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Riemann-Hilbert approach and N-soliton formula for coupled derivative Schrödinger equation

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The coupled derivative Schrödinger equation is studied in the framework of the Riemann-Hilbert problem and a compact N-soliton solution formula is found. Taking advantage of this result, some properties for single soliton solution and asymptotic analysis of N-soliton solution are explored. As a by-product, a coupled Fokas-Lenells equation together its N-soliton solution is presented. © *2012 American Institute of Physics*. [http://dx.doi.org/10.1063/1.4732464]

I. INTRODUCTION

The integrable systems are differential or difference equations with rich mathematical structures and wide physics and engineering applications. In particular, they often have multi-soliton solutions. Among many integrable systems, the nonlinear Schrödinger (NLS) equation has been recognized as a ubiquitous mathematical model, which may be adopted to describe the evolution of a slowly varying wave packet in a nonlinear wave system. It plays an important role in nonlinear optics,[3] water waves,[4] and plasma physics. However, in some physical situations, two or more wave packets of different carrier frequencies appear simultaneously. This type of wave interactions could be modeled by the coupled NLS equations.[3]

Another integrable system of NLS type, the derivative nonlinear Schrödinger (DNLS) equation[9]

$$\mathrm{i}q_t = q_{xx} + \frac{2\mathrm{i}\sigma}{3}(q^2 q^*)_x, \quad \sigma = \pm 1 \tag{1}$$

has also attracted considerable attention. This system emerges as a model for Alfvén wave propagation along the magnetic field.[16] The inverse scattering transformation was applied to this equation and the soliton solution was constructed by Kaup and Newell.[9] Many researches have been conducted for it and then lots of results have been achieved.[8,10,18] The associated two-component extension of DNLS equation, namely, the coupled derivative nonlinear Schrödinger (cDNLS) equation proposed by Dodd and Morris,[15] reads as

$$\mathrm{i}q_{1,t} = [q_{1,x} + \frac{2\mathrm{i}}{3}q_1(|q_1|^2 + \sigma|q_2|^2)]_x, \tag{2a}$$

$$\mathrm{i}q_{2,t} = [q_{2,x} + \frac{2\mathrm{i}}{3}q_2(|q_1|^2 + \sigma|q_2|^2)]_x, \tag{2b}$$

which is relevant in the theory of polarized Alfvén waves and the propagation of the ultra-short pulse. The system of equations (2) was studied within the framework of inverse scattering transformation. More recently, Hirota's direct method was developed and in particular two-soliton solutions were constructed for this system.[22] Its N-soliton solutions were constructed by means of Darboux transformation very recently.[14] We observe that the N-soliton solutions constructed are the ratios of $3N \times 3N$ determinants, which may not be convenient to practical purpose.

Finally, we should point out the integrable properties for N-component derivative Schrödinger equations had been studied by Hisakado and Wadati,[6,7] they constructed the gauge transformation between N-component derivative Schrödinger equations and N-component Schrödinger equations. Tsuchida and Wadati[19,20] considered the gauge transformation between different kinds of matrix derivative-type Schrödinger equations. In our work, we construct the Riemann-Hilbert problem for two-component derivative Schrödinger equations directly, rather than depend with the gauge transformation.

The aim of this paper is to construct a compact representation for the N-soliton solutions of the cDNLS system and present an asymptotic analysis for these solutions. To this end, we will take the well-known Riemann-Hilbert approach,[1,2] which relays on the Riemann-Hilbert problem rather than the Gel'fand-Levitan-Marchenko equation. This approach enables us to find simple formula for the cDNLS equation. As a by-product, we obtain a coupled Fokas-Lenells system and its multi-soliton solutions. We remark that, for the single-component Fokas-Lenells equation, Lenells[13] found its N-soliton solutions via dressing method. However, it is not clear how to generalize his results to the coupled Fokas-Lenells system.

The paper is organized as follows. In Sec. II, we derive the cDNLS hierarchy from a third-order spectral problem. The inverse scattering method is applied to this third-order spectral problem and corresponding Riemann-Hilbert problem is formulated in Sec. III. Section IV intends to find simple and compact N-soliton formula. Then this formula is employed to study the asymptotic behavior of the N soliton interactions. In Sec. V, we derive a coupled Fokas-Lenells system and give its N-solition solution formula. Finally, Sec. VI contains some remarks.

II. cDNLS HIERARCHY

In this section, we will derive the cDNLS hierarchy which may lead to a coupled Fokas-Lenells system in Sec. V. To have a clear picture, we consider first the derivation of the DNLS hierarchy which has the following spectral problem:

$$\phi_x = U\phi, \quad U = \begin{pmatrix} -\mathrm{i}\alpha\zeta^2 & \mathrm{i}\zeta q \\ \mathrm{i}\zeta r & \mathrm{i}\zeta^2 \end{pmatrix}, \tag{3a}$$

$$\phi_t = V\phi, \quad V = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{3b}$$

where α is a real number, ζ is the spectral parameter, $q = q(x, t)$ and $r = r(x, t)$ are field variables, A, B, C, and D are the quantities depending on field variables and their derivatives and ζ.

The corresponding zero-curvature equation or the compatibility condition of (3)

$$U_t - V_x + [U, V] = 0 \tag{4}$$

implies

$$-A_x + \mathrm{i}\zeta qC - \mathrm{i}\zeta Br = 0, \tag{5a}$$

$$-D_x - \mathrm{i}\zeta qC + \mathrm{i}\zeta Br = 0, \tag{5b}$$

$$\mathrm{i}\zeta q_t - B_x - \mathrm{i}(\alpha+1)\zeta^2 B + \mathrm{i}\zeta qD - \mathrm{i}\zeta Aq = 0, \tag{5c}$$

$$\mathrm{i}\zeta r_t - C_x + \mathrm{i}(\alpha+1)\zeta^2 C + \mathrm{i}\zeta rA - \mathrm{i}\zeta Dr = 0. \tag{5d}$$

Using Eqs. (5a) and (5b), Eqs. (5c) and (5d) may be rewritten as

$$\mathrm{i}\zeta\begin{pmatrix} q_t \\ r_t \end{pmatrix} - \begin{pmatrix} B_x \\ C_x \end{pmatrix} + \zeta^2 L\begin{pmatrix} B \\ C \end{pmatrix} + \mathrm{i}\zeta(D_0 - A_0)\begin{pmatrix} q \\ -r \end{pmatrix} = 0, \tag{6}$$

where D_0 and A_0 are integration constants, and

$$L=-\mathrm{i}(\alpha+1)\sigma_3-2\sigma_3\begin{pmatrix}q\\ r\end{pmatrix}\partial^{-1}(r,q)\sigma_3,\quad \sigma_3=\mathrm{diag}\{1,-1\}.$$

To obtain the evolution equations, we expand

$$\begin{pmatrix}B\\ C\end{pmatrix}=\sum_{n=1}^{N}\begin{pmatrix}b_i\\ c_i\end{pmatrix}\zeta^{2i-1} \tag{7}$$

and let $A_0=-\alpha D_0=-\frac{\alpha\beta}{1+\alpha}\zeta^{2N}$ with β a constant. It follows from (6) and (7) that

$$L\begin{pmatrix}b_N\\ c_N\end{pmatrix}+\mathrm{i}\beta\begin{pmatrix}q\\ -r\end{pmatrix}=0, \tag{8a}$$

$$-\begin{pmatrix}b_i\\ c_i\end{pmatrix}_x+L\begin{pmatrix}b_{i-1}\\ c_{i-1}\end{pmatrix}=0, \tag{8b}$$

$$\mathrm{i}\begin{pmatrix}q_t\\ r_t\end{pmatrix}-\begin{pmatrix}b_1\\ c_1\end{pmatrix}_x=0. \tag{8c}$$

With the help of (8a) and (8b), we find that (8c) may be reformulated as

$$\mathrm{i}\begin{pmatrix}q_t\\ r_t\end{pmatrix}=\frac{\beta}{1+\alpha}(\partial_x L^{-1})^{N-1}\begin{pmatrix}q_x\\ r_x\end{pmatrix}, \tag{9}$$

where

$$L^{-1}=\frac{\mathrm{i}\sigma_3}{1+\alpha}-\frac{2}{(1+\alpha)^2}\begin{pmatrix}q\\ r\end{pmatrix}\partial^{-1}(r,q).$$

The first nontrivial flow in the hierarchy (9) is

$$\mathrm{i}\begin{pmatrix}q_t\\ r_t\end{pmatrix}=\begin{pmatrix}q_x+\frac{2\mathrm{i}}{3}q^2r\\ -r_x+\frac{2\mathrm{i}}{3}qr^2\end{pmatrix}_x, \tag{10}$$

where we took $\alpha=2$, $\beta=-(1+\alpha)^2\mathrm{i}$. The reduction $r=\sigma q^*$ for (10) yields the DNLS equation (2a). The explicit forms for U and V in the present case are

$$U=\mathrm{i}\zeta^2\sigma_0+\mathrm{i}\zeta Q,\quad \sigma_0=\begin{pmatrix}-2&0\\ 0&1\end{pmatrix},\quad Q=\begin{pmatrix}0&q\\ r&0\end{pmatrix}, \tag{11a}$$

$$V=-3\mathrm{i}\zeta^4\sigma_0+V_2,\quad V_2=-3\mathrm{i}\zeta^3Q+\mathrm{i}\zeta^2Q^2+\zeta(\sigma_3Q_x+\frac{2\mathrm{i}}{3}Q^3). \tag{11b}$$

Now we move to the multi-component extension of the DNLS equation. To this end, we merely modify above matrices U and V with

$$\sigma_0=\begin{pmatrix}-2&0_{1\times N}\\ 0_{N\times 1}&I\end{pmatrix},\quad \sigma_3=\begin{pmatrix}1&0_{1\times N}\\ 0_{N\times 1}&-I\end{pmatrix},\quad Q=\begin{pmatrix}0&\mathbf{q}^T\\ \mathbf{r}&0_{N\times N}\end{pmatrix}, \tag{12}$$

where $\mathbf{q}=(q_1,q_2,\ldots,q_N)^T$, $\mathbf{r}=(r_1,r_2,\ldots,r_N)^T$, $I\equiv I_{N\times N}$ is the $N\times N$ identity matrix, $0_{1\times N}$ $1\times N$ zero matrix, $0_{N\times 1}$ $N\times 1$ zero matrix, and $0_{N\times N}$ $N\times N$ zero matrix. By a direct calculation, the compatibility condition (4) leads to interesting nonlinear evolution equations. Letting A be a non-degenerate Hermite matrix and considering the reduction $\mathbf{r} = A^T\mathbf{q}^*$, one finds the following N-component system:

$$\mathbf{q}_t = [\mathbf{q}_x + \frac{2\mathrm{i}}{3}\mathbf{q}\mathbf{q}^\dagger A\mathbf{q}]_x. \tag{13}$$

The two-component DNLS or coupled DNLS system considered in Ref. 15 is the first nontrivial case – the case $N = 2$, which will be our main concern in the rest part of the paper. For convenience, we list down the explicit spectral problem here

$$\Phi_x = U\psi, \quad U = \mathrm{i}\zeta^2\sigma_0 + \mathrm{i}\zeta Q, \tag{14a}$$

$$\Phi_t = V\Phi, \quad V = -3\mathrm{i}\zeta^4\sigma_0 + V_2, \quad V_2 = -3\mathrm{i}\zeta^3 Q + \mathrm{i}\zeta^2 Q^2 + \zeta(\sigma_3 Q_x + \frac{2\mathrm{i}}{3}Q^3), \tag{14b}$$

where

$$\sigma_0 = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & q_1 & q_2 \\ r_1 & 0 & 0 \\ r_2 & 0 & 0 \end{pmatrix}.$$

and r_1, r_2 fulfill the following reductions:

$$r_1 = q_1^*, \quad r_2 = \sigma q_2^*, \quad \sigma = \pm 1. \tag{15}$$

III. INVERSE SCATTERING FOR cDNLS

In this section, we consider the scattering and inverse scattering problems for (2) and work out the associated Riemann-Hilbert formulation. These results will lay the ground for the construction of the N-soliton solutions of cDNLS system.

A. Analytical solution for spectral problem

Assume that the functions q_1, q_2, r_1, r_2 decay to zero sufficiently fast as $x \to \pm\infty$. It will be convenient for us to write the spectral equation (14) in terms of the matrix $J = \Phi E_1^{-1}$, where $E_1 = \exp[\mathrm{i}(\zeta^2 x - 3\zeta^4 t)\sigma_0]$ is a solution of spectral equation at $x \to \pm\infty$. Hence, the spectral problem (14) we shall deal with is written as

$$J_x = \mathrm{i}\zeta^2[\sigma_0, J] + \mathrm{i}\zeta QJ, \tag{16a}$$

$$J_t = -3\mathrm{i}\zeta^4[\sigma_0, J] + V_2 J. \tag{16b}$$

The Jost solutions $J_\pm$ of the spectral equation (16a) obey the asymptotic condition $J_\pm \to I$ as $x \to \pm\infty$. They solve the following integral equations:

$$J_\pm = I + \mathrm{i}\zeta\int_{\pm\infty}^{x} \exp[\mathrm{i}\zeta^2\sigma_0(x-y)]QJ_\pm\exp[-\mathrm{i}\zeta^2\sigma_0(x-y)]\mathrm{d}y. \tag{17}$$

These integral equations of Volterra type allow us to prove the existence and uniqueness of the Jost solutions through standard process. Partitioning $J_\pm$ into columns, namely, $J_\pm = (J_\pm^{[1]}, J_\pm^{[2]}, J_\pm^{[3]})$, then the column vectors $J_-^{[1]}$, $J_+^{[2]}$, and $J_+^{[3]}$ are continuous for $\zeta \in \mathbb{C}_+ \cup \mathbb{R} \cup \mathrm{i}\mathbb{R}$ and analytic for $\zeta \in \mathbb{C}_+$; $J_+^{[1]}$, $J_-^{[2]}$, and $J_-^{[3]}$ are continuous for $\zeta \in \mathbb{C}_- \cup \mathbb{R} \cup \mathrm{i}\mathbb{R}$ and analytical for $\zeta \in \mathbb{C}_-$, where

$$\mathbb{C}_+ = \left\{\zeta \,|\, \arg\zeta \in (0, \frac{\pi}{2}) \cup (\pi, \frac{3\pi}{2})\right\}, \quad \mathbb{C}_- = \left\{\zeta \,|\, \arg\zeta \in (\frac{\pi}{2}, \pi) \cup (\frac{3\pi}{2}, 2\pi)\right\}.$$

Since both J_+E and J_-E are solutions of spectral problem (14), they must be linearly related by a matrix $S(\zeta)$ – the so-called scattering matrix. That is,

$$J_-E = J_+ES(\zeta), \quad \zeta \in \mathbb{R} \cup \mathrm{i}\mathbb{R}, \tag{18}$$

where $E = \exp(\mathrm{i}\zeta^2 x\sigma_0)$ and $S(\zeta) = (s_{ij})_{3\times 3}$.

It follows from Abel's formula and $\mathrm{tr}(Q) = 0$ that the determinants of $J_\pm$ are independent of x, so the evaluations of $\det(J_\pm)$ at $x = \pm\infty$ show that

$$\det J_\pm = 1.$$

Furthermore, from (18) we obtain $\det(S) = 1$. Evaluation of $S(\zeta)$ as $x \to +\infty$ gives

$$S(\zeta) = \lim_{x\to+\infty} E^{-1}J_-E = I + \mathrm{i}\zeta \int_{-\infty}^{+\infty} E^{-1}(QJ_-)(x)E\mathrm{d}x, \quad \zeta \in \mathbb{R} \cup \mathrm{i}\mathbb{R}. \tag{19}$$

Thanks to the analytic property of J_-, s_{11} allows analytic extension to $\mathbb{C}_+$, s_{22}, s_{23}, s_{32}, and s_{33} can be analytically extended to $\mathbb{C}_-$.

In order to construct the Riemann-Hilbert problem, we define the matrix function

$$\Phi_+ = (J_-^{[1]}, J_+^{[2]}, J_+^{[3]}),$$

which is analytic in $\zeta \in \mathbb{C}_+$. Taking account of (18), we have

$$\Phi_+ = J_+ES_+E^{-1} = J_+E\begin{pmatrix} s_{11} & 0 & 0 \\ s_{21} & 1 & 0 \\ s_{31} & 0 & 1 \end{pmatrix}E^{-1},$$

therefore $\det(\Phi_+) = s_{11}$. To find the boundary condition of Φ_+ as $\zeta \to \infty$, we consider the following asymptotic expansion:

$$\Phi_+ = \Phi_+^{(0)} + \frac{1}{\zeta}\Phi_+^{(1)} + \frac{1}{\zeta^2}\Phi_+^{(2)} + O(\frac{1}{\zeta^3}). \tag{20}$$

Substituting (20) into (16) and equating terms with like powers of ζ, we find

$$[\sigma_1, \Phi_+^{(0)}] = 0, \quad [\sigma_1, \Phi_+^{(1)}] = -Q\Phi_+^{(0)}, \quad [\sigma_1, \Phi_+^{(2)}] = \Phi_{+,x}^{(0)} - \mathrm{i}Q\Phi_+^{(1)},$$

which lead to

$$\Phi_{+,x}^{(0)} = -\frac{\mathrm{i}}{3}\sigma_3 Q^2\Phi_+^{(0)}. \tag{21}$$

This means $\Phi_+ \to \Phi_+^{(0)}$ as $\zeta \to \infty$ and $\Phi_+^{(0)}$ satisfies Eq. (21). Furthermore, we have $\det(\Phi_+^{(0)}) = \exp(\mathrm{i}\eta)$ by the boundary condition and (21), where $\eta = -\frac{1}{3}\int_{-\infty}^{+\infty}(q_1r_1 + q_2r_2)\mathrm{d}x$.

To obtain the analytic counterpart of Φ_+ in $\mathbb{C}_-$, we consider the adjoint equation of spectral equation (16a),

$$K_x = \mathrm{i}\zeta^2[\sigma_0, K] - \mathrm{i}\zeta KQ. \tag{22}$$

It is easy to see that $J_\pm^{-1}$ solve above adjoint equation (22) and satisfy the boundary condition $J_\pm^{-1} \to I$ as $x \to \pm\infty$. Let $(J_\pm^{-1})^{[k]}$ be the kth row of the matrices $J_\pm^{-1}$, then

$$J_\pm = \begin{pmatrix} (J_\pm^{-1})^{[1]} \\ (J_\pm^{-1})^{[2]} \\ (J_\pm^{-1})^{[3]} \end{pmatrix}.$$

Employing the same techniques as above, we can show that

$$\Phi_-^{-1} = \begin{pmatrix} (J_-^{-1})^{[1]} \\ (J_+^{-1})^{[2]} \\ (J_+^{-1})^{[3]} \end{pmatrix}$$

are analytic for $\zeta \in \mathbb{C}_-$. Also

$$J_-^{-1} = ERE^{-1}J_+^{-1},$$

where $R = S^{-1} \equiv (r_{ij})_{3\times 3}$. Therefore, we have

$$\Phi_-^{-1} = E\begin{pmatrix} r_{11} & r_{12} & r_{13} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}E^{-1}J_+^{-1},$$

and $\det \Phi_-^{-1} = r_{11}$. Assuming $\Phi_- \to \Phi_-^{(0)}$ as $\zeta \to \infty$ and expanding Φ_-, we find that $\Phi_-^{(0)}$ also solves Eq. (21). Furthermore both $\Phi_+^{(0)}$ and $\Phi_-^{(0)}$ enjoy the same boundary condition as $x \to \pm\infty$, thus we have $\Phi_+^{(0)} = \Phi_-^{(0)} \equiv \Phi_0$.

Hence we find two matrix functions Φ_+ and Φ_- which are analytic in $\mathbb{C}_+$ and $\mathbb{C}_-$, respectively. They satisfy

$$\Phi_-^{-1}\Phi_+ = G = E\begin{pmatrix} 1 & r_{12} & r_{13} \\ s_{21} & 1 & 0 \\ s_{31} & 0 & 1 \end{pmatrix}E^{-1}, \quad \zeta \in \mathbb{R}\cup \mathrm{i}\mathbb{R}. \tag{23}$$

with boundary condition

$$\Phi_\pm \to \Phi_0 \quad \text{as} \quad \zeta \to \infty. \tag{24}$$

In the rest of the subsection, we consider the involution property such that the interesting reductions may be taken account of. The Hermitian of the spectral equation (16) reads as

$$(J_\pm^\dagger)_x = \mathrm{i}\zeta^2[\sigma_0, J_\pm^\dagger] - \mathrm{i}\zeta J_\pm^\dagger Q^\dagger \tag{25}$$

which yields

$$(B^{-1}J_\pm^\dagger B)_x = \mathrm{i}\zeta^2[\sigma_0, B^{-1}J_\pm^\dagger B] - \mathrm{i}\zeta B^{-1}J_\pm^\dagger BQ, \quad B = \mathrm{diag}\{1, 1, \sigma\}, \tag{26}$$

where $Q^\dagger = BQB^{-1}$ is used. Recalling that $J_\pm^{-1}$ fulfill the adjoint equation (22) and the boundary conditions, we have the following involution relation:

$$J_\pm^\dagger = BJ_\pm^{-1}B^{-1},$$

which in turn gives

$$\Phi_+^\dagger(\zeta^*) = B\Phi_-^{-1}(\zeta)B^{-1}. \tag{27}$$

In view of the relation $J_-E = J_+ES$, we have the involution property of the scattering matrix

$$S^\dagger(\zeta^*) = BS^{-1}(\zeta)B^{-1}. \tag{28}$$

The similar analysis shows that the Jost solutions satisfy another symmetry relation

$$J_\pm(-\zeta) = \sigma_3 J_\pm(\zeta)\sigma_3.$$

It follows that

$$\Phi_\pm(-\zeta) = \sigma_3\Phi_\pm(\zeta)\sigma_3, \tag{29}$$

and

$$S(-\zeta) = \sigma_3 S(\zeta)\sigma_3. \tag{30}$$

Next, we study the property of s_{11}, which plays an important role in later analysis. From (28), we obtain the relations

$$(r_{12}(\zeta), r_{13}(\zeta)) = (s_{21}^*(\zeta^*), s_{31}^*(\zeta^*))A, \qquad r_{11}(\zeta) = s_{11}^*(\zeta^*), \tag{31}$$

$$\begin{pmatrix} r_{21}(\zeta) \\ r_{31}(\zeta) \end{pmatrix} = A^{-1}\begin{pmatrix} s_{12}^*(\zeta^*) \\ s_{13}^*(\zeta^*) \end{pmatrix}, \qquad A = \begin{pmatrix} 1 & 0 \\ 0 & \sigma \end{pmatrix}, \tag{32}$$

and (30) gives us

$$s_{1i}(-\zeta) = -s_{1i}(\zeta), \qquad s_{i1}(-\zeta) = -s_{i1}(\zeta), \quad i = 2, 3, \tag{33}$$

$$s_{jj}(\zeta) = s_{jj}(-\zeta), \qquad s_{23}(\zeta) = s_{23}(-\zeta), \quad j = 1, 2, 3, \tag{34}$$

$$s_{32}(\zeta) = s_{32}(-\zeta). \tag{35}$$

Suppose that $\zeta_1 \in \mathbb{C}_+$ is a zero of s_{11}, then by means of (33), $-\zeta_1$ is a zero as well. The relation (31) indicates that r_{11} has two zeros, namely $\pm\zeta_1^*$. Since s_{11} is an even function of ζ, we may assume $s_{11} = s_{11}(\lambda)$ with $\lambda = \zeta^2$. In general case, we take s_{11} as

$$s_{11} = \exp(\mathrm{i}\eta)h(\lambda)\prod_{k=1}^{N}\frac{\zeta^2-\zeta_k^2}{\zeta^2-\zeta_k^{*2}}. \tag{36}$$

The relation $RS = I$ yields $r_{11}(\zeta)s_{11}(\zeta) + r_{12}(\zeta)s_{21}(\zeta) + r_{13}(\zeta)s_{31}(\zeta) = 1$ and substituting (31) into it, we have $s_{11}^*(\zeta^*)s_{11}(\zeta) + s_{21}^*(\zeta^*)s_{21}(\zeta) + \sigma s_{31}^*(\zeta^*)s_{31}(\zeta) = 1$. Therefore, for $\zeta \in \mathbb{R}$, we get

$$|s_{11}|^2 + |s_{21}|^2 + \sigma|s_{31}|^2 = 1, \tag{37}$$

and for $\zeta \in \mathrm{i}\mathbb{R}$, we achieve

$$|s_{11}|^2 - |s_{21}|^2 - \sigma|s_{31}|^2 = 1. \tag{38}$$

According to the Cauchy theorem, the factor $h(\lambda)$ in (36) for $\mathrm{Im}(\lambda) > 0$ is represented as

$$h(\lambda) = \exp\left[\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{0}\frac{\ln(1+|s_{21}|^2+\sigma|s_{31}|^2)}{\xi-\lambda}\mathrm{d}\xi + \frac{1}{2\pi\mathrm{i}}\int_{0}^{\infty}\frac{\ln(1-|s_{21}|^2-\sigma|s_{31}|^2)}{\xi-\lambda}\mathrm{d}\xi\right],$$

while for $\mathrm{Im}(\lambda) = 0$, we have $h(\lambda) = \lim_{\epsilon\to 0^+} h(\lambda + \mathrm{i}\epsilon)$.

B. Solutions for Riemann-Hilbert problem

In this subsection, we first consider the regular Riemann-Hilbert problem, i.e., $\det(\Phi_+) = s_{11} \neq 0$ and $\det(\Phi_-^{-1}) = r_{11} \neq 0$ in their analytic domain. For convenience, we introduce $\Psi_\pm = \Phi_0^{-1}\Phi_\pm$ and rewrite the Riemann-Hilbert problem and the boundary condition as

$$\Psi_-^{-1}\Psi_+ = G, \quad \zeta \in \mathbb{R}\cup\mathrm{i}\mathbb{R}, \tag{39}$$

and

$$\Psi_\pm \to I \quad \text{as} \quad \zeta \to \infty. \tag{40}$$

By Plemelj formula, the formal solution of this problem reads as (cf. Ref. 1, p. 590, Eq. (7.5.25))

$$\Psi_- = I + \frac{1}{2\pi\mathrm{i}}\int_\Gamma \frac{\Psi_-(\xi)\widehat{G}(\xi)}{\xi-\zeta}\mathrm{d}\xi, \quad \zeta \in \mathbb{C}_-,$$

where $\widehat{G} = G - I$ and $\Gamma = [0, \infty) \cup (\mathrm{i}\infty, 0] \cup [0, -\infty) \cup (-\mathrm{i}\infty, 0]$.

In what follows, we shall prove that the solution of regular Riemann-Hilbert problem (39) ($\det\Psi_\pm \neq 0$) and the canonical normalization condition (40) is unique. The argument is standard and goes as follows. Suppose that $\Psi_\pm$ and $\widetilde{\Psi}_\pm$ are two sets of solutions of (39), then $\Psi_-^{-1}\Psi_+ = \widetilde{\Psi}_-^{-1}\widetilde{\Psi}_+$, thus

$$\Psi_+\widetilde{\Psi}_+^{-1} = \Psi_-\widetilde{\Psi}_-^{-1}, \quad \zeta \in \mathbb{R}\cup\mathrm{i}\mathbb{R}. \tag{41}$$

Since $\Psi_+\widetilde{\Psi}_+^{-1}$ is analytic in $\mathbb{C}_+$ and $\Psi_-\widetilde{\Psi}_-^{-1}$ is analytic in $\mathbb{C}_-$, and on the curve $\mathbb{R}\cup\mathrm{i}\mathbb{R}$, they are equal to each other, we can define a matrix function in the whole plane by virtue of analytic continuation. Due to the boundary condition (40), this analytic function approaches the unit matrix I as $\zeta \to \infty$. Thus, we obtain

$$\Psi_+\widetilde{\Psi}_+^{-1} = \Psi_-\widetilde{\Psi}_-^{-1} = I$$

in the whole plane by Liouville's theorem. Therefore, $\Psi_\pm = \widetilde{\Psi}_\pm$, which shows the solution for Riemann-Hilbert problem (39) is unique.

Now we move to the Riemann-Hilbert problem with simple zeros. From above section we know that $\det \Phi_+ = s_{11}$ and $\det \Phi_-^{-1} = r_{11}$. Since s_{11} is analytical in $\mathbb{C}_+$ and the symmetry relation (34), we can suppose that zeros of s_{11} are $\{\pm\zeta_j \in \mathbb{C}_+, 1 \leq j \leq N\}$. It follows that symmetry relation (31) implies $\{\pm\zeta_j^* \in \mathbb{C}_-, 1 \leq j \leq N\}$ are the zeros of r_{11}. In this case, both $\ker(\Phi_+(\pm\zeta_k))$ and $\ker\left(\Phi_+(\pm\zeta_k^*)\right)$ det804 TD0 Tc(]TJ8/F5 1 Tf9.5876 0 TD0 Tc()Tj/F7 1 Tf6.9712im4128544.87233.884 618.6271 Tm()Tj/F1 1 related the analytic function as $\zeta \to +\infty$.[5] However, for the equations of the derivative Schrödinger type, it is more convenient to work in the following way. From (17) we have

$$J_{\pm}(\zeta = 0) = I.$$

Also we have $\Phi_{\pm}(\zeta = 0) = I$. Expanding $\Phi_{+}(\zeta)$ as $\zeta \to 0$

$$\Phi_{+}(\zeta) = I + \Phi_{+}^{(1)}\zeta + \Phi_{+}^{(2)}\zeta^2 + o(\zeta^2), \tag{48}$$

and substituting (48) into (16), we obtain

$$Q = -\mathrm{i}\Phi_{+,x}^{(1)}, \tag{49}$$

which is the formula we are looking for.

D. Scattering data evolution

From the solutions of the Riemann-Hilbert problem (52), we see that the scattering data needed to solve this Riemann-Hilbert problem and reconstruct the potential are

$$\{s_{21}, s_{31}, \zeta \in \mathbb{R} \cup \mathrm{i}\mathbb{R}; \pm\zeta_k, \pm\zeta_k^*, |v_k\rangle, \langle v_k|\}.$$

Noticing that J_- satisfies the temporal part of spectral equation

$$J_{-,t} = -3\mathrm{i}\zeta^4[\sigma_0, J_-] + V_2 J_-, \tag{50}$$

we have

$$\left(E_1^{-1} J_- E_1\right)_t = E_1^{-1} V_2 J_- E_1, \quad E_1 = \exp[\mathrm{i}(\zeta^2 x - 3\zeta^4 t)\sigma_0]. \tag{51}$$

Assuming that q_1, q_2 have sufficient decay at infinity, we have $V_2 \to 0$ as $x \to \pm\infty$. Evaluating of (51) at $x \to +\infty$ leads to $(\exp[3\mathrm{i}\zeta^4 t\sigma_0] S \exp[-3\mathrm{i}\zeta^4 t\sigma_0])_t = 0$. Therefore, we obtain

$$\begin{aligned}
&s_{11,t} = s_{22,t} = s_{23,t} = 0, && s_{32,t} = s_{33,t} = 0,\\
&s_{12}(t;\zeta) = s_{12}(0;\zeta)\exp(9\mathrm{i}\zeta^4 t), && s_{21}(t;\zeta) = s_{21}(0;\zeta)\exp(-9\mathrm{i}\zeta^4 t),\\
&s_{13}(t;\zeta) = s_{13}(0;\zeta)\exp(9\mathrm{i}\zeta^4 t), && s_{31}(t;\zeta) = s_{31}(0;\zeta)\exp(-9\mathrm{i}\zeta^4 t).
\end{aligned}$$

Furthermore, the Riemann-Hilbert problem becomes

$$\Phi_-^{-1}\Phi_+ = G = E_1 \begin{pmatrix} 1 & r_{12} & r_{13} \\ s_{21} & 1 & 0 \\ s_{31} & 0 & 1 \end{pmatrix} E_1^{-1}, \quad \zeta \in \mathbb{R} \cup \mathrm{i}\mathbb{R}, \tag{52}$$

with the boundary condition

$$\Phi_{\pm} \to \Phi_0 \quad \text{as} \quad \zeta \to \infty. \tag{53}$$

Thus, the analytic matrices $\Phi_{\pm}$ solve the temporal part of spectral problem (50).

To get the explicit formulae for vector $|v_j\rangle$, we differentiate the equation $\Phi_+(\zeta_j)|v_j\rangle = 0$ in x and t and find

$$|v_j\rangle_x = \mathrm{i}\zeta_j^2\sigma_0|v_j\rangle + \alpha(x)|v_j\rangle, \tag{54a}$$

$$|v_j\rangle_t = -3\mathrm{i}\zeta_j^4\sigma_0|v_j\rangle + \beta(t)|v_j\rangle, \tag{54b}$$

where $\alpha(x)$ and $\beta(t)$ are arbitrary func18 228525130./F7 1 T9326 -1.7724 Thus,9.9(f)6(if)24.8(fe(f)-0hary)-a

β

IV. SOLITON SOLUTIONS

With above analysis, we are now ready to construct a more compact formula of the N-soliton solutions for the cDNLS system (2). It is well known that the soliton solutions correspond to the vanishing of scattering coefficients, i.e., $s_{21} = s_{31} = 0$. Thus, we intend to solve the corresponding Riemann-Hilbert problem (45) and (46): $\phi_{\pm} = \Phi_0$.

We notice that (20) and (44) imply $\Phi_0 = \phi_+$. On the other hand, (48) yields

$$\Phi_0 = \left(\Gamma|_{\zeta=0}\right)^{-1}. \tag{55}$$

and

$$|z_j\rangle = \begin{pmatrix} \widehat{\alpha}_j & 0 & 0 \\ 0 & -\widehat{\alpha}_j^* & 0 \\ 0 & 0 & -\widehat{\alpha}_j^* \end{pmatrix} |w_j\rangle,$$

$$|w_j\rangle = \left[\widehat{T}_{j-1}(\eta)|_{\eta=\eta_j} \cdots \widehat{T}_1(\eta)|_{\eta=\eta_j}\right] |v_j\rangle, \quad \langle w_j| = |w_j\rangle^\dagger,$$

$$\widehat{\alpha}_j = \frac{\eta_j^2 - \eta_j^{*2}}{\eta_j(\langle w_j|B\sigma_3|w_j\rangle - \langle w_j|B|w_j\rangle) - \eta_j^*(\langle w_j|B\sigma_3|w_j\rangle + \langle w_j|B|w_j\rangle)}.$$

Direct calculation shows that

$$(\widetilde{T}_i|_{\zeta=0})^{-1}\widetilde{T}_i = \widehat{T}_i, \quad \widetilde{T}_i^{-1}\widetilde{T}_i|_{\zeta=0} = \widehat{T}_i^{-1}.$$

From above discussion, we may take the following more convenient forms for $(\Gamma|_{\zeta=0})^{-1}\Gamma$ and its inverse, i.e.,

$$(\Gamma|_{\zeta=0})^{-1}\Gamma = I - \sum_{j=1}^{N}\left[\frac{D_j}{\eta - \eta_j^*} - \frac{\sigma_3 D_j \sigma_3}{\eta + \eta_j^*}\right],$$

and

$$\Gamma^{-1}(\Gamma|_{\zeta=0}) = I - \sum_{j=1}^{N}\left[\frac{B^{-1}D_j^\dagger B}{\eta - \eta_j} - \frac{\sigma_3 B^{-1} D_j^\dagger B \sigma_3}{\eta + \eta_j}\right],$$

where $D_i = |x_i\rangle\langle y_i|$. These equations enable us to have

$$\Phi_+^{(1)} = \sum_{j=1}^{N}(\sigma_3 D_j \sigma_3 - D_j),$$

from it, taking the relevant matrix entries (1, 2) and (1, 3) yields

$$q_1 = \mathrm{i}\sum_{j=1}^{N}\left[\left(D_j - \sigma_3 D_j \sigma_3\right)_{12}\right]_x, \tag{59}$$

$$q_2 = \mathrm{i}\sum_{j=1}^{N}\left[\left(D_j - \sigma_3 D_j \sigma_3\right)_{13}\right]_x. \tag{60}$$

To determine the matrices D_j, we consider $(\Gamma|_{\zeta=0})^{-1}\Gamma(\zeta)\Gamma(\zeta)^{-1}(\Gamma|_{\zeta=0}) = I$ and its Taylor expansion at $\eta = \eta_l$, then we obtain

$$\left[I - \sum_{j=1}^{N}\left(\frac{|x_j\rangle\langle y_j|B}{\eta_l - \eta_j^*} - \frac{\sigma_3|x_j\rangle\langle y_j|B\sigma_3}{\eta_l + \eta_j^*}\right)\right]|y_l\rangle = 0, \quad l = 1, 2, \ldots N.$$

This system and the conditions (42) indicates that we could assume $|y_l\rangle = |v_l\rangle$. Then $|x_j\rangle$'s obey

$$|y_l\rangle_1 = \sum_{j=1}^{N}|x_j\rangle_1(M_{lj}), \tag{61}$$

where $|y_l\rangle_1$ and $|x_j\rangle_1$ are the first components for the vectors $|y_l\rangle$ and $|x_j\rangle$, respectively, and

$$M_{lj} = \frac{\eta_j^*(\langle y_j|B|y_l\rangle + \langle y_j|B\sigma_3|y_l\rangle) + \eta_l(\langle y_j|B|y_l\rangle - \langle y_j|B\sigma_3|y_l\rangle)}{\eta_l^2 - \eta_j^{*2}}.$$

Solving (61), we obtain

$$\begin{pmatrix} |x_1\rangle_1 \\ |x_2\rangle_1 \\ \vdots \\ |x_N\rangle_1 \end{pmatrix}$$

A. Single-soliton solutions

To obtain the single soliton solution, we set $N = 1$ in formula (63). The solution so obtained by the Riemann-Hilbert method, which is consistent with Darboux transformation,[14] reads as

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \mathrm{i}\frac{\zeta_1^{*2}-\zeta_1^2}{|\zeta_1|^2}\left[\frac{1}{\zeta_1 e^{\gamma_1-\mathrm{i}\beta_1}+\zeta_1^*(|a_1|^2+\sigma|b_1|^2)e^{-\gamma_1-\mathrm{i}\beta_1}}\right]_x \begin{pmatrix} a_1^* \\ \sigma b_1^* \end{pmatrix}, \tag{64}$$

or

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \frac{6\zeta_{1R}\zeta_{1I}}{\delta_1\sqrt{|\zeta_1|^2\cosh^2(\gamma_1-\ln\delta_1)-\zeta_{1I}^2}}\begin{pmatrix} |a_1|e^{\mathrm{i}\phi_1} \\ \sigma|b_1|e^{\mathrm{i}\psi_1} \end{pmatrix}, \tag{65}$$

where $\delta_1 = \sqrt{|a_1|^2+\sigma|b_1|^2}$ and

$$\phi_1 = -\frac{\mathrm{i}}{2}\ln\left[\frac{a_1^*(\zeta_1\delta_1^2 e^{-\gamma_1}+\zeta_1^* e^{\gamma_1})^3}{a_1(\zeta_1^*\delta_1^2 e^{-\gamma_1}+\zeta_1 e^{\gamma_1})^3}\right]+\beta_1+\frac{3}{2}\pi,$$

$$\psi_1 = -\frac{\mathrm{i}}{2}\ln\left[\frac{b_1^*(\zeta_1\delta_1^2 e^{-\gamma_1}+\zeta_1^* e^{\gamma_1})^3}{b_1(\zeta_1^*\delta_1^2 e^{-\gamma_1}+\zeta_1 e^{\gamma_1})^3}\right]+\beta_1+\frac{3}{2}\pi.$$

Thus, the velocity for the single soliton is $v_1 = 6[\zeta_{1R}^2-\zeta_{1I}^2]$, and its center both for $|q_1|^2$ and $|q_2|^2$ locates on the line

$$x-6v_1t-\frac{\ln\delta_1}{12m_1}=0.$$

The amplitudes associated with $|q_1|^2$ and $|q_2|^2$ are given by

$$A(q_1)=\frac{36|a_1|^2\zeta_{1I}^2}{|a_1|^2+\sigma|b_1|^2},\quad A(q_2)=\frac{36|b_1|^2\zeta_{1I}^2}{|a_1|^2+\sigma|b_1|^2},$$

respectively. We remark that this soliton solution has two interesting properties which distinguish it from the standard NLS soliton. First, the soliton has nonzero phase difference at its limits. Indeed,

$$\frac{\zeta_1^* e^{\gamma_1}+\zeta_1(|a_1|^2+\sigma|b_1|^2)e^{-\gamma_1}}{[\zeta_1 e^{\gamma_1}+\zeta_1^*(|a_1|^2+\sigma|b_1|^2)e^{-\gamma_1}]^2} \to \begin{cases} \frac{\zeta_1^*}{\zeta_1^2}e^{-\gamma_1}, \ \gamma_1\to+\infty \\ \frac{\zeta_1}{\zeta_1^{*2}}e^{\gamma_1}, \ \ \gamma_1\to-\infty \end{cases}$$

it follows that

$$\arg(q_1(\gamma_1\to-\infty))-\arg(q_1(\gamma_1\to+\infty)) = 6\arg(\zeta_1)\neq 0,$$

$$\arg(q_2(\gamma_1\to-\infty))-\arg(q_2(\gamma_1\to+\infty)) = 6\arg(\zeta_1)\neq 0.$$

Second, the important invariant of the cDNLS equation, namely, number of particles $\int_{-\infty}^{+\infty}(|q_1|^2 + |q_2|^2)\mathrm{d}x$, has the upper limit

$$\int_{-\infty}^{+\infty}(|q_1|^2+|q_2|^2)\mathrm{d}x = 6\arg(\zeta_1)\frac{|a_1|^2+|b_1|^2}{|a_1|^2+\sigma|b_1|^2} < 3\pi\frac{|a_1|^2+|b_1|^2}{|a_1|^2+\sigma|b_1|^2}.$$

We first consider the case $t\to -\infty$. The analysis relays on (57), i.e.,

$$\Phi_+ = \widehat{T}_k\widehat{T}_N\cdots\widehat{T}_{k+1}\widehat{T}_{k-1}\cdots\widehat{T}_1. \tag{66}$$

For $|y_l\rangle$ $(1 < l < N)$, we assume

$$\text{for}\, i\le k: |y_i\rangle = \begin{pmatrix} \exp(\gamma_i+\mathrm{i}\beta_i)\\ a_i\\ b_i\end{pmatrix}, \quad \text{and for}\, j>k: |y_j\rangle = \begin{pmatrix} 1\\ a_j\exp(-\gamma_j-\mathrm{i}\beta_j)\\ b_j\exp(-\gamma_j-\mathrm{i}\beta_j)\end{pmatrix}, \tag{67}$$

which provide

$$\text{for } i<k:\ |y_i\rangle_- \equiv \lim_{t\to-\infty}|y_i\rangle = \begin{pmatrix}0\\ a_i\\ b_i\end{pmatrix}, \quad \text{and for } j>k:\ |y_j\rangle_- \equiv \lim_{t\to-\infty}|y_j\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix}. \tag{68}$$

With the help of above $|y_i\rangle_-$, we arrive at

$$\widehat{T}_i = \begin{cases} I+\frac{\eta_i^{*2}-\eta_i^2}{\eta^2-\eta_i^{*2}}\begin{pmatrix}0 & 0_{1\times 2}\\ 0_{2\times 1} & \widehat{P}_i\end{pmatrix}, & \text{if } i<k\\[2ex] I+\frac{\eta_i^{*2}-\eta_i^2}{\eta^2-\eta_i^{*2}}\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}, & \text{if } i>k\end{cases}$$

where

$$\widehat{P}_i = \frac{|w_i\rangle\langle w_i|A}{\langle w_i|A|w_i\rangle}, \quad |w_i\rangle = \left(I+\frac{\eta_{i-1}^{*2}-\eta_{i-1}^2}{\eta_i^2-\eta_{i-1}^{*2}}\widehat{P}_{i-1}\right)\cdots\left(I+\frac{\eta_1^{*2}-\eta_1^2}{\eta_i^2-\eta_1^{*2}}\widehat{P}_1\right)|y_i\rangle,$$

$$A = \begin{pmatrix}1&0\\0&\sigma\end{pmatrix}, \quad |y_i\rangle = \begin{pmatrix}a_i\\ b_i\end{pmatrix}, \quad |w_1\rangle = |y_1\rangle, \quad \langle w_i| = |w_i\rangle^\dagger.$$

For $i=k$ we have

$$\widehat{T}_k = I+\frac{\widehat{C}_k}{\eta-\eta_k^*}-\frac{\sigma_3\widehat{C}_k\sigma_3}{\eta+\eta_k^*}, \quad \widehat{C}_k = |z_k\rangle\langle w_k|B,$$

where

$$|z_k\rangle = \begin{pmatrix}\widehat{\alpha}_k & 0 & 0\\ 0 & -\widehat{\alpha}_k{}^* & 0\\ 0&0&-\widehat{\alpha}_k{}^*\end{pmatrix}|w_k\rangle, \quad |w_k\rangle = \left(\widehat{T}_N\widehat{T}_{N-1}\cdots\widehat{T}_{k+1}\widehat{T}_{k-1}\cdots\widehat{T}_1\right)|_{\eta=\eta_k}|y_k\rangle,$$

$$\widehat{\alpha}_k = \frac{\eta_k^2-\eta_k^{*2}}{\eta_k(\langle w_k|B\sigma_3|w_k\rangle-\langle w_k|B|w_k\rangle)-\eta_k^*(\langle w_k|B\sigma_3|w_k\rangle+\langle w_k|B|w_k\rangle)}.$$

Thus using (66) and taking the limit we are lead to

$$\begin{pmatrix} q_{1,k}(t\to-\infty)\\ q_{2,k}(t\to-\infty)\end{pmatrix} = \frac{6\zeta_{kR}\zeta_{kI}}{\delta_{k,-}\sqrt{|\zeta_k|^2\cosh^2(\gamma_k-\ln\delta_{k,-})-\zeta_{kI}^2}}\begin{pmatrix}|\widehat{a}_k|e^{\mathrm{i}\phi_{k,-}}\\ |\widehat{b}_k|e^{\mathrm{i}\psi_{k,-}}\end{pmatrix}, \tag{69}$$

where $\delta_{k,-} = \sqrt{|\widehat{a_k}|^2 + \sigma|\widehat{b_k}|^2}$,

$$\phi_{k,-} = -\frac{\mathrm{i}}{2}\ln\left[\frac{\widehat{a_k}^*(\zeta_k\delta_{k,-}^2 e^{-\gamma_k} + \zeta_k^* e^{\gamma_k})^3}{\widehat{a_k}(\zeta_k^*\delta_{k,-}^2 e^{-\gamma_k} + \zeta_k e^{\gamma_k})^3}\right] + \beta_k + \frac{3}{2}\pi,$$

$$\psi_{k,-} = -\frac{\mathrm{i}}{2}\ln\left[\frac{\widehat{b_k}^*(\zeta_k\delta_{k,-}^2 e^{-\gamma_k} + \zeta_k^* e^{\gamma_k})^3}{\widehat{b_k}(\zeta_k^*\delta_{k,-}^2 e^{-\gamma_k} + \zeta_k e^{\gamma_k})^3}\right] + \beta_k + \frac{3}{2}\pi,$$

and

$$\begin{pmatrix}\widehat{a_k}\\ \widehat{b_k}\end{pmatrix} = \frac{\eta_k^2 - \eta_N^{*2}}{\eta_k^2 - \eta_N^2}\cdots\frac{\eta_k^2 - \eta_{k+1}^{*2}}{\eta_k^2 - \eta_{k+1}^2}\left(I + \frac{\eta_{k-1}^{*2} - \eta_{k-1}^2}{\eta_k^2 - \eta_{k-1}^{*2}}\widehat{P}_{k-1}\right)\cdots\left(I + \frac{\eta_1^{*2} - \eta_1^2}{\eta_k^2 - \eta_1^{*2}}\widehat{P}_1\right)\begin{pmatrix}a_k\\ b_k\end{pmatrix}.$$

Therefore, the relevant data for the kth soliton follow. Its velocity is $v_k = 6[\zeta_{kR} - \zeta_{kI}]$, the centers both for $|q_{1,k}(t \to -\infty)|^2$ and $|q_{2,k}(t \to -\infty)|^2$ locate on the line

$$x - 6v_k t - \frac{\ln\delta_{k,-}}{12m_k} = 0,$$

and the amplitudes related to $|q_1(t \to -\infty)|^2$ and $|q_2(t \to -\infty)|^2$ are given by

$$A(q_{1,k}(t \to -\infty)) = \frac{36|\widehat{a_k}|^2\zeta_{kI}^2}{|\widehat{a_k}|^2 + \sigma|\widehat{b_k}|^2}, \quad A(q_{2,k}(t \to -\infty)) = \frac{36|\widehat{b_k}|^2\zeta_{kI}^2}{|\widehat{a_k}|^2 + \sigma|\widehat{b_k}|^2},$$

respectively.

Next we turn to the case $t \to +\infty$ and take

$$\text{for } i < k: \ |y_i\rangle = \begin{pmatrix}1\\ a_i\exp(-\gamma_i - \mathrm{i}\beta_i)\\ b_i\exp(-\gamma_i - \mathrm{i}\beta_i)\end{pmatrix}, \quad \text{and for } j \ge k: \ |y_j\rangle = \begin{pmatrix}\exp(\gamma_j + \mathrm{i}\beta_j)\\ a_j\\ b_j\end{pmatrix}, \tag{70}$$

so that

$$\text{for } i < k: \ |y_i\rangle_+ \equiv \lim_{t\to-\infty}|y_i\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad \text{and for } j \ge k: \ |y_j\rangle_+ \equiv \lim_{t\to-\infty}|y_j\rangle = \begin{pmatrix}0\\ a_j\\ b_j\end{pmatrix}. \tag{71}$$

Above formulas may be employed to find the limit of Φ_+ represented by

$$\Phi_+ = \widetilde{T}_k\widetilde{T}_N\cdots\widetilde{T}_{k+1}\widetilde{T}_{k-1}\cdots\widetilde{T}_1, \tag{72}$$

Indeed, $\widetilde{T}_i$'s are given by

$$\widetilde{T}_i = \begin{cases} I + \frac{\eta_i^{*2} - \eta_i^2}{\eta^2 - \eta_i^{*2}}\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, & \text{if } i < k\\ I + \frac{\eta_i^{*2} - \eta_i^2}{\eta^2 - \eta_i^{*2}}\begin{pmatrix}0 & 0_{1\times 2}\\ 0_{2\times 1} & \widetilde{P}_i\end{pmatrix}, & \text{if } i > k\end{cases}$$

where

$$\widetilde{P}_j = \frac{|w_j\rangle\langle w_j|A}{\langle w_j|A|w_j\rangle}, \quad \langle w_j| = |w_j\rangle^\dagger, \quad |y_j\rangle = \begin{pmatrix}a_j\\ b_j\end{pmatrix},$$

$$|w_j\rangle = \left(I + \frac{\eta_{j-1}^{*2} - \eta_{j-1}^2}{\eta_j^2 - \eta_{j-1}^{*2}}\widetilde{P_{j-1}}\right)\cdots\left(I + \frac{\eta_{k+1}^{*2} - \eta_{k+1}^2}{\eta_j^2 - \eta_{k+1}^{*2}}\widetilde{P_{k+1}}\right)|y_j\rangle.$$

Finally, $|y_k\rangle$ reads as

$$\widetilde{T}_k = I + \frac{\widetilde{C}_k}{\eta - \eta_k^*} - \frac{\sigma_3 \widetilde{C}_k \sigma_3}{\eta + \eta_k^*}, \quad \widetilde{C}_k = |z_k\rangle\langle w_k| B,$$

$$|z_k\rangle = \begin{pmatrix} \widetilde{\alpha_k} & 0 & 0 \\ 0 & -\widetilde{\alpha_k}^* & 0 \\ 0 & 0 & -\widetilde{\alpha_k}^* \end{pmatrix} |w_k\rangle, \quad |w_k\rangle = \left(\widetilde{T}_N \widetilde{T}_{N-1} \cdots \widetilde{T}_{k+1} \widetilde{T}_{k-1} \cdots \widetilde{T}_1\right)|_{\eta=\eta_k} |y_k\rangle,$$

$$\widetilde{\alpha_k} = \frac{\eta_k^2 - \eta_k^{*2}}{\eta_k(\langle w_k|B\sigma_3|w_k\rangle - \langle w_k|B|w_k\rangle) - \eta_k^*(\langle w_k|B\sigma_3|w_k\rangle + \langle w_k|B|w_k\rangle)}.$$

Therefore, by means of (72) we obtain

$$\begin{pmatrix} q_{1,k}(t \to +\infty) \\ q_{2,k}(t \to +\infty) \end{pmatrix} = \frac{6\zeta_{kR}\zeta_{kI}}{\delta_{k,+}\sqrt{|\zeta_k|^2 \cosh^2(\gamma_k - \ln \delta_{k,+}) - \zeta_{kI}^2}} \begin{pmatrix} |\widetilde{a_k}| e^{\mathrm{i}\phi_{k,+}} \\ |\widetilde{b_k}| e^{\mathrm{i}\psi_{k,+}} \end{pmatrix}, \tag{73}$$

where $\delta_{k,+} = \sqrt{|\widetilde{a_k}|^2 + \sigma|\widetilde{b_k}|^2}$,

$$\phi_{k,+} = -\frac{\mathrm{i}}{2} \ln \left[\frac{\widetilde{a_k}^*(\zeta_k \delta_{k,+}^2 e^{-\gamma_k} + \zeta_k^* e^{\gamma_k})^3}{\widetilde{a_k}(\zeta_k^* \delta_{k,+}^2 e^{-\gamma_k} + \zeta_k e^{\gamma_k})^3} \right] + \beta_k + \frac{3}{2}\pi,$$

$$\psi_{k,+} = -\frac{\mathrm{i}}{2} \ln \left[\frac{\widetilde{b_k}^*(\zeta_k \delta_{k,+}^2 e^{-\gamma_k} + \zeta_k^* e^{\gamma_k})^3}{\widetilde{b_k}(\zeta_k^* \delta_{k,+}^2 e^{-\gamma_k} + \zeta_k e^{\gamma_k})^3} \right] + \beta_k + \frac{3}{2}\pi,$$

and

$$\begin{pmatrix} \widetilde{a_k} \\ \widetilde{b_k} \end{pmatrix} = \frac{\eta_k^2 - \eta_{k-1}^{*2}}{\eta_k^2 - \eta_{k-1}^2} \cdots \frac{\eta_k^2 - \eta_1^{*2}}{\eta_k^2 - \eta_1^2} \left(I + \frac{\eta_N^{*2} - \eta_N^2}{\eta_k^2 - \eta_N^{*2}} \widetilde{P}_N \right) \cdots \left(I + \frac{\eta_{k+1}^{*2} - \eta_{k+1}^2}{\eta_k^2 - \eta_{k+1}^{*2}} \widetilde{P}_{k+1} \right) \begin{pmatrix} a_k \\ b_k \end{pmatrix}.$$

Equation (73) allows us to read off the relevant data for the kth soliton at this limit. Explicitly, its velocity is $v_k = 6[\zeta_{kR} - \zeta_{kI}]$, the centers of $|q_{1,\,k}(t \to +\infty)|^2$ and $|q_{2,\,k}(t \to +\infty)|^2$ are along the line

$$x - 6v_k t - \frac{\ln \delta_{k,+}}{12 m_k} = 0,$$

and the amplitudes for both $|q_1(t \to +\infty)|^2$ and $|q_2(t \to +\infty)|^2$ are given by

$$A(q_{1,k}(t \to +\infty)) = \frac{36|\widetilde{a_k}|^2 \zeta_{kI}}{|\widetilde{a_k}|^2 + \sigma|\widetilde{b_k}|^2}, \quad A(q_{2,k}(t \to +\infty)) = \frac{36|\widetilde{b_k}|^2 \zeta_{kI}}{|\widetilde{a_k}|^2 + \sigma|\widetilde{b_k}|^2},$$

respectively.

Therefore, the position variation of the kth soliton (either for q_1 or q_2) is

$$\Delta x_k = \frac{1}{12\zeta_{kR}\zeta_{kI}} \ln \left| \frac{|\widehat{a_k}|^2 + \sigma|\widehat{b_k}|^2}{|\widetilde{a_k}|^2 + \sigma|\widetilde{b_k}|^2} \right|,$$

and the amplitude of $|q_1|^2$ changes from

$$\frac{36|\widehat{a_k}|^2 \zeta_{kI}}{|\widehat{a_k}|^2 + \sigma|\widehat{b_k}|^2} \quad \text{to} \quad \frac{36|\widetilde{a_k}|^2 \zeta_{kI}}{|\widetilde{a_k}|^2 + \sigma|\widetilde{b_k}|^2},$$

and for $|q_2|^2$ it changes from

$$\frac{36|\widehat{b_k}|^2 \zeta_{kI}}{|\widehat{a_k}|^2 + \sigma|\widehat{b_k}|^2} \quad \text{to} \quad \frac{36|\widetilde{b_k}|^2 \zeta_{kI}}{|\widetilde{a_k}|^2 + \sigma|\widetilde{b_k}|^2}.$$

As a final remark, it is pointed out that, since the DNLS system is a reduction of the cDNLS system, we may obtain the asymptotic behavior for the former. In fact, assuming $b_k = 0$ ($k = 1, 2, \ldots, N$), we have from (69)

$$q_{1,k}(t \to -\infty) = \frac{6\zeta_{kR}\zeta_{kI}e^{\mathrm{i}\phi_{k,-}}}{\delta_{k,-}\sqrt{|\zeta_k|^2\cosh^2(\gamma_k - \ln\delta_{k,-}) - \zeta_{kI}^2}}, \tag{74}$$

where $\delta_{k,-} = |\widehat{a_k}|$,

$$\phi_{k,-} = -\frac{\mathrm{i}}{2}\ln\left[\frac{\widehat{a_k}^*(\zeta_k\delta_{k,-}^2e^{-\gamma_k} + \zeta_k^*e^{\gamma_k})^3}{\widehat{a_k}(\zeta_k^*\delta_{k,-}^2e^{-\gamma_k} + \zeta_ke^{\gamma_k})^3}\right] + \beta_k + \frac{3}{2}\pi,$$

$$\widehat{a_k} = \frac{\eta_k^2 - \eta_N^{*2}}{\eta_k^2 - \eta_N^2}\cdots\frac{\eta_k^2 - \eta_{k+1}^{*2}}{\eta_k^2 - \eta_{k+1}^2}\frac{\eta_k^2 - \eta_{k-1}^2}{\eta_k^2 - \eta_{k-1}^{*2}}\cdots\frac{\eta_k^2 - \eta_1^2}{\eta_k^2 - \eta_1^{*2}}a_k.$$

And from (73) we get

$$q_{1,k}(t \to +\infty) = \frac{6\zeta_{kR}\zeta_{kI}e^{\mathrm{i}\phi_{k,+}}}{\delta_{k,+}\sqrt{|\zeta_k|^2\cosh^2(\gamma_k - \ln\delta_{k,+}) - \zeta_{kI}^2}}, \tag{75}$$

where $\delta_{k,+} = |\widetilde{a_k}|$,

$$\phi_{k,+} = -\frac{\mathrm{i}}{2}\ln\left[\frac{\widehat{a_k}^*(\zeta_k\delta_{k,+}^2e^{-\gamma_k} + \zeta_k^*e^{\gamma_k})^3}{\widehat{a_k}(\zeta_k^*\delta_{k,+}^2e^{-\gamma_k} + \zeta_ke^{\gamma_k})^3}\right] + \beta_k + \frac{3}{2}\pi,$$

$$\widetilde{a_k} = \frac{\eta_k^2 - \eta_N^2}{\eta_k^2 - \eta_N^{*2}}\cdots\frac{\eta_k^2 - \eta_{k+1}^2}{\eta_k^2 - \eta_{k+1}^{*2}}\frac{\eta_k^2 - \eta_{k-1}^{*2}}{\eta_k^2 - \eta_{k-1}^2}\cdots\frac{\eta_k^2 - \eta_1^{*2}}{\eta_k^2 - \eta_1^2}a_k.$$

The position variation of kth soliton is simply written as

$$\Delta x_k = \frac{1}{6\zeta_{kR}\zeta_{kI}}\left[\sum_{j=k+1}^{N}\ln\left|\frac{(\zeta_k^2 - \zeta_j^{*2})\zeta_j^2}{(\zeta_k^2 - \zeta_j^2)\zeta_j^{*2}}\right| - \sum_{j=1}^{k-1}\ln\left|\frac{(\zeta_k^2 - \zeta_j^{*2})\zeta_j^2}{(\zeta_k^2 - \zeta_j^2)\zeta_j^{*2}}\right|\right].$$

In particular, when $N = 2$ and $v_1 = v_2$, namely, two-soliton solution case with specific velocities, the width for two-soliton changes periodically with the time, as for the nonlinear Schrödinger equation,[5] this solution is called a "breather." The temporal period of this breather is $\pi/[18|m_1^2 - m_2^2|]$.

V. COUPLED FOKAS-LENELLS EQUATIONS AND N-SOLITON SOLUTION

The purpose of this section is to derive a coupled Fokas-Lenells equation and give its simple N-soliton solutions. In fact, the Fokas-Lenells equation itself is related by a gauge transformation to the first negative member of the integrable hierarchy of the derivative NLS equation.[11] The initial-boundary value problem for the Fokas-Lenells equation on the half-line was studied by Lenells and Fokas in Ref. 12. A simple N-bright-soliton solution was given by Lenells[13] and the N-dark soliton solution was obtained by means of Bäcklund transformation.[21] We first consider the negative flow for the Kaup-Newell hierarchy and recall the derivation of the Fokas-Lenells equation. Thus, instead of (7) we do the following expanding:

$$\begin{pmatrix} B \\ C \end{pmatrix} = \sum_{i=1}^{N}\begin{pmatrix} b_i \\ c_i \end{pmatrix}\zeta^{1-2i}, \tag{76}$$

and $A_0 = -\alpha D_0 = -\zeta^{-2N}\beta\mathrm{i}$, then we obtain the hierarchy from (6)

$$\mathrm{i}\begin{pmatrix} q \\ r \end{pmatrix}_t + \beta(1+\alpha)(L\partial_x^{-1})^N\sigma_3\begin{pmatrix} q \\ r \end{pmatrix} = 0. \tag{77}$$

Introducing potentials

$$\begin{pmatrix} q \\ r \end{pmatrix} = \begin{pmatrix} u_x \\ v_x \end{pmatrix}$$

and taking $N = 1$, $\alpha = 2$ and $\beta = \frac{1}{3}$, we arrive at the first negative flow

$$u_{xt} = 3u - 2\mathrm{i}uvu_x, \tag{78a}$$

$$v_{xt} = 3v + 2\mathrm{i}uvv_x, \tag{78b}$$

which, under the reduction $v = u^*$, is nothing but the Fokas-Lenells equation[21]

$$u_{xt} - 3u + 2\mathrm{i}|u|^2 u_x = 0. \tag{79}$$

Remark: In Refs. 11–13, the Fokas-Lenells equation was given as

$$\psi_{\xi\tau} + \psi - \psi_{\xi\xi} - 2\mathrm{i}\psi_\xi - \mathrm{i}|\psi|^2\psi_\xi = 0, \tag{80}$$

which, by the following gauge and coordinate transformations

$$u(x,t) = \frac{\sqrt{2}}{2}\psi(\xi,\tau)e^{-2\mathrm{i}\tau}, \quad x = \frac{1}{3}(\xi + \tau), \quad t = -\tau$$

may be converted into (79).

The Lax pair for this system (78) are

$$\Phi_x = U\Phi, \qquad U = \mathrm{i}\zeta^2\sigma_0 + \mathrm{i}\zeta U_{1x}, \tag{81a}$$

$$\Phi_t = V\Phi, \qquad V = -\frac{1}{3\zeta^2}\mathrm{i}\sigma_0 + \frac{1}{\zeta}\sigma_3 U_1 - \mathrm{i}\sigma_3 U_1^2, \tag{81b}$$

where

$$U_1 = \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix}.$$

As in the case of the N-component DNLS discussed in Sec. I, we modify the relevant matrices and introduce

$$\sigma_0 = \begin{pmatrix} -2 & 0_{1\times N} \\ 0_{N\times 1} & I_{N\times N} \end{pmatrix}, \quad U_1 = \begin{pmatrix} 0 & \mathbf{v}^T \\ \mathbf{u} & 0_{N\times N} \end{pmatrix},$$

where $\mathbf{u} = (u_1, u_2, \ldots, u_N)^T$ and $\mathbf{v} = (v_1, v_2, \ldots, v_N)^T$, then the N-component Fokas-Lenells equations are resulted from the compatibility condition (81). With the further reduction relation $\mathbf{v} = A^T\mathbf{u}^*$, we obtain

$$\mathbf{u}_{xt} - 3\mathbf{u} + \mathrm{i}(\mathbf{u}_x\mathbf{u}^\dagger A\mathbf{u} + \mathbf{u}\mathbf{u}^\dagger A\mathbf{u}_x) = 0. \tag{82}$$

For simplicity, we merely consider the simplest non-trivial case, the coupled Fokas-Lenells system. The matrices U and V in the spectral problem are given explicitly by

$$U = \mathrm{i}\zeta^2\sigma_0 + \mathrm{i}\zeta U_{1,x}, \tag{83a}$$

$$V = -\frac{1}{3\zeta^2}\mathrm{i}\sigma_0 + \frac{1}{\zeta}\sigma_3 U_1 - \mathrm{i}\sigma_3 U_1^2, \tag{83b}$$

where

$$U_1 = \begin{pmatrix} 0 & u_1 & u_2 \\ v_1 & 0 & 0 \\ v_2 & 0 & 0 \end{pmatrix}.$$

The corresponding coupled Fokas-Lenells equation under the reduction $v_1 = u_1^*$, $v_2 = \sigma u_2^*$ reads as

$$u_{1,xt} = 3u_1 - \mathrm{i}(2|u_1|^2 u_{1,x} + \sigma u_2^* u_1 u_{2,x} + \sigma |u_2|^2 u_{1,x}), \tag{84a}$$

$$u_{2,xt} = 3u_2 - \mathrm{i}(2\sigma |u_2|^2 u_{2,x} + u_1^* u_2 u_{1,x} + |u_1|^2 u_{2,x}). \tag{84b}$$

Since the coupled Fokas-Lenells system shares the same spatial part of the spectral problem with the cDNLS system, we can easily find the former's N-soliton solution, which is given by

$$u_1 = -\mathrm{i}\frac{\widetilde{M}_1}{\widetilde{M}}, \quad u_2 = -\mathrm{i}\sigma \frac{\widetilde{M}_2}{\widetilde{M}}, \tag{85}$$

where

$$\widetilde{M}_1 = \begin{vmatrix} \widetilde{M}_{11} & \cdots & \widetilde{M}_{1N} & \exp(\gamma_1 + \mathrm{i}\beta_1) \\ \vdots & \vdots & \vdots & \vdots \\ \widetilde{M}_{N1} & \cdots & \widetilde{M}_{NN} & \exp(\gamma_N + \mathrm{i}\beta_N) \\ a_1^* & \cdots & a_N^* & 0 \end{vmatrix}, \qquad \widetilde{M}_2 = \begin{vmatrix} \widetilde{M}_{11} & \cdots & \widetilde{M}_{1N} & \exp(\gamma_1 + \mathrm{i}\beta_1) \\ \vdots & \vdots & \vdots & \vdots \\ \widetilde{M}_{N1} & \cdots & \widetilde{M}_{NN} & \exp(\gamma_N + \mathrm{i}\beta_N) \\ b_1^* & \cdots & b_N^* & 0 \end{vmatrix},$$

and $\widetilde{M} = \left|(\widetilde{M}_{lj})_{N\times N}\right|$ with

$$\widetilde{M}_{lj} = \frac{\zeta_j^* \zeta_l [\zeta_l^2 e^{\gamma_j + \gamma_l + \mathrm{i}(\beta_l - \beta_j)} + \zeta_l^{*2}(a_j^* a_l + b_j^* b_l)]}{\zeta_j^{*2} - \zeta_l^2},$$

$$\gamma_j = 6m_j(x + v_j t), \quad \beta_j = -3n_j(x - v_j t), \quad v_j^{-1} = 3[\mathrm{Re}^2(\zeta_j) + \mathrm{Im}^2(\zeta_j)],$$

$$m_j = \mathrm{Re}(\zeta_j)\mathrm{Im}(\zeta_j), \ \ n_j = -3[\mathrm{Re}^2(\zeta_j) - \mathrm{Im}^2(\zeta_j)].$$

VI. DISCUSSIONS

The inverse scattering method has been applied to the cDNLS system and by studying the associated Riemann-Hilbert problem, we have successfully constructed a simple representation for the N-soliton solutions for this system. It is remarked that we merely considered the simple zeros for s_{11} of the scattering matrix. The more general case – the case of multiple zeros would lead to more general solutions and may be studied in the future.

For cDNLS system, we were considering the solutions with vanishing boundary conditions. A modification of our analysis may supply certain solutions corresponding non-vanishing boundary conditions. What we need to do is to seek the Jost solutions of the spectral problem (14) with q_1 and q_2 as plane wave solutions.

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