tau\\ \quad -\int_t^1 S^+(t-\tau)[G_{R(\theta_m\omega)}(v_m(\tau))-G_{R(\theta_m\omega)}(u_m(\tau))\\ \quad -DG_{R(\theta_m\omega)}(u_m(\tau))(v_m(\tau)-u_m(\tau))]d\theta_m\omega(\tau), & m\ge m_1.\end{cases}$$

Taking $\gamma \le \gamma_0$, we estimate all of the above terms. For $m \ge m_1$, by assumption $(\mathbf{A}_2^k),(\mathbf{A}_3^k)$ and the definitions of $L_F(R)$ and $L_G(R)$, we have the following inequalities:

$$|F_{R(\theta_{i-1}\omega)}(v_{i-1})-F_{R(\theta_{i-1}\omega)}(u_{i-1})-DF_{R(\theta_{i-1}\omega)}(u_{i-1})(v_{i-1}-u_{i-1})| \le L_F(R(\theta_{i-1}\omega))|v_{i-1}-u_{i-1}|, \tag{3.27a}$$

$$|F_{R(\theta_{i-1}\omega)}(v_{i-1})-F_{R(\theta_{i-1}\omega)}(u_{i-1})-DF_{R(\theta_{i-1}\omega)}(u_{i-1})(v_{i-1}-u_{i-1})|_{-\beta} \le L_F(R(\theta_{i-1}\omega))|v_{i-1}-u_{i-1}|, \tag{3.27b}$$

$$\|G_{R(\theta_{i-1}\omega)}(v_{i-1})-G_{R(\theta_{i-1}\omega)}(u_{i-1})-DG_{R(\theta_{i-1}\omega)}(u_{i-1})(v_{i-1}-u_{i-1})\|_{L_2(\mathcal{B};\mathcal{B})} \le L_G(R(\theta_{i-1}\omega))|v_{i-1}-u_{i-1}|, \tag{3.27c}$$

$$\|G_{R(\theta_{i-1}\omega)}(v_{i-1})-G_{R(\theta_{i-1}\omega)}(u_{i-1})-DG_{R(\theta_{i-1}\omega)}(u_{i-1})(v_{i-1}-u_{i-1})\|_{L_2(\mathcal{B};\mathcal{B}_{-\beta})} \le L_G(R(\theta_{i-1}\omega))|v_{i-1}-u_{i-1}|, \tag{3.27d}$$

$$\begin{aligned}&\|G_{R(\theta_{i-1}\omega)}(v_{i-1}(r))-G_{R(\theta_{i-1}\omega)}(u_{i-1}(r))-DG_{R(\theta_{i-1}\omega)}(u_{i-1}(r))[v_{i-1}(r)-u_{i-1}(r)]\\ &\quad -G_{R(\theta_{i-1}\omega)}(v_{i-1}(q))+G_{R(\theta_{i-1}\omega)}(u_{i-1}(q))\\ &\quad +DG_{R(\theta_{i-1}\omega)}(u_{i-1}(q))[v_{i-1}(q)-u_{i-1}(q)]\|_{L_2(\mathcal{B};\mathcal{B}_{-\beta})}\\ &\quad \le L_G(R)(r-q)^\beta\|v_{i-1}-u_{i-1}\|_{\beta,-\beta}.\end{aligned} \tag{3.27e}$$

Replacing $F(u),G(u)$ as in Lemma 2.11 and Theorem 2.12 by $F_{R(\theta_{i-1}\omega)}(v_{i-1}(\tau))-F_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))-DF_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))(v_{i-1}(\tau)-u_{i-1}(\tau))$ and $G_{R(\theta_{i-1}\omega)}(v_{i-1}(\tau))-$

$G_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))-DG_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))(v_{i-1}(\tau)-u_{i-1}(\tau))$ respectively, we obtain the following inequalities from (3.27a), (3.27b), (3.27c), (3.27d) and (3.27e):

$$\begin{aligned}
&\Big|\int_0^1 S(1-\tau)[F_{R(\theta_{i-1}\omega)}(v_{i-1}(\tau))-F_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))\\
&\qquad\qquad - DF_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))(v_{i-1}(\tau)-u_{i-1}(\tau))]d\tau\Big|\\
&\quad\le L_F(R(\theta_{i-1}\omega))\|v_{i-1}-u_{i-1}\|_{\beta,-\beta},
\end{aligned}$$

$$\begin{aligned}
&\Big\|\int_0^{\cdot} S(\cdot-\tau)[F_{R(\theta_{i-1}\omega)}(v_{i-1}(\tau))-F_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))\\
&\qquad\qquad - DF_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))(v_{i-1}(\tau)-u_{i-1}(\tau))]d\tau\Big\|_{\beta,-\beta}\\
&\quad\le L_F(R(\theta_{i-1}\omega))\|v_{i-1}-u_{i-1}\|_{\beta,-\beta},
\end{aligned}$$

$$\begin{aligned}
&\Big\|\int_0^1 S(1-\tau)[G_{R(\theta_{i-1}\omega)}(v_{i-1}(\tau))-G_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))\\
&\qquad\qquad - DG_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))(v_{i-1}(\tau)-u_{i-1}(\tau))]d\tau\Big\|_{L_2(\mathcal{B};\mathcal{B})}\\
&\quad\le L_G(R(\theta_{i-1}\omega))\|v_{i-1}-u_{i-1}\|_{\beta,-\beta},
\end{aligned}$$

and

$$\begin{aligned}
&\Big\|\int_0^{\cdot} S(\cdot-\tau)[G_{R(\theta_{i-1}\omega)}(v_{i-1}(\tau))-G_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))\\
&\qquad\qquad - DG_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))(v_{i-1}(\tau)-u_{i-1}(\tau))]d\tau\Big\|_{\beta,-\beta}\\
&\quad\le L_G(R(\theta_{i-1}\omega))\|v_{i-1}-u_{i-1}\|_{\beta,-\beta}.
\end{aligned}$$

Hence, for any $\epsilon>0$ and large enough m_1, $\mathcal{I}_2^-$ and $\mathcal{I}_2^+$ can be estimated by

$$\begin{aligned}
\sup_{m\ge m_1} e^{m(\kappa+\gamma)}\|\mathcal{I}_2^-\|_{\beta,-\beta} &\le K\|U-V\|_{\mathcal{H}_{\kappa+2\gamma_0}}\sup_{m\ge m_1}\sum_{i=0}^{m-1} e^{\check{\mu}(m-i-1)+(\kappa+\gamma)m-(\kappa+2\gamma_0)i}\\
&\le 2c_S K e^{(\gamma-2\gamma_0)m_1}\sup_{m\ge m_1}\sum_{i=0}^{m-1} e^{\check{\mu}(m-i-1)+(\kappa+2\gamma_0)(m-i)}|\xi-\xi_0|\\
&\le \frac{\epsilon}{4}|\xi-\xi_0|,
\end{aligned}$$

and

$$\begin{aligned}\sup_{m\geq m_1} e^{m(\kappa+\gamma)}\|\mathcal{I}_2^+\|_{\beta,-\beta} &\leq K\|U-V\|_{\mathcal{H}_{\kappa+2\gamma_0}} \sup_{m\geq m_1}\sum_{i=m} e^{\hat{\mu}(m-i)+(\kappa+\gamma)m-(\kappa+2\gamma_0)i}\\ &\leq 2c_S K e^{(\gamma-2\gamma_0)m_1}\sup_{m\geq m_1}\sum_{i=m} e^{\hat{\mu}(m-i)+(\kappa+2\gamma_0)(m-i)}|\xi-\xi_0|\\ &\leq \frac{\epsilon}{4}|\xi-\xi_0|.\end{aligned}$$

For the case of $1\leq m\leq m_1$, one can directly apply the result of the case of $m>m_1$ for $\mathcal{I}_2^+$, that is for large enough m_1 it holds that

$$\begin{aligned}\sup_{1\leq m<m_1} e^{m(\kappa+\gamma)}\|\mathcal{I}_2^+\|_{\beta,-\beta} &\leq 2c_S K e^{(\gamma-2\gamma_0)m_1}|\xi-\xi_0| \sup_{m\leq m_1}\sum_{i=m_1} e^{(\check{\mu}+\kappa+\gamma)(m-i)}\\ &\leq \frac{\epsilon}{4}|\xi-\xi_0|.\end{aligned}$$

However, the method applied above cannot be used for $\mathcal{I}_1^-$ and $\mathcal{I}_1^+$ when $1\leq m\leq m_1$. Fortunately, we can use the continuity of F, G and Γ to obtain a su ciently small quantity. More precisely, for any $\epsilon>0$, we denote by

$$\begin{aligned}\mathcal{E}_1 := \max\Bigg\{&\frac{\epsilon}{4c_S^2\sup_{m\leq m_1}\sum_{i=m}^{m_1-1} e^{\hat{\mu}(m-i)+(\kappa+\gamma)m-(\kappa+2\gamma_0)i}(1+C_{\alpha,\beta}|||\theta_i\omega|||_{\beta',0,1})},\\ &\frac{\epsilon}{4c_S^2\sup_{m\leq m_1}\sum_{i=1}^{m} e^{\check{\mu}(m-i)+(\kappa+\gamma)m-(\kappa+2\gamma_0)i}(1+C_{\alpha,\beta}|||\theta_i\omega|||_{\beta',0,1})}\Bigg\}.\end{aligned}$$

By the continuity of DF, DG and $\Gamma(\omega,\xi)$ with respect to ξ, for given $\epsilon>0$, there exists a constant $\epsilon_1>0$ such that when $|\xi-\xi_0|\leq\epsilon_1$, the following inequalities are valid:

$$\begin{aligned}\sup_{i\leq m\leq m_1,\tau\in[0,1]}\max\Big\{&\|DF_{R(\theta_{i-1}\omega)}(\tau v_{i-1}+(1-\tau)u_{i-1})-DF_{R(\theta_{i-1}\omega)}(u_{i-1})\|_{L(\mathcal{B};\mathcal{B})}\\ &\|DF_{R(\theta_{i-1}\omega)}(\tau v_{i-1}+(1-\tau)u_{i-1})-DF_{R(\theta_{i-1}\omega)}(u_{i-1})\|_{L(\mathcal{B};\mathcal{B}_{-\beta})}\\ &\big\|DG_{R(\theta_{i-1}\omega)}(\tau v_{i-1}+(1-\tau)u_{i-1})-DG_{R(\theta_{i-1}\omega)}(u_{i-1})\big\|_{L(\mathcal{B};L_2(\mathcal{B};\mathcal{B}_{-\beta}))}\\ &\big\|DG_{R(\theta_{i-1}\omega)}(\tau v_{i-1}+(1-\tau)u_{i-1})-DG_{R(\theta_{i-1}\omega)}(u_{i-1})\big\|_{L(\mathcal{B};L_2(\mathcal{B};\mathcal{B}))}\Big\}\\ \leq{}&\mathcal{E}_1.\end{aligned}$$

Hence we further have

$$\max\Big\{|F_{R(\theta_{i-1}\omega)}(v_{i-1})-F_{R(\theta_{i-1}\omega)}(u_{i-1})-DF_{R(\theta_{i-1}\omega)}(u_{i-1})(v_{i-1}-u_{i-1})|,$$
$$|F_{R(\theta_{i-1}\omega)}(v_{i-1})-F_{R(\theta_{i-1}\omega)}(u_{i-1})-DF_{R(\theta_{i-1}\omega)}(u_{i-1})(v_{i-1}-u_{i-1})|_{-\beta},$$
$$\|G_{R(\theta_{i-1}\omega)}(v_{i-1})-G_{R(\theta_{i-1}\omega)}(u_{i-1})-DG_{R(\theta_{i-1}\omega)}(u_{i-1})(v_{i-1}-u_{i-1})\|_{L_2(\mathcal{B};\mathcal{B}_{-\beta})},$$
$$\|G_{R(\theta_{i-1}\omega)}(v_{i-1})-G_{R(\theta_{i-1}\omega)}(u_{i-1})-DG_{R(\theta_{i-1}\omega)}(u_{i-1})(v_{i-1}-u_{i-1})\|_{L_2(\mathcal{B};\mathcal{B})}\Big\}$$
$$\leq \mathcal{E}_1|v_{i-1}-u_{i-1}|,$$

and additionally,

$$\begin{aligned}&\|G_{R(\theta_{i-1}\omega)}(v_{i-1}(r))-G_{R(\theta_{i-1}\omega)}(u_{i-1}(r))-DG_{R(\theta_{i-1}\omega)}(u_{i-1}(r))[v_{i-1}(r)-u_{i-1}(r)]\\&\quad -G_{R(\theta_{i-1}\omega)}(v_{i-1}(q))+G_{R(\theta_{i-1}\omega)}(u_{i-1}(q))\\&\qquad +DG_{R(\theta_{i-1}\omega)}(u_{i-1}(q))[v_{i-1}(q)-u_{i-1}(q)]\|_{L_2(\mathcal{B};\mathcal{B}_{-\beta})}\\&\leq \mathcal{E}_1(r-q)^{\beta}\|v_{i-1}-u_{i-1}\|_{\beta,-\beta}.\end{aligned}$$

Hence we further obtain

$$\begin{aligned}&\Big|\int_0^1 S(1-\tau)[F_{R(\theta_{i-1}\omega)}(v_{i-1}(\tau))-F_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))\\&\qquad\qquad -DF_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))(v_{i-1}(\tau)-u_{i-1}(\tau))]d\tau\Big|\\&\quad\leq \mathcal{E}_1c_S\|v_{i-1}-u_{i-1}\|_{\beta,-\beta},\\&\Big\|\int_0^{\cdot} S(\cdot-\tau)[F_{R(\theta_{i-1}\omega)}(v_{i-1}(\tau))-F_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))\\&\qquad\qquad -DF_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))(v_{i-1}(\tau)-u_{i-1}(\tau))]d\tau\Big\|_{\beta,-\beta}\\&\quad\leq \mathcal{E}_1c_S\|v_{i-1}-u_{i-1}\|_{\beta,-\beta},\\&\Big\|\int_0^1 S(1-\tau)[G_{R(\theta_{i-1}\omega)}(v_{i-1}(\tau))-G_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))\\&\qquad\qquad -DG_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))(v_{i-1}(\tau)-u_{i-1}(\tau))]d\theta_{i-1}\omega(\tau)\Big\|_{L_2}\end{aligned}$$

$$- DG_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))(v_{i-1}(\tau) - u_{i-1}(\tau))]d\theta_{i-1}\omega(\tau)\Big\|_{\beta,-\beta}$$

$$\leq \mathcal{E}_1 c_S C_{\alpha,\beta} |||\theta_{i-1}\omega|||_{\beta,0,1} \|v_{i-1} - u_{i-1}\|_{\beta,-\beta}.$$

Finally, $\mathcal{I}_1^-$ and $\mathcal{I}_1^+$ can be estimated by

$$\sup_{1\leq m<m_1} e^{m(\kappa+\gamma)} \|\mathcal{I}_1^-\|_{\beta,-\beta} \tag{3.28a}$$

$$\leq \mathcal{E}_1 c_S \|V - U\|_{\mathcal{H}_{\kappa+2\gamma_0}} \sup_{m\leq m_1} \sum_{i=0}^{m} e^{\check{\mu}(m-i)+(\kappa+\gamma)m-(\kappa+2\gamma_0)i}(1 + C_{\alpha,\beta}|||\theta_i\omega|||_{\beta',0,1})$$

$$\sup_{1\leq m<m_1} e^{m(\kappa+\gamma)} \|\mathcal{I}_1^+\|_{\beta,-\beta} \tag{3.28b}$$

$$\leq \mathcal{E}_1 c_S \|V - U\|_{\mathcal{H}_{\kappa+2\gamma_0}} \sup_{m\leq m_1} \sum_{i=m}^{m_1-1} e^{\hat{\mu}(m-i)+(\kappa+\gamma)m-(\kappa+2\gamma_0)i}(1 + C_{\alpha,\beta}|||\theta_i\omega|||_{\beta',0,1}).$$

All of the above quantities are bounded by $\frac{\epsilon}{2}|\xi - \xi_0|$. In conclusion, there exist $m_1 > 0$ large enough and $\epsilon_1 > 0$ such that for $|\xi - \xi_0| \leq \epsilon_1$, we have

$$\|\mathcal{I}(V - U)\|_{\mathcal{H}_{\kappa+\gamma}} \leq \epsilon|\xi - \xi_0|,$$

which proves the claim. □

According to the definition of the solution to (3.6), we have $D_\xi U = \{D_\xi u_i\}_{i\in\mathbb{Z}^+}$ with

$$\begin{aligned}
D_\xi u_m(t) = S^-(t) &+ \sum_{i=1}^{m} S^-(t + m - i) \int_0^1 S^-(1-\tau) DF_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau)) D_\xi u_{i-1}(\tau) d\tau \\
&+ \sum_{i=1}^{m} S^-(t + m - i) \int_0^1 S^-(1-\tau) DG_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau)) D_\xi u_{i-1}(\tau) d\theta_{i-1}\omega(\tau) \\
&+ \int_0^t S^-(t-\tau) DF_{R(\theta_m\omega)}(u_m(\tau)) D_\xi u_m(\tau) d\tau \\
&+ \int_0^t S^-(t-\tau) DG_{R(\theta_m\omega)}(u_m(\tau)) D_\xi u_m(\tau) d\theta_m\omega(\tau) \\
&- \sum_{i=m+2} S^+(t + m - i) \int_0^1 S^+(1-\tau) DF_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau)) D_\xi u_{i-1}(\tau) d\tau \\
&- \sum_{i=m+2} S^+(t + m - i) \int_0^1 S^+(1-\tau) DG_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau)) D_\xi u_{i-1}(\tau) d\theta_{i-1}\omega(\tau)
\end{aligned} \tag{3.29}$$

$$- \int_t^1 S^+(t-\tau) DF_{R(\theta_m\omega)}(u_m(\tau)) D_\xi u_m(\tau) d\tau$$

$$- \int_t^1 S^+(t-\tau) DG_{R(\theta_m\omega)}(u_m(\tau)) D_\xi u_m(\tau) d\theta_m\omega(\tau).$$

Combining this with the estimate on $\mathcal{L}$, one can directly obtain that

$$\sup_{i\in\mathbb{Z}^+} e^{(\kappa+\gamma)i} \|D_\xi u_i\|_{\beta,-\beta} \leq c_S + \frac{1}{2} \sup_{i\in\mathbb{Z}^+} e^{(\kappa+\gamma)i} \|D_\xi u_i\|_{\beta,-\beta} \leq 2c_S,$$

for any $\gamma \in [0, \gamma_0]$. Hence $D_\xi u$ is a bounded linear operator on $\mathcal{B}^-$. Lastly we have to show the continuity with respect to ξ. By the definition of $\mathcal{L}$, we can express the di erence of $D_\xi u$ and $D_\xi v$ as

$$D_\xi v_m(t) - D_\xi u_m(t) = [\mathcal{L}(D_\xi V - D_\xi U)]_m(t) + [\mathcal{T}]_m(t).$$

By (3.25), $D_\xi u_m(t)$ is continuous with respect to ξ if $\|\mathcal{T}\|_{\kappa+\gamma} \to 0$ as $\xi \to \xi_0$. To this aim, for an $m_1 > 0$ to be determined later, we rewrite $\mathcal{T}$ as follows:

$$[\mathcal{T}]_m(t) = \mathcal{T}_1^-(t) + \mathcal{T}_2^-(t) + \mathcal{T}_1^+(t) + \mathcal{T}_2^+(t),$$

where

$$\mathcal{T}_1^-(t) = \begin{cases} \sum_{i=1}^m S^-(t+m-i) \int_0^1 S^-(1-\tau)[DF_{R(\theta_{i-1}\omega)}(v_{i-1}(\tau)) \\ \quad -DF_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))] D_\xi v_{i-1}(\tau) d\tau \\ \quad + \sum_{i=1}^m S^-(t+m-i) \int_0^1 S^-(1-\tau)[DG_{R(\theta_{i-1}\omega)}(v_{i-1}(\tau)) \\ \quad -DG_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))] D_\xi v_{i-1}(\tau) d\theta_{i-1}\omega(\tau) \\ \quad + \int_0^t S^-(t-\tau)[DF_{R(\theta_m\omega)}(v_m(\tau)) - DF_{R(\theta_m\omega)}(u_m(\tau))] D_\xi v_m(\tau) d\tau \\ \quad + \int_0^t S^-(t-\tau)[DG_{R(\theta_m\omega)}(v_m(\tau)) - DG_{R(\theta_m\omega)}(u_m(\tau))] D_\xi v_m(\tau) d\theta_m\omega(\tau), & 1 \leq m < m_1', \\ 0, & m \geq m_1', \end{cases}$$

$$\mathcal{T}_2^-(t) = \begin{cases} 0, & 1 \leq m < m_1', \\ \sum_{i=1}^m S^-(t+m-i) \int_0^1 S^-(1-\tau)[DF_{R(\theta_{i-1}\omega)}(v_{i-1}(\tau)) \\ \quad -DF_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))] D_\xi v_{i-1}(\tau) d\tau \\ \quad + \sum_{i=1}^m S^-(t+m-i) \int_0^1 S^-(1-\tau)[DG_{R(\theta_{i-1}\omega)}(v_{i-1}(\tau)) \\ \quad -DG_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))] D_\xi v_{i-1}(\tau) d\theta_{i-1}\omega(\tau) \\ \quad + \int_0^t S^-(t-\tau)[DF_{R(\theta_m\omega)}(v_m(\tau)) - DF_{R(\theta_m\omega)}(u_m(\tau))] D_\xi v_m(\tau) d\tau \\ \quad + \int_0^t S^-(t-\tau)[DG_{R(\theta_m\omega)}(v_m(\tau)) - DG_{R(\theta_m\omega)}(u_m(\tau))] D_\xi v_m(\tau) d\theta_m\omega(\tau), & m \geq m_1', \end{cases}$$

$$\mathcal{T}_1^+(t) = \begin{cases} -\sum_{i=m+2}^{m_1'} S^+(t+m-i) \int_0^1 S^+(1-\tau)[DF_{R(\theta_{i-1}\omega)}(v_{i-1}(\tau)) \\ \quad -DF_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))] D_\xi v_{i-1}(\tau) d\tau \\ \quad - \sum_{i=m+2}^{m_1'} S^+(t+m-i) \int_0^1 S^+(1-\tau)[DG_{R(\theta_{i-1}\omega)}(v_{i-1}(\tau)) \\ \quad -DG_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))] D_\xi v_{i-1}(\tau) d\theta_{i-1}\omega(\tau) \\ \quad - \int_t^1 S^+(t-\tau)[DF_{R(\theta_m\omega)}(v_m(\tau)) - DF_{R(\theta_m\omega)}(u_m(\tau))] D_\xi v_m(\tau) d\tau \\ \quad - \int_t^1 S^+(t-\tau)[DG_{R(\theta_m\omega)}(v_m(\tau)) - DG_{R(\theta_m\omega)}(u_m(\tau))] D_\xi v_m(\tau) d\theta_m\omega(\tau), & 1 \leq m < m_1', \\ 0, & m \geq m_1', \end{cases}$$

$$\mathcal{T}_2^+(t) = \begin{cases} -\sum_{i=m_1'+2}^{\infty} S^+(t+m-i)\int_0^1 S^+(1-\tau)[DF_{R(\theta_{i-1}\omega)}(v_{i-1}(\tau)) \\ \quad -DF_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))]D_\xi v_{i-1}(\tau)d\tau \\ \quad -\sum_{i=m_1'+2}^{\infty} S^+(t+m-i)\int_0^1 S^+(1-\tau)[DG_{R(\theta_{i-1}\omega)}(v_{i-1}(\tau)) \\ \quad -DG_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))]D_\xi v_{i-1}(\tau)d\theta_{i-1}\omega(\tau) \\ \quad -\int_0^1 S^+(t+m-m_1'-\tau)[DF_{R(\theta_{m_1'}\omega)}(u_{m_1'}(\tau)) \\ \quad -DF_{R(\theta_{m_1'}\omega)}(v_{m_1'}(\tau))]D_\xi v_{m_1'}(\tau)d\tau \\ \quad -\int_0^1 S^+(t+m-m_1'-\tau)[DG_{R(\theta_{m_1'}\omega)}(v_{m_1'}(\tau)) \\ \quad -DG_{R(\theta_{m_1'}\omega)}(u_{m_1'}(\tau))]D_\xi v_{m_1'}d\theta_{m_1'}\omega(\tau), & 1 \le m < m_1', \\ -\sum_{i=m+2}^{\infty} S^+(t+m-i)\int_0^1 S^+(1-\tau)[DF_{R(\theta_{i-1}\omega)}(v_{i-1}(\tau)) \\ \quad -DF_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))]D_\xi v_{i-1}(\tau)d\tau \\ \quad -\sum_{i=m+2}^{\infty} S^+(t+m-i)\int_0^1 S^+(1-\tau)[DG_{R(\theta_{i-1}\omega)}(v_{i-1}(\tau)) \\ \quad -DG_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))]D_\xi v_{i-1}(\tau)d\theta_{i-1}\omega(\tau) \\ \quad -\int_t^1 S^+(t-\tau)[DF_{R(\theta_m\omega)}(v_m(\tau)) - DF_{R(\theta_m\omega)}(u_m(\tau))]D_\xi v_m(\tau)d\tau \\ \quad -\int_t^1 S^+(t-\tau)[DG_{R(\theta_m\omega)}(v_m(\tau)) - DG_{R(\theta_m\omega)}(u_m(\tau))]D_\xi v_m(\tau)d\theta_m\omega(\tau), & m \ge m_1'. \end{cases}$$

Lemma 3.12. *Assume that* $(\mathbf{A_2^k})$, $(\mathbf{A_2^k})$ *and* (3.17) *are satisfied. Then for* $\gamma \in [0, \gamma_0)$ *it holds*

$$\lim_{\xi \to \xi_0} \|\mathcal{T}\|_{\mathcal{H}_{\kappa+\gamma}} = 0.$$

Proof. We now take $\gamma < \gamma_0$, for $m \ge m_1$. Due to $(\mathbf{A_2^k}), (\mathbf{A_3^k})$ and by the definition of $L_F(R), L_G(R)$, we have

$$|[DF_{R(\theta_{i-1}\omega)}(v_{i-1}) - DF_{R(\theta_{i-1}\omega)}]D_\xi u_{i-1}| \le L_F(R(\theta_{i-1}\omega))|D_\xi u_{i-1}|, \tag{3.30a}$$

$$|[DF_{R(\theta_{i-1}\omega)}(v_{i-1}) - DF_{R(\theta_{i-1}\omega)}(u_{i-1})]D_\xi u_{i-1}|_{-\beta} \le L_F(R(\theta_{i-1}\omega))|D_\xi u_{i-1}|, \tag{3.30b}$$

$$\|[DG_{R(\theta_{i-1}\omega)}(v_{i-1}) - DG_{R(\theta_{i-1}\omega)}(u_{i-1})]D_\xi u_{i-1}\|_{L_2(\mathcal{B};\mathcal{B}_{-\beta})} \le L_G(R(\theta_{i-1}\omega))|D_\xi u_{i-1}|, \tag{3.30c}$$

$$\|[DG_{R(\theta_{i-1}\omega)}(v_{i-1}) - DG_{R(\theta_{i-1}\omega)}(u_{i-1})]D_\xi u_{i-1}\|_{L_2(\mathcal{B};\mathcal{B})} \le L_G(R(\theta_{i-1}\omega))|D_\xi u_{i-1}|, \tag{3.30d}$$

$$\begin{aligned} &\|[DG_{R(\theta_{i-1}\omega)}(v_{i-1}(r)) - DG_{R(\theta_{i-1}\omega)}(u_{i-1}(r))]D_\xi u_{i-1}(r) \\ &\quad - [DG_{R(\theta_{i-1}\omega)}(v_{i-1}(q)) - DG_{R(\theta_{i-1}\omega)}(u_{i-1}(q))]D_\xi u_{i-1}(q)\|_{L_2(\mathcal{B};\mathcal{B}_{-\beta})} \\ &\le L_G(R(\theta_{i-1}\omega))(r-q)^\beta \|D_\xi u_{i-1}\|_{\beta,-\beta}. \end{aligned} \tag{3.30e}$$

Hence we further obtain that

$$\begin{aligned} &\left|\int_0^1 S(1-\tau)[DF_{R(\theta_{i-1}\omega)}(v_{i-1}(\tau)) - DF_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))]D_\xi u_{i-1}(\tau)d\tau\right| \\ &\quad \le L_F(R(\theta_{i-1}\omega))\|D_\xi u_{i-1}(\tau)\|_{\beta,-\beta}, \end{aligned} \tag{3.31a}$$

$$\left\|\int_0^{\cdot} S(\cdot-\tau)[DF_{R(\theta_{i-1}\omega)}(v_{i-1}(\tau)) - DF_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))]D_\xi u_{i-1}(\tau)d\tau\right\|_{\beta,-\beta} \tag{3.31b}$$

$$\leq L_F(R(\theta_{i-1}\omega))\|D_\xi u_{i-1}\|_{\beta,-\beta},$$

$$\left\|\int_0^1 S(1-\tau)[DG_{R(\theta_{i-1}\omega)}(v_{i-1}(\tau)) - DG_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))]D_\xi u_{i-1}(\tau)d\theta_{i-1}\omega(\tau)\right\|_{L_2(\mathcal{B};\mathcal{B})} \tag{3.31c}$$

$$\leq C_{\alpha,\beta}L_G(R(\theta_{i-1}\omega))|||\theta_{i-1}\omega|||_{\beta',0,1}\|D_\xi u_{i-1}\|_{\beta,-\beta},$$

$$\left\|\int_0^{\cdot} S(\cdot-\tau)[DG_{R(\theta_{i-1}\omega)}(v_{i-1}(\tau)) - DG_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))]D_\xi u_{i-1}(\tau)d\theta_{i-1}\omega(\tau)\right\|_{\beta,-\beta} \tag{3.31d}$$

$$\leq C_{\alpha,\beta}L_G(R(\theta_{i-1}\omega))|||\theta_{i-1}\omega|||_{\beta',0,1}\|D_\xi u_{i-1}\|_{\beta,-\beta}.$$

By similar computation as for $\mathcal{J}_R$ and using (3.31a), (3.31b), (3.31c) and (3.31d), we derive

$$\|\mathcal{T}_2^-\|_{\mathcal{H}_{\kappa+\gamma}} \leq K\|D_\xi U\|_{\mathcal{H}_{\kappa+\gamma_0}} \sup_{m\geq m_1'}\sum_{i=0}^{m} e^{\check{\mu}(m-i)+(\kappa+\gamma)m-(\kappa+\gamma_0)i},$$

$$\|\mathcal{T}_2^+\|_{\mathcal{H}_{\kappa+\gamma}} \leq K\|D_\xi U\|_{\mathcal{H}_{\kappa+\gamma_0}} \sup_{m\geq m_1'}\sum_{i=m} e^{\hat{\mu}(m-i)+(\kappa+\gamma)m-(\kappa+\gamma_0)i},$$

and

$$\|\mathcal{T}_2^+\|_{\mathcal{H}_{\kappa+\gamma}} \leq K \sup_{m\leq m_1'}\sum_{i=m_1'} e^{\hat{\mu}(m-i)+(\kappa+\gamma)m-(\kappa+\gamma_0)i}\|D_\xi U\|_{\mathcal{H}_{\kappa+\gamma_0}}.$$

Then for any $\epsilon > 0$, there exists a large enough m_1 such that both of them are bounded by $\frac{\epsilon}{4}\|D_\xi U\|_{\mathcal{H}_{\kappa+\gamma_0}}$.

For other terms when $1 \leq m \leq m_1$, we use the continuity of DF, DG and $\Gamma(\omega,\xi)$ with respect to ξ. For any $\epsilon > 0$ we denote by

$$\mathcal{E}_1 := \max\left\{\frac{\epsilon}{4c_S \sup_{m\leq m_1'}\sum_{i=0}^{m} e^{\check{\mu}(m-i)+(\kappa+\gamma)m-(\kappa+\gamma_0)i}(1 + C_{\alpha,\beta}|||\theta_i\omega|||_{\beta',0,1})},\right.$$
$$\left.\frac{\epsilon}{4c_S \sup_{m\leq m_1'}\sum_{i=m}^{m_1'-1} e^{\hat{\mu}(m-i)+(\kappa+\gamma)m-(\kappa+\gamma_0)i}(1 + C_{\alpha,\beta}|||\theta_i\omega|||_{\beta',0,1})}\right\}.$$

Due to the continuity of the derivatives of F and G, there exists a constant $\epsilon_1 > 0$ such that when $|\xi - \xi_0| \leq \epsilon_1$, it holds

$$\sup_{i\leq m\leq m_1'} \max\left\{\|DF_{R(\theta_{i-1}\omega)}(v_{i-1}) - DF_{R(\theta_{i-1}\omega)}(u_{i-1})\|_{L(\mathcal{B};\mathcal{B})},\right.$$

$$\left.\begin{array}{l}\|DF_{R(\theta_{i-1}\omega)}(v_{i-1}) - DF_{R(\theta_{i-1}\omega)}(u_{i-1})\|_{L(\mathcal{B};\mathcal{B}_{-\beta})}\\ \left\|DG_{R(\theta_{i-1}\omega)}(v_{i-1}) - DG_{R(\theta_{i-1}\omega)}(u_{i-1})\right\|_{L(\mathcal{B};L_2(\mathcal{B};\mathcal{B}_{-\beta}))}\\ \left\|DG_{R(\theta_{i-1}\omega)}(v_{i-1}) - DG_{R(\theta_{i-1}\omega)}(u_{i-1})\right\|_{L(\mathcal{B};L_2(\mathcal{B};\mathcal{B}))}\end{array}\right\} \leq \mathcal{E}_1,$$

and

$$\begin{aligned}&\|[DG_{R(\theta_{i-1}\omega)}(v_{i-1}(r)) - DG_{R(\theta_{i-1}\omega)}(u_{i-1}(r))]D_\xi u_{i-1}(r)\\ &\quad - [DG_{R(\theta_{i-1}\omega)}(v_{i-1}(q)) - DG_{R(\theta_{i-1}\omega)}(u_{i-1}(q))]D_\xi u_{i-1}(q)\|_{L_2(\mathcal{B};\mathcal{B}_{-\beta})}\\ &\leq \mathcal{E}_1(r-q)^\beta \|D_\xi u_{i-1}\|_{\beta,-\beta}.\end{aligned}$$

Hence we obtain

$$\left|\int_0^1 S(1-\tau)[DF_{R(\theta_{i-1}\omega)}(v_{i-1}(\tau)) - DF_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))]D_\xi u_{i-1}(\tau)d\tau\right|,$$

and

$$\left\|\int_0^\cdot S(\cdot-\tau)[DF_{R(\theta_{i-1}\omega)}(v_{i-1}(\tau)) - DF_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))]D_\xi u_{i-1}(\tau)d\tau\right\|_{\beta,-\beta}$$

are bounded by $\mathcal{E}_1\|D_\xi u_{i-1}\|_{\beta,-\beta}$, and

$$\left\|\int_0^1 S(1-\tau)[DG_{R(\theta_{i-1}\omega)}(v_{i-1}(\tau)) - DG_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))]D_\xi u_{i-1}(\tau)d\theta_{i-1}\omega(\tau)\right\|_{L_2(\mathcal{B};\mathcal{B})},$$

and

$$\left\|\int_0^\cdot S(\cdot-\tau)[DG_{R(\theta_{i-1}\omega)}(v_{i-1}(\tau)) - DG_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))]D_\xi u_{i-1}(\tau)d\theta_{i-1}\omega(\tau)\right\|_{\beta,-\beta}$$

are bounded by $\mathcal{E}_1|||\theta_{i-1}\omega|||_{\beta',0,1}\|D_\xi u_{i-1}\|_{\beta,-\beta}$. Hence

$$\begin{aligned}&\|\mathcal{T}_1^-\|_{\mathcal{H}_{\kappa+\gamma}} \leq \mathcal{E}_1 c_S\|D_\xi U\|_{\mathcal{H}_{\kappa+\gamma_0}} \sup_{m\leq m_1'}\sum_{i=0}^m e^{\check{\mu}(m-i)+(\kappa+\gamma)m-(\kappa+\gamma_0)i}(1+C_{\alpha,\beta}|||\theta_i\omega|||_{\beta',0,1}),\\ &\|\mathcal{T}_1^+\|_{\mathcal{H}_{\kappa+\gamma}}\\ &\quad\leq \mathcal{E}_1 c_S\|D_\xi U\|_{\mathcal{H}_{\kappa+\gamma_0}} \sup_{m\leq m_1'}\sum_{i=m}^{m_1'-1} e^{\hat{\mu}(m-i)+(\kappa+\gamma)m-(\kappa+\gamma_0)(i-1)}(1+C_{\alpha,\beta}|||\theta_i\omega|||_{\beta',0,1})\end{aligned}$$

and all of them are bounded by $\frac{\epsilon}{2}\|D_\xi U\|_{\mathcal{H}_{\kappa+\gamma_0}}$. In conclusion, there exists a constant $m_1 > 0$ large enough and ϵ_1 (as above) such that for $|\xi - \xi_0| \leq \epsilon_1$, we have

$$\|\mathcal{T}\|_{\mathcal{H}_{\kappa+\gamma}} \leq \epsilon\|D_\xi U\|_{\mathcal{H}_{\kappa+\gamma_0}}.$$

For $\gamma \in [0, \gamma_0)$ this implies that $\mathcal{T} \to 0$ as $\xi \to \xi_0$. $\square$

3.3.2. C^k-smoothness

Now we prove that U is in $C^k(\mathcal{B}^-; \mathcal{H}_\kappa)$ for $k \geq 2$ by induction. Suppose that U is of C^j-class as an operator from $\mathcal{B}^-$ to $\mathcal{H}_{j\kappa+2\gamma*}$ for all $1 \leq j \leq k-1$, $\gamma* \in \big[0, \frac{\gamma_0}{2}\big)$. By a direct computation, we know that $D_\xi^{k-1} u_m$ satisfies

$$\begin{aligned}
&D_\xi^{k-1} u_m(t) \\
&\quad = \sum_{i=1}^{m} S^-(t+m-i) \int_0^1 S^-(1-\tau) DF_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau)) D_\xi^{k-1} u_{i-1}(\tau) d\tau \\
&\qquad + \sum_{i=1}^{m} S^-(t+m-i) \int_0^1 S^-(1-\tau) DG_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau)) D_\xi^{k-1} u_{i-1}(\tau) d\theta_{i-1}\omega(\tau) \\
&\qquad + \int_0^t S^-(t-\tau) DF_{R(\theta_m\omega)}(u_m(\tau)) D_\xi^{k-1} u_m(\tau) d\tau \\
&\qquad + \int_0^t S^-(t-\tau) DG_{R(\theta_m\omega)}(u_m(\tau)) D_\xi^{k-1} u_m(\tau) d\theta_m\omega(\tau) \\
&\qquad - \sum_{i=m+2} S^+(t+m-i) \int_0^1 S^+(1-\tau) DF_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau)) D_\xi^{k-1} u_{i-1}(\tau) d\tau \\
&\qquad - \sum_{i=m+2} S^+(t+m-i) \int_0^1 S^+(1-\tau) DG_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau)) D_\xi^{k-1} u_{i-1}(\tau) d\theta_{i-1}\omega(\tau) \\
&\qquad - \int_t^1 S^+(t-\tau) DF_{R(\theta_m\omega)}(u_m(\tau)) D_\xi^{k-1} u_m(\tau) d\tau \\
&\qquad - \int_t^1 S^+(t-\tau) DG_{R(\theta_m\omega)}(u_m(\tau)) D_\xi^{k-1} u_m(\tau) d\theta_m\omega(\tau) + [\mathcal{S}^{k-1}(\omega, \xi_0)]_m(t),
\end{aligned} \tag{3.32}$$

where

$$\begin{aligned}
&[\mathcal{S}^{k-1}(\omega, \xi_0)]_m(t) \\
&\quad = \sum_{i=1}^{m} S^-(t+m-i) \int_0^1 S^-(1-\tau) \mathcal{F}^{k-1}_{R(\theta_{i-1}\omega)}(u_{i-1})(\tau) d\tau
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{i=1}^{m} S^-(t+m-i) \int_0^1 S^-(1-\tau)\mathcal{G}^{k-1}_{R(\theta_{i-1}\omega)}(u_{i-1})(\tau)d\theta_{i-1}\omega(\tau) \\
&+ \int_0^t S^-(t-\tau)\mathcal{F}^{k-1}_{R(\theta_m\omega)}(u_m)(\tau)d\tau + \int_0^t S^-(t-\tau)\mathcal{G}^{k-1}_{R(\theta_m\omega)}(u_m)(\tau)d\theta_m\omega(\tau) \\
&- \sum_{i=m+2} S^+(t+m-i) \int_0^1 S^+(1-\tau)\mathcal{F}^{k-1}_{R(\theta_{i-1}\omega)}(u_{i-1})(\tau)d\tau \\
&- \sum_{i=m+2} S^+(t+m-i) \int_0^1 S^+(1-\tau)\mathcal{G}^{k-1}_{R(\theta_{i-1}\omega)}(u_{i-1})(\tau)d\theta_{i-1}\omega(\tau) \\
&- \int_t^1 S^+(t-\tau)\mathcal{F}^{k-1}_{R(\theta_m\omega)}(u_m)(\tau)d\tau - \int_t^1 S^+(t-\tau)\mathcal{G}^{k-1}_{R(\theta_m\omega)}(u_m)(\tau)d\theta_m\omega(\tau),
\end{aligned}
$$

with

$$
\mathcal{F}^{k-1}_{R(\theta_{i-1}\omega)}(u_{i-1})(\tau) = \sum_{l=0}^{k-3} \binom{k-2}{l} D_\xi^{k-2-l}(DF_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau)))D_\xi^{l+1}u_{i-1}(\tau),
$$

$$
\mathcal{G}^{k-1}_{R(\theta_{i-1}\omega)}(u_{i-1})(\tau) = \sum_{l=0}^{k-3} \binom{k-2}{l} D_\xi^{k-2-l}(DG_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau)))D_\xi^{l+1}u_{i-1}(\tau).
$$

Then

$$
D_\xi\mathcal{F}^{k-1}_{R(\theta_{i-1}\omega)}(u_{i-1})(\tau) = \sum_{l=0}^{k-3} \binom{k-2}{l} \Big[D_\xi^{k-1-l}(DF_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau)))D
$$

$$
\begin{aligned}
&= \sum_{l=0}^{k-3} \binom{k-2}{l} \Big\{ D_\xi^{k-2-l} \left(DF_{R(\theta_{i-1}\omega)}(v_{i-1}) - DF_{R(\theta_{i-1}\omega)}(u_{i-1}) \right) D_\xi^{l+1} v_{i-1} \\
&\quad + D_\xi^{k-2-l}(DF_{R(\theta_{i-1}\omega)}(u_{i-1})) \left[D_\xi^{l+1} v_{i-1} - D_\xi^{l+1} u_{i-1} \right] \\
&- \left[D_\xi^{k-1-l}(DF_{R(\theta_{i-1}\omega)}(u_{i-1})) D_\xi^{l+1} u_{i-1} + D_\xi^{k-2-l}(DF_{R(\theta_{i-1}\omega)}(u_{i-1})) D_\xi^{l+2} u_{i-1} \right] (\xi - \xi_0) \Big\} \\
&= \sum_{l=0}^{k-3} \binom{k-2}{l} \Big\{ D_\xi^{k-2-l} \left(D^2 F_{R(\theta_{i-1}\omega)}(w_{i-1}) D_\xi w_{i-1} (\xi - \xi_0) \right) D_\xi^{l+1} v_{i-1} \\
&\quad + D_\xi^{k-2-l}(DF_{R(\theta_{i-1}\omega)}(u_{i-1})) \left[D_\xi^{l+2} w_{i-1} (\xi - \xi_0) \right] \\
&- \left[D_\xi^{k-1-l}(DF_{R(\theta_{i-1}\omega)}(u_{i-1})) D_\xi^{l+1} u_{i-1} + D_\xi^{k-2-l}(DF_{R(\theta_{i-1}\omega)}(u_{i-1})) D_\xi^{l+2} u_{i-1} \right] (\xi - \xi_0) \Big\} \\
&= \sum_{l=0}^{k-3} \binom{k-2}{l} \Big\{ D_\xi^{k-2-l} \left(D^2 F_{R(\theta_{i-1}\omega)}(w_{i-1}) D_\xi w_{i-1} - D^2 F_{R(\theta_{i-1}\omega)}(u_{i-1}) D_\xi u_{i-1} \right) \\
&\quad \times (\xi - \xi_0) D_\xi^{l+1} v_{i-1} \\
&\quad + D_\xi^{k-2-l}(DF_{R(\theta_{i-1}\omega)}(u_{i-1})) \left[D_\xi^{l+2} w_{i-1} - D_\xi^{l+2} u_{i-1} \right] (\xi - \xi_0) \\
&\quad - D_\xi^{k-1-l}(DF_{R(\theta_{i-1}\omega)}(u_{i-1})) \left[D_\xi^{l+1} v_{i-1} - D_\xi^{l+1} u_{i-1} \right] (\xi - \xi_0) \Big\},
\end{aligned}
$$

where $w_{i-1} = r v_{i-1} + (1 - r) u_{i-1}$, $w_{i-1} = r\ v_{i-1} + (1 - r\ ) u_{i-1}$, and $w_{i-1} = r\ \ v_{i-1} + (1 - r\ \ ) u_{i-1}$ for some $r, r\ , r\ \ \in [0, 1]$. The case of G is similar, so we omit the details. We do not intend to write down the expressions for D_ξ^{k-2-l} but analyze its construction which is enough to estimate it. For example, applying the chain rule to $D_\xi^{k-2-l}(DF_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau)))$ (respectively, $D_\xi^{k-2-l}(DG_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))))$ we find that each term in $\mathcal{F}_{R(\theta_{i-1}\omega)}^{k-1}(u_{i-1})$ (respectively, $\mathcal{G}_{R(\theta_{i-1}\omega)}^{k-1}(u_{i-1})$) contains factors $D^{l_1} F_{R(\theta_{i-1}\omega)}(u_{i-1})$ (respectively, $D^{l_1} G_{R(\theta_{i-1}\omega)}(u_{i-1})$) for some $l_1 = 1, 2, \cdots, k-2$, and at least two derivatives D^{l_2}

$$\begin{aligned}
&\le C_{k,\gamma*} L_F(R(\theta_{i-1}\omega))|\xi - \xi_0| \Big(e^{-(k\kappa+2\gamma*)(i-1)} |v_{i-1}(\tau) - u_{i-1}(\tau)| \\
&\quad \vee e^{-[(k-1)\kappa+2\gamma*](i-1)} |D_\xi v_{i-1}(\tau) - D_\xi u_{i-1}(\tau)| \vee \\
&\quad \cdots \vee e^{-(\kappa+2\gamma*)(i-1)} |D_\xi^{k-1} v_{i-1}(\tau) - D_\xi^{k-1} u_{i-1}(\tau)| \Big),
\end{aligned} \tag{3.35}$$

and

$$\begin{aligned}
&\|\mathcal{G}^{k-1}_{R(\theta_{i-1}\omega)}(v_{i-1})(\tau) - \mathcal{G}^{k-1}_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau)) - D_\xi \mathcal{G}^{k-1}_{R(\theta_{i-1}\omega)}(v_{i-1})(\tau)\|_{L^{k-1}(\mathcal{B}^-;L_2(\mathcal{B};\mathcal{B}_{-\beta}))} \\
&\quad \le C_{k,\gamma*} L_G(R(\theta_{i-1}\omega))|\xi - \xi_0| \Big(e^{-[k\kappa+2\gamma*](i-1)} |v_{i-1}(\tau) - u_{i-1}(\tau)|_{-\beta} \\
&\quad \vee e^{-[(k-1)\kappa+2\gamma*](i-1)} |D_\xi v_{i-1}(\tau) - D_\xi u_{i-1}(\tau)|_{-\beta} \vee \\
&\quad \cdots \vee e^{-(\kappa+2\gamma*)(i-1)} |D_\xi^{k-1} v_{i-1}(\tau) - D_\xi^{k-1} u_{i-1}(\tau)|_{-\beta} \Big),
\end{aligned} \tag{3.36}$$

and

$$\begin{aligned}
&\|\mathcal{G}^{k-1}_{R(\theta_{i-1}\omega)}(v_{i-1})(r) - \mathcal{G}^{k-1}_{R(\theta_{i-1}\omega)}(u_{i-1}(r)) - D_\xi \mathcal{G}^{k-1}_{R(\theta_{i-1}\omega)}(v_{i-1})(r) \\
&\quad - \mathcal{G}^{k-1}_{R(\theta_{i-1}\omega)}(v_{i-1})(q) + \mathcal{G}^{k-1}_{R(\theta_{i-1}\omega)}(u_{i-1}(q)) + D_\xi \mathcal{G}^{k-1}_{R(\theta_{i-1}\omega)}(v_{i-1})(q)\|_{L^{k-1}(\mathcal{B}^-;L_2(\mathcal{B};\mathcal{B}_{-\beta}))} \\
&\quad \le C_{k,\gamma*} L_G(R(\theta_{i-1}\omega))(r-q)^\beta |\xi - \xi_0| \Big(e^{-[k\kappa+2\gamma*](i-1)} \|v_{i-1} - u_{i-1}\|_{\beta,-\beta} \\
&\quad \vee e^{-[(k-1)\kappa+2\gamma*](i-1)} \|D_\xi v_{i-1} - D_\xi u_{i-1}\|_{\beta,-\beta} \vee \\
&\quad \cdots \vee e^{-(\kappa+2\gamma*)(i-1)} \|D_\xi^{k-1} v_{i-1} - D_\xi^{k-1} u_{i-1}\|_{\beta,-\beta} \Big),
\end{aligned} \tag{3.37}$$

where $C_{k,\gamma*}$ only depends on $\|D_\xi^j U\|_{j\kappa+\gamma}$ for $i = 0, 1, 2, \cdots, k-1$ and $\gamma \in [0, 2\gamma*]$.

Recalling the definition of $\mathcal{L}$, we denote by

$$\begin{aligned}
&[\mathcal{I}(D_\xi^{k-1} V - D_\xi^{k-1} U)]_m(t) \\
&:= [D_\xi^{k-1} V - D_\xi^{k-1} U]_m(t) - [\mathcal{L}(D_\xi^{k-1} V - D_\xi^{k-1} U)]_m(t) \\
&= \sum_{i=1}^m S^-(t+m-i) \int_0^1 S^-(1-\tau) \Big[DF_{R(\theta_{i-1}\omega)}(v_{i-1}(\tau)) \\
&\qquad\qquad - DF_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau)) \Big] D_\xi^{k-1} v_{i-1}(\tau) d\tau \\
&\quad + \sum_{i=1}^m S^-(t+m-i) \int_0^1 S^-(1-\tau) \Big[DG_{R(\theta_{i-1}\omega)}(v_{i-1}(\tau))
\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad\qquad\qquad - DG_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))\Big] D_\xi^{k-1} v_{i-1}(\tau)\omega(\tau) \\
&+ \int_0^t S^-(t-\tau)[DF_{R(\theta_m\omega)}(v_m(\tau)) - DF_{R(\theta_m\omega)}(u_m(\tau))]D_\xi^{k-1}v_m(\tau)d\tau \\
&+ \int_0^t S^-(t-\tau)[DG_{R(\theta_m\omega)}(v_m(\tau)) - DG_{R(\theta_m\omega)}(u_m(\tau))]D_\xi^{k-1}v_m(\tau)d\theta_m\omega(\tau) \qquad (3.38)\\
&- \sum_{i=m+2} S^+(t+m-i)\int_0^1 S^+(1-\tau)\Big[DF_{R(\theta_{i-1}\omega)}(v_{i-1}(\tau)) \\
&\qquad\qquad\qquad\qquad - DF_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))\Big] D_\xi^{k-1} v_{i-1}(\tau)d\tau \\
&- \sum_{i=m+2} S^+(t+m-i)\int_0^1 S^+(1-\tau)\Big[DG_{R(\theta_{i-1}\omega)}(v_{i-1}(\tau)) \\
&\qquad\qquad\qquad\qquad - DG_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))\Big] D_\xi^{k-1} v_{i-1}(\tau)d\theta_{i-1}\omega(\tau) \\
&- \int_t^1 S^+(t-\tau)[DF_{R(\theta_m\omega)}(v_{i-1}(\tau)) - DF_{R(\theta_m\omega)}(u_m(\tau))]D_\xi^{k-1}v_m(\tau)d\tau \\
&- \int_t^1 S^+(t-\tau)[DG_{R(\theta_m\omega)}(v_m(\tau)) - DG_{R(\theta_m\omega)}(u_m(\tau))]D_\xi^{k-1}v_m(\tau)d\theta_m\omega(\tau) \\
&+ [\mathcal{S}^{k-1}(\omega,\xi)]_m(t) - [\mathcal{S}^{k-1}(\omega,\xi_0)]_m(t).
\end{aligned}
$$

To obtain the continuous di erentiability of $D_\xi^{k-1}u$, we decompose $\mathcal{I}$ as follows:

$$
\begin{aligned}
&[\mathcal{I}(D_\xi^{k-1}V - D_\xi^{k-1}U)]_m(t) \\
&\quad = [\mathcal{S}^{k-1}(\omega,\xi)]_m(t) - [\mathcal{S}^{k-1}(\omega,\xi_0)]_m(t) - D_\xi[\mathcal{S}^{k-1}(\omega,\xi_0)]_m(t)(\xi-\xi_0) \\
&\qquad + D_\xi[\mathcal{S}^{k-1}(\omega,\xi_0)]_m(t)(\xi-\xi_0) + \mathcal{I}^1(t) + \mathcal{I}^2(t) + \mathcal{I}^3(t) + \mathcal{I}^4(t)(\xi-\xi_0),
\end{aligned}
$$

and rewrite it as

$$
\begin{aligned}
&D_\xi^{k-1}v_m(t) - D_\xi^{k-1}u_m(t) - (id-\mathcal{L})^{-1}(D_\xi[\mathcal{S}^{k-1}(\omega,\xi_0)]_m(t)(\xi-\xi_0) + \mathcal{I}^4(\xi-\xi_0)) \\
&\quad = (id-\mathcal{L})^{-1}\{[\mathcal{S}^{k-1}(\omega,\xi)]_m(t) - [\mathcal{S}^{k-1}(\omega,\xi_0)]_m(t) - D_\xi[\mathcal{S}^{k-1}(\omega,\xi_0)]_m(t)(\xi-\xi_0) \\
&\qquad + \mathcal{I}^1(t) + \mathcal{I}^2(t) + \mathcal{I}^3(t)\},
\end{aligned} \qquad (3.39)
$$

where

$$
\begin{aligned}
\mathcal{I}^1(t) = &\sum_{i=1}^{m} S^-(t+m-i)\int_0^1 S^-(1-\tau)[DF_{R(\theta_{i-1}\omega)}(v_{i-1}(\tau)) - DF_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))\\
&- D^2F_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))(v_{i-1}(\tau)-u_{i-1}(\tau))]D_\xi^{k-1}u_{i-1}(\tau)d\tau\\
&+ \sum_{i=1}^{m} S^-(t+m-i)\int_0^1 S^-(1-\tau)[DG_{R(\theta_{i-1}\omega)}(v_{i-1}(\tau)) - DG_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))\\
&- D^2G_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))(v_{i-1}(\tau)-u_{i-1}(\tau))]D_\xi^{k-1}u_{i-1}(\tau)\omega(\tau)\\
&+ \int_0^t S^-(t-\tau)[DF_{R(\theta_m\omega)}(v_m(\tau)) - DF_{R(\theta_m\omega)}(u_m(\tau))\\
&- D^2F_{R(\theta_m\omega)}(u_m(\tau))(v_m(\tau)-u_m(\tau))]D_\xi^{k-1}u_m(\tau)d\tau\\
&+ \int_0^t S^-(t-\tau)[DG_{R(\theta_m\omega)}(v_m(\tau)) - DG_{R(\theta_m\omega)}(u_m(\tau))\\
&- D^2G_{R(\theta_m\omega)}(u_m(\tau))(v_m(\tau)-u_m(\tau))]D_\xi^{k-1}u_m(\tau)d\theta_m\omega(\tau)\\
&- \sum_{i=m+2} S^+(t+m-i)\int_0^1 S^+(1-\tau)[DF_{R(\theta_{i-1}\omega)}(v_{i-1}(\tau)) - DF_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))\\
&- D^2F_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))(v_{i-1}(\tau)-u_{i-1}(\tau))]D_\xi^{k-1}u_{i-1}(\tau)d\tau\\
&- \sum_{i=m+2} S^+(t+m-i)\int_0^1 S^+(1-\tau)[DG_{R(\theta_{i-1}\omega)}(v_{i-1}(\tau)) - DG_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))\\
&- D^2G_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))(v_{i-1}(\tau)-u_{i-1}(\tau))]D_\xi^{k-1}u_{i-1}(\tau)d\theta_{i-1}\omega(\tau)\\
&- \int_t^1 S^+(t-\tau)[DF_{R(\theta_m\omega)}(v_{i-1}(\tau)) - DF_{R(\theta_m\omega)}(u_m(\tau))\\
&- D^2F_{R(\theta_m\omega)}(u_m(\tau))(v_m(\tau)-u_m(\tau))]D_\xi^{k-1}u_m(\tau)d\tau\\
&- \int_t^1 S^+(t-\tau)[DG_{R(\theta_m\omega)}(v_m(\tau)) - DG_{R(\theta_m\omega)}(u_m(\tau))\\
&- D^2G_{R(\theta_m\omega)}(u_m(\tau))(v_m(\tau)-u_m(\tau))]D_\xi^{k-1}u_m(\tau)d\theta_m\omega(\tau),
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{I}^2(t) = &\sum_{i=1}^{m} S^-(t+m-i)\int_0^1 S^-(1-\tau)[D^2F_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))(v_{i-1}(\tau)-u_{i-1}(\tau))\\
&- D^2F_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))D_\xi u_{i-1}(\tau)(\xi-\xi_0)]D_\xi^{k-1}u_{i-1}(\tau)d\tau
\end{aligned}
$$

$$+\sum_{i=1}^{m} S^{-}(t+m-i)\int_{0}^{1} S^{-}(1-\tau)[D^2G_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))(v_{i-1}(\tau)-u_{i-1}(\tau))$$

$$-D^2G_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))D_\xi u_{i-1}(\tau)(\xi-\xi_0)]D_\xi^{k-1}u_{i-1}(\tau)\omega(\tau)$$

$$+\int_{0}^{t} S^{-}(t-\tau)[D^2F_{R(\theta_m\omega)}(u_m(\tau))(v_m(\tau)-u_m(\tau))$$

$$-D^2F_{R(\theta_m\omega)}(u_m(\tau))D_\xi u_m(\tau)(\xi-\xi_0)]D_\xi^{k-1}u_m(\tau)d\tau$$

$$+\int_{0}^{t} S^{-}(t-\tau)[D^2G_{R(\theta_m\omega)}(u_m(\tau))(v_m(\tau)-u_m(\tau))$$

$$-D^2G_{R(\theta_m\omega)}(u_m(\tau))D_\xi u_m(\tau)(\xi-\xi_0)]D_\xi^{k-1}u_m(\tau)d\theta_m\omega(\tau)$$

$$-\sum_{i=m+2} S^{+}(t+m-i)\int_{0}^{1} S^{+}(1-\tau)[D^2F_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))(v_{i-1}(\tau)-u_{i-1}(\tau))$$

$$-D^2F_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))D_\xi u_{i-1}(\tau)(\xi-\xi_0)]D_\xi^{k-1}u_{i-1}(\tau)d\tau$$

$$-\sum_{i=m+2} S^{+}(t+m-i)\int_{0}^{1} S^{+}(1-\tau)[D^2G_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))(v_{i-1}(\tau)-u_{i-1}(\tau))$$

$$-D^2G_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))D_\xi u_{i-1}(\tau)(\xi-\xi_0)]D_\xi^{k-1}u_{i-1}(\tau)d\theta_{i-1}\omega(\tau)$$

$$-\int_{t}^{1} S^{+}(t-\tau)[D^2F_{R(\theta_m\omega)}(u_m(\tau))(v_m(\tau)-u_m(\tau))$$

$$-D^2F_{R(\theta_m\omega)}(u_m(\tau))D_\xi u_m(\tau)(\xi-\xi_0)]D_\xi^{k-1}u_m(\tau)d\tau$$

$$-\int_{t}^{1} S^{+}(t-\tau)[D^2G_{R(\theta_m\omega)}(u_m(\tau))(v_m(\tau)-u_m(\tau))$$

$$-D^2G_{R(\theta_m\omega)}(u_m(\tau))D_\xi u_m(\tau)(\xi-\xi_0)]D_\xi^{k-1}u_m(\tau)d\theta_m\omega(\tau),$$

$\mathcal{I}^3(t)$

$$=\sum_{i=1}^{m} S^{-}(t+m-i)\int_{0}^{1} S^{-}(1-\tau)[DF_{R(\theta_{i-1}\omega)}(v_{i-1}(\tau))-DF_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))]$$

$$\times[D_\xi^{k-1}v_{i-1}(\tau)-D_\xi^{k-1}u_{i-1}(\tau)]d\tau$$

$$+\sum_{i=1}^{m} S^{-}(t+m-i)\int_{0}^{1} S^{-}(1-\tau)[DG_{R(\theta_{i-1}\omega)}(v_{i-1}(\tau))-DG_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))]$$

$$\times [D_\xi^{k-1} v_{i-1}(\tau) - D_\xi^{k-1} u_{i-1}(\tau)] d\theta_{i-1}\omega(\tau)$$

$$+ \int_0^t S^-(t-\tau)[DF_{R(\theta_m\omega)}(v_m(\tau)) - DF_{R(\theta_m\omega)}(u_m(\tau))][D_\xi^{k-1} v_m(\tau) - D_\xi^{k-1} u_m(\tau)] d\tau$$

$$+ \int_0^t S^-(t-\tau)[DG_{R(\theta_m\omega)}(v_m(\tau)) - DG_{R(\theta_m\omega)}(u_m(\tau))]$$

$$\times [D_\xi^{k-1} v_m(\tau) - D_\xi^{k-1} u_m(\tau)] d\theta_m\omega(\tau)$$

$$- \sum_{i=m+2} S^+(t+m-i) \int_0^1 S^+(1-\tau)[DF_{R(\theta_{i-1}\omega)}(v_{i-1}(\tau)) - DF_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))]$$

$$\times [D_\xi^{k-1} v_{i-1}(\tau) - D_\xi^{k-1} u_{i-1}(\tau)] d\tau$$

$$- \sum_{i=m+2} S^+(t+m-i) \int_0^1 S^+(1-\tau)[DG_{R(\theta_{i-1}\omega)}(v_{i-1}(\tau)) - DG_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))]$$

$$\times [D_\xi^{k-1} v_{i-1}(\tau) - D_\xi^{k-1} u_{i-1}(\tau)] d\theta_{i-1}\omega(\tau)$$

$$- \int_t^1 S^+(t-\tau)[DF_{R(\theta_m\omega)}(v_m(\tau)) - DF_{R(\theta_m\omega)}(u_m(\tau))][D_\xi^{k-1} v_m(\tau) - D_\xi^{k-1} u_m(\tau)] d\tau$$

$$- \int_t^1 S^+(t-\tau)[DG_{R(\theta_m\omega)}(v_m(\tau)) - DG_{R(\theta_m\omega)}(u_m(\tau))]$$

$$\times [D_\xi^{k-1} v_m(\tau) - D_\xi^{k-1} u_m(\tau)] d\theta_m\omega(\tau),$$

and

$$\begin{aligned}
&\mathcal{I}^4(t)(\xi-\xi_0) \\
&= \sum_{i=1}^m S^-(t+m-i) \\
&\quad \times \int_0^1 S^-(1-\tau) D^2 F_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau)) D_\xi u_{i-1}(\tau)(\xi-\xi_0) D_\xi^{k-1} u_{i-1}(\tau) d\tau \\
&\quad + \sum_{i=1}^m S^-(t+m-i) \int_0^1 S^-(1-\tau) D^2 G_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau)) D_\xi u_{i-1}(\tau) \\
&\quad \times (\xi-\xi_0) D_\xi^{k-1} u_{i-1}(\tau) d\theta_{i-1}\omega(\tau)
\end{aligned}$$

$$+ \int_0^t S^-(t-\tau) D^2 F_{R(\theta_m\omega)}(u_m(\tau)) D_\xi u_m(\tau)(\xi-\xi_0) D_\xi^{k-1} u_m(\tau) d\tau$$

$$+ \int_0^t S^-(t-\tau) D^2 G_{R(\theta_m\omega)}(u_m(\tau)) D_\xi u_m(\tau)(\xi-\xi_0) D_\xi^{k-1} u_m(\tau) d\theta_m\omega(\tau)$$

$$- \sum_{i=m+2} S^+(t+m-i)$$

$$\times \int_0^1 S^+(1-\tau) D^2 F_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau)) D_\xi u_{i-1}(\tau)(\xi-\xi_0) D_\xi^{k-1} u_{i-1}(\tau) d\tau$$

$$- \sum_{i=m+2} S^+(t+m-i) \int_0^1 S^+(1-\tau) D \ 1$$

$$
\begin{aligned}
&+ \int_0^t S^-(t-\tau)[\mathcal{F}^{k-1}_{R(\theta_m\omega)}(v_m)(\tau) - \mathcal{F}^{k-1}_{R(\theta_m\omega)}(u_m)(\tau) - D_\xi(\mathcal{F}^{k-1}_{R(\theta_m\omega)}(u_m))(\tau)(\xi-\xi_0)]d\tau \\
&+ \int_0^t S^-(t-\tau)[\mathcal{G}^{k-1}_{R(\theta_m\omega)}(v_m)(\tau) - \mathcal{G}^{k-1}_{R(\theta_m\omega)}(u_m)(\tau) \\
&\quad - D_\xi(\mathcal{G}^{k-1}_{R(\theta_m\omega)}(u_m))(\tau)(\xi-\xi_0)]d\theta_m\omega(\tau) \\
&- \sum_{i=m+2} S^+(t+m-i)\int_0^1 S^+(1-\tau)[\mathcal{F}^{k-1}_{R(\theta_{i-1}\omega)}(v_{i-1})(\tau) - \mathcal{F}^{k-1}_{R(\theta_{i-1}\omega)}(u_{i-1})(\tau) \\
&- D_\xi(\mathcal{F}^{k-1}_{R(\theta_{i-1}\omega)}(u_{i-1}))(\tau)(\xi-\xi_0)]d\tau \\
&- \sum_{i=m+2} S^+(t+m-i)\int_0^1 S^+(1-\tau)[\mathcal{G}^{k-1}_{R(\theta_{i-1}\omega)}(v_{i-1})(\tau) - \mathcal{G}^{k-1}_{R(\theta_{i-1}\omega)}(u_{i-1})(\tau) \\
&- D_\xi(\mathcal{G}^{k-1}_{R(\theta_{i-1}\omega)}(u_{i-1}))(\tau)(\xi-\xi_0)]d\theta_{i-1}\omega(\tau) \\
&- \int_t^1 S^+(t-\tau)[\mathcal{F}^{k-1}_{R(\theta_m\omega)}(v_m)(\tau) - \mathcal{F}^{k-1}_{R(\theta_m\omega)}(u_m)(\tau) - D_\xi(\mathcal{F}^{k-1}_{R(\theta_m\omega)}(u_m))(\tau)(\xi-\xi_0)]d\tau \\
&- \int_t^1 S^+(t-\tau)[\mathcal{G}^{k-1}_{R(\theta_m\omega)}(v_m)(\tau) - \mathcal{G}^{k-1}_{R(\theta_m\omega)}(u_m)(\tau) \\
&\quad - D_\xi(\mathcal{G}^{k-1}_{R(\theta_m\omega)}(u_m))(\tau)(\xi-\xi_0)]d\theta_m\omega(\tau).
\end{aligned}
$$

Based on the estimates of (3.35), (3.36), (3.37) and the continuous dependence on ξ of $D^l_\xi U$, for $l = 0, 1, \cdots, k-1$, we know that for any $\epsilon > 0$, there exists an $\epsilon_k > 0$ such that when $|\xi - \xi_0| \le \epsilon_k$, we have

$$
\max_{0\le l\le k-1} \sup_{i\in\mathbb{Z}^+} \|D^l_\xi v_i - D^l_\xi u_i\|_{\beta,-\beta} \le \mathcal{E}_k,
$$

where

$$
\frac{\epsilon}{\mathcal{E}_k} = KC_{k,\gamma*} \sup_{m\in\mathbb{Z}^+}\left[\sum_{i=0}^{m} e^{\check{\mu}(m-i)+(k\kappa+\gamma)m-(k\kappa+2\gamma*)i} + \sum_{i=m} e^{\hat{\mu}(m-i)+(k\kappa+\gamma)m-(k\kappa+2\gamma*)i}\right].
$$

Then

$$
\begin{aligned}
\sup_{m\in\mathbb{Z}^+} e^{m(k\kappa+\gamma)}\|\Delta\mathcal{S}(\xi,\xi_0)\|_{\beta,-\beta} \le KC_{k,\gamma*}\mathcal{E}_k|\xi-\xi_0| \sup_{m\in\mathbb{Z}^+}\Bigg[\sum_{i=0}^{m} e^{\check{\mu}(m-i)+(k\kappa+\gamma)m-(k\kappa+2\gamma*)i} \\
+ \sum_{i=m} e^{\hat{\mu}(m-i)+(k\kappa+\gamma)m-(k\kappa+2\gamma*)i}\Bigg]
\end{aligned}
$$

$$\leq \epsilon|\xi - \xi_0|.$$

Applying similar procedures to

$$
-\int_t^1 S^+(t-\tau)DF_{R(\theta_m\omega)}(u_m(\tau))D_\xi^k u_m(\tau)d\tau
$$
$$
-\int_t^1 S^+(t-\tau)DG_{R(\theta_m\omega)}(u_m(\tau))D_\xi^k u_m(\tau)d\theta_m\omega(\tau) + [\mathcal{S}^k(\omega,\xi_0)]_m(t).
$$

By $(\mathbf{A_2^k})$ and $(\mathbf{A_3^k})$ and (3.17), it could be checked that there exists a constant $M > 0$ such that

$$
\|\mathcal{S}^k(\omega,\xi_0)\|_{\mathcal{H}_{k\kappa+\kappa}} \le M.
$$

Hence regarding the estimate of $\mathcal{L}$, we have

$$
\|D_\xi^k U\|_{\mathcal{H}_{k\kappa+\kappa}} \le M + \frac{1}{2}\|D_\xi^k U\|_{\mathcal{H}_{k\kappa+\kappa}} \le 2M
$$

for any $\gamma \in [0, 2\gamma*]$. For the continuity of $D_\xi^k U$ with respect with ξ, recalling the definition of $\mathcal{L}$, we express the di erence of $D_\xi^k u$ and $D_\xi^k v$ as

$$
\begin{aligned}
D_\xi^k v_m(t) - D_\xi^k u_m(t) = {} & [\mathcal{L}(D_\xi^k V(t) - D_\xi^k U)]_m(t) + [\mathcal{T}^k]_m(t) + [\mathcal{S}^k(\omega,\xi)]_m(t) \\
& - [\mathcal{S}^k(\omega,\xi_0)]_m(t),
\end{aligned}
$$

where

$$
\begin{aligned}
&[\mathcal{T}^k]_m(t) \\
&= \sum_{i=1}^m S^-(t+m-i)\int_0^1 S^-(1-\tau)[DF_{R(\theta_{i-1}\omega)}(v_{i-1}(\tau)) \\
&\quad - DF_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))]D_\xi^k v_{i-1}(\tau)d\tau \\
&\quad + \sum_{i=1}^m S^-(t+m-i)\int_0^1 S^-(1-\tau)[DG_{R(\theta_{i-1}\omega)}(v_{i-1}(\tau)) \\
&\quad - DG_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))]D_\xi^k v_{i-1}(\tau)d\theta_{i-1}\omega(\tau) \\
&\quad + \int_0^t S^-(t-\tau)[DF_{R(\theta_m\omega)}(v_m(\tau)) - DF_{R(\theta_m\omega)}(u_m(\tau))]D_\xi^k v_m(\tau)d\tau \\
&\quad + \int_0^t S^-(t-\tau)[DG_{R(\theta_m\omega)}(v_m(\tau)) - DG_{R(\theta_m\omega)}(u_m(\tau))]D_\xi^k v_m(\tau)d\theta_m\omega(\tau) \quad (3.41)
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{i=m+2} S^{+}(t+m-i) \int_{0}^{1} S^{+}(1-\tau)[DF_{R(\theta_{i-1}\omega)}(v_{i-1}(\tau)) \\
&- DF_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))]D_{\xi}^{k} v_{i-1}(\tau)d\tau \\
&- \sum_{i=m+2} S^{+}(t+m-i) \int_{0}^{1} S^{+}(1-\tau)[DG_{R(\theta_{i-1}\omega)}(v_{i-1}(\tau)) \\
&- DG_{R(\theta_{i-1}\omega)}(u_{i-1}(\tau))]D_{\xi}^{k} v_{i-1}(\tau)d\theta_{i-1}\omega(\tau) \\
&- \int_{t}^{1} S^{+}(t-\tau)[DF_{R(\theta_{m}\omega)}(v_{m}(\tau)) - DF_{R(\theta_{m}\omega)}(u_{m}(\tau))]D_{\xi}^{k} v_{m}(\tau)d\tau \\
&- \int_{t}^{1} S^{+}(t-\tau)[DG_{R(\theta_{m}\omega)}(v_{m}(\tau)) - DG_{R(\theta_{m}\omega)}(u_{m}(\tau))]D_{\xi}^{k} v_{m}(\tau)d\theta_{m}\omega(\tau).
\end{aligned}
$$

Then similar to the estimate of $\mathcal{T}$, one can derive by a straightforward computation that $\|\mathcal{T}^k\|_{\mathcal{H}_{k\kappa+\gamma}} \to 0$ as $\xi \to \xi_0$ for $\gamma \in [0, \gamma*)$. Consequently, regarding the previous deliberations, the proof of the C^k-smoothness of the local stable manifold is completed.

Data a ailabilit

No data was used for the research described in the article.

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