behaviors, among which the projections of local attractor and repeller of the associated skew product semiflow have been extensively studied, see for instance Aragao-Costa, Caraballo, Carvalho & Langa [1], Caraballo, Jara, Langa & Liu [5], and Bortolan, Caraballo, Carvalho & Langa [3]. In particular, it was proved that a non-autonomous gradient-like evolution process possesses a Morse decomposition on the pullback attractor, and admits a Lyapunov function to describe a decaying energy level on the evolution of trajectories in [1]. As to the Morse decompositions of uniform attractors, the concept of lifted-invariance was provided to deal with the non-invariance of uniform attractor under non-autonomous systems [4].

In general, dynamical systems are often subject to random influences, such as uncertain parameters, internal agitation, external fluctuations and fluctuating initial conditions. One well-studied general framework for the study of systems driven by noise is the theory of random dynamical systems (RDS), see the monograph by Arnold [2] and the references listed therein. In this framework, many efforts have been done corresponding to Morse decomposition of random attractors, as for instance Crauel, Duc & Siegmund [10], Liu [15–17] in the case of autonomous systems. The Morse decomposition theory was further developed and the equivalence between the existence of Morse decomposition and the existence of Lyapunov function under strong conditions was presented in Liu, Ji & Su [18]. Afterward, Caraballo, Langa & Liu [7] studied the properties of the gradient infinite-dimensional random dynamical systems on pullback attractors. Recently, a strong Lyapunov function was constructed by Ju, Qi & Wang [13] to study the Morse decomposition and its stability.

The situation becomes more complicated when taking non-autonomous effect into consideration since there are two base flows: random base flow and non-autonomous base flow. To characterize the long-term dynamics, Wang [19,20] established some criteria for existence of pullback attractor for non-autonomous random dynamical systems (NRDS). Recently, Cui & Langa [12] introduced uniform random attractor to study the uniform asymptotic dynamics of NRDS with respect to the compact time-dependent symbol space. They provided the existence and uniqueness theorem of uniform random attractor, discussed the relationship with the cocycle attractor of NRDS and the pullback attractor of the corresponding skew product semiflow, and further showed that uniform random attractor uniformly forward attracts proper random sets in probability. This inspired us to study the forward dynamics of NRDS on uniform random attractor. It is noteworthy that uniform random attractors have non-invariance, which is completely different from the pullback attractors of random dynamical systems.

In this paper, we aim to investigate the Morse decomposition of uniform random attractors, together with the Lyapunov function and the stability of deterministic Morse decomposition under a small random disturbance. To this end, the first step is to define Morse sets according to local attractor-repeller pairs, inspired by Kloeden & Rasmussen [14]. However, it is impossible to define directly them in uniform random attractors by the so-called local attractor-repeller pairs because of the non-invariance of uniform random attractor. More precisely, the uniform repulsion can not be established since the NRDS is irreversible on such attractor. In order to avoid this embarrassing situation, we will define the Morse sets on uniform random attractor by means of the Morse sets on pullback attractor of the skew product semiflow, inspired by Bortolan, Carvalho & Langa [4]. In addition, we prove the existence of Morse decomposition of uniform random attractors and establish a measurable Lyapunov function on such attractor. Moreover, we construct a random Morse decomposition for disturbed dynamical system based on deterministic local attractors and prove random Morse sets converge to deterministic ones as the noise intensity tends to zero.

The remainder of this paper is organized as follows. We firstly recall some preliminary results from [2,17,12] in Section 2. We then define the Morse decomposition of random uniform attractor for a jointly continuous NRDS, obtain Lyapunov function for uniform random attractor and further present the stability of Morse decomposition under a small random disturbance in Section 3. An illustrated example is presented in Section 4 to justify the obtained result in Section 3. Finally, we discuss several related topics of Morse decomposition in random framework in Section 5.

2. Preliminaries

In this section, we fix the basic notations that we use in this paper and extract some basic definitions and results from [2,17,12].

2.1. Morse decomposition for random semiflow

Let (X, d_X) be a complete metric space with metric d_X, and $\mathcal{B}(X)$ the Borel σ-algebra of X. The Hausdorff semi-distance on X is given by

$$d_X(A, B) = \sup_{x\in A} \inf_{y\in B} d_X(x, y),\ A, B \subset X.$$

Denote $\mathcal{O}_\epsilon(A)$ by the ϵ-neighborhood of A, that is to say

$$\mathcal{O}_\epsilon(A) = \{x \in X; d_X(x, A) < \epsilon\}.$$

Definition 2.1. An RDS ϕ on the measurable space $(X, \mathcal{B}(X))$ over a metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \{\vartheta_t\}_{t\in\mathbb{R}^+})$ with time $\mathbb{R}^+$ is a mapping

$$\phi : \mathbb{R}^+ \times \Omega \times X \to X, \quad (t, \omega, x) \mapsto \phi(t, \omega, x),$$

with the following properties:

(i) $\phi(\cdot, \omega, x)$ is an identity operator on X.
(ii) $\phi(t, \vartheta_s\omega, \phi(s, \omega, x)) = \phi(t + s, \omega, x)$.
(iii) $(t, \omega, x) \mapsto \phi(t, \omega, x)$ is $(\mathcal{B}(\mathbb{R}^+), \mathcal{F}, \mathcal{B}(X))$-measurable.

Herein, we assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is a measure space, and $\mathbb{P}$ is ϑ_t-invariant for $t \in \mathbb{R}$ and ergodic under ϑ.

A set-valued mapping $\omega \mapsto D(\omega) \subset X, \omega \in \Omega$ is called a random closed (or compact) if $D(\omega), \omega \in \Omega$ is a closed (or compact) subset of X and for every $x \in X$, the function $\omega \mapsto d_X(x, D(\omega))$ is measurable. We call D open random set if its complement D^c is random closed.

Definition 2.2. A random set K is forward

Moreover, it is invariant if for any $t \geq 0$

$$\phi(t, \omega, K(\omega)) = K(\vartheta_t \omega), \quad \omega \in \Omega.$$

In order to describe the backward trends of semiflows in random invariant sets, we introduce the concepts of backward trajectory and complete trajectory for random semiflow, whose properties can be found in [17].

Definition 2.3. Let $\mathcal{M}$ be the set of all X-valued random variables and x a random variable in $\mathcal{M}$. A mapping $\xi : \mathbb{R}^- \to \mathcal{M}$ is called a backward trajectory of ϕ through random variable x if for all $\omega \in \Omega$, the following cocycle property holds:

$$\xi(0, \omega) = x(\omega), \xi(t+s, \omega) = \phi(s, \vartheta_t \omega)\xi(t, \omega), \quad t \leq 0, s \geq 0, t+s \leq 0.$$

ξ is called a complete trajectory of ϕ through x if for all $\omega \in \Omega$, the following cocycle property holds:

$$\xi(0, \omega) = x(\omega), \xi(t+s, \omega) = \phi(s, \vartheta_t \omega)\xi(t, \omega), \quad t \in \mathbb{R}, s \geq 0.$$

For a given complete trajectory ξ, denote Ω_ξ, Ω^*_ξ by the omega-limit set and the alpha-limit of ξ respectively, namely,

$$\Omega_\xi(\omega) = \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} \xi(t, \vartheta_{-t}\omega)}, \quad \Omega^*_\xi(\omega) = \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} \xi(-t, \vartheta_t \omega)}.$$

Now, let us recall the Morse decomposition of a compact random invariant set $S \subset X$ under random semiflow ϕ, which is useful to study the Morse decomposition of uniform random attractor of NRDS through the associated skew product semiflow. Let us start with the definitions of local attractor and repeller.

Definition 2.4. Assume that $A \subset S$ is a compact and invariant random set under ϕ. If there exists a froward invariant random open neighborhood U of A such that

$$\Omega_U(\omega) =: \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} \phi(t, \vartheta_{-t}\omega, U(\vartheta_{-t}\omega))} = A(\omega).$$

Then we call A a local attractor of ϕ. The basin of attraction of A is defined by

$$B(A)(\omega) = \{\chi \in S(\omega) : \phi(t, \omega)\chi \in U(\vartheta_t \omega), \quad \text{for some} \quad t \geq 0\}.$$

The repeller A^* corresponding to A is defined by

$$A^*(\omega) = S(\omega) \setminus B(A)(\omega), \quad \omega \in \Omega.$$

(A, A^*) is called an attractor-repeller pair.

Notice that $B(A)$ is an invariant random open set which is almost surely independent of the choice of U, see [15, Lemma 3.2].

Definition 2.5. Assume that $(A_i, A_i^*), i = 1, \cdots, n$ are attractor-repeller pairs of S with

$$\emptyset = A_0 \subsetneq A_1 \subsetneq \cdots \subsetneq A_{n-1} \subsetneq A_n = S,$$

and

$$\emptyset = A_n^* \subsetneq A_{n-1}^* \subsetneq \cdots \subsetneq A_1^* \subsetneq A_0^* = S.$$

Then the family $\{M_i\}_{i=1}^n$ of subsets of S, defined by

$$M_i(\omega) = A_i(\omega) \cap A_{i-1}^*(\omega), i = 1, 2, \cdots, n,$$

is called a Morse decomposition of S, and each M_i is called Morse set.

By the definition of Morse set, it is easy to check that $M_i, i = 1, \cdot, n$ are non-empty, invariant, pairwise disjoint random compact sets. Since S is a invariant random set in X, for any random variable $x \in S$, there exists a complete trajectory ξ through x lying on S, see [17, Corollary 4.2]. Throughout this paper, when we say a complete trajectory through x, we refer to the complete trajectory lying on S.

Definition 2.6. An invariant random set M is called isolated if there exists a random neighborhood U of M such that for each random variable x with the property that the complete trajectory $\xi(t, \omega), t \in \mathbb{R}, \omega \in \Omega$ through x remains in U almost surely, then the orbit always belongs to M almost surely, i.e. the inclusion

$$\xi(t, \omega) \in U(\vartheta_t \omega), t \in \mathbb{R}$$

implies $x(\omega) \in M(\omega)$ for almost all $\omega \in \Omega$.

Let $B(A^*)(\omega) := S \setminus B(A)(\omega), \omega \in \Omega$. Based on the fact that $\Omega_\xi \subset A$ if $x \in B(A)$ and $\Omega_\xi^* \subset A^*$ if $x \in B(A^*)$, see [17, Lemma 4.5], one can easily check that $M_i, i = 1, \cdots, n$ are isolated following analogous proof procedures with [10, Lemma 5.1].

Lemma 2.1. *[17, Theorem 4.1] Suppose that $\{M_i\}_{i=1}^n$ is a Morse decomposition of S, given by attractor-repeller pairs $(A_i, A_i^*), 1 \le i \le n$. Then the compact random set, given by*

(2) *If ξ is a complete trajectory through x satisfying that $\Omega_\xi \subset M_i$ almost surely and $\Omega^*_\xi \subset M_j$ almost surely for some $1 \le i, j \le n$, then $i \le j$. Moreover, $i = j$ if and only if ξ lies on M_i.*
(3) *If random variables $\xi_1, \cdots, \xi_p \in S$ satisfy that for some $1 \le j_0, \cdots, j_p \le n$, $\Omega_{\xi_k} \subset M_{j_{k-1}}$ and $\Omega^*_{\xi_k} \subset M_{j_k}$ for $k = 1, \cdots, n$, then $j_0 \le j_p$. Furthermore, $j_0 < j_p$ if and only if ξ_k does not lie on M with positive probability for some k, otherwise $j_0 = \cdots = j_p$.*

2.2. NRDS and uniform random attractors

Let (Σ, d_Σ) be a complete metric space. For simplicity, we assume the base space Σ is compact directly from now on. An operator θ on $\mathbb{R} \times \Sigma$ is called a base flow on Σ if it satisfies:

(1) θ_0 is an identity operator on Σ;
(2) $\theta_t(\theta_s\sigma) = \theta_{t+s}\sigma$ for any $t, s \in \mathbb{R}$;
(3) $(t, \sigma) \mapsto \theta_t\sigma$ is continuous;
(4) Σ is invariant under θ, i.e. $\theta_t(\Sigma) = \Sigma, t \in \mathbb{R}$.

Definition 2.7. A mapping $\Phi : \mathbb{R}^+ \times \Omega \times \Sigma \times X \to X$ is called a non-autonomous random dynamical system (NRDS) when

- Φ is $(\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{F} \otimes \mathcal{B}(\Sigma) \otimes \mathcal{B}(X); \mathcal{B}(X))$-measurable;
- $\Phi(0, \omega, \sigma, x)$ is an identity operator on X;
- Φ satisfies the cocycle property, i.e.

$$\Phi(t, \vartheta_s\omega, \theta_s\sigma, \Phi(s, \omega, \sigma, x)) = \Phi(t + s, \omega, \sigma, x), t, s \in \mathbb{R}^+.$$

Herein, we call $\{\theta_t : t \in \mathbb{R}\}$ and $\{\vartheta_t : t \in \mathbb{R}\}$ the driving systems of Φ.

Let $\mathcal{D}_X$ be a collection of random sets in X satisfying: (i) neighborhood-closed, i.e. for each $D \in \mathcal{D}_X$ there exists an $\epsilon > 0$ such that the closed ϵ-neighborhood $\mathcal{O}_\epsilon(D) \in \mathcal{D}_X$; (ii) inclusion-closed, i.e. if $D \in \mathcal{D}_X$, each random set smaller than D belongs to $\mathcal{D}_X$. An X-valued set $A = \{A_\sigma\}_{\sigma\in\Sigma}$ is said to be non-autonomous random set if A_σ is random for any $\sigma \in \Sigma$.

Definition 2.8. A random set $\mathcal{A} = \{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ is called the $\mathcal{D}_X$-uniform random attractor of NRDS Φ if $\mathcal{A} \in \mathcal{D}_X$ and it is the minimal compact uniformly $\mathcal{D}_X$-pullback attracting set in $\mathcal{D}_X$, that is

$$\lim_{t\to\infty} \sup_{\sigma\in\Sigma} d_X(\Phi(t, \vartheta_{-t}\omega, \theta_{-t}\sigma, D(\vartheta_{-t}\omega)), \mathcal{A}(\omega)) = 0, \omega \in \Omega, D \in \mathcal{D}_X.$$

Definition 2.9. A non-autonomous random set $A = \{A_\sigma\}_{\sigma\in\Sigma}$ is called a $\mathcal{D}_X$-cocycle attractor for an NRDS Φ if

(1) A is compact;
(2) A is $\mathcal{D}_X$-pullback attracting;
(3) A is invariant under Φ, that is

$$\Phi(t, \omega, \sigma, A_\sigma(\omega)) = A_{\theta_t\sigma}(\vartheta_t\omega), t \in \mathbb{R}, \omega \in \Omega, \sigma \in \Sigma.$$

(4) A is the minimal among all the closed non-autonomous random sets which are $\mathcal{D}_X$-pullback attracting.

Denote $\mathbb{X} = X \times \Sigma$ by the skew product space with the metric

$$d_{\mathbb{X}}(\chi_1, \chi_2) = d_X(x, y) + d_\Sigma(\sigma_x, \sigma_y),$$

where $\chi_1 = (x, \sigma_x)$ and $\chi_2 = (y, \sigma_y)$ are in $\mathbb{X}$. The associated skew product semiflow $\{\Pi(t, \omega) : t \in \mathbb{R}^+, \omega \in \Omega\}$ is given by

$$\Pi(t, \omega)(x, \sigma) = (\Phi(t, \omega, \sigma, x), \theta_t\sigma), \sigma \in \Sigma, x \in X.$$

Denote a universal $\mathcal{D}_{\mathbb{X}}$ by the *proper* random set if $\mathbb{B}$ is random and $P_\sigma\mathbb{B}(\omega) \neq \emptyset$, $P_X\mathbb{B} \in \mathcal{D}_X$, $\sigma \in \Sigma, \omega \in \Omega$, where $P_\sigma\mathbb{B}$ is the σ-section of $\mathbb{B}$ and $P_X\mathbb{B}$ is the union of all σ-sections of $\mathbb{B}$.

Definition 2.10. A proper random set $\mathbb{A}$ is called the $\mathcal{D}_{\mathbb{X}}$-attractor of skew product semiflow Π if

1. $\mathbb{A}$ is a compact set in $\mathbb{X}$;
2. $\mathbb{A}$ pullback attracts any proper set in $\mathcal{D}_{\mathbb{X}}$, that is

$$\lim_{t\to\infty} d_{\mathbb{X}}(\Pi(t, \vartheta_{-t}\omega, \mathbb{D}(\vartheta_{-t}\omega)), \mathbb{A}(\omega)) = 0, \mathbb{D} \in \mathcal{D}_{\mathbb{X}};$$

3. $\mathbb{A}$ is invariant under Π, that is

$$\Pi(t, \omega, \mathbb{A}(\omega)) = \mathbb{A}(\vartheta_t\omega), t \in \mathbb{R}^+.$$

Definition 2.11. [11, Definition 5.3] Given any $\sigma \in \Sigma$, a mapping $\eta : \mathbb{R} \times \Omega \to X$ is called a σ-driven complete trajectory of NRDS Φ if

$$\eta(t, \omega) = \Phi(t - s, \vartheta_s\omega, \theta_s\sigma, \eta(s, \omega)), t \geq s \in \mathbb{R}, \omega \in \Omega.$$

We say it is through random variable x when $\eta(0, \omega) = x(\omega)$, $\omega \in \Omega$. If there exists a random set $B \in \mathcal{D}_X$ such that $\cup_{t\in\mathbb{R}}\eta(t, \cdot) \subset B(\cdot)$, then η is called a $\mathcal{D}_X$-complete trajectory.

Lemma 2.2. *[12, Corollary 4.5, Theorem 4.15, and Proposition 4.19] Suppose Φ is a jointly continuous NRDS both in Σ and X. Then the uniform attractor $\mathcal{A}$ of the NRDS exists if and only if so does the $\mathcal{D}_{\mathbb{X}}$-random attractor $\mathbb{A}$ of Π. Moreover, If Φ has uniform attractor $\mathcal{A}$, then*

$$\mathcal{A}(\omega) = P_X\mathbb{A}(\omega) = \cup_{\sigma\in\Sigma}A_\sigma(\omega), \forall\omega \in \Omega,$$

and

$$\mathcal{A}(\vartheta_t\omega) = \{\eta(t, \omega) : \eta \text{ is a } \mathcal{D}_X\text{-complete trajectory of } \Phi, \omega \in \Omega\}, t \in \mathbb{R}.$$

For a given $\mathcal{D}_X$-complete trajectory η lying on $\mathcal{A}$, let $\xi(s,\cdot) = (\eta(s,\cdot), \theta_s\sigma)$ with $s \in \mathbb{R}, \sigma \in \Sigma$. Then for any $t \geq 0, \omega \in \Omega$,

$$\begin{aligned}\Pi(t, \vartheta_s\omega, \xi(s,\omega)) &= (\Phi(t, \vartheta_s\omega, \theta_s\sigma, \eta(s,\omega)), \theta_t\theta_s\sigma)\\ &= (\eta(t+s,\omega), \theta_{t+s}\sigma)\\ &= \xi(t+s,\omega),\end{aligned}$$

and $\xi(0,\omega) = (\eta(0,\omega), \sigma) = (x(\omega), \sigma), \omega \in \Omega$. Hence ξ is a complete $\mathcal{D}_{\mathbb{X}}$-complete trajectory of Π through (x,σ), naturally, lying on $\mathbb{A}$. Conversely, Let ξ be a complete trajectory of Π through $(x,\sigma) \in \mathbb{A}$ lying on $\mathbb{A}$, then $\eta = P_X\xi$ is the σ-driven complete trajectory through x lying on $\mathcal{A}$ since

$$\eta(t+s,\omega) = \Phi(t, \vartheta_s\omega, \theta_s\sigma, \eta(s,\omega)), \omega \in \Omega, t \geq 0, s \in \mathbb{R}.$$

3. Main results

3.1. Morse decomposition of uniform random attractor

Let $\mathcal{A} \in \mathcal{D}_X$ and $A = \{A_\sigma\}_{\sigma\in\Sigma}$ be the uniform attractor and $\mathcal{D}_X$-cocycle attractor of Φ respectively, and $\mathbb{A} \in \mathcal{D}_{\mathbb{X}}$ the $\mathcal{D}_{\mathbb{X}}$-random attractor of Π. Assume that the random compact invariant sets family $\{\mathbb{M}_i\}_{i=1}^n$ is a Morse decomposition of $\mathbb{A}$ with $P_X\mathbb{M}_i \cap P_X\mathbb{M}_j = \emptyset, i \neq j$. Herein we denote

$$\mathbb{M} = \bigcup_{i=1}^n \mathbb{M}_i, M = \bigcup_{i=1}^n M_i,$$

where

$$M_i := \{x \in \mathcal{A}; (x,\sigma) \in \mathbb{M}_i \text{ for some } \sigma \in \Sigma\}. \tag{1}$$

To cover the shortage of non-invariance of uniform random attractor, the concept of lifted-invariance is introduced as follows:

Definition 3.1. Let Φ be a jointly continuous NRDS in both Σ and X. We say that a random variable x is lifted if there exists a $\sigma \in \Sigma$ and a σ-driven complete trajectory η through x. Let

$$A := \{x;\ x \text{ is lifted random variable}\},$$

then A is called lifted-invariant random set if all complete trajectories through random variables in A under Φ remain in A. Moreover, if there exists a neighborhood U of A such that A is the maximal lifted-invariant random set in U, i.e. if there is a lifted random variable $x \in U$, the σ-driven complete trajectory η through x remains in U, then $x \in A$, we call A isolated lifted-invariant set.

Lemma 2.2 shows that $\mathcal{A}$ is a lifted-invariant random set.

Lemma 3.1. *Assume that* $\{\mathbb{M}_i\}_{i=1}^n$ *is a Morse decomposition of the* $\mathcal{D}_{\mathbb{X}}$*-random attractor* $\mathbb{A}$*. Then* $M_i, i = 1, \cdots, n$ *given in (1) are nonempty, mutually disjoint and isolated lifted-invariant random sets.*

Proof. First of all, the statement that $\{M_i\}_{i=1}^n$ is a nonempty mutually disjoint sets family can be easily checked since $\mathbb{M}_i \in \mathcal{D}_{\mathbb{X}}$ is nonempty and $P_X\mathbb{M}_i \cap P_X\mathbb{M}_i = \emptyset$ for any $i \neq j$ as supposed.

By means of the definition of M_i, for any given $x \in M_i$, there is a $\sigma \in \Sigma$ such that $(x, \sigma) \in \mathbb{M}_i$. Then according to the invariance of $\mathbb{M}_i$ under Π, there exists a unique complete trajectory ξ of Π through (x, σ) which remains in $\mathbb{M}_i$. Consequently, the mapping $\eta(t, \cdot) = P_X\xi(t, \cdot)$ is the σ-driven complete trajectory of Φ through x and remains in M_i. Hence M_i is lifted-invariant random set.

Last but not least, it is known that $\mathbb{M}_i$ is an isolated invariant set from Subsection 2.2. Then one can find a neighborhood of $\mathbb{M}_i$ denoting by $\mathbb{U}_i$ such that $\mathbb{M}_i$ is the biggest invariant set in $\mathbb{U}_i$. Then set

$$U_i = \{x \in X : (x, \sigma) \in \mathbb{U}_i, \quad \text{for some } \ \sigma \in \Sigma\}$$

is a neighborhood of M_i. Since for any $x \in U_i$, $\sigma \in \Sigma$ such that $(x, \sigma) \in \mathbb{U}_i$, the complete trajectory $\xi(t, \cdot) = (\eta(t, \cdot), \theta_t\sigma)$ through $(x(\cdot), \sigma)$ remains in $\mathbb{M}_i$, for any $t \in \mathbb{R}$, that is, the σ-driven complete trajectory η is in M_i all the time. Thus we can conclude that M_i is an isolated lifted-invariant random set. $\square$

Now, we introduce the Morse decomposition of the uniform random attractor for NRDS Φ, which is an extension from deterministic Morse decomposition of uniform attractor to random case, see [4, Definition 3.26].

Definition 3.2. Let Φ be a jointly continuous NRDS in both Σ and X, $\mathcal{A}$ the uniform random attractor of Φ and $\{M_i\}_{i=1}^n$ a mutually disjoint isolated lifted-invariant random subsets family of $\mathcal{A}$. We call $\{M_i\}_{i=1}^n$ a Morse decomposition of $\mathcal{A}$ if for any given random variable $x \in \mathcal{A}, \sigma \in \Sigma$ such that $(x, \sigma) \in \mathbb{A}$, the σ-driven complete trajectory η of Φ through x satisfies that:

(i) there is a sequence $\{t_n\}_{n\in\mathbb{Z}}$ such that for any $\epsilon > 0$,

$$\lim_{t_n\to\pm\infty} \mathbb{P}\{d_X(\eta(t_n, \omega), M(\vartheta_{t_n}\omega)) > \epsilon\} = 0;$$

(ii) if $(x, \sigma) \in \mathbb{M}_i$ for some $1 \le i \le n$, then $\eta(t, \cdot)$ remains in M_i; if $\Omega_\xi \subset \mathbb{M}_i$ and $\Omega_\xi^* \subset \mathbb{M}_j$, for $1 \le i, j \le n$, there is a sequence $\{t_n\}_{n\in\mathbb{Z}}$ such that for any $\epsilon > 0$,

$$\lim_{t_n\to\infty} \mathbb{P}\{d_X(\eta(t_n, \omega), M_i(\vartheta_{t_n}\omega)) > \epsilon\} = 0,$$

and

$$\lim_{t_n\to-\infty} \mathbb{P}\{d_X(\eta(t_n, \omega), M_j(\vartheta_{t_n}\omega)) > \epsilon\} = 0,$$

with $1 \le i \le j \le n$.

Theorem 3.1. *Let* $\{\mathbb{M}_i\}_{i=1}^n$ *be a Morse decomposition of the random attractor* $\mathbb{A}$ *of* Π

with $1 \le i < j \le n$. Hence we summarize that $\{M_i\}_{i=1}^n$ is a Morse decomposition of $\mathcal{A}$ according to Definition 3.2. $\square$

Remark 3.1. As all known that convergence in probability is weaker than pathwise convergence, the case of [18]. When we adopt pathwise convergence to define pullback local attractor and repeller of the associated random attractor of skew product semiflow, we always expect to study the pathwise dynamics. However, it may happen that the time on Ω is different from the time on Σ in general cases. Precisely, assume that $(x, \sigma) \in \mathbb{A} \in \mathcal{D}_{\mathbb{X}}$ is pullback attracted by $\mathbb{B} \in \mathcal{D}_{\mathbb{X}}$ under Π, then we have

$$\lim_{t\to\infty} d_{\mathbb{X}}((\Pi(t, \vartheta_{-t}\omega)(x(\vartheta_{-t}\omega), \sigma)), \mathbb{B}(\omega)) = 0, \ \omega \in \Omega,$$

where only ω-fiber is pulled back to $\vartheta_{-t}\omega$. This formulate does not indicate pathwise behavior of the σ-driven complete orbit through x, but it implies the attraction in probability. Hence, it allows us to define the local attractor and repeller in the sense of probability, rather than pathwise convergence.

We do find, however, that the pathwise convergence holds for some special situations. In this respect, an example will be provided in Section 4, where the Morse set M_1 is pullback attracting, similar to the cases in [15–18,5].

3.2. Lyapunov function on uniform random attractor

Definition 3.3. Assume that $\mathcal{A}$, $\mathbb{A}$ are the uniform random attractor of Φ and pullback attractor of Π respectively, and $\{M_i\}_{i=1}^n$ is a Morse decomposition of $\mathcal{A}$ given by the Morse decomposition $\{\mathbb{M}_i\}_{i=1}^n$ of $\mathbb{A}$. A function $V : \Omega \times \Sigma \times X \to [0, +\infty)$ is called a Lyapunov function of Φ on $\mathcal{A}$ if

(1) Given any variable $x \in \mathcal{A}$, $\sigma \in \Sigma$ such that $(x, \sigma) \in \mathbb{A}$, the map

$$t \mapsto V(\vartheta_t\omega, \theta_t\sigma, \Phi(t, \omega, \sigma, x(\omega))), t \ge 0$$

is non-increasing.

(2) For any $(x, \sigma) \in \mathbb{M}_i$, $V(\vartheta_t\omega, \theta_t\sigma, \Phi(t, \vartheta_t\omega, \theta_t\sigma, x(\omega)))$ is constant, $i = 1, \cdots, n$.

(3) If $V(\vartheta_t\omega, \theta_t\sigma, \Phi(t, \omega, \sigma, x(\omega))) = const, t \ge 0, \omega \in \Omega$, then $x \in M$ and $(x, \sigma) \in \mathbb{M}$.

Theorem 3.2. *Assume that $\mathcal{A}$, $\mathbb{A}$ are the uniform random attractor of Φ and pullback attractor of Π respectively, and $\{M_i\}_{i=1}^n$ is a Morse decomposition of $\mathcal{A}$ given by the Morse decomposition $\{\mathbb{M}_i\}_{i=1}^n$ of $\mathbb{A}$. Then Φ has a Lyapunov function of on $\mathcal{A}$.*

Proof. We separate this proof into three steps:

Step 1. For any $(x, \sigma) \in \mathbb{A}$, denote $\tau_i(\omega, \sigma, x(\omega))$ by the minimal positive time of $(x(\omega), \sigma)$ entering into the forward invariant neighborhood $\mathbb{U}_i(\omega)$ of $\mathbb{A}_i(\omega)$ under Π, that is

$$\tau_i(\omega, \sigma, x(\omega)) = \inf\{t \in \mathbb{R}^+ : \Pi(t, \omega, \sigma, x(\omega)) \in \mathbb{U}_i(\vartheta_t\omega)\}. \tag{2}$$

Hence it enjoys three basic properties:

(i) if $(x, \sigma) \in \mathbb{A}_i$, then $\tau_i(\omega, \sigma, x(\omega)) = 0$ with probability one;
(ii) if $(x, \sigma) \in \mathbb{A}_i^*$, then $\tau_i(\omega, \sigma, x(\omega)) = \infty$ with probability one;
(iii) if $(x, \sigma) \in \mathbb{A} \setminus (\mathbb{A}_i \cup \mathbb{A}_i^*)$, then $0 < \tau_i(\omega, \sigma, x(\omega)) < \infty$ with probability one.

In addition, τ_i is a measurable non-increasing function, which will be proved in Lemma 3.2 for non-interruption reading.

Step 2. We further define the function $F_i : (\Omega, \Sigma, X) \to \mathbb{R}^+$ by

F_i

$$F_i(\vartheta_t\omega, \theta_t\sigma, \Phi(t, \omega, \theta, x(\omega))) = \cdots = F_n(\vartheta_t\omega, \theta_t\sigma, \Phi(t, \omega, \theta, x(\omega))) = 0,$$

and

$$F_1(\vartheta_t\omega, \theta_t\sigma, \Phi(t, \omega, \theta, x(\omega))) = \cdots = F_{i-1}(\vartheta_t\omega, \theta_t\sigma, \Phi(t, \omega, \theta, x(\omega))) = 1.$$

Hence we further obtain

$$V(\vartheta_t\omega, \theta_t\sigma, \Phi(t, \omega, \theta, x(\omega))) = \sum_{i=1}^{n} F_i(\vartheta_t\omega, \theta_t\sigma, \Phi(t, \omega, \theta, x(\omega))) = i-1,$$

- If there is a constant c such that

$$V(\vartheta_t\omega, \theta_t\sigma, \Phi(t, \omega, \sigma, x(\omega))) = c, t \geq 0,$$

then $(x, \sigma) \in \mathbb{M}$.
Indeed, if $(x, \sigma) \notin \mathbb{A} \setminus \mathbb{M}$, then one could find the minimal $i \in \{1, \cdots, n\}$ such that $(x, \sigma) \notin \mathbb{A}_{i-1} \cup \mathbb{A}^*_{i-1}$. Hence

$$F_{i-1}(\vartheta_t\omega, \theta_t\sigma, \Phi(t, \omega, \sigma, x(\omega))) < F_{i-1}(\omega, \sigma, x(\omega))$$

with probability one, which implies that $V(\vartheta_t\omega, \theta_t\sigma, \Phi(t, \omega, \sigma, x(\omega)))$ is a decreasing function with respect to $t > 0$. That is a contradiction with our assumption. Hence $(x, \sigma) \in \mathbb{M}$.

In conclusion, V is a Lyapunov function on $\mathcal{A}$ of Φ. □

Lemma 3.2. *Provided with τ_i given in (2), it is a measurable non-increasing mapping along orbits of Φ.*

Proof. First, we consider the measurability of τ_i. Observe that $\mathbb{U}_i$ is a random open set, the mapping $\omega \mapsto d_{\mathbb{X}}((x, \sigma), \mathbb{U}^c{}_i(\omega))$ is measurable for any $(x, \sigma) \in \mathbb{X}$. Note that the mapping $(x, \sigma) \mapsto d_{\mathbb{X}}((x, \sigma), \mathbb{U}^c{}_i(\omega))$ is continuous, then $(\omega, \sigma, x) \mapsto d_{\mathbb{X}}((\sigma, x), \mathbb{U}^c{}_i(\omega))$ is $\mathcal{F} \otimes \mathcal{B}(\Sigma) \otimes \mathcal{B}(X)$-measurable, see [8, Lemma III.14]. Hence for any $t \geq 0$, the mapping

$$(\omega, \sigma, x) \mapsto d_{\mathbb{X}}((\Phi(t, \omega, \sigma, x), \theta_t\sigma), \mathbb{U}^c{}_i(\vartheta_t\omega))$$

is $\mathcal{F} \otimes \mathcal{B}(\Sigma) \otimes \mathcal{B}(X)$-measurable. As a result, for any given $a > 0$ the set

$$\begin{aligned}\{\omega : \tau_i(\omega, \sigma, x) > a\} &= \bigcap_{0 \leq t \leq a} \{\omega : d_{\mathbb{X}}((\Phi(t, \omega, \sigma, x), \theta_t\sigma), \mathbb{U}^c{}_i(\vartheta_t\omega)) = 0\} \\ &= \bigcap_{t \in \mathbb{Q}, 0 \leq t \leq a} \{\omega : d_{\mathbb{X}}((\Phi(t, \omega, \sigma, x), \theta_t\sigma), \mathbb{U}^c{}_i(\vartheta_t\omega)) = 0\}\end{aligned} \tag{4}$$

is measurable, where the last equality follows from the forward invariance of $\mathbb{U}_i$ under Π. This completes the proof of the measurability of $\tau_i(\omega, \sigma, x)$.

It remains to show τ_i is non-increasing. By the definition of τ_i, for any $t > 0$, $(x, \sigma) \notin \mathbb{A}_i \cup \mathbb{A}_i^*$, one has the inequality

$$\begin{aligned}
&\tau_i(\vartheta_t\omega, \theta_t\sigma, \Phi(t, \omega, \sigma, x(\omega))) \\
&= \inf\{s \in \mathbb{R}^+, (\Phi(t+s, \omega, \sigma, x(\omega)), \theta_{t+s}\sigma) \in \mathbb{U}_i(\vartheta_{t+s}\omega)\} \\
&= \begin{cases} 0, & t \geq \tau_i(\omega, \sigma, x(\omega)) \\ \tau_i(\omega, \sigma, x(\omega)) - t, & t < \tau_i(\omega, \sigma, x(\omega)) \end{cases} \\
&< \tau_i(\omega, \sigma, x(\omega)).
\end{aligned} \tag{5}$$

Then $0 \leq \tau_i(\vartheta_t\omega, \theta_t\sigma, \Phi(t, \omega, \sigma, x(\omega))) < \infty$. If $(x, \sigma) \in \mathbb{A}_i \cup \mathbb{A}_i^*$, then

$$\tau_i(\vartheta_t\omega, \theta_t\sigma, \Phi(t, \omega, \sigma, x(\omega))) = \tau_i(\omega, \sigma, x(\omega))$$

with probability one. Thus $\tau_i(\vartheta_t\omega, \theta_t\sigma, \Phi(t, \omega, \sigma, x(\omega))) \leq \tau_i(\omega, \sigma, x(\omega))$ with probability one, meaning that $\tau_i(\omega, \sigma, x(\omega))$ is not increasing under $t \in \mathbb{R}^+$. □

3.3. Stability of Morse decompositions

In this subsection, we focus on the approximation of Morse decomposition for an NRDS with a small random disturbance. Herein when the random disturbance vanishes, the system becomes a deterministic non-autonomous dynamical system.

Let Φ be a non-autonomous dynamical system on X with a driving system $\{\theta_t\}_{t\in\mathbb{R}}$ defined on a compact metric space Σ, and Φ^μ $(0 < \mu \ll 1)$ be a disturbed NRDS on the metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \{\vartheta_t\}_{t\in\mathbb{R}})$, where μ represents the intensity of random disturbance. Denote $\{\Pi(t) : t \in \mathbb{R}^+\}$ and $\{\Pi^\mu(t, \omega) : t \in \mathbb{R}^+, \omega \in \Omega\}$ by the skew product semiflow of Φ^μ and Φ respectively. Assume that the following conditions hold:

(A1) B is an arbitrary bounded subset of X,

$$\lim_{\mu\to 0} \sup_{\sigma\in\Sigma} d_X(\varphi^\mu(t, \vartheta_{-t}\omega, \theta_{-t}\sigma, x), \varphi(t, \theta_{-t}\sigma, x)) = 0, x \in B, t \in \mathbb{R}^+.$$

(A2) Π, Π^μ have global attractor $\mathbb{A}$ and pullback random attractor $\mathbb{A}^\mu \in \mathcal{D}_{\mathbb{X}}$ respectively. Moreover,

$$\lim_{\mu\to 0} d_{\mathbb{X}}(\mathbb{A}^\mu(\omega), \mathbb{A}) = 0, \omega \in \Omega.$$

Note that the speed of convergence is independent with both t and $x \in B$ in **(A1)**. As a result of **(A2)**, Φ and Φ^μ possess uniform attractor $\mathcal{A}$ and uniform random attractor $\mathcal{A}^\mu$ respectively.

Lemma 3.3. *Under these two assumptions **(A1)** and **(A2)**, the following three statements hold:*

(1) *For any given proper determinate bounded set $\mathbb{B} \subset X$,*

$$\lim_{\mu\to 0} d_{\mathbb{X}}(\Pi^\mu(t, \vartheta_{-t}\omega, \chi), \Pi(t, \chi)) = 0, \chi \in \mathbb{B}, t \in \mathbb{R}^+.$$

(2) *For any $\omega \in \Omega$, it holds*

$$\lim_{\mu \to 0} d_X(\mathcal{A}^\mu(\omega), \mathcal{A}) = 0.$$

Proof. **(1)** Given any bounded subset $\mathbb{B} \subset \mathbb{X}$, let $B = P_X \mathbb{B}$. Then it is a bounded subset of X. Choosing an element $\chi = (x, \sigma) \in \mathbb{B}$ arbitrarily, we can obtain from **(A1)** that

$$\begin{aligned} d_{\mathbb{X}}(\Pi^\mu(t, \vartheta_{-t}\omega, \chi), \Pi(t, \chi)) &= d_X(\varphi^\mu(t, \vartheta_{-t}\omega, \sigma, x), \varphi(t, \sigma)x) \\ &\le \sup_{\sigma \in \Sigma} d_X(\varphi^\mu(t, \vartheta_{-t}\omega, \theta_{-t}\sigma, x), \varphi(t, \theta_{-t}\sigma, x)), \end{aligned}$$

where the last term converges to 0 as $\omega \in \Omega$, $\mu \to 0$.

(2) Based on the result of **(A2)**, we complete this proof with contradiction. Assume there is an $\epsilon_0 > 0$, $x \in \mathcal{A}^\mu$ such that

$$d_X(x(\omega), \mathcal{A}) > \epsilon_0.$$

Note that there exists a $\sigma \in \Sigma$ such that $(x, \sigma) \in \mathbb{A}^\mu$, then

$$d_{\mathbb{X}}((x(\omega), \sigma), \mathbb{A}) \ge d_{\mathbb{X}}((x(\omega), \sigma), \mathcal{A} \times \Sigma) = d_X(x(\omega), \mathcal{A}) > \epsilon_0.$$

That is a contradiction with **(A2)**. So $\mathcal{A}^\mu(\omega)$ converges to $\mathcal{A}$ as $\mu \to 0$. □

Lemma 3.4. *[16, Lemma 3.1] Assume $U(\omega)$ is a random open set and an invariant compact random set $A(\omega) \subset U(\omega)$ satisfies that $\Omega_U(\omega) = A(\omega)$, then the open set*

$$\tilde{U}(\omega) := \mathrm{int}\overline{\bigcup_{t \ge 0} \Phi(t, \vartheta_{-t}\omega), U(\vartheta_{-t}\omega)}$$

is a forward invariant random set with the same properties as $U(\omega)$,

Theorem 3.3. *Assume that $\mathbb{A}_1 \subset \mathbb{A}$ is a local attractor of Π in the global attractor $\mathbb{A}$. Then the disturbed random dynamical system Π^μ possesses a local attractor $\mathbb{A}_1^\mu \subset \mathbb{A}^\mu$ converging to $\mathbb{A}_1$ as $\mu \to 0$.*

Proof. On the one hand, by the definition of local attractor, there exists an $\epsilon_1 > 0$ such that for any $0 \le \epsilon \le \epsilon_1$ such that $\mathbb{A}_1$ attracts $\mathcal{O}_{\epsilon_1}(\mathbb{A}_1)$. Then one can find a $T = T(\epsilon) > 0$ such that for any $t \ge T$,

$$d_{\mathbb{X}}(\Pi(t, \cup_{\sigma \in \Sigma}\{\sigma\} \times \mathcal{O}_{\epsilon_1}(P_\sigma \mathbb{A}_1)), \mathbb{A}_1) \le \frac{\epsilon}{4}.$$

On the other hand, Lemma 3.3 **(1)** shows that there exists a $\mu_1(\omega) > 0$ such that for $\mu \in [0, \mu_1(\omega))$

$$d_{\mathbb{X}}(\Pi^\mu(t, \vartheta_{-t}\omega, \chi), \Pi(t, \chi)) \le \frac{\epsilon}{4},$$

for any $\chi \in \cup_{\sigma\in\Sigma}\{\sigma\} \times \mathcal{O}_{\epsilon_1}(P_\sigma \mathbb{A}_1), t \geq 0$. Hence for $t \geq T$, $\mu \in [0, \mu_1(\omega))$, we have

$$\begin{aligned}
&d_{\mathbb{X}}(\Pi^\mu(t, \vartheta_{-t}\omega, \chi), \mathbb{A}_1)\\
&\quad\leq d_{\mathbb{X}}(\Pi^\mu(t, \vartheta_{-t}\omega, \chi), \Pi(t, \chi)) + d_{\mathbb{X}}(\Pi(t, \cup_{\sigma\in\Sigma}\{\sigma\} \times \mathcal{O}_{\epsilon_1}(P_\sigma \mathbb{A}_1)), \mathbb{A}_1)\\
&\quad\leq \frac{\epsilon}{4} + \frac{\epsilon}{4}\\
&\quad= \frac{\epsilon}{2},
\end{aligned}$$

that is

$$\begin{aligned}
&\bigcup_{\sigma\in\Sigma}\{\sigma\} \times \Phi^\mu(t, \vartheta_{-t}\omega, \theta_{-t}\sigma, \mathcal{O}_{\epsilon_1}(P_{\theta_{-t}\sigma}\mathbb{A}_1))\\
&\quad= \bigcup_{\sigma\in\Sigma} \Pi^\mu(t, \vartheta_{-t}\omega, \theta_{-t}\sigma \times \mathcal{O}_{\epsilon_1}(P_{\theta_{-t}\sigma}\mathbb{A}_1))\\
&\quad= \Pi^\mu(t, \vartheta_{-t}\omega, \bigcup_{\sigma\in\Sigma}\{\theta_{-t}\sigma\} \times \mathcal{O}_{\epsilon_1}(P_{\theta_{-t}\sigma}\mathbb{A}_1))\\
&\quad= \Pi^\mu(t, \vartheta_{-t}\omega, \bigcup_{\sigma'\in\Sigma}\{\sigma'\} \times \mathcal{O}_{\epsilon_1}(P_{\sigma'}\mathbb{A}_1))\\
&\quad\subset \mathcal{O}_{\frac{\epsilon}{2}}(\mathbb{A}_1).
\end{aligned} \tag{6}$$

Herein, the property $\theta_t\Sigma = \Sigma, t \in \mathbb{R}$ is taken into account in the last term.

In view of (6), we define a random set $\mathbb{A}_1^\mu$ given by

$$\mathbb{A}_1^\mu(\omega) := \cup_{\sigma\in\Sigma}\{\sigma\} \times \bigcap_{s\geq 0}\overline{\bigcup_{t\geq s}\Phi^\mu(t, \vartheta_{-t}\omega, \theta_{-t}\sigma, \mathcal{O}_{\epsilon_1}(P_{\theta_{-t}\sigma}\mathbb{A}_1))}, \tag{7}$$

in which $\cup_{\sigma\in\Sigma}\{\sigma\} \times \mathcal{O}_{\epsilon_1}(P_\sigma\mathbb{A}_1) \in \mathcal{D}_{\mathbb{X}}$ is pullback attracted by $\mathbb{A}^\mu$ under Π^μ since $\mathbb{A}^\mu$ is the $\mathcal{D}_{\mathbb{X}}$-pullback attractor of Π^μ. It is obvious that $\mathbb{A}_1^\mu$ is random compact and invariant subset of $\mathbb{A}^\mu$. Additionally, for any $\chi = (\sigma, x) \in \mathbb{A}_1^\mu(\omega)$, there exist a sequence $\{t_m\}_{m\in\mathbb{Z}^+} \to \infty$ as $m \to \infty$ and $\{x_m\}_{m\in\mathbb{Z}^+} \subset \mathcal{O}_{\epsilon_1}(P_{\theta_{-t_m}\sigma}\mathbb{A}_1)$ such that

$$x = \lim_{m\to\infty} \Phi^\mu(t_m, \vartheta_{-t_m}\omega, \theta_{-t_m}\sigma, x_m).$$

On the base of the triangle inequality, denoting $\chi_m = (x_m, \theta_{-t_m}\sigma)$, one has

$$\begin{aligned}
d_{\mathbb{X}}(\chi, \mathbb{A}_1) &= \lim_{m\to\infty} d_{\mathbb{X}}(\Pi^\mu(t_m, \vartheta_{-t_m}\omega, \chi_m), \mathbb{A}_1)\\
&\leq \lim_{m\to\infty} d_{\mathbb{X}}(\Pi^\mu(t_m, \vartheta_{-t_m}\omega, \chi_m), \Pi(t_m, \chi_m))\\
&\quad + \lim_{m\to\infty} d_{\mathbb{X}}(\Pi(t_m, \chi_m), \mathbb{A}_1).
\end{aligned}$$

Therefore, one further obtains that

$$\begin{aligned}
d_{\mathbb{X}}(\mathbb{A}_1^\mu(\omega), \mathbb{A}_1) = & \sup_{\chi \in \mathbb{A}_1^\mu(\omega)} d_{\mathbb{X}}(\chi, \mathbb{A}_1) \\
\leq & \sup_{\chi \in \mathbb{A}_1^\mu(\omega)} \lim_{m\to\infty} d_{\mathbb{X}}(\Pi^\mu(t_m, \vartheta_{-t_m}\omega, \chi_m), \Pi(t_m, \chi_m)) \\
& + \sup_{\chi \in \mathbb{A}_1^\mu(\omega)} \lim_{m\to\infty} d_{\mathbb{X}}(\Pi(t_m, \chi_m), \mathbb{A}_1) \\
\leq & \sup_{\chi \in \mathbb{A}_1^\mu(\omega)} \lim_{m\to\infty} d_{\mathbb{X}}(\Pi^\mu(t_m, \vartheta_{-t_m}\omega, \chi_m), \Pi(t_m, \chi_m)) + \frac{\epsilon}{4} \\
\leq & \sup_{\xi \in \mathcal{O}_\epsilon(\mathbb{A}_1)} d_{\mathbb{X}}(\Pi^\mu(t, \vartheta_{-t}\omega, \xi), \Pi(t, \xi)) + \frac{\epsilon}{4} \\
\leq & \frac{\epsilon}{2}
\end{aligned}$$

when $\mu \in [0, \mu_1(\omega))$, where the third inequality comes from

$$\begin{aligned}
d_{\mathbb{X}}(\Pi(t_m, \chi_m), \mathbb{A}_1) &\leq d_{\mathbb{X}}\left(\Pi\left(t_m, \bigcup_{\sigma' \in \Sigma} \{\sigma'\} \times \mathcal{O}_{\epsilon_1}(P_{\sigma'}\mathbb{A}_1)\right), \mathbb{A}_1\right) \\
&\leq d_{\mathbb{X}}(\Pi(t_m, \mathcal{O}_{\epsilon_1}(\mathbb{A}_1), \mathbb{A}_1) \\
&\leq \frac{\epsilon}{4}
\end{aligned}$$

for $t_m > T(\epsilon)$. Hence by the arbitrariness of $\epsilon > 0$, one has $\mathbb{A}_1^\mu(\omega)$ converges to $\mathbb{A}_1$ as $\mu \to 0$.

In view of Lemma 3.4, we define

$$\mathbb{U}_1^\mu(\omega) := \cup_{\sigma\in\Sigma}\{\sigma\} \times \mathrm{int}\overline{\bigcup_{t\geq 0} \Phi^\mu(t, \vartheta_{-t}\omega, \theta_{-t}\sigma, \mathcal{O}_{\epsilon_1}(P_{\theta_{-t}\sigma}\mathbb{A}_1))}, \ \omega \in \Omega. \tag{8}$$

One can easily check that $\mathbb{U}_1^\mu$ is a forward invariant neighborhood of $\mathbb{A}_1^\mu$ by the form of $\mathbb{U}_1^\mu$. Furthermore, The omega limit set of it is given by

$$\begin{aligned}
&\Omega_{\mathbb{U}_1^\mu(\omega)} \\
&= \cup_{\sigma\in\Sigma}\{\sigma\} \times \bigcap_{T\geq 0} \overline{\bigcup_{s\geq T} \Phi^\mu(s, \vartheta_{-s}\omega, \theta_{-s}\sigma, P_{\theta_{-s}\sigma}\mathbb{U}_1^\mu(\vartheta_{-s}\omega))} \\
&= \cup_{\sigma\in\Sigma}\{\sigma\} \times \bigcap_{T\geq 0} \overline{\bigcup_{s\geq T} \Phi^\mu(s, \vartheta_{-s}\omega, \theta_{-s}\sigma, \bigcup_{t\geq 0} \Phi^\mu(t, \vartheta_{-t-s}\omega), \theta_{-t-s}\sigma, P_{\theta_{-t-s}\sigma}\mathbb{A}_1)} \\
&= \cup_{\sigma\in\Sigma}\{\sigma\} \times \bigcap_{T\geq 0} \overline{\bigcup_{s+t\geq T} \Phi^\mu(s+t, \vartheta_{-s-t}\omega, \theta_{-t-s}\sigma, P_{\theta_{-t-s}\sigma}\mathbb{A}_1)} \\
&= \mathbb{A}_1^\mu(\omega).
\end{aligned} \tag{9}$$

Hence $\mathbb{A}_1^\mu$ is a local attractor of Π^μ in $\mathbb{A}^\mu$ with forward invariant neighborhood $\mathbb{U}_1^\mu$. □

Lemma 3.5. *Suppose that* $(\mathbb{A}_1, \mathbb{A}_1^*)$ *is a local attractor-repeller pair* for large $t \geq 0$. That is for any fixed t large enough, there exists a $\xi \in \mathcal{O}_\epsilon(\mathbb{A}_1) \subset \mathcal{O}_{\epsilon_1}(\mathbb{A}_1)$ such that

$$\Pi^\mu(t, \vartheta_{-t}\omega, \chi) = \xi.$$

By Lemma 3.3, for any $\epsilon > 0$ (independent with t), one has

$$d_{\mathbb{X}}(\Pi^\mu(t, \vartheta_{-t}\omega, \chi), \Pi(t, \chi)) \leq \epsilon, \omega \in \Omega.$$

Then

$$\begin{aligned} d_{\mathbb{X}}(\xi, \Pi(t, \chi) &\leq d_{\mathbb{X}}(\xi, \Pi^\mu(t, \vartheta_{-t}\omega, \chi)) + d_{\mathbb{X}}(\Pi^\mu(t, \vartheta_{-t}\omega, \chi), \Pi(t, \chi)) \\ &\leq \epsilon. \end{aligned}$$

Note that t is fixed but ϵ is arbitrary, hence one must have $\xi = \Pi(t, \chi)$. That is a contradiction with $B(\mathbb{A}_1)$ is invariant under Π since $\xi \in B(\mathbb{A}_1)$, while $\chi \in \mathbb{A}_2 \setminus B(\mathbb{A}_1)$. Thus χ can not be attracted by $\mathbb{A}_1^\mu$. Then we complete this proof. □

Theorem 3.4. *Let $\mathbb{M} = \{\mathbb{M}_i\}_{i=1}^n$ be a Morse decomposition of $\mathbb{A}$ induced by attractor-repeller pairs $(\mathbb{A}_i, \mathbb{A}_i^*)$, where*

$$\emptyset = \mathbb{A}_0 \subsetneq \mathbb{A}_1 \subsetneq \cdots \subsetneq \mathbb{A}_n = \mathbb{A}.$$

Then the random disturbed global attractor $\mathbb{A}^\mu$ possesses a Morse decomposition $\mathbb{M}^\mu = \{\mathbb{M}_i^\mu\}_{i=1}^n$ with

$$\lim_{\mu \to 0} d_{\mathbb{X}}(\mathbb{M}_i^\mu(\omega), \mathbb{M}_i) = 0, \omega \in \Omega, i = 1, \cdots, n.$$

Additionally, denoting $M_i = P_X \mathbb{M}_i, i = 1, \cdots, n$, and

$$M_i^\mu = \{x \text{ is a } X\text{-random variable} : (\sigma, x) \in \mathbb{M}_i \text{ for some } \sigma \in \Sigma\},$$

it holds that

$$\lim_{\mu \to 0} d(M_i^\mu(\omega), M_i) = 0, i = 1, \cdots, n.$$

Proof. For each $\mathbb{A}_i^\mu$, let $\mathbb{A}_i^{\mu,*} = \mathbb{A}^\mu \setminus B(\mathbb{A}_i^\mu)$, $\mathbb{M}_i^\mu = \mathbb{A}_i^\mu \cap \mathbb{A}_{i-1}^{\mu,*}, i = 1, \cdots, n$. By Definition 2.5, it holds that $\mathbb{M}^\mu = \left\{\mathbb{M}_i^\mu\right\}_{i=1}^n$ is a Morse decomposition of $\mathbb{A}^\mu$. Theorem 3.1 further proves that $M^\mu = \{M_i^\mu\}_{i=1}^n$ is a Morse decomposition of $\mathcal{A}^\mu$. So it remains to prove the approximation of M_i^μ.

Denote $\epsilon_0 = \min\{\epsilon_1, \cdots, \epsilon_n\}$, $\mu_0(\omega) = \min\{\mu_1(\omega), \mu_2(\omega), \cdots, \mu_n(\omega)\}$, in which $\epsilon_i, \mu_i(\omega)$, $i = 2, \cdots, n$ are defined similarly to ϵ_1 and $\mu_1(\omega)$ in Theorem 3.3. For any $0 < \epsilon < \epsilon_0$, denote $\mathbb{V}_i = \mathcal{O}_\epsilon(\mathbb{A}_i) \setminus \mathcal{O}_\epsilon(\mathbb{A}_{i-1}^*)$. One can easily check that $\mathbb{V}_i$ is a bounded subset of $B(\mathbb{A}_{i-1})$ and then is a subset of $B(\mathbb{A}_{i-1}^{\mu})$ for small enough μ because of Lemma 3.5. Considering $\mu < \mu_0(\omega)$, one has

$$\mathbb{M}_i^{\mu}(\omega) = \mathbb{A}_i^{\mu}(\omega) \setminus B(\mathbb{A}_{i-1}^{\mu})(\omega) \subset \mathcal{O}_\epsilon(\mathbb{A}_i) \setminus B(\mathbb{A}_{i-1}), \omega \in \Omega.$$

Since $\mathbb{V}_i = \mathcal{O}_\epsilon(\mathbb{A}_i) \setminus \mathcal{O}_\epsilon(\mathbb{A}_{i-1}^*) \subset B(\mathbb{A}_{i-1})$, one can further have

$$\begin{aligned} \mathcal{O}_\epsilon(\mathbb{A}_i) \setminus B(\mathbb{A}_{i-1}) &\subset \mathcal{O}_\epsilon(\mathbb{A}_i) \setminus (\mathcal{O}_\epsilon(\mathbb{A}_i) \setminus \mathcal{O}_\epsilon(\mathbb{A}_{i-1}^*)) \\ &= \mathcal{O}_\epsilon(\mathbb{A}_i) \cap \mathcal{O}_\epsilon(\mathbb{A}_{i-1}^*) \\ &\subset \mathcal{O}_\epsilon(\mathbb{A}_i \cap \mathbb{A}_{i-1}^*) \\ &= \mathcal{O}_\epsilon(\mathbb{M}_i), \end{aligned}$$

that is $\mathbb{M}_i^{\mu}(\omega) \subset \mathcal{O}_\epsilon(\mathbb{M}_i), \omega \in \Omega$. Thus

$$\lim_{\mu \to 0} d_{\mathbb{X}}(\mathbb{M}_i^{\mu}(\omega), \mathbb{M}_i) = 0, i = 1, \cdots, n.$$

Last but not least, since $\lim_{\mu \to 0} d_{\mathbb{X}}(\mathbb{M}_i^{\mu}(\omega), \mathbb{M}_i) = 0$, for any $x \in M_i^{\mu}$, there is a $\sigma \in \Sigma$ such that $(x, \sigma) \in \mathbb{M}_i^{\mu}$. Then it can be deduced that

$$\begin{aligned} d_X(x(\omega), M_i) = d_{\mathbb{X}}((x(\omega), \sigma), M_i \times \Sigma) &\le d_{\mathbb{X}}((x(\omega), \sigma), \mathbb{M}_i) \\ &\le d_{\mathbb{X}}(\mathbb{M}_i^{\mu}(\omega), \mathbb{M}_i), \end{aligned}$$

which converges to 0 as $\mu \to 0$. Hence we complete this proof. □

4. An example

Consider a non-autonomous stochastic system of Stratonovich type

$$\begin{cases} dU(t) = \big(U(t) - g(t)U^3(t)\big)\, dt + \varsigma U(t) \circ dW(t), \\ U(t_0) = U_0, \end{cases} \tag{12}$$

where $t \in \mathbb{R}^+$, $U_0 \in \mathbb{R}$, $\varsigma > 0$ and $W(t)$ is a standard one-dimensional Wiener process. Support that $g(t)$ is periodic function on $t \in [0, 2\pi]$ and there exist $0 < g_0 \le g_1 < \infty$ such that $g(t) \in [g_0, g_1]$. For example, one can take $g(t) = 2 + \cos(t)$.

To our purpose, we first construct a metric dynamical system following the standard procedure. In fact, let us denote $\Omega = C_0(\mathbb{R}, \mathbb{R})$ by the collection of continuous mappings from $\mathbb{R}$ to $\mathbb{R}$ with value zero at zero equipped with the compact-open topology, $\mathcal{F}$ the Borel-σ-algebra of Ω, $\mathbb{P}$ the Wiener measure and $\vartheta_t \omega(\cdot) = \omega(t + \cdot) - \omega(t)$ the Wiener shift. In this situation, it was shown in [2, Appendix A.3] that $\mathbb{P}$ is ergodic and invariant for ϑ_t and thus the metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \{\vartheta_t\}_{t \in \mathbb{R}})$ is ergodic. Let us also identify $W(\cdot, \omega)$ and $\omega(\cdot)$. Besides, we shall use the cohomology method based on the stationary Ornstein-Uhlenbeck process.

We next pause to present some nice properties of the mentioned process.

Lemma 4.1. *[6, Lemma 4.1]*

(1) *There exists a* $\{\vartheta_t\}_{t\in\mathbb{R}}$*-invariant set* $\tilde{\Omega}\subset\Omega$ *of full measure with sublinear growth:*

$$\lim_{t\to\pm\infty}\frac{|\omega(t)|}{|t|}=0 \quad \forall\omega\in\tilde{\Omega}.$$

(2) *For* $\omega\in\tilde{\Omega}$ *the random variable*

$$z(\omega)=-\int_{-\infty}^{0}e^{\tau}\omega(\tau)d\tau$$

exists and generates a unique stationary solution

$$z(\vartheta_t\omega)=-\int_{-\infty}^{0}e^{\tau}\vartheta_t\omega(\tau)d\tau=-\int_{-\infty}^{0}e^{\tau}\omega(\tau+t)d\tau+\omega(t)$$

solving

$$dz=-zdt+d\omega.$$

Moreover, the mapping $t\mapsto z(\vartheta_t\omega)$ *is continuous.*

(3) *On* $\tilde{\Omega}$*, it holds*

$$\lim_{t\to\pm\infty}\frac{|z(\vartheta_t\omega)|}{|t|}=0 \text{ and } \lim_{t\to\pm\infty}\frac{1}{t}\int_{0}^{t}z(\vartheta_\tau\omega)d\tau=0.$$

Now we are ready to focus on (12) and perform the change of variable

$$V(t)=U(t)e^{-\varsigma z(\vartheta_t\omega)}$$

resulting in the following random system

$$\begin{cases}\frac{dV(t)}{dt}=(1+\varsigma z(\vartheta_t\omega))\,V(t)-g(t)e^{2\varsigma z(\vartheta_t\omega)}V^3(t),\\ V(t_0)=V_0\end{cases}\tag{13}$$

with $V_0=U_0e^{-\varsigma z(\vartheta_{t_0}\omega)}$.

Let us denote the hull of g as

$$\Sigma:=\overline{\{g(t_0+\cdot),t_0\in[0,2\pi]\}},$$

define $\theta_t:\Sigma\to\Sigma$ by

$$\theta_t p(\cdot)=p(t+\cdot)\quad\text{for all}\quad p\in\Sigma,t\in\mathbb{R}.$$

Then Σ is a compact metric space due to the periodicity of g. Since (13) is Bernoulli differential equation, it can be solved analytically using an integrating factor. Thus by the solution to (13), one can define an NRDS ϕ as

$$\phi(t,\omega,p,V_0)=V(t+t_0,t_0,\vartheta_{-t_0}\omega,p,V_0)$$

on $\mathbb{R}^+\times\Omega\times\Sigma\times\mathbb{R}$ with

$$\begin{aligned}\frac{1}{\phi^2(t,\omega,p,V_0)}=&\frac{1}{V_0^2}e^{-2t-2\varsigma\int_0^t z(\vartheta_\tau\omega)d\tau}\\&+2\int_0^t p(s)e^{-2(t-s)+2\varsigma z(\vartheta_s\omega)-2\varsigma\int_s^t z(\vartheta_\tau\omega)d\tau}ds,\ t\geq 0.\end{aligned}$$

Replacing ω and p by $\vartheta_{-t}\omega$ and $\theta_{-t}p$, respectively, it yields

$$\begin{aligned}\frac{1}{\phi^2(t,\vartheta_{-t}\omega,\theta_{-t}p,V_0)}=&\frac{1}{V_0^2}e^{-2t-2\varsigma\int_{-t}^0 z(\vartheta_\tau\omega)d\tau}\\&+2\int_{-t}^0 p(s)e^{2s+2\varsigma z(\vartheta_s\omega)-2\varsigma\int_s^0 z(\vartheta_\tau\omega)d\tau}ds.\end{aligned}$$

By Lemma 4.1, we get

$$\lim_{t\to+\infty}\frac{1}{\phi^2(t,\vartheta_{-t}\omega,\theta_{-t}p,V_0)}=2\int_{-\infty}^0 p(s)e^{2s+2\varsigma z(\vartheta_s\omega)-2\varsigma\int_s^0 z(\vartheta_\tau\omega)d\tau}ds.$$

To construct a conjugated NRDS for (12), we define the homeomorphism $T:\Omega\times\mathbb{R}\to\mathbb{R}$ by

$$T(\omega,x)=e^{-\varsigma z(\omega)}x$$

whose inverse is given by

$$T^{-1}(\omega,x)=e^{\varsigma z(\omega)}x.$$

Then it follows from [6, Lemma 2.2] that the mapping

$$\varphi(t,\omega,p,U_0)=U(t+t_0,t_0,\vartheta_{-t_0}\omega,g,U_0)=T^{-1}(\vartheta_t\omega,\phi(t,\omega,p,T(\omega,U_0))),$$

$t\in\mathbb{R}^+$ defines an NRDS for the original equation (12). Denote by

$$\frac{1}{\xi}$$

$$-\infty$$

Note that $\pm\xi(\vartheta_t\omega, \theta_t p)$ are stable stationary solutions of (12) since

$$\frac{1}{\varphi^2(t, \omega, p, \xi(\omega, p))} = \frac{1}{\xi^2(\vartheta_t\omega, \theta_t p)},$$

and 0 is an unstable stationary solution of (12). Hence the non-autonomous random set $A = \{A_p\}_{p\in P}$ given by

$$A_p(\omega) = \{-\xi(\omega, p), 0, \xi(\omega, p)\}$$

is the cocycle attractor of system (12). As a consequence, the pullback attractor of the associated skew product semiflow of φ is given by

$$\mathbb{A}(\omega) = \bigcup_{p\in P}\{p\} \times \{-\xi(\omega, p), 0, \xi(\omega, p)\}$$

with

$$\mathbb{A}_0^*(\omega) = \mathbb{A}(\omega), \mathbb{A}_1^*(\omega) = \bigcup_{p\in P}\{p\} \times \{0\}, \mathbb{A}_2^*(\omega) = \emptyset,$$
$$\mathbb{A}_0(\omega) = \emptyset, \mathbb{A}_1(\omega) = \bigcup_{p\in P}\{p\} \times \{-\xi(\omega, p), \xi(\omega, p)\}, \mathbb{A}_2(\omega) = \mathbb{A}(\omega),$$

as the corresponding family of local repeller and local attractors. Herein, the forward invariant neighborhood of $\mathbb{A}_1$ is given by

$$\mathbb{U}_1(\omega) = \bigcup_{p\in P}\{p\} \times [-\xi(\omega, p), \xi(\omega, p)] \setminus \{0\}.$$

So, we get

$$\mathbb{M}_1(\omega) = \bigcup_{p\in P}\{p\} \times \{-\xi(\omega, p), \xi(\omega, p)\}, \mathbb{M}_2(\omega) = \bigcup_{p\in P}\{p\} \times \{0\}$$

as a Morse decomposition of the associated skew product semiflow. Furthermore, the uniform random attractor for (12) is given by

$$\mathcal{A}(\omega) = \bigcup_{p\in\Sigma} A_p(\omega) = \bigcup_{t_0\in[0,2\pi]} \{-\xi(\omega, \theta_{t_0}g), 0, \xi(\omega, \theta_{t_0}g)\}.$$

Thus we get

$$M_1(\omega) = \bigcup_{t_0\in[0,2\pi]} \left\{-\xi(\omega, \theta_{t_0}g), \xi(\omega, \theta_{t_0}g)\right\}, M_2(\omega) = \{0\}, \omega \in \Omega$$

as the desired Morse decomposition.

5. Discussion

We conclude this paper by discussing several existing and possible issues about the Morse decomposition in random setting. Notice that the Morse decomposition of uniform random attractors in flow case under the convergence in probability is straightforward and can be obtained by simply rereading the paper carefully and combining the results of [10]. In this respect, an alternative definition corresponding to Definition 3.2 is given as follows.

Definition 5.1. Let Φ be a jointly continuous NRDS in both Σ and X, $\mathcal{A}$ the uniform random attractor of Φ and $\{M_i\}_{i=1}^n$ a mutually disjoint isolated lifted-invariant random subsets family of $\mathcal{A}$. We call $\{M_i\}_{i=1}^n$ a Morse decomposition of $\mathcal{A}$ if for any given random variable $x \in \mathcal{A}$ and $\sigma \in \Sigma$ with $(x, \sigma) \in \mathbb{A}$, the σ-driven complete trajectory η of Φ through x satisfies that:

(i) for any $\epsilon > 0$, $\lim_{t\to\pm\infty} \mathbb{P}\{d_X(\eta(t, \vartheta_t\omega), M(\vartheta_t\omega)) > \epsilon\} = 0$;
(ii) if $(x, \sigma) \in \mathbb{M}_i$ for some $1 \le i \le n$, then $\eta(t, \cdot)$ remains in M_i; if for some $1 \le i, j \le n$, (x, σ) satisfies that

$$\lim_{t\to\infty} \mathbb{P}\{d_{\mathbb{X}}(\xi(t, \omega), \mathbb{M}_i(\vartheta_t\omega)) > \epsilon\} = 0,$$

and

$$\lim_{t\to-\infty} \mathbb{P}\{d_{\mathbb{X}}(\xi(t, \omega), \mathbb{M}_j(\vartheta_t\omega)) > \epsilon\} = 0,$$

for any $\epsilon > 0$, then

$$\lim_{t\to\infty} \mathbb{P}\{d_X(\eta(t, \omega), M_i(\vartheta_t\omega)) > \epsilon\} = 0,$$

and

$$\lim_{t\to-\infty} \mathbb{P}\{d_X(\eta(t, \omega), M_j(\vartheta_t\omega)) > \epsilon\} = 0,$$

with $1 \le i \le j \le n$.

Theorem 5.1. *Let $\{\mathbb{M}_i\}_{i=1}^n$ be a Morse decomposition of the random attractor $\mathbb{A}$ of skew product flow Π in probability. Then $\{M_i\}_{i=1}^n$ given in (1) is a Morse decomposition of the uniform random attractor $\mathcal{A}$.*

Proof. We can proceed it by an analogous argument to the proof of Theorem 3.1, so we omit it here. □

We now collect the related results on the Morse decomposition in random setting and distinguish them in eight situations, listed in Table 1.

Besides, the two cases of uniform random attractor in pathwise are still open since there is a major obstacle discussed in Remark 3.1. As for the case of random attractors for semiflows in probability, it is less obvious and merits further investigation to extend the results of [10] from the random flow case to the random semiflow case.

Table 1
Summary of Morse decomposition in random setting.

Convergence type	Random attractor		Uniform random attractor	
	Flow	Semiflow	Flow	Semiflow
Pathwise	[18, Theorem 5.1]	[17, Theorem 4.1]	Open	Open
Probability	[10, Theorem 5.2]	Open	Theorem 5.1	Theorem 3.1

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