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(C)

ABSTRACT. … C_0 …

1. INTRODUCTION

I … a … d … a a a … c a c - … a … a ab Ba ac ac $\mathcal{H}$

$$du(t) = (Au(t) + F(u(t)))dt + u(t) \circ dW(t), \tag{1.1}$$

A -d d d, … a a $F : \overline{D(A)} \to \mathcal{H}$ a ba L c L c c a L a d $F(0) = 0$, W a - d d a- a d W c a bab ac a d a d a S a c c a c d a . S c a c a … a d a a a c a a d a a a (SPDE), d d a a d d a ca (RDS) c c c a a c d d a . H , c c a RDS a a a d c b ca c a a d d a d [9]. T a d d d c ca d a c a d d a ca b a b a - (a) a ac . O b a d c a c a a b ac a d a a a d a c a , c a ac a B a a a 1/2. A … a c c a ab a d a d a d a ac a b ab d a c a SPDE a a H d c d c [10, 11]. H , a ac d c … ca S … a ca a b a c d.

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2. Preliminaries

2.1. [illegible]. I[illegible] [2]. L[illegible] $(\Omega, \mathcal{F}, \mathbb{P})$ b[illegible] a[illegible] $\mathcal{H}$ [illegible] Ba[illegible] $\|\cdot\|$ [illegible]. W[illegible] $\mathcal{B}(\mathbb{R})$, $\mathcal{B}(\mathbb{R}^+)$ a[illegible]d $\mathcal{B}(\mathcal{H})$ [illegible] B[illegible] $\mathbb{R}$, $\mathbb{R}^+$ a[illegible]d $\mathcal{H}$, [illegible].

D[illegible] 2.1. T[illegible] $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$ [illegible]:

(i) [illegible] $\theta : \mathbb{R} \times \Omega \to \Omega$ [illegible] $(\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}, \mathcal{F})$-[illegible];
(ii) $\theta_0 = \mathrm{id}_\Omega$, [illegible] Ω, a[illegible]d $\theta_{t+s} = \theta_t \circ \theta_s$, [illegible] $s, t \in \mathbb{R}$;
(iii) $\theta_t\mathbb{P} = \mathbb{P}$ [illegible] $t \in \mathbb{R}$.

Remark 2.2. T[illegible] $\Omega = C_0(\mathbb{R}, \mathbb{R})$ [illegible] $\mathbb{R}$ [illegible] $\mathbb{R}$ [illegible], $\mathcal{F}$ [illegible] B[illegible]-σ-a[illegible] Ω, $\mathbb{P}$ [illegible] W[illegible] $\theta_t\omega(\cdot) = \omega(t + \cdot) - \omega(t)$ [illegible] W[illegible]. I[illegible] [2] [illegible] $\mathbb{P}$ [illegible] θ_t a[illegible]d [illegible] $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$ [illegible].

D[illegible] 2.3. A [illegible] $\phi(t, \omega, x) : \mathbb{R}^+ \times \Omega \times \mathcal{H} \to \mathcal{H}$ [illegible] (RDS) [illegible] $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$ [illegible]

(i) ϕ [illegible] $(\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{F} \otimes \mathcal{B}(\mathcal{H}), \mathcal{B}(\mathcal{H}))$-[illegible];
(ii) [illegible] $\phi(t, \omega) := \phi(t, \omega, \cdot) : \mathcal{H} \to \mathcal{H}$ [illegible] θ_t:

$$\phi(0, \omega) = \mathrm{id}_{\mathcal{H}}, \quad \forall \omega \in \Omega,$$

$$\phi(t + s, \omega) = \phi(t, \theta_s\omega) \circ \phi(s, \omega), \quad t, s \in \mathbb{R}^+ \text{ and } \forall \omega \in \Omega.$$

2.2. **I[illegible].** T[illegible] Ca[illegible] A[illegible]d [1] [illegible] [1, 18, 22].

A[illegible] $(\mathcal{H}, \|\cdot\|)$ [illegible] Ba[illegible] $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ b[illegible] $\rho(A) \neq 0$ [illegible] $\overline{D(A)} \neq \mathcal{H}$. W[illegible] $\overline{D(A)} = \mathcal{H}_0$ a[illegible]d [illegible] A [illegible] $\mathcal{H}_0$ a[illegible]

$$A_0 x \doteq Ax \quad \text{a[illegible] } x \in D(A_0) \doteq \{y \in D(A) : Ay \in \mathcal{H}_0\}.$$

D[illegible] 2.4. A [illegible] $\{S(t)\}_{t\geq 0}$ [illegible] $\mathcal{H}$ [illegible]:

(i) $S(0) = 0$;
(ii) [illegible] $t \mapsto S(t)x$ [illegible] $[0, \infty)$ [illegible] $x \in \mathcal{H}$;
(iii) [illegible]

$$S(t)S(s) = \int_0^t (S(\tau + s) - S(\tau))d\tau, \qquad \text{a[illegible] } t, s \geq 0.$$

I[illegible] $S(t)S(s) = S(s)S(t)$ [illegible] $t, s \geq 0$ [illegible]

$$S(t)S(s) = \left(\int_0^{t+s} - \int_0^t - \int_0^s\right) S(\tau)d\tau.$$

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[illegible] $\mathcal{H}_0$, [illegible] $\omega_A \in \mathbb{R}$ [illegible] $M \geq 1$, [illegible] $(\omega_A, \infty) \subset \rho(A)$ [illegible]

$$\left\|(\lambda I - A)^{-k}\right\|_{\mathcal{L}(\mathcal{H}_0)} \leq \frac{M}{(\lambda - \omega_A)^k} \qquad \text{[illegible] } \lambda > \omega_A \text{ [illegible] } k \geq 1. \tag{2.1}$$

[illegible] A [illegible]

$$\lim_{\lambda \to \infty} (\lambda I - A)^{-1} x = 0, \qquad \text{[illegible] } x \in \mathcal{H}. \tag{2.2}$$

[illegible] [22], [illegible] (2.1) [illegible] (2.2) [illegible] A [illegible] $S(t)$ [illegible] $\mathcal{H}$ [illegible] A_0 [illegible] $T(t)$ [illegible] $\mathcal{H}_0$. [illegible] $T(t)$ [illegible] A [illegible] A_0:

$$\sigma(A) = \sigma(A_0), \tag{2.3}$$

[illegible] [18, [illegible] 2.1].

[illegible] $\mathcal{H}_0$; [illegible] $\mathcal{H}$ [illegible] [18, [illegible] 3.5] [illegible] $\mathcal{H}_0$ [illegible] $\mathcal{H}$.

L[illegible] 2.5. *Let $P_0 : \mathcal{H}_0 \to \mathcal{H}_0$ be a bounded linear projection, such that*

$$P_0(\lambda I - A_0)^{-1} = (\lambda I - A_0)^{-1} P_0 \quad \text{for } \lambda > \omega_A$$

and

$$P_0 \mathcal{H}_0 \subset D(A_0) \quad \text{with } A_0|_{P_0\mathcal{H}_0} \text{ bounded.}$$

Then there exists a unique bounded projection $P : \mathcal{H} \to \mathcal{H}$ satisfying

(i) $P|_{\mathcal{H}_0} = P_0$;
(ii) $P(\mathcal{H}) \subset \mathcal{H}_0$;
(iii) $P(\lambda I - A)^{-1} = (\lambda I - A)^{-1} P$ *for all* $\lambda > \omega_A$.

[illegible] 2.6.

D[illegible] 2.6. [illegible] $(S(t))_{t \geq 0}$ [illegible] $[0, t]$ [illegible]

$$V^\infty(S, 0, t) = \text{[illegible]} \left\{ \left| \sum_{i=1}^n \left(S(t_i) - S(t_{i-1})\right) x_i \right| \right\} < \infty,$$

[illegible] $0 = t_0 < \cdots < t_n = t$ [illegible] $x_i \in \mathcal{H}$ [illegible] $\|x_i\| = 1$ [illegible] $i \in \{1, \ldots, n\}$.

2.3. **C[illegible]** [illegible]

$$dz = -z dt + d\omega, \tag{2.4}$$

[illegible] [7, [illegible] 2.1] [illegible]

L[illegible] 2.7.

(1) *There exists a $(\theta_t)_{t \in \mathbb{R}}$-invariant set $\Omega \in \mathcal{B}(C_0(\mathbb{R}, \mathbb{R}))$ of full measure with sublinear growth:*

$$\lim_{t \to \pm\infty} \frac{|\omega(t)|}{|t|} = 0, \quad \forall \omega \in \Omega.$$

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(2) *For* $\omega \in \Omega$ *the random variable*

$$z(\omega) = -\int_{-\infty}^{0} e^{\tau}\omega(\tau)d\tau$$

exists and generates a unique stationary solution of (2.4) *given by*

$$\Omega \times \mathbb{R} \ni (\omega, t) \mapsto z(\theta_t\omega) = -\int_{-\infty}^{0} e^{\tau}\theta_t\omega(\tau)d\tau = -\int_{-\infty}^{0} e^{\tau}\omega(\tau + t)d\tau + \omega(t).$$

Moreover, the mapping $t \mapsto z(\theta_t\omega)$ *is continuous.*

(3) *On* Ω, *it holds*

$$\lim_{t\to\pm\infty} \frac{|z(\theta_t\omega)|}{|t|} = 0 \text{ and } \lim_{t\to\pm\infty} \frac{1}{t}\int_0^t z(\theta_\tau\omega)d\tau = 0.$$

F , $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$, $\mathcal{F}$ ac a a Ω. F c , d a . W d c a

(2.5) $$(x, \omega) = xe^{-z(\omega)}$$

a d a

(2.6) $$^{-1}(x, \omega) = xe^{z(\omega)}$$

$x \in \mathcal{H}_0$ a d $\omega \in \Omega$. F [15, L a 3.2], a a a b d b (2.7) a d (2.8), d d b c .

2.4. **E**. W a (1.1)

(2.7) $$du(t) = (Au(t) + F(u(t)))dt + u(t) \circ d\omega(t)$$

a a ab Ba ac ac $(\mathcal{H}, \|\cdot\|)$. H A a -d d d - a a (2.1) a d (2.2) a d a a a d $(S_A(t))_{t\geq 0}$. I a A_0 a a a c ac a a c $(T(t))_{t\geq 0}$ H_0. A , $\omega(t)$ ca ca d a a da d W c , $F : \mathcal{H}_0 \to \mathcal{H}$ a ba L c c a c L c c a L a d $F(0) = 0$, a d a a $u(0) \in H_0$.

B a a $v = e^{-z(\theta_t\omega)}u(t, \omega, x)$, (2.7) d c

(2.8) $$\dot{v}(t) = Av(t) + z(\theta_t\omega)v(t) + e^{-z(\theta_t\omega)}F\left(e^{z(\theta_t\omega)}v(t)\right),$$

$z(\theta_t\omega)$ d d (2.4). F a a , d $G(\omega, v) = e^{-z(\omega)}F(e^{z(\omega)}v)$ a d c c d c d :

(H1) $(S_A(t))_{t\geq 0}$ a a d a (2.1) a d (2.2), a d a a b d d - a a $V^{\infty}(S_A, 0, t)$ $[0, t]$.

(H2) A_0 a a d - a d c $\alpha > \beta$ a d b d $K \geq 1$. Na , a b d c $P_0^s \in \mathcal{L}(\mathcal{H}_0)$ c a

( ) $P_0^sT(t) = T(t)P_0^s$ a $t \geq 0$;

( ) $P_0^u = \mathrm{d}_{\mathcal{H}_0} - P_0^s$ a - a c , a d $(T(t)|_{P_0^u})_{t\geq 0}$ ca b d d a C_0- ;

( ) a a a d:

(2.9) $$\|T(t)P_0^sx\| \leq Ke^{\beta t}\|P_0^sx\|, \quad \forall\ t \geq 0 \text{ a d } x \in \mathcal{H}_0,$$

(2.10) $$\|T(t)P_0^ux\| \leq Ke^{\alpha t}\|P_0^ux\|, \quad \forall\ t \leq 0 \text{ a d } x \in \mathcal{H}_0;$$

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(**H3**) $F:\mathcal{H}_0\to\mathcal{H}$ ba L c c ,

$$\|F(u_1)-F(u_2)\|\le L\|u_1-u_2\|,\quad \forall u_1,u_2\in\mathcal{H}_0,$$

L L c c a a d $F(0)=0$.

B (**H2**), ca $\mathcal{H}_0$ a d c c d a a b ac

$$\mathcal{H}_0=\mathcal{H}_0^u\oplus\mathcal{H}_0^s,$$

$\mathcal{H}_0^u=P_0^u\mathcal{H}_0$ a d $\mathcal{H}_0^s=P_0^s\mathcal{H}_0$. T b L a 2.5, a $P^u:\mathcal{H}\to\mathcal{H}$ P_0^u $\mathcal{H}_0$ $\mathcal{H}$. T $P^u=$ d$_{\mathcal{H}}-P^s$, $\mathcal{H}^s=P^s\mathcal{H}$, $\mathcal{H}^u=P^u\mathcal{H}$ a d a a $A^u=A|_{\mathcal{H}^u}$, $A^s=A|_{\mathcal{H}^s}$, $A_0^u=A|_{\mathcal{H}_0^u}$, $A_0^s=A|_{\mathcal{H}_0^s}$ c . C , a :

$$\mathcal{H}=\mathcal{H}^u\oplus\mathcal{H}^s.$$

M , A_0^i a A^i $\mathcal{H}_0^i=\overline{D(A^i)}$ $i=s,u$. I a [18, L a 3.6] a $\mathcal{H}_0^u=\mathcal{H}^u$ a d $\mathcal{H}_0^s=\mathcal{H}^s\cap\mathcal{H}_0$. I add , d C_0- $T_{A_0^s}$ a d b A_0^s a d C_0- $e^{A_0^u}$ a d b A_0^u. L $\varepsilon>0$ d, d c a c c a c a

$$C(\varepsilon,\zeta)=\frac{2\varepsilon\ \text{a}\ \{1,e^{-\zeta\tau_\varepsilon}\}}{1-e^{(\omega_A-\zeta)\tau_\varepsilon}} \tag{2.11}$$

$\tau_\varepsilon>0$ $MV^\infty(S_A,0,\tau_\varepsilon)<\varepsilon$ a d $\zeta>\omega_A$.

T c , ac [22] a d a d a d a a a d , d a d d a d ca b d [22].

L 2.8. *The following assertions are valid:*

( ) *Assume* (**H1**) *and* (**H3**) *hold; equation* (2.8) *possesses a unique global integrated solution on* $\mathcal{H}_0$ *given by*

$$\begin{aligned} v(t,\omega,x)=&T(t)e^{\int_0^t z(\theta_\tau\omega)d\tau}x+\lim_{\lambda\to\infty}\int_0^t T(t-r)e^{\int_r^t z(\theta_\tau\omega)d\tau}\\ &\times\lambda(\lambda I-A)^{-1}G(\theta_r\omega,v(r,\omega,x))dr. \end{aligned} \tag{2.12}$$

Moreover, this solution operator in (2.12) *generates an RDS* $\phi:\mathbb{R}^+\times\Omega\times\mathcal{H}_0\to\mathcal{H}_0$.

( ) *Assume* (**H1**)–(**H3**) *hold and for some:*

3. Existence of Lipschitz invariant foliations

I [illegible] (2.7) b[illegible] L[illegible] -P[illegible] d. T[illegible] L[illegible] (2.8) a d [illegible] $^{-1}$ [illegible] (2.7).

N[illegible] $(\mathcal{H}_0, \|\cdot\|)$ [illegible] a Ba[illegible] ac[illegible] $v(t,\omega,x)$ [illegible] (2.8) [illegible] $\mathcal{H}_0$ [illegible] $v(0,\omega,x) = x$. F[illegible] $\gamma \in (\beta,\alpha)$, [illegible] Ba[illegible]

$$\mathcal{C}_\gamma^+(\mathbb{R}^+;\mathcal{H}_0) = \left\{ \varphi \in \mathcal{C}([0,+\infty),\mathcal{H}_0) \middle| \ {}_{t\in[0,+\infty)} e^{-\gamma t - \int_0^t z(\theta_\tau\omega)d\tau}\|\varphi(t)\| < +\infty \right\}$$

[illegible]

$$\|\varphi\|_\gamma^+ = {}_{t\in[0,+\infty)} e^{-\gamma t - \int_0^t z(\theta_\tau\omega)d\tau}\|\varphi(t)\|$$

a d

$$\mathcal{C}_\gamma^-(\mathbb{R}^-;\mathcal{H}_0) = \left\{ \varphi \in \mathcal{C}((-\infty,0],\mathcal{H}_0) \middle| \ {}_{t\in(-\infty,0]} e^{-\gamma t - \int_0^t z(\theta_\tau\omega)d\tau}\|\varphi(t)\| < +\infty \right\}$$

[illegible]

$$\|\varphi\|_\gamma^- = {}_{t\in(-\infty,0]} e^{-\gamma t - \int_0^t z(\theta_\tau\omega)d\tau}\|\varphi(t)\|.$$

T[illegible]

$$\mathcal{W}^{ss}(x,\omega) = \left\{ \acute{x} \in \mathcal{H}_0 \middle| v(t,\omega,\acute{x}) - v(t,\omega,x) \in \mathcal{C}_\gamma^+(\mathbb{R}^+;\mathcal{H}_0) \right\},$$
$$\mathcal{W}^{uu}(x,\omega) = \left\{ \acute{x} \in \mathcal{H}_0 \middle| v(t,\omega,\acute{x}) - v(t,\omega,x) \in \mathcal{C}_\gamma^-(\mathbb{R}^-;\mathcal{H}_0) \right\}.$$

W[illegible] $\mathcal{W}^{ss}(x,\omega)$ a d $\mathcal{W}^{uu}(x,\omega)$ [illegible] $x \in \mathcal{H}_0$ [illegible] (2.7), [illegible] F[illegible]

$$(S_A \diamond G_x(v))(t) = {}_{\lambda\to\infty} \int_0^t T(t-r)e^{\int_r^t z(\theta_\tau\omega)d\tau}\lambda(\lambda I - A)^{-1}G(\theta_r\omega, v(r,\omega,x))dr,$$

a d [illegible] [17, P[illegible] 2.14] [illegible]

$$\|(S_A \diamond G_x(v))(t)\| \le C(\beta,\gamma) {}_{r\in[0,t]} e^{\gamma(t-r)+\int_r^t z(\theta_\tau\omega)d\tau}\|G(\theta_r\omega, v(r,\omega,x))\|. \tag{3.1}$$

I [illegible] $\mathcal{W}^{ss}(x,\omega)$ [illegible] a L[illegible] c[illegible].

[illegible] **3.1.** *Assume* **(H1)**–**(H3)** *hold. If the gap condition*

$$KL\left(C(\beta,\gamma)\|P^s\|_{\mathcal{L}(\mathcal{H})} + \frac{\|P^u\|_{\mathcal{L}(\mathcal{H})}}{\alpha-\gamma}\right) < 1, \tag{3.2}$$

is satisfied, then there exists a Lipschitz invariant stable foliation for (2.8) *whose leaf is given by*

$$\mathcal{W}^{ss}(x,\omega) = \{\eta + l^s(\omega,\eta,x) | \eta \in \mathcal{H}_0^s\},$$

where $x \in \mathcal{H}_0$ *and* $l^s(\omega,\eta,x)$ *is measurable in* (ω,η,x) *and Lipschitz continuous in* η *with*

$$Lip\ l^s(\omega,\cdot,x) \le \frac{K^2L\|P^s\|_{\mathcal{L}(\mathcal{H})}\|P^u\|_{\mathcal{L}(\mathcal{H})}}{(\alpha-\gamma)\left[1 - KL\left(C(\beta,\gamma)\|P^s\|_{\mathcal{L}(\mathcal{H})} + \frac{\|P^u\|_{\mathcal{L}(\mathcal{H})}}{\alpha-\gamma}\right)\right]}.$$

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Moreover, if

$$(K+1)KL\left(C(\beta,\gamma)\|P^s\|_{\mathcal{L}(\mathcal{H})}+\frac{\|P^u\|_{\mathcal{L}(\mathcal{H})}}{\alpha-\gamma}\right)<1, \tag{3.3}$$

then $\mathcal{W}^{ss}(x,\omega)\cap\mathcal{M}^u(\omega)$ *contains a unique point. Furthermore,* $\hat{\mathcal{W}}^{ss}(x,\omega) = {}^{-1}(\mathcal{W}^{ss}(x,\omega),\omega)$ *is a leaf of invariant stable foliation for the RDS generated by* (2.7).

Proof. T c ba d L a -P d.

Step 1. W c a a $x\in\mathcal{W}^{ss}(x,\omega)$ a d a c $\psi(\cdot)\in\mathcal{C}^+_\gamma(\mathbb{R}^+;\mathcal{H}_0)$ a a $\psi(0)=x-x$ a

$$\begin{aligned}\psi(t)=&T_{A_0^s}(t)e^{\int_0^t z(\theta_\tau\omega)d\tau}\eta+P_0^s(S_A\diamond\ G_x^\eta(\nu,\psi))(t)\\&+\int_{+\infty}^t e^{A_0^u(t-r)+\int_r^t z(\theta_\tau\omega)d\tau}P^u\ G(\theta_r\omega,\nu(r,\omega,x),\psi(r))dr,\end{aligned}\tag{3.4}$$

$\eta=P_0^s(x-x)$ a d

$$\begin{aligned}G(\theta_r\omega,\nu(r,\omega,x),\psi(r))=&G(\theta_r\omega,\nu(r,\omega,x)+\psi(r))-G(\theta_r\omega,\nu(r,\omega,x)),\\(S_A\diamond\ G_x^\eta(\nu,\psi))(t)=&(S_A\diamond G_x(\nu+\psi))(t)-(S_A\diamond G_x(\nu))(t).\end{aligned}$$

T c a , a $x\in\mathcal{W}^{ss}(x,\omega)$. L $\psi(t)=\nu(t,\omega,x)-\nu(t,\omega,x)$. U a a c a a (2.12), $0<t<\iota$ a a

$$P^s\nu(t,\omega,x)=T_{A_0^s}(t)e^{\int_0^t z(\theta_\tau\omega)d\tau}P_0^s$$

T[illegible] (3.4) b[illegible] $J^s(\psi,\omega,\eta,x)$:

$$
\begin{aligned}
J^s(\psi,\omega,\eta,x) =& T_{A_0^s}(t)e^{\int_0^t z(\theta_\tau\omega)d\tau}\eta + P_0^s(S_A \diamond G_x^\eta(\nu,\psi))(t) \\
&- \int_t^{+\infty} e^{A_0^u(t-r)+\int_r^t z(\theta_\tau\omega)d\tau} P^u G(\theta_r\omega, \nu(r,\omega,x), \psi(r))dr.
\end{aligned}
$$

N[illegible]

$$
\begin{aligned}
& e^{-\gamma t-\int_0^t z(\theta_\tau\omega)d\tau}\|J^s(\psi,\omega,\eta,x)\| \\
& \leq K\|\eta\| + C(\beta,\gamma)\|P^s\|_{\mathcal{L}(\mathcal{H})} \sup_{r\in[0,t]} e^{-\gamma r-\int_0^t z(\theta_\tau\omega)d\tau+\int_r^t z(\theta_\tau\omega)d\tau} \\
& \times \| G(\theta_r\omega, \nu(r,\omega,x), \psi(r))\| + K\|P^u\|_{\mathcal{L}(\mathcal{H})} \int_t^{+\infty} e^{-\gamma t-\int_0^t z(\theta_\tau\omega)d\tau} \\
& \times e^{\alpha(t-r)+\int_r^t z(\theta_\tau\omega)d\tau}\| G(\theta_r\omega, \nu(r,\omega,x), \psi(r))\|dr \\
& \leq K\|\eta\| + LC(\beta,\gamma)\|P^s\|_{\mathcal{L}(\mathcal{H})}\|\psi\|_\gamma^+ + KL\|P^u\|_{\mathcal{L}(\mathcal{H})}\int_t^{+\infty} e^{(\alpha-\gamma)(t-r)}dr\|\psi\|_\gamma^+ \\
& \leq K\|\eta\| + KL\left(C(\beta,\gamma)\|P^s\|_{\mathcal{L}(\mathcal{H})} + \frac{\|P^u\|_{\mathcal{L}(\mathcal{H})}}{\alpha-\gamma}\right)\|\psi\|_\gamma^+,
\end{aligned}
$$

[illegible] $J^s(\cdot,\omega,\eta,x)$ [illegible] $\mathcal{C}_\gamma^+(\mathbb{R}^+;\mathcal{H}_0)$ [illegible]. F[illegible] $\psi,\psi\in\mathcal{C}_\gamma^+(\mathbb{R}^+;\mathcal{H}_0)$, [illegible]

$$
\|J^s(\psi,\omega,\eta,x) - J^s(\psi,\omega,\eta,x)\|_\gamma^+ \leq KL\left(C(\beta,\gamma)\|P^s\|_{\mathcal{L}(\mathcal{H})} + \frac{\|P^u\|_{\mathcal{L}(\mathcal{H})}}{\alpha-\gamma}\right)\|\psi-\psi\|_\gamma^+.
$$

T[illegible] $J^s(\cdot,\omega,\eta,x)$ [illegible] (ω,η,x). B[illegible] $J^s(\cdot,\omega,\eta,x)$ [illegible] $\psi^*(\cdot,\omega,\eta,x)$. S[illegible] $J^s : \mathcal{C}_\gamma^+(\mathbb{R}^+;\mathcal{H}) \to \mathcal{C}_\gamma^+(\mathbb{R}^+;\mathcal{H}_0)$ a[illegible]

$$
\begin{aligned}
J^s(\psi^*)(t) =& P_0^s(S_A \diamond G_x^\eta(\nu,\psi^*))(t) \\
&+ \int_{+\infty}^t e^{A_0^u(t-r)+\int_r^t z(\theta_\tau\omega)d\tau} P^u G(\theta_r\omega, \nu(r,\omega,x), \psi^*(r))dr.
\end{aligned}
$$

F[illegible] $(I-J)$ [illegible] $(I-J)^{-1}$ [illegible] L[illegible]. T[illegible] L[illegible] η. Na[illegible], [illegible] $\eta_1,\eta_2\in\mathcal{H}_0^s$, [illegible]d

$$
\|\psi^*(t,\omega,\eta_1,x) - \psi^*(t,\omega,\eta_2,x)\|_\gamma^+ \leq \frac{K}{1-L\|J\|}\|\eta_1-\eta_2\|. \tag{3.5}
$$

B[illegible], $\psi^*(0,\omega,\eta,x)$ [illegible]

T a , d a $l^s : \Omega \times \mathcal{H}_0^s \times \mathcal{H}_0 \to \mathcal{H}_0^u$ a

$$l^s(\omega, \eta, x) = P_0^u x + P_0^u \psi^*(0, \omega, \eta - P_0^s x, x),$$

$\eta \in \mathcal{H}_0^s$. T $l^s(\omega, \eta, x)$ a ab (ω, η, x). B ac a $x \in \mathcal{W}^{ss}(x, \omega)$ a d $\psi^*(\cdot) \in \mathcal{C}_\gamma^+(\mathbb{R}^+; \mathcal{H}_0)$ a d a (3.4), a a $x \in \mathcal{W}^{ss}(x, \omega)$ a d $\eta \in \mathcal{H}_0^s$ c a

$$x = x + \eta + P^u \psi^*(0, \omega, \eta, x).$$

L $y_s = P_0^s x + \eta$, a

$$x + \eta + P^u \psi^*(0, \omega, \eta, x) = P^u x + y_s + P^u \psi^*(0, \omega, y_s - P_0^s x, x).$$

R ac y_s b η ab , b a

$$x = P^u x + \eta + P^u \psi^*(0, \omega, \eta - P_0^s x, x) = \eta + l^s(\omega, \eta, x),$$

c d ca d (3.6). W c c L c c . I ac , b (3.5), a $\eta_1, \eta_2 \in \mathcal{H}_0^s$,

$$\begin{aligned} &\|l^s(\omega, \eta_1, x) - l^s(\omega, \eta_2, x)\| \\ &\quad \le \frac{K^2 L \|P^s\|_{\mathcal{L}(\mathcal{H})} \|P^u\|_{\mathcal{L}(\mathcal{H})}}{(\alpha - \gamma)\left[1 - KL\left(C(\beta, \gamma)\|P^s\|_{\mathcal{L}(\mathcal{H})} + \frac{\|P^u\|_{\mathcal{L}(\mathcal{H})}}{\alpha - \gamma}\right)\right]} \|\eta_1 - \eta_2\|. \end{aligned}$$

Step 4. W a c a a $\mathcal{W}^{ss}(x, \omega) \cap \mathcal{M}^u(\omega)$ c a a .

I ac , $x \in \mathcal{W}^{ss}(x, \omega) \cap \mathcal{M}^u(\omega)$, $\xi \in \mathcal{H}_0^u$ a d $\eta \in \mathcal{H}_0^s$ c a

$$x = \xi + h^u(\omega, \xi) = \eta + l^s(\omega, \eta, x),$$

c

$$\xi = l^s(\omega, \eta, x) \text{ a d } \eta = h^u(\omega, \xi). \tag{3.7}$$

B L c c h^u a d l^s a d c d (3.3), a a L $h^u(\omega, \cdot) \cdot$ L $l^s(\omega, \cdot, x) < 1$; a

$$\eta = h^u(\omega, l^s(\omega, \eta, x))$$

a a a d $\mathcal{W}^{ss}(x, \omega)$ a a c $\mathcal{M}^u(\omega)$. M , d $\mathcal{W}^{ss}(x, \omega)$ a

$$_{t \to \infty} \|v(t, \omega, x) - v(t, \omega, x)\| \le -\frac{2}{C(\beta, \gamma)\|P^s\|_{\mathcal{L}(\mathcal{H})} + \frac{\|P^u\|_{\mathcal{L}(\mathcal{H})}}{\alpha - \gamma}}.$$

U a a [15, T 4.2] a d (3.7), a

$$\text{a } l^s(\omega, \cdot, x) = \text{ a } l^s(\omega, \cdot, \xi(x) + h^u(\omega, \xi(x)))$$

a d a $l^s(\omega, \cdot, x_1) =$ a $l^s(\omega, \cdot, x_2)$ a $l^s(\omega, \cdot, x_1) \cap$ a $l^s(\omega, \cdot, x_2) \neq \emptyset$. H c

$$\mathcal{H}_0 = \cup_{\xi \in \mathcal{H}_0^u} \text{ a } l^s(\omega, \cdot, \xi + h^u(\omega, \xi))$$

a L c a .

Step 5. W a a c $\mathcal{W}^{ss}(x, \omega)$. I d , a τ- a $v(\tau, \omega, \cdot)$ a $\mathcal{W}^{ss}(x, \omega)$ a $\mathcal{W}^{ss}(v(\tau, \omega, x), \theta_\tau \omega)$.

T , a $x \in \mathcal{W}^{ss}(x, \omega)$, a a $v(\cdot, \omega, x) - v(\cdot, \omega, x) \in \mathcal{C}_\gamma^+(\mathbb{R}^+; \mathcal{H}_0)$, c d ca a $v(\cdot + \tau, \omega, x) - v(\cdot + \tau, \omega, x) \in \mathcal{C}_\gamma^+(\mathbb{R}^+; \mathcal{H}_0)$ $\tau \in \mathbb{R}$. T

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[illegible] $v(t+\tau,\omega,z) = v(t,\theta_\tau\omega, v(\tau,\omega,z))$ [illegible] $z \in \mathcal{H}_0$, [illegible]

$$v(\tau,\omega,x) \in \mathcal{W}^{ss}(v(\tau,\omega,x),\theta_\tau\omega).$$

Step 6. [illegible] $\hat{\mathcal{W}}^{ss}(x,\omega) = {}^{-1}(\mathcal{W}^{ss}(x,\omega),\omega)$ [illegible] RDS [illegible] (2.7).

[illegible] u [illegible] v, [illegible]

$$\begin{aligned} u(t,\omega,\hat{\mathcal{W}}^{ss}(x,\omega)) &= {}^{-1}(v(t,\omega, \ (\hat{\mathcal{W}}^{ss}(x,\omega),\omega)),\theta_t\omega) \\ &= {}^{-1}(v(t,\omega,\mathcal{W}^{ss}(x,\omega)),\theta_t\omega) \\ &\subset {}^{-1}(\mathcal{W}^{ss}(v(t,\omega,x),\theta_t\omega),\theta_t\omega) \\ &\subset \hat{\mathcal{W}}^{ss}(v(t,\omega,x),\theta_t\omega). \end{aligned}$$

[illegible] $x_i \in \hat{\mathcal{W}}^{ss}(x,\omega)$, [illegible] $y_i \in \mathcal{W}^{ss}(x,\omega)$ [illegible] $x_i = e^{z(\omega)}y_i$ [illegible] $i = 1,2$. [illegible] $u(t,\omega,x_1) - u(t,\omega,x_2) = e^{z(\omega)}(v(t,\omega,y_1) - v(t,\omega,y_2))$. [illegible] $t \mapsto z(\omega)$ [illegible] 2.5. [illegible] $^{-1}$ [illegible] $\hat{\mathcal{W}}^{ss}(x,\omega)$ [illegible] (2.7). [illegible]

$$\begin{aligned} \hat{\mathcal{W}}^{ss}(x,\omega) &= \{ \ {}^{-1}(\eta + l^s(\omega,\eta,x),\omega) | \eta \in \mathcal{H}_0^s \} \\ &= \left\{ e^{z(\omega)}(\eta + l^s(\omega,\eta,x)) | \eta \in \mathcal{H}_0^s \right\} \\ &= \left\{ \eta + e^{z(\omega)} l^s(\omega, e^{-z(\omega)}\eta, x) | \eta \in \mathcal{H}_0^s \right\}. \end{aligned}$$

[illegible] □

[illegible]

[illegible] **3.2.** *Under the assumptions of Theorem* 3.1, *there exists a Lipschitz invariant unstable foliation for* (2.8) *whose leaf is given by*

$$\mathcal{W}^{uu}(x,\omega) = \{\eta + l^u(\omega,\eta,x) | \eta \in \mathcal{H}_0^u\},$$

where $x \in \mathcal{H}_0$ *and* $l^u(\omega,\eta,x)$ *is measurable in* (ω,η,x) *and Lipschitz continuous in* η. *Moreover,* $\mathcal{W}^{uu}(x,\omega) \cap \mathcal{M}^s(\omega)$ *contains a unique point and* $\hat{\mathcal{W}}^{uu}(x,\omega) = {}^{-1}(\mathcal{W}^{uu}(x,\omega),\omega)$ *is a leaf of invariant unstable foliation for the RDS generated by* (2.7).

Proof. [illegible] 3.1. [illegible] $C_\gamma^-(\mathbb{R}^-;\mathcal{H}_0)$, [illegible] (3.4). [illegible]

$$\begin{aligned} \psi(t) =& e^{A_0^u(t)} e^{\int_0^t z(\theta_\tau\omega)d\tau}\eta + \int_0^t e^{A_0^u(t-r)+\int_r^t z(\theta_\tau\omega)d\tau} P^u \quad G(\theta_r\omega, v(r,\omega,x),\psi(r))dr \\ &+ \lim_{\lambda\to\infty} \int_{-\infty}^t T_{A_0^s}(t-r) e^{\int_r^t z(\theta_\tau\omega)d\tau} \lambda(\lambda I - A)^{-1} \quad G(\theta_r\omega, v(r,\omega,x),\psi(r))dr. \end{aligned}$$

[illegible] 3.1. □

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4. Illustrative example

C d c a c S a c a ab c a a a

$$
\begin{cases}
\dfrac{\partial u(t,x)}{\partial t} = \left(\dfrac{\partial^2 u(t,x)}{\partial x^2} + \delta u(t,x) + f(u(t,x))\right) dt + u(t,x) \circ dW(t), \\
-\dfrac{\partial u(t,0)}{\partial x} = g_0(u(t,\cdot)), \\
\dfrac{\partial u(t,1)}{\partial x} = g_1(u(t,\cdot)), \\
u(0,\cdot) = u_0 \in L^2(0,1),
\end{cases}
\tag{4.1}
$$

$t > 0, x \in (0,1), \delta > 0, f : L^2(0,1) \to L^2(0,1)$ a d $g_0, g_1 : L^2(0,1) \to \mathbb{R}$ a a b a a c , c a a d b ba L c c . H , $L^2(0,1)$ a L^2- ac a $(0,1)$. I d a $T_{(\mathcal{A}+\delta I)_0}(t)$ c a c ac a a c b d d a a $\mathcal{H}_0$. I d , a (**H1**) a d. T d - a d c c d $(\mathcal{A}+\delta I)_0$, add a a a $\delta \neq \pi k$ a $k \in \mathbb{N}$. I a , c $(\mathcal{A}+\delta I)_0$ d c a a a a a a d 0. D a - a a a d a a $(\mathcal{A}+\delta I)_0$ b λ^+ a d λ^-, c . T a $\alpha = \lambda^+ -$ * a d $\beta = \lambda^- +$ * a * > 0 c a c a . T a (**H2**) ab d. A , a $f(0) = g_0(0) = g_1(0) = 0$ a a (**H3**).

H c b a d T 3.1 a d 3.2 a a cab ca a d c L c ab ab a (4.1).

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