High-Order Solutions and Generalized Darboux Transformations of Derivative Nonlinear Schrödinger Equations

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B ea Vf ce t a t tech e, t V d Vf ge e a ed Da bV t a fV a V a e cV t d ed fV the de a e V ea Sch Vd ge e a V (DNLS). The e t a fV a V a e hV t V ead t Vt V V t V fV a fV DNLS t e Vf d e a t . A a ca V , e e a d ffe e t t e Vf h gh-V de V t V a e ca c a ed fV th e a V .

1. Introduction

The de a e V ea Sch Vd ge e a V (DNLS) [1, 2]

$$u_t + u_{xx} + (|u|^2 u)_x = 0, \tag{1}$$

ha a h ca a ca V , e ec a ace a a h c a d V ea Vt c . I e de c be a -a t de V ea A f e a e a V -β a a, V aga g t d a a e V a a a a g e t V the a b e t ag d c e d. I a hV tha the DNLS a V Vde a ge-a t de ag d Vh d Vd a c (MHD) a e a h gh-β a a V aga g a a a b t a a g e t V the a b e t ag d c e d. I V ea Vt c , the Vd ed V ea Sch Vd ge e a V [3], h ch ga ge e a e t t V DNLS, a e t he t he V Vf t a hVt fe t V ec V d V ea e Vt ca b e , he the ed a d h Vf the e bec V e c V a ab e t h the ca e f e e c a d the effed Vf e f- t ee e g Vf the e hV d be t a e t V accV t.

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H gh-Vde VtV de c be the te ad V bd ee *N* VtV Vf e a a t de b t ha g a at c a ch [4]. I the te VVg Vf e e catte g t a fV aV (IST), the cV e V d tV t e- Ve VtV . I the ca e Vf the KVte eg-de V e e aV (KdV), the Ve t be e, that the ea V h h gh-Vde V g a VtV dV Vf e t. I deed e cV d Vta t e- Ve V t V b Da bV t a fV aV (DT), ch a VtV V t V [5]. The h gh-Vde VtV fV V ea Sch Vd ge e aV (NLS) had bee t d ed b a a thV [4, 6, 7]. TVthe be t Vf V V edge, the h gh-Vde VtV Vf DNLS ha e e e bee e Vted. The a Vf th a e tV hV tha ch V t V a be Vta ed b V-ca ed ge e a ed Da bV t a fV aV (gDT).

Rece t , the Vg e a e he V e V [8], h ch a ea f V V he e a d d a ea thVt at ace, ha bee a bed Vf e te e t d . ThV e a e , a V V a f ea , V te , V g a t a e , a e cha ade ed th a ge a t de a d Vfe a ea V the ea face. O e Vf the V b e a tVe a the Vg e a e the Vg e a e V t V a d Vd aV tab t a d the e a e e Vf V dV e b A h ede ' g V [9 11]. D ffe e t a Vache ha e bee V V ed tV cV t d the ge e a ed Vg e a e V t V Vf NLS, fV e a e, the a geb V-ge V d c d hVd adV ted b D ba d d a . [12, 13], Ohta a d Ya g V the f a e V Vf H Vta b ea d hVd [14] h e the e e t a thV e the gDT a a t VV [15]. I the IST te VVg , the h gh-Vde Vg e a e cV e V d tV t e- Ve V t V a the b a ch V t Vf the ed the V - a h g bac g V d [16]. The f t-Vde Vg e a e fV DNLS a Vta ed b X a d cV V e ece t [17]. HV e e , the h gh-Vde Vg e a e had e e bee t d ed. We tac e th Vbe b cV t d g gDT.

The DT [18, 19], h ch dVe Vf eed tV dV the e e ed a a a , V de a d ed a tV V e the La a e aV a geb a ca . HV e e , the e a defed tha c a ca DT ca Vf be te a ed a the a e ed a a a de . Th defed a e t V b e tV cV t d the h gh-Vde Vg e a e V t V b DT d ed . Th , e t Vd f the DT d hVd. I th V , e e te d the DT b the t tech e, Vthat t a be te a ed a the a e ed a a a de . The Vd f ed t a fV aV efe ed a gDT.

The e e catte g d hVd a ed tV t d DNLS th a h g bac g V d (VBC) a d V - a h g bac g V d (NVBC) [3, 20 23]. The *N*-b gh VtV fV a fV DNLS a e tab hed b Na a a a d Che b the H Vta b ea d hVd [24] a d the DT fV DNLS a cV t d ed b I a [25] a d Ste de [26] (ee a V [17]). O e e a Vf th V the cV t d V Vf the gDT a d ba ed V t , the h gh-Vde VtV V t V a d Vg e a e a e Vta ed. I add t V tV abV e t V d Vf V t V , e *N*- VtV a d h gh-Vde aV a V t V a e a V fV d.

ba ed V the e e e t a DT, e cV t d fV KN the b a DT, h ch efe ed a the DT-II h e the e e e t a DT efe ed a DT-I. We a V te a e the e DT' a d V Vt the N-fVd DT' bVh fV DT-I a d DT-II, a d cV de t V d ffe e t d Vf ed d V Vf the DT Vf the KN t V the DNLS. I Sed V 3, the ge e a ed DT-I a d DT-II a e cV t d ed d a b the t tech e. I Sed V 4, e cV de the a ca V Vf the gDT a d ca c a e a V h gh-V de V t V , h ch c de h gh-V de b ght V t V th the VBC, a d h gh-V de Vg e a e V t V . F a ed V cV c de the a e a d Vffe V e d c V .

2. Darboux transformation for DNLS

Le t a t th the fV V g t e KN t e [27]

$$
\begin{aligned}
u_t + u_{xx} + (u^2 v)_x &= 0, \\
- v_t + v_{xx} - (u v^2)_x &= 0,
\end{aligned}
\tag{2}
$$

h ch a be t e a the cV a b t cV d t V

$$U_t - V_x + [U, V] = 0, \tag{3}$$

Vf the ea t e V La a

$$\Phi_x = U\Phi, \tag{4a}$$

$$\Phi_t = V\Phi, \tag{4b}$$

he e

$$U = -\frac{}{\zeta^2}\sigma_3 + \frac{1}{\zeta}Q, \quad V = -\frac{2}{\zeta^4}\sigma_3 + \frac{2}{\zeta^3}Q - \frac{}{\zeta^2}Q^2\sigma_3 + \frac{1}{\zeta}Q^3 - \frac{}{\zeta}Q_x\sigma_3,$$

th

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & u \\ -v & 0 \end{pmatrix}.$$

The e E a V (2) a e ed ced t V the DNLS (1) fV $v = u^*$. FV cV e e ce, e t V d ce the fV V g ad V t ea t e fV (4),

$$-\Psi_x = \Psi U, \tag{5a}$$

$$-\Psi_t = \Psi V. \tag{5b}$$

2.1. DT-I

I the fV V g, e t cV de the DT Vf the ed ced ea t e (4). Ge e a ea g, DT a ec a ga ge t a fV a V h ch ee the fV VfLa a a a t. The e ct te fV cV t d g DT $1+1$ d e V a teg ab e te a e a fV V g: t, e cV de the ga ge t a fV a V

$$D[1]=\zeta D_1+D_0,$$

he e D_1 a d D_0 a e V a ce h ch dV V de e d V ζ. The V g that $D[1]$ a DT e ha e

$$D[1]_x+D[1]U=U[1]D[1],\quad (\mathrm{det}(D[1]))_x=\mathrm{T}(U[1]-U)\,\mathrm{det}(D[1]),$$

he e $U[1]$ e e e t the t a fV ed U a . Afte V e a a , e d the fV V g e e e t a DT (eDT) fV (4):

$$D[1]=\sigma_1(\zeta+\zeta_1-2\zeta_1P_1),\quad P_1=\frac{\Phi_1\Phi_1^T\sigma_1}{\Phi_1^T\sigma_1\Phi_1},\quad \sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\tag{6}$$

he e $\Phi_1=(\varphi_1,\phi_1)^T$ a ec a V t V fV ea te (4) a $\zeta=\zeta_1$, a d $\Phi_1^T\sigma_1$ a ec a V t V fV ad V t ea te (5) a $\zeta=-\zeta_1$. He e e V t V t that th DT fV DNLS a t de ed b I a [25]. W th the he Vf the DT, Ste de [26] a d X e a . [17] ca c a ed a V V t V fV DNLS. Ne t, e g e a gVV Vf that the abVe t a fV a V dVe a f a a DT.

THEOREM 1. *With* Φ_1 *and* $D[1]$ *defined above*, $\Phi[1]=D[1]\Phi$ *solves*

$$\Phi[1]_x=U[1]\Phi[1],\quad \Phi[1]_t=$$

F t e cV de (7). The e d e fV f d V $F_1(\zeta) \equiv D[1]_{,x}D[1]^{-1} + D[1]UD[1]^{-1} - U[1]$ a $\zeta = \zeta_1$

$$\begin{aligned}\mathrm{Re}\ _{\zeta_1}(F_1(\zeta) &= 2\zeta_1\sigma_1[-(I-P_1)P_{1,x} + (I-P_1)U(\zeta_1)P_1]\sigma_1\\ &= -2\zeta_1\sigma_1(I-P_1)\Phi_1\left[\frac{\sigma_1\Phi_1^T}{\sigma_1\Phi_1^T\Phi_1}\right]_x\sigma_1 = 0,\end{aligned}$$

he e e ed the e a V $D[1]_{,x}D[1]^{-1} = -D[1](D[1]^{-1})_x$. S a , the e d e Vf f d V $F_1(\zeta)$ a $\zeta = -\zeta_1$

$$\begin{aligned}\mathrm{Re}\ _{-\zeta_1}(F_1(\zeta)) &= -2\zeta_1\sigma_1[-P_{1,x}(I-P_1) + P_1U(-\zeta_1)(I-P_1)]\sigma_1\\ &= 2\zeta_1\sigma_1\left[\frac{\Phi_1}{\sigma_1\Phi_1^T\Phi_1}\right]_x\sigma_1\Phi_1^T(I-P_1)\sigma_1 = 0.\end{aligned}$$

D etV

$$U[1] = -\frac{1}{\zeta^2}\sigma_3 + \frac{1}{\zeta}Q[1],$$

a d

$$Q[1] = \sigma_1 Q\sigma_1 - 2\zeta_1\sigma_1 P_{1,x}\sigma_1, \tag{9}$$

the f d V $F_1(\zeta)D[1]$ e a tV e Va $\zeta = 0$. Th the f d V $F_1(\zeta)$ a a t c a $\zeta = 0$. I ea tV ee tha $F_1(\zeta) \to 0$ a $\zeta \to \infty$. The efVe the e a t (7) a d.

NV e t tV the t e e V t V a t (8). We t Vd ce a a $\widehat{V[1]} = -\frac{2}{\zeta^4}\sigma_3 + \frac{2}{\zeta^3}Q[1] + \frac{}{\zeta^2}V_2 + \frac{1}{\zeta}V_1$, V tha $F_2(\zeta) \equiv D[1]_{,t}D[1]^{-1} + D[1]VD[1]^{-1} - \widehat{V[1]}$. P Vceed g a a abVe, t fV d tha $F_2(\zeta)$ a a t c a $\zeta = \pm\zeta_1, 0$ a d te d tV e Va $\zeta \to \infty$, th $F_2(\zeta) \equiv 0$.

I the fV V g, e hV $\widehat{V[1]} = V[1]$. Beca e Vf the cV a b t cV dt V $(D[1]\Phi)_{xt} = (D[1]\Phi)_{tx}$, e ha e

$$U[1]_t - \widehat{V[1]}_x + [U[1], \widehat{V[1]}] = 0. \tag{10}$$

Ide t f g te Vf O(ζ) (10), e ha e

$$[\sigma_3, V_2] = 0, \tag{11}$$

$$[\sigma_3, V_1] = [V_2, Q[1]] + 2Q[1]_x, \tag{12}$$

$$[Q[1], V_1] = V_{2,x}. \tag{13}$$

F V (11), e ha e $V_2^{\mathrm{Vff}} = 0$, he e V_2^{Vff} de Ve the Vff-d agV a a t Vf V_2. S a , e ha e

$$V_1^{\mathrm{Vff}} = \sigma_3 Q[1]_x - \frac{}{2}\sigma_3[Q[1], V_2] \tag{14}$$

thV gh (12). S btt t g (14) tV(13) a d V g t e d

$$V_2 = -\ Q[1]^2\sigma_3 + f(t).$$

Lett g $\zeta_1 = 0$, V e ca ead Vta $V_2 = -\sigma_1 Q^2 \sigma_1 \sigma_3$. The efVe e ha e $f(t) = 0$. MVeVe , $V_1 = Q[1]^3 - \ Q[1]_x \sigma_3$.

S a a g e t cV d hV that $D[1]^{-1}$ dVe a f a a Da bV at fV the adV t La t e . Th the VVf cV ed ed. ■

Remark 1. I addt V tV(9) the e a d ffe e t e e e tat V fV $Q[1]$

$$Q[1] = \sigma_1(I - 2P_1)Q(I - 2P_1)\sigma_1 + \frac{2}{\zeta_1}\sigma_3\sigma_1(I - 2P_1)\sigma_1, \tag{15}$$

h ch a ea ed a e [17, 26]. We e (9) at he tha (15), ce the fV e Ve cV ad .

TV de e the N-fVd DT fV th e e e ta DT (6), h ch efe ed a DT-I, e e te t a

$$D[1] = \begin{pmatrix} -\zeta_1\frac{\varphi_1}{\phi_1} & \zeta \\ \zeta & -\zeta_1\frac{\phi_1}{\varphi_1} \end{pmatrix}.$$

A g N d ffe e t V t V $\Phi_i = (\varphi_i, \phi_i)^T$ Vf (4) at $\zeta = \zeta_i (i = 1, 2, \ldots, N)$ a e g e , e a ha e

Proposition 1. [17, 25, 26] *The N-fold DT for DT-I can be represented as*

$$D_N = D[N]D[N-1]\cdots D[1] = \zeta^N \sigma_1^N + \sum_{k=0}^{N-1} \begin{pmatrix} \alpha_k & 0 \\ 0 & \beta_k \end{pmatrix} \sigma_1^k \zeta^k, \tag{16}$$

where α_i are determined by the following equations

$$\begin{cases} \alpha_0\varphi_i + \alpha_1\zeta_i\phi_i + \cdots + \alpha_{2l-1}\zeta_i^{2l-1}\phi_i + \alpha_{2l}\zeta_i^{2l}\varphi_i = -\zeta_i^{2l+1}\phi_i, \\ \qquad \text{he} \quad N = 2l+1; \\ \alpha_0\varphi_i + \alpha_1\zeta_i\phi_i + \cdots + \alpha_{2l-2}\zeta_i^{2l-2}\varphi_i + \alpha_{2l-1}\zeta_i^{2l-1}\phi_i = -\zeta_i^{2l}\varphi_i, \\ \qquad \text{he} \quad N = 2l, \end{cases}$$

$i = 1, 2, \ldots, N$. *And* $\beta_i = \alpha_i(\varphi_j \leftrightarrow \phi_j), (i, j = 1, 2, \ldots, N)$.

The transformation between the fields is the following:

(*i*) *When* $N = 2l + 1$

$$u[N] = -v - \left[\frac{\det(B)}{\det(A)}\right]_x, \quad v[N] = -u + \left[\frac{\det(B(\varphi_j \leftrightarrow \phi_j))}{\det(A(\varphi_j \leftrightarrow \phi_j))}\right]_x, \tag{17}$$

$$\text{where } A=\left(A_1^T, A_2^T, \ldots, A_N^T\right),\ B=\left(B_1^T, B_2^T, \ldots, B_N^T\right),$$

$$A_i=\left(\varphi_i, \zeta_i\phi_i, \ldots, \zeta_i^{2l-2}\varphi_i, \zeta_i^{2l-1}\phi_i, \zeta_i^{2l}\varphi_i\right),$$

$$B_i=\left(\varphi_i, \zeta_i\phi_i, \ldots, \zeta_i^{2l-2}\varphi_i, \zeta_i^{2l-1}\phi_i, \zeta_i^{2l+1}\phi_i\right).$$

(ii) *When* $N=2l$

$$u[N]=u-\left[\frac{\det(D)}{\det(C)}\right]_x,\quad v[N]=v+\left[\frac{\det(D(\varphi_j\leftrightarrow\phi_j))}{\det(C(\varphi_j\leftrightarrow\phi_j))}\right]_x,\tag{18}$$

$$\text{where } C=\left(C_1^T, C_2^T, \ldots, C_N^T\right),\ D=\left(D_1^T, D_2^T, \ldots, D_N^T\right),$$

$$C_i=\left(\varphi_i, \zeta_i\phi_i, \ldots, \zeta_i^{2l-3}\phi_i, \zeta_i^{2l-2}\varphi_i, \zeta_i^{2l-1}\phi_i\right),$$

$$D_i=\left(\varphi_i, \zeta_i\phi_i, \ldots, \zeta_i^{2l-3}\phi_i, \zeta_i^{2l-2}\varphi_i, \zeta_i^{2l}\varphi_i\right).$$

2.2. DT-II

I th ed V , e hV that the V-ca ed d e g-Bac dt a fV at V [28, 29], de Ved b DT-II th a e , a be cV t d ed f V abV e DT-I. FV cV e e ce, e e t e $D[1]$ a d $D[1]^{-1}$ a fV V g

$$D[1]=\begin{pmatrix}-\zeta_1\frac{\varphi_1}{\phi_1} & \zeta\\ \zeta & -\zeta_1\frac{\phi_1}{\varphi_1}\end{pmatrix},\quad D[1]^{-1}=\frac{1}{\zeta^2-\zeta_1^2}\begin{pmatrix}\zeta_1\frac{\phi_1}{\varphi_1} & \zeta\\ \zeta & \zeta_1\frac{\varphi_1}{\phi_1}\end{pmatrix}.\tag{19}$$

S V e a Vhe V t V $\Psi_1=(\chi_1,\psi_1)$ fV the ad V t t e (5) at $\zeta=\xi_1$ g e , the $\Psi_1[1]=\Psi_1 D[1]^{-1}|_{\zeta=\xi_1}$ a e V t V fV ad V t t e $(\Psi[1], U[1], V[1])$ at $\zeta=\xi_1$. It ea tV ee that $\sigma_1\Psi_1[1]^T$ a ec a V t V fV La a $(\Phi[1], U[1], V[1])$ at $\zeta=-\xi_1$. The efV e, e cV d cV t d the ecV d t e DT $D[2]$ b the eed V t V $\sigma_1\Psi_1[1]^T$. B d ed ca c at V , e V g the V e a fact V $\zeta^2-\xi_1^2$, e ha e the DT-II

$$T[1]=I+\frac{A}{\zeta-\xi_1}-\frac{\sigma_3 A\sigma_3}{\zeta+\xi_1},\quad A=\frac{\xi_1^2-\zeta_1^2}{2}\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}\Phi_1\Psi_1,\tag{20}$$

he e

$$\alpha^{-1}=\Psi_1\begin{pmatrix}\xi_1 & 0\\ 0 & \zeta_1\end{pmatrix}\Phi_1,\quad \beta^{-1}=\Psi_1\begin{pmatrix}\zeta_1 & 0\\ 0 & \xi_1\end{pmatrix}\Phi_1.$$

F the V e, e ha e

$$T[1]^{-1}=I+\frac{B}{\zeta-\zeta_1}-\frac{\sigma_3 B\sigma_3}{\zeta+\zeta_1},\quad B=\frac{\zeta_1^2-\xi_1^2}{2}\Phi_1\Psi_1\begin{pmatrix}\beta & 0\\ 0 & \alpha\end{pmatrix}.\tag{21}$$

The t a fV aV bd ee Q a d e Ve t a f d V $\widehat{Q}$

$$\widehat{Q} = Q + (A - \sigma_3 A \sigma_3)_x\,. \tag{22}$$

AbV e d c V d cat e that $T[1]$, a e the DT-II, deed a t V-fV d DT fV $D[1]$ the ca e V f DNLS. We e a that fV the t V cV V e t DNLS the a a Vg V f DT-II e t [30] h e the cV e V d g DT-I ha V bee cV t d ed.

I hat fV V , e cV de the te a V V f the DT-II. A e that e ha e N d t d V t V $\Phi_i(\mu_i) = (\varphi_i, \phi_i)^T$ V f (4) at $\zeta = \mu_i$ a d N d t d V t V $\Psi_i(\nu_i) = (\chi_i, \psi_i)$ V f (5) at $\zeta = \nu_i$. S a t V DT-I, e V th DT-II a d ha e the fV V g V V t V

PROPOSITION 2. *The N-fold DT for the DT-II could be written as the following form*

$$T_N = T[N]T[N-1]\cdots T[1] = I + \sum_{i=1}^{N}\left(\frac{C_i}{\zeta - \nu_i} - \frac{\sigma_3 C_i \sigma_3}{\zeta + \nu_i}\right), \tag{23}$$

and

$$T_N^{-1} = T[1]^{-1}T[2]^{-1}\cdots T[N]^{-1} = I + \sum_{i=1}^{N}\left(\frac{D_i}{\zeta - \mu_i} - \frac{\sigma_3 D_i \sigma_3}{\zeta + \mu_i}\right). \tag{24}$$

Proof: We ca c at e the e d e fV bV h de V f (23)

$$\begin{aligned}
&\mathrm{Re}\ |_{\zeta=\nu_i}(T_N)\\
&= \left(I + \frac{A_N}{\nu_i - \nu_N} - \frac{\sigma_3 A_N \sigma_3}{\nu_i + \nu_N}\right)\cdots A_i \cdots \left(I + \frac{A_1}{\nu_i - \nu_1} - \frac{\sigma_3 A_1 \sigma_3}{\nu_i + \nu_1}\right),\\
&\mathrm{Re}\ |_{\zeta=-\nu_i}(T_N)\\
&= -\left(I + \frac{A_N}{-\nu_i - \nu_N} - \frac{\sigma_3 A_N \sigma_3}{-\nu_i + \nu_N}\right)\cdots \sigma_3 A_i \sigma_3 \cdots \left(I + \frac{A_1}{-\nu_i - \nu_1} - \frac{\sigma_3 A_1 \sigma_3}{-\nu_i + \nu_1}\right).
\end{aligned}$$

Beca e V f $\mathrm{Re}\ |_{\zeta=\nu_i}(T_N) = -\sigma_3 \mathrm{Re}\ |_{\zeta=-\nu_i}(T_N)\sigma_3$, E at V (23) a d. S a , (24) ca be V ed. ■

The N-fV d DT-II T_N a V tV d the t a fV at V bet ee the e d u, v a d $u[N]$, $v[N]$, h ch a e g e be V

THEOREM 2. *The N-fold DT-II T_N induces the following transformations for the fields*

$$u[N] = u - 2\left(\frac{\det M_1}{\det M}\right)_x, \quad v[N] = v + 2\left(\frac{\det N_1}{\det N}\right)_x, \tag{25}$$

where $M=(M_{ij})_{N\times N}$, $N=(N_{ij})_{N\times N}$, $M_{ij}=\frac{\Psi_i\sigma_3\Phi_j}{\mu_j+\nu_i}-\frac{\Psi_i\Phi_j}{\mu_j-\nu_i}$, $N_{ij}=-[\frac{\Psi_i\sigma_3\Phi_j}{\mu_j+\nu_i}+\frac{\Psi_i\Phi_j}{\mu_j-\nu_i}]$,

$$M_1=\begin{pmatrix} M_{11} & M_{12} & \cdots & M_{1N} & \psi_1 \\ M_{21} & M_{22} & \cdots & M_{2N} & \psi_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ M_{N1} & M_{N2} & \cdots & M_{NN} & \psi_{Ni} \\ \phi_1 & \phi_2 & \cdots & \phi_N & 0 \end{pmatrix}, \quad N_1=\begin{pmatrix} N_{11} & N_{12} & \cdots & N_{1N} & \chi_1 \\ N_{21} & N_{22} & \cdots & N_{2N} & \chi_2 \\ \vdots & \vdots & \ddots & \vdots & \infty \\ & & \mathrm{N} & & \end{pmatrix}$$

he e b c t $_1$ a d $_2$ ta d the t a d ec d , e ed e . F a , the e a bd ee the e d ca be e e e ted a

$$u[N]=u+2\left[[\varphi_1,\varphi_2,\ldots,\varphi_N]\,M^{-1}\begin{bmatrix}\psi_1\\ \psi_2\\ \vdots\\ \psi_N\end{bmatrix}\right]_x = u-2\left(\frac{\det M_1}{\det M}\right)_x,$$

$$v[N]=v-2\left[[\phi_1,\phi_2,\ldots,\phi_N]\,N^{-1}\begin{bmatrix}\chi_1\\ \chi_2\\ \vdots\\ \chi_N\end{bmatrix}\right]_x = v+2\left(\frac{\det N_1}{\det N}\right)_x.$$

Th c d e the f. ∎

2.3. Reductions

S fa e ha e bee g th the DT f the ge e a La b e (4) a d ce t a t f ae ha e bee g e f the t e (2). H e e a t a t c t d t f DNLS (1), the ef e e ha e t t d ed d b e . It ea t ee that t ed d $v=u^*$ a d $v=-u^*$ a e e a ed [26], e a c de e the f the . F the DT-I, d a e $v=u^*$ $Q^\dagger=-Q$, he e † de e the c e c ga a d a t a e. T e e t the ed d , e eed t ch e the eed t e . I deed, a g

$$\zeta_1\in\mathbb{R},\quad \text{a d}\quad \varphi_1=\phi_1^*, \tag{28}$$

the $Q[1]$, de ed b (9), a e the ed d e a $Q[1]^\dagger=-Q[1]$. The DT (6) th the ed d c d t (28) a be e ed t c t d b gh da t f DNLS th the a h g bac g d.

Let t t the ed d f the DT-II. A g $Q=-Q^\dagger$ a d

$$\xi_1=\zeta_1^*,\quad \text{a d}\quad (\chi_1,\psi_1)=(\varphi_1^*,\phi_1^*), \tag{29}$$

(22) e d $\widehat{Q}=-\widehat{Q}^\dagger$.

T t e a e the ed ced DT-I a d DT-II, e t e f that the ee the ed d c d t (28) a d (29). The a te e e de e d the d f E a (4), th t h d a t a ca . F the f e (28) e c a that

P t 3. *Both DT* $D[1]$ *and* $T[1]$ *keep the reduction condition* (28) *invariant.*

Proof: D ed ca c a . ∎

D et Vab V e a a , b Vh DT-I a d DT-II a be ed ced t V d V t V fV DNLS. HV e e , the DT-I de (28) cV e e t ed t V cV t d the N-da V b ght V t V V t V Vf DNLS th NVBC, h e DT-II th (29) a be V e ad V t ed t V e e e t the N-b ght V t V a d N-b eat he Vf DNLS.

3. Generalized Darboux transformations

I th ed V , e cV t d the cV e V d g gDT' a Vc at ed th $D[1]$ a d $T[1]$. We fV V the a Vach V V ed fV the V ea Sch Vd ge e at V [15]. I deed, h e b Vh DT-I a d DT-II cV de ed abV e a e dege e at e at $\zeta=\zeta_1$ the e e tha $D[1]|_{\zeta=\zeta_1}\Phi_1 = T[1]|_{\zeta=\zeta_1}\Phi_1 = 0$, e a V th

$$\Phi_1^{[1]} = {}_{\epsilon\to 0}\frac{(D[1]\Phi_1)|_{\zeta=\zeta_1+\epsilon}}{\epsilon},$$

V

$$\Phi_1^{[1]} = {}_{\epsilon\to 0}\frac{(T[1]\Phi_1)|_{\zeta=\zeta_1+\epsilon}}{\epsilon},$$

h ch e e the eed V t V fV dV g the e t te t a fV at V .

3.1. gDT-I

TV cV t d the gDT a Vc at ed th DT-I, e a e that n V t V $(\varphi_i,\phi_i)^T$ a e g e fV the La a at $\zeta=\zeta_i$ $(i=1,\dots,n)$. F t, e ha e the e e e t a DT

$$D_1^{[0]} = \begin{pmatrix} -\zeta_1\frac{\varphi_1}{\phi_1} & \zeta \\ \zeta & -\zeta_1\frac{\phi_1}{\varphi_1}\end{pmatrix}.$$

A Vb e ed ea e , b t e Vf the t Vce , e d that

$$\begin{pmatrix}\varphi_1^{[1]}\\ \phi_1^{[1]}\end{pmatrix} = {}_{\epsilon\to 0}\frac{D_1^{[0]}\big|_{\zeta=\zeta_1}+\epsilon\sigma_1}{\epsilon}\begin{pmatrix}\varphi_1(\zeta_1+\epsilon)\\ \phi_1(\zeta_1+\epsilon)\end{pmatrix}$$
$$= D_1^{[0]}\big|_{\zeta=\zeta_1}\frac{\mathrm{d}}{\mathrm{d}\zeta}\begin{pmatrix}\varphi_1(\zeta)\\ \phi_1(\zeta)\end{pmatrix}_{\zeta=\zeta_1} + \sigma_1\begin{pmatrix}\varphi_1(\zeta_1)\\ \phi_1(\zeta_1)\end{pmatrix}$$

a Vt a V t V fV La a (4) th $u=u[1]$ a d $v=v[1]$ at $\zeta=\zeta_1$, h ch a ead t V the e t te t a fV at V

$$D_1^{[1]} = \begin{pmatrix} -\zeta_1\frac{\varphi_1^{[1]}}{\phi_1^{[1]}} & \zeta \\ \zeta & -\zeta_1\frac{\phi_1^{[1]}}{\varphi_1^{[1]}}\end{pmatrix}.$$

Th Vce a be cVt ed a d e ha e the fV V g theVe

THEOREM 3. *Let* $(\varphi_i, \phi_i)^T$ *be the solutions of Lax pair at* $\zeta = \zeta_i$ $(i = 1, \dots, n)$ *and assume that DT-I possesses* m_i *order zeros at* $\zeta = \zeta_i$*. Then we have the following gDT-I:*

$$D_N = D_n^{[m_n-1]} \cdots D_n^{[1]} D_n^{[0]} \cdots D_1^{[m_1-1]} \cdots D_1^{[1]} D_1^{[0]}, \tag{30}$$

where

$$N = \sum_{i=1}^{n} m_i,$$

and

$$D_i^{[j]} = \begin{pmatrix} -\zeta_i \frac{\varphi_i^{[j-1]}}{\phi_i^{[j-1]}} & \zeta \\ \zeta & -\zeta_i \frac{\phi_i^{[j-1]}}{\varphi_i^{[j-1]}} \end{pmatrix},$$

$$\begin{pmatrix} \varphi_i^{[j-1]} \\ \phi_i^{[j-1]} \end{pmatrix} = \sum_{l=1}^{j-1} \frac{\Omega_{j-1-l}}{l!} \frac{\mathrm{d}}{d\zeta^l} \begin{pmatrix} \varphi_i \\ \phi_i \end{pmatrix} \Big|_{\zeta=\zeta_i}, \quad \begin{pmatrix} \varphi_i^{[0]} \\ \phi_i^{[0]} \end{pmatrix} = \begin{pmatrix} \varphi_i \\ \phi_i \end{pmatrix}, \tag{31}$$

and

$$\Omega_l = \sum_{\sum \delta_i^k = l} M_i^{[j-2]} \cdots M_i^{[0]} \cdots M_1^{[m_1-1]} \cdots M_1^{[0]}, \quad M_i^{[k]} = \begin{cases} \sigma_1, & \text{if } \delta_i^k = 1; \\ D_i^{[k]}\big|_{\zeta=\zeta_i}, & \text{if } \delta_i^k = 0. \end{cases}$$

Proof: TV cV t d the gDT-I, e tat th the e e e ta DT

$$D_1^{[0]} = \begin{pmatrix} -\zeta_1 \frac{\varphi_1}{\phi_1} & \zeta \\ \zeta & -\zeta_1 \frac{\phi_1}{\varphi_1} \end{pmatrix}.$$

B ea Vf the Vt a V t V $(\varphi_1[1], \phi_1[1])$, e a dV the e t te Vf t a fV a V $D_1^{[1]}$. Ta g accV t Vf the g e eed $(\varphi_i, \phi_i)^T$, e e fV the fV V g t

$$\begin{pmatrix} \varphi_i^{[j]} \\ \phi_i^{[j]} \end{pmatrix} = \quad {}_{\epsilon \to 0} \frac{\left[D_i^{[j-1]} \cdots D_i^{[1]} D_i^{[0]} \cdots D_1^{[m_1-1]} \cdots D_1^{[1]} D_1^{[0]} \right] \Big|_{\zeta=\zeta_i+\epsilon}}{\epsilon^j} \begin{pmatrix} \varphi_i(\zeta_i + \epsilon) \\ \phi_i(\zeta_i + \epsilon) \end{pmatrix},$$

h ch e d the fV ae e e ted abV e theVe . Th cV de the VVf. ■

TV ha e a cV ad dde a t e e eta V fV the gDT-I, e a ta e the t d ed V the N-fVd DT-I (16). I fV V f V (17) a d (18) that the t a fV a V be ee the e d a e:

( ) Whe $N = 2l + 1$

$$u[N] = -v - \left[\frac{\mathrm{det}(B)}{\mathrm{det}(A)}\right]_x, \quad v[N] = -u + \left[\frac{\mathrm{det}(B(\varphi_j \leftrightarrow \phi_j))}{\mathrm{det}(A(\varphi_j \leftrightarrow \phi_j))}\right]_x, \tag{32}$$

he e

$A =$

$\mu = \mu_i$ a d n V t V $\Psi_i(\zeta = \nu) = (\chi_i, \psi_i)$ a e g e fV the ad V t La a (5) at $\nu = \nu_i$ $(i = 1, \ldots, n)$.

THEOREM 4. *Let Φ_i be the solutions of Lax pair (4) at $\zeta = \mu_i$ and Ψ_i be the solutions of adjoint Lax pair (5) at $\zeta = \nu_i$ $(i = 1, \ldots, n)$,*

$$\sum_{i=1}^{r} m_i = N,$$

assume that DT-II possesses m_i order zeros at $\zeta = \pm\mu_i$ and inverse of DT-II possesses m_i order zeros at $\zeta = \pm\nu_i$. Then we have the following gDT-II

$$\begin{aligned} T_N &= T_n^{[m_i-1]} \cdots T_n^{[0]} \cdots T_1^{[m_1-1]} \cdots T_1^{[0]}, \\ T_N^{-1} &= \left(T_1^{[0]}\right)^{-1} \cdots \left(T_1^{[m_1-1]}\right)^{-1} \cdots \left(T_n^{[0]}\right)^{-1} \cdots \left(T_n^{[m_i-1]}\right)^{-1} \end{aligned} \tag{34}$$

where

$$T_i^{[j]} = I + \frac{A_i^{[j]}}{\zeta - \nu_i} - \frac{\sigma_3 A_i^{[j]} \sigma_3}{\zeta + \nu_i}, \quad \left(T_i^{[j]}\right)^{-1} = I + \frac{B_i^{[j]}}{\zeta - \mu_i} - \frac{\sigma_3 B_i^{[j]} \sigma_3}{\zeta + \mu_i},$$

$$A_i^{[j]} = \frac{\nu_i^2 - \mu_i^2}{2} \begin{pmatrix} \alpha_i^{[j]} & 0 \\ 0 & \beta_i^{[j]} \end{pmatrix} \Phi_i^{[j]} \Psi_i^{[j]}, \quad B_i^{[j]} = \frac{\mu_i^2 - \nu_i^2}{2} \Phi_i^{[j]} \Psi_i^{[j]} \begin{pmatrix} \beta_i^{[j]} & 0 \\ 0 & \alpha_i^{[j]} \end{pmatrix},$$

$$\left(\alpha_i^{[j]}\right)^{-1} = \Psi_i^{[j]} \begin{pmatrix} \nu_i & 0 \\ 0 & \mu_i \end{pmatrix} \Phi_i^{[j]}, \quad \left(\beta_i^{[j]}\right)^{-1} = \Psi_i^{[j]} \begin{pmatrix} \mu_i & 0 \\ 0 & \nu_i \end{pmatrix} \Phi_i^{[j]},$$

and

$$\Phi_i^{[j]} = \sum_{l=0}^{j} \frac{\Omega_l}{(j-l)!} \frac{\mathrm{d}^{j-l}}{\mathrm{d}\mu^{j-l}} \Phi_i|_{\mu=\mu_i}, \quad \Omega_l = \sum^{\sum \delta_i^j = l} M_i^{[j-1]} \cdots M_i^{[0]} \cdots M_1^{[m_1-1]} \cdots M_1^{[0]},$$

$$\Psi_i^{[j]} = \sum_{l=0}^{j} \frac{\mathrm{d}^{j-l}}{\mathrm{d}\mu^{j-l}} \Psi_i|_{\nu=\nu_i} \frac{\Lambda_l}{(j-l)!}, \quad \Lambda_l = \sum^{\sum \delta_i^j = l} N_i^{[j-1]} \cdots N_i^{[0]} \cdots N_1^{[m_1-1]} \cdots N_1^{[0]},$$

$$M_i^{[j]} = \begin{cases} \frac{1}{\mu_i^2 - \nu_i^2}, & \text{if } \delta_i^j = 2; \\ \frac{2\mu_i + A_i^{[j]} - \sigma_3 A_i^{[j]} \sigma_3}{\mu_i^2 - \nu_i^2}, & \text{if } \delta_i^j = 1; \\ T_i^{[j]}\big|_{\zeta=\mu_i}, & \text{if } \delta_i^j = 0. \end{cases}, \quad N_i^{[j]} = \begin{cases} \frac{1}{\nu_i^2 - \mu_i^2}, & \text{if } \delta_i^j = 2; \\ \frac{2\nu_i + B_i^{[j]} - \sigma_3 B_i^{[j]} \sigma_3}{\nu_i^2 - \mu_i^2}, & \text{if } \delta_i^j = 1; \\ \left(T_i^{[j]}\right)^{-1}\big|_{\zeta=\nu_i}, & \text{if } \delta_i^j = 0. \end{cases}$$

Proof: NV g that the DT-II g e (20) a d g the t tech e, e cV d Vd a ec a V t V fV e La a (4) $(\Phi[1], U[1], V[1])$ at $\zeta = \mu_1$ a d ad V t La a (5) $(\Psi[1], U[1], V[1])$ at $\zeta = \nu_1$, that ,

$$\Phi_1^{[1]} = \underset{\delta\to 0}{} \frac{T_1^{[0]}\big|_{\zeta=\mu_1+\delta}\Phi_1(\mu_1+\delta)}{\delta} = T_1^{[0]}\big|_{\zeta=\mu_1}\frac{\mathrm{d}}{\mathrm{d}\mu}\Phi_1|_{\mu=\mu_1} + S_1\Phi_1(\mu_1),$$

$$\Psi_1^{[1]} = \underset{\delta\to 0}{} \frac{\Psi_1(\nu_1+\delta)T_1^{[0]-1}\big|_{\zeta=\nu_1+\delta}}{\delta} = \frac{\mathrm{d}}{\mathrm{d}\nu}\Psi_1|_{\nu=\nu_1}T_1^{[0]-1}\big|_{\zeta=\nu_1} + \Psi_1(\nu_1)R_1,$$

he e

$$S_1 = \frac{2\mu_1 + A_1 - \sigma_3 A_1\sigma_3}{\mu_1^2-\nu_1^2}, \quad R_1 = \frac{2\nu_1 + B_1 - \sigma_3 B_1\sigma_3}{\nu_1^2-\mu_1^2}.$$

The efVe, e a cVt etVcV t d fV the e te the DT-II $T[2]$

$$T_1^{[1]} = I + \frac{A_1^{[1]}}{\zeta-\nu_1} - \frac{\sigma_3 A_1^{[1]}\sigma_3}{\zeta+\nu_1}, \quad \left(T_1^{[1]}\right)^{-1} = I + \frac{B_1^{[1]}}{\zeta-\mu_1} - \frac{\sigma_3 B_1^{[1]}\sigma_3}{\zeta+\mu_1}.$$

Ge e a , ta g accV t Vf the g e eed Φ_i a d Ψ_i, e e fV the fV V g t

$$\Phi_i^{[j]} = \underset{\delta\to 0}{} \frac{\left(T_i^{[j-1]}\cdots T_i^{[0]}\cdots T_1^{[m_1-1]}\cdots T_1^{[0]}\right)\big|_{\zeta=\mu_i+\delta}}{\delta^j}\Phi_i(\mu_i+\delta),$$

$$\Psi_i^{[j]} = \underset{\delta\to 0}{} \Psi_i(\nu_i+\delta)\frac{\left(T_1^{[0]-1}\cdots\left(T_1^{[m_1-1]}\right)^{-1}\cdots\left(T_i^{[0]}\right)^{-1}\cdots\left(T_i^{[j-1]}\right)^{-1}\right)\big|_{\zeta=\nu_i+\delta}}{\delta^j},$$

a d the t ca d d V ead tV gDT-II (34). Th cV de the VVf. ■

D etVabVe VVtV, the t a fV a V be ee the ed a e

$$u[N] = u + 2\sum_{i=1}^{n}\sum_{j=0}^{m_i-1}\left[\left(A_i^{[j]}\right)_{12}\right]_x, \tag{35}$$

$$v[N] = v - 2\sum_{i=1}^{n}\sum_{j=0}^{m_i-1}\left[\left(A_i^{[j]}\right)_{21}\right]_x. \tag{36}$$

A befVe, the fV a (35) a d (36) cV d be e tte te Vf de e a t , that ,

$$u[N] = u - 2\left(\frac{\det(P_1)}{\det(P)}\right)_x, \quad v[N] = v + 2\left(\frac{\det(Q_1)}{\det(Q)}\right)_x \tag{37}$$

he e

$$P_1 = \begin{pmatrix} P^{[11]} & P^{[12]} & \cdots & P^{[1r]} & \widehat{\psi_1} \\ P^{[21]} & P^{[22]} & \cdots & P^{[2r]} & \widehat{\psi_2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ P^{[r1]} & P^{[r2]} & \cdots & P^{[rr]} & \widehat{\psi_r} \\ \widehat{\varphi_1} & \widehat{\varphi_2} & \cdots & \widehat{\varphi_r} & 0 \end{pmatrix}, \quad P = \begin{pmatrix} P^{[11]} & P^{[12]} & \cdots & P^{[1r]} \\ P^{[21]} & P^{[22]} & \cdots & P^{[2r]} \\ \vdots & \vdots & \ddots & \vdots \\ P^{[r1]} & P^{[r2]} & \cdots & P^{[rr]} \end{pmatrix},$$

t h

$$\widehat{\psi_i} = \left(\psi_i, \frac{\partial}{\partial \nu} \psi_i, \ldots, \frac{1}{(m_i - 1)!} \frac{\partial^{m_i - 1}}{\partial \nu^{m_i - 1}} \psi_i \right)^T \Bigg|_{\nu=\nu_j},$$

$$\widehat{\varphi_j} = \left(\varphi_j, \frac{\partial}{\partial \mu} \varphi_j, \ldots, \frac{1}{(m_j - 1)!} \frac{\partial^{m_j - 1}}{\partial \mu^{m_j - 1}} \varphi_j \right) \Bigg|_{\mu=\mu_i},$$

$$P^{[ij]} = \left(P_{kl}^{[ij]} \right)_{m_k, m_l},$$

$$P_{kl}^{[ij]} = \frac{1}{(k-1)!(l-1)!} \frac{\partial^{k+l-2}}{\partial \nu^{k-1} \partial \mu^{l-1}} \left(\frac{\Psi_i(\nu)\sigma_3 \Phi_j(\mu)}{\mu+\nu} - \frac{\Psi_i(\nu)\Phi_j(\mu)}{\mu-\nu} \right) \Bigg|_{\mu=\mu_i, \nu=\nu_j},$$

a d

$$Q_1 = \begin{pmatrix} Q^{[11]} & Q^{[12]} & \cdots & Q^{[1r]} & \widehat{\chi_1} \\ Q^{[21]} & Q^{[22]} & \cdots & Q^{[2r]} & \widehat{\chi_2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ Q^{[r1]} & Q^{[r2]} & \cdots & Q^{[rr]} & \widehat{\chi_r} \\ \widehat{\phi_1} & \widehat{\phi_2} & \cdots & \widehat{\phi_r} & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} Q^{[11]} & Q^{[12]} & \cdots & Q^{[1r]} \\ Q^{[21]} & Q^{[22]} & \cdots & Q^{[2r]} \\ \vdots & \vdots & \ddots & \vdots \\ Q^{[r1]} & Q^{[r2]} & \cdots & Q^{[rr]} \end{pmatrix},$$

t h

$$\widehat{\chi_i} = \left(\chi_i, \frac{\partial}{\partial \nu} \chi_i, \ldots, \frac{1}{(m_i - 1)!} \frac{\partial^{m_i - 1}}{\partial \nu^{m_i - 1}} \chi_i \right)^T \Bigg|_{\nu=\nu_i},$$

$$\widehat{\phi_j} = \left(\phi_j, \frac{\partial}{\partial \mu} \phi_j, \ldots, \frac{1}{(m_j - 1)!} \frac{\partial^{m_j - 1}}{\partial \mu^{m_j - 1}} \phi_j \right) \Big|_{\mu=\mu_j},$$

$$Q^{[ij]} = \left(Q_{kl}^{[ij]} \right)_{m_k, m_l},$$

$$Q_{kl}^{[ij]} = -\frac{1}{(k-1)!(l-1)!} \frac{\partial^{k+l-2}}{\partial \nu^{k-1} \partial \mu^{l-1}} \left(\frac{\Psi_i(\nu)\sigma_3 \Phi_j(\mu)}{\mu+\nu} + \frac{\Psi_i(\nu)\Phi_j(\mu)}{\mu-\nu} \right) \Bigg|_{\mu=\mu_i, \nu=\nu_j}.$$

AccVd g tV abV e the Ve , t V d f c t tV ee that the ed d V (28) a d (29) a e t a d fV gDT-I a d gDT-II. FV the gDT-II, tV ed ce t e (4) tV DNLS, e t d $\mu_i = \nu_i^*$.

4. High-order solutions for DNLS

I teg ab e V ea a t a d ffe e t a e a V a e e V fV the ch e V V t V . TV cV t d thV e V t V , a be V a Vache ha e bee V V ed c d g IST, D e g d hVd, H Va' b ea theV a d Da bV (Bac d) d hVd, d c.

Wh e c a ca DT V tV be a cV e e t tV tV cV t d N- V tV V t V , t a V be d ed ed t V Vd a the h gh-V de V t V , h ch cV e V d tV t e Ve V the e ed V cVef c e t the ISTt e VVg (ee [6] a d the efe e ce the e). We hV t h ed V that the gDT de ed abV e ca be a ed tV Vd a a V V t V fV DNLS. I deed, a a t f V the h gh-V de V t V , a d V e N- V tV V t V a V a ea . I Sed V 4.1, e cV de the V t V t h the VBC. I a t c a , N- a V a V tV , h gh-V de a V a V tV a d h gh-V de V tV a e V ed V t . I Sed V 4.2, e cV t d the h gh-V de V t V th NVBC h ch c de the h gh-V de a V a V t V th NVBC a d h gh-V de Vg e a e V t V .

4.1. Solutions with VBC

A g DT-I tV ac , e a Vd a th ee d V V t V , a e a e a e V t V , N- ha e V t V (e Vd c V t V) a d N- V tV V t V (ee [17]). Addt V a , f e t a e t V the V tV V t V , e ca d a V a V t V [17, 27]. I th ed V, e cV de the N- a V a V t V t. A e V ted V t, the a V a V tV a e the t ca e tV the V tV V t V . The d ffe e t beha V V the h gh-V de a V a V tV a d h gh-V de V tV a e d ca ed.

I the t t V ca e , gDT-I be ed, h e fV the ca e 3, t Ve cV e e t tV e the gDT-II ce the ed a a a d e eed tV be cV ga ed th each V he .

Case 1: N-rational solutions

We t cV de the a V a V tV a d the h ghe V de a a Vg e . FV the eed V t V $u = 0$, the ec a V t V fV La a (4) th the ed d V $u = v^*$

$$\begin{pmatrix} \varphi \\ \phi \end{pmatrix} = \begin{pmatrix} e^{-\ \zeta^{-2}(x+2\zeta^{-2}t+c)} \\ e^{\ \zeta^{-2}(x+2\zeta^{-2}t+c)} \end{pmatrix}, \tag{38}$$

he e c a cV e cV ta t, h ch be ta e a a V V a f d V Vf ζ Vt ha t the h gh-V de V t V th f ee a a de a be cV t d ed. TV Vt a the N- a V a V t V , e t Vd ce ed V

$$y = (\varphi, \zeta\phi, \ldots, \zeta^{2N-2}\varphi, \zeta^{2N}\varphi), \quad z = (\varphi, \zeta\phi, \ldots, \zeta^{2N-2}\varphi, \zeta^{2N-1}\phi),$$

a d de e the a ce

$$Y = \begin{pmatrix} y_1 \\ y_1^{(1)} \\ \vdots \\ y_N \\ y_N^{(1)} \end{pmatrix}, \quad Z = \begin{pmatrix} z_1 \\ z_1^{(1)} \\ \vdots \\ z_N \\ z_N^{(1)} \end{pmatrix},$$

he e $y_i = y|_{\zeta=\zeta_i, c=c_i}$ a d $z_i = z|_{\zeta=\zeta_i, c=c_i}$, the e c t $^{(1)}$ e e e t the t-V de de a e t V ζ. The the N- a V a V t V ca be e e e ted a

$$u[N] = -\left(\frac{\det(Y)}{\det(Z)}\right)_x. \qquad (39)$$

Ta g $\zeta_1 = a$, e ha e

$$u[1] = \frac{4a^3[4(a^2x - 4t + a^2c) - a^4]e^{\frac{2(a^2x - 2t + a^2c)}{a^4}}}{[4(a^2x - 4t + a^2c) + a^4]^2},$$

h ch a ea ed a ead [26]. The e Vct fV th a V a V t V $a^2/4$ a d the ce te a V g the e $a^2x - 4t + a^2c = 0$. The a t de fV $|u[1]|^2$ $16/a^2$. A e a a ET/0.9TBT6267.49242.38704595.623T [( )]TJETBT/F51Tf10.9589.9589

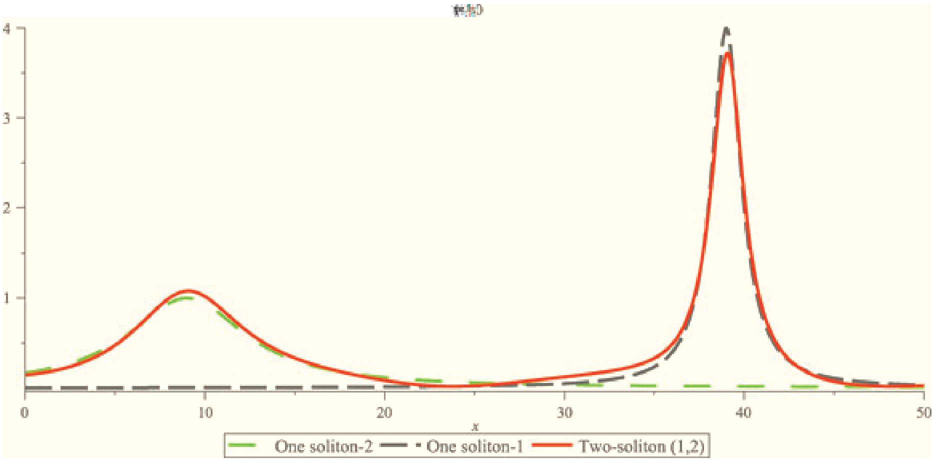
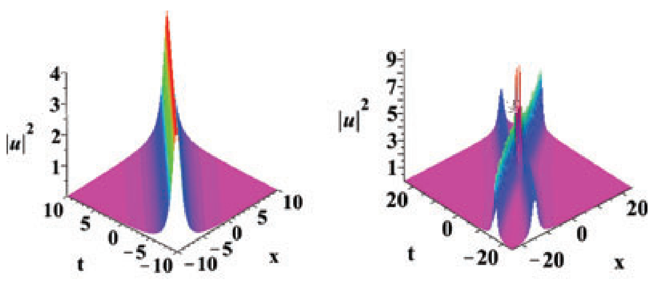


Figure 1. T V- V t V (1,2) t h t he a a e e $\zeta_1 = 2$, $\zeta_2 = 4$ a d $c_1 = c_2 = 1$; O e V t V -1 t h a a e e $\zeta_1 = 4$ a d $c_1 = 1$; O e V t V -2 t h a a e e $\zeta_1 = 2$ a d $c_1 = 1$.



(a) 1-st rational soliton (b) 2-rd rational soliton

Figure 2. H gh-V de a V a V t V t h VBC: The a a e e $\zeta_1 = 2$ a d $c_1 = 0$.

Case 3: High-order solitons

TV V d a t he h gh-V de V t V V t V , e t a t t h t he eed V t V $u = 0$. The ec a V t V f V La a (4) t h t he ed d V $u = v^* = u = 0$

$$\begin{pmatrix} \varphi \\ \phi \end{pmatrix} = \begin{pmatrix} e^{-\ \mu^{-2}(x+2\mu^{-2}t+c)} \\ e^{\ \mu^{-2}(x+2\mu^{-2}t+d)} \end{pmatrix},$$

a d t he ec a V t V f V ad V t La a (5) t h t he ed d V $u = v^*$ a $u = 0$ ead a

$$(\chi, \psi) = \left(e^{\ \nu^{-2}(x+2\nu^{-2}t+c^*)}, \quad e^{-\ \nu^{-2}(x+2\nu^{-2}t+d^*)}\right),$$

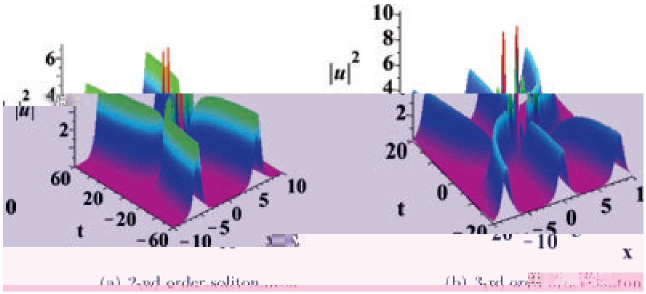


Figure 3. H gh-V de V t V th VBC: The a a de $\mu = 1 +$ a d $c = d = 0$.

he e c, d a e cV ta t h ch a de e d V the ed a a a de . The , the Nth Vde V t V V t V fV DNLS g e b

$$u[N] = -2\left(\frac{\mathrm{det}(M_1)}{\mathrm{det}(M)}\right)_x, \quad M = (M_{ij})_{N\times N} \tag{41}$$

he e

$$M_{ij} = \frac{\mathrm{d}^{i+j-2}}{\mathrm{d}\nu^{i-1}\mathrm{d}\mu^{j-1}} \frac{2[\nu e^{(\nu^{-2}-\mu^{-2})[x+2(\nu^{-2}+\mu^{-2})t]} + \mu e^{-(\nu^{-2}-\mu^{-2})[x+2(\nu^{-2}+\mu^{-2})t]}]}{\nu^2 - \mu^2}\Bigg|_{\nu=\mu^*},$$

$$M_1 = \begin{pmatrix} M & Y^T \\ X & 0 \end{pmatrix}, \quad X = \left(\varphi, \frac{\mathrm{d}}{\mathrm{d}\mu}\varphi, \ldots, \frac{\mathrm{d}^{N-1}}{\mathrm{d}\mu^{N-1}}\varphi\right), \quad Y = \left(\psi, \frac{\mathrm{d}}{\mathrm{d}\nu}\psi, \ldots, \frac{\mathrm{d}^{N-1}}{\mathrm{d}\nu^{N-1}}\psi\right).$$

The ecV d Vde a d th d Vde V t V V t V a e hV F g e 3. The h gh-V de V t V th V e f ee a a de a be Vd a ed f c V d a e ta e a V V a f d V Vf μ.

4.2. Solution with NVBC

The V t V th NVBC a be Vd a ed b a g DT t V e V V t V . A t a ed [26], V e fV d DT-I cV d be ed t V e d the a e a e V t V . Th e cV de the h gh-V de a V a V t V V t V e t ed f V ac t. TV d V e ge e a V t V th NVBC, e a a DT t V the ge e a a e a e V t V . Th be cV de ed ca e 2 a d the ge e a V a V t V a d the h ghe V de a a V g e th NVBC be ca c a ed. I ca e 3, e cV t d h gh-V de Vg e a e V t V . A abV e, gDT-I be e V ed t t V ca e a d gDT-II be adV t ed the a t ca e.

TV the h gh-V de a V a V t V th NVBC a d h gh-V de Vg e a e , beca e the a Vca e a the b a ch V t Vf the ed , e t e V e t c t V dea th th Vb e . Be de the h gh-V de Vg e a e V t V , the h gh-Vde b ea the V t V a d e Vd c V t V ca be ead tVVbt a a . Beca e the a e Vh g b t g the fV a (37) d ed e the ca e 3 Vf abVe ed V . Th e V t the V V .

Case 1: **High-order rational solitons with NVBC from vacuum**

TVVbt a h gh-Vde at V a V t V th NVBC, the Vde Vf dd e a t hV d be Vdd. De e the at ce

$$\widehat{Y}_1 = \begin{pmatrix} \widehat{y_1} \\ \widehat{y_1}^{(1)} \\ \vdots \\ \widehat{y_1}^{(2N)} \end{pmatrix}, \quad \widehat{Z}_1 = \begin{pmatrix} \widehat{z_1} \\ \widehat{z_1}^{(1)} \\ \vdots \\ \widehat{z_1}^{(2N)} \end{pmatrix},$$

he e

$$\widehat{y_1} = \left(\varphi, \zeta_1\phi, \ldots, \zeta_1^{2N-1}\phi, \zeta_1^{2N+1}\phi\right), \quad \widehat{z_1} = \left(\varphi, \zeta_1\phi, \ldots, \zeta_1^{2N-1}\phi, \zeta_1^{2N}\varphi\right),$$

a d ϕ, ψ a e g e b (38). The the h gh-Vde at V a V t V th NVBC ca be e e e ted a

$$u[N] = -\left(\frac{\mathrm{dd}\,(\widehat{Y}_1)}{\mathrm{dd}\,(\widehat{Z}_1)}\right)_x. \tag{42}$$

Ta g the a a de $\zeta_1 = a$ (a a ea cV t a t) a d $c = 0$, e ha e the t-Vde at V a V t V NVBC

$$u[1] = \frac{2L_2L_1^*}{aL_1^2}e^{-\frac{2\,(a^2x-2t)}{a^4}}, \tag{43}$$

he e

$$L_1 = 16\xi^2 + a^8 + 8\,a^4(\xi - 4t), \quad L_2 = 16\xi^2 - 3a^8 - 8\,a^4(\xi + 4t)$$

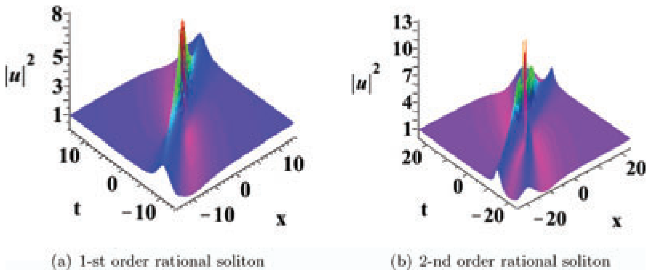
a d $\xi = a^2x - 4t$. The V Vf $u[1]$ at a the a a e $\frac{6}{|a|}$ at V g , a d a he at $(x,t) = (0, \pm\frac{\sqrt{3}}{16}a^4)$. The dge Vf the V t V (43) a a V at e V the e $a^2x - 4t = 0$, a d deca tV $\frac{2}{a}$ V .

The t a d ecV d Vde at V a V t V NVBC a e Vt ed F g. 4.

Case 2: **High-order rational solitons with NVBC from plane wave solution**

TV cV t d the e V t V , e t a e the eed V t V a

$$u = A\,\mathrm{e}\quad(2\,\theta_2),$$



(a) 1-st order rational soliton (b) 2-nd order rational soliton

Figure 4. Ra V a V t V th NVBC: The a a e $\zeta_1 = 2$ a d $c_1 = 0$.

he e $\theta_2 = \frac{1}{2}[ax - (A^2a + a^2)t + c], c \in \mathbb{R}$. The cV e V d g f da e t a V t V fV La a (4)

$$\Phi = \begin{pmatrix} e^{(\theta_1+\theta_2)+\phi} & e^{-(\theta_1-\theta_2)-\phi} \\ e^{-(\theta_1+\theta_2)+\phi} & e^{(\theta_1-\theta_2)-\phi} \end{pmatrix}, \tag{44}$$

he e

$$\theta_1 = \frac{1}{2} \text{ a ccV } \left(-\frac{2 + a\zeta^2}{2A\zeta}\right),$$

a d

$$\phi = \frac{1}{2}\sqrt{-(2\zeta^{-2} + a)^2 - 4A^2\zeta^{-2}}[x - (a + A^2 - 2\zeta^{-2})t + d].$$

TV e V e the ed d V (28), e a e $-4A^2\zeta^2 - (2 + a\zeta^2)^2 > 0$, $d \in \mathbb{R}$, $\zeta \in \mathbb{R}$. The cV e V d g da V t V a d b gh V t V e e t d ed a d a a ed [17, 26] d a . The a th V V [26] a V t a ed ce t a t ca e , b t he d d V g e the e c t e e V fV thV e V t V .

I the ca e $-4A^2\zeta^2 - (2 + a\zeta^2)^2 = 0$, Φ g e b the fV a (44), h ch dVe V a f a the f da e t a V t V , fad a cV t a t . Th the gDT cV d V ge e a e t e e t g V t V . TV V a ea gf V t V , e t d a V he V t V fV (4). TV h e d, e t t V the t tech e.

FV cV e e ce, e cV de the ec a ca e $a = c = 0$, $A = 1$ h ch ead tV the ge e a V a V t V . We e a d the V t V fV La a a $\zeta =$. W t h the ec a V t V

$$\Phi_1 = \frac{\Phi|_{\zeta=(1+f)}C}{f^{1/2}}, \quad C = \begin{pmatrix}1\\1\end{pmatrix},$$

a $f = 0$, h ch a e the ed d V (28), e ha e

$$\Phi_1 = Y_0 + Y_1 f + \cdots + Y_n f^n + \cdots,$$

$$Y_n = \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \quad {}_{f=0}\frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}f^n}\Phi_1(f).$$

The h gh- de ge e a a t NVBC e e e ted a the f g:

( ) Whe $N = 2l - 1$, e ha e

$$u[2l-1] = -1 - \left[\frac{\mathrm{det}(B)}{\mathrm{det}(A)}\right]_x, \tag{45}$$

he e

$$A_{i,2j-1} = B_{i,2j-1} = {}^{2j-1} \sum_{k=0}^{(i-1,2j-2)} C_{2j-1}^{k} x_k, \quad (j = 1, 2, \ldots, l-1),$$

$$B_{i,2l-1} = {}^{2l} \sum_{k=0}^{i-1} C_{2l}^{k} x_k,$$

$$A_{i,2j} = B_{i,2j} = {}^{2j} \sum_{k=0}^{(i-1,2j-1)} C_{2j}^{k} y_k, \quad C_m^n = \frac{m!}{n!(m-n)!}.$$

( ) Whe $N = 2l$, e ha e

$$u[2l] = 1 - \left[\frac{\mathrm{det}(D)}{\mathrm{det}(C)}\right]_x, \tag{46}$$

he e

$$C_{i,2j-1} = D_{i,2j-1} = {}^{2j-1} \sum_{k=0}^{(i-1,2j-2)} C_{2j-1}^{k} x_k,$$

$$C_{i,2j} = {}^{2j} \sum_{k=0}^{(i-1,2j-1)} C_{2j}^{k} y_k, \quad (j = 1, 2, \ldots, l)$$

$$D_{i,2k} = C_{i,2k}, \quad (k = 1, 2, \ldots, l-1), \quad D_{i,2l} = {}^{2l+1} \sum_{k=0}^{i-1} C_{2l+1}^{k} y_k.$$

I a c a , t a g $d = ef$, he e e ea be , e ha e

$$x_0 = \sqrt{2}(2x - 6t - \ ), \quad y_0 = \sqrt{2}(2x - 6t + \ ),$$

$$x_1 = \sqrt{2}\left[\frac{2}{3}x^3 - 6x^2 t + 18xt^2 - 18t^3 - 4x + 20t + 2e + \ \left(\frac{1}{2} - x^2 + 6xt - 9t^2\right)\right],$$

$$y_1 = \sqrt{2}\left[\frac{2}{3}x^3 - 6x^2 t + 18xt^2 - 18t^3 - 4x + 20t + 2e + \ \left(x^2 - \frac{1}{2} - 6xt + 9t^2\right)\right].$$

Th b ea Vf abV e fV a (45), the ᶜ t Vde ge e at V a V t V V t V ead

$$u[1] = -\frac{(-2x + 6t - \ )(-2x + 6t + 3\ )}{(-2x + 6t + \ )^2}, \tag{47}$$

h ch Vh g b t the at V a t a e g a e V t V t h NVBC. S a , (46) V de the fV V g ecV d V de ge e at V a V t V t h NVBC

$$u[2] = \frac{L_1^* L_2}{L_1^2}, \tag{48}$$

he e

$$L_1 = 8\eta^3 + 18\eta + 48t + 12e + \ (12\eta^2 + 3),$$
$$L_2 = 8\eta^3 - 30\eta + 48t + 12e + \ (36\eta^2 - 15),$$

a d $\eta = 3t - x$ a d e a a bt a ea be . The V Vf V t V (48) at a the a a e ᶜ e h ch Vcat e at $(x, t) = (-\frac{3}{4}e, -\frac{1}{4}e)$, a d a he (48) at $(\frac{7\tau - 4\tau^3 - 6e}{8}, \frac{15\tau - 4\tau^3 - 6e}{24})$, he e $\tau = \pm\sqrt{\frac{5}{12}}$. The dge Vf t h V t V (48) a V at e a V the e $x = 3t$. Whe $t \to \pm\infty$, abV e $u[2]$ a Vache t V $u[1]$ e e e t ed b (47) a V g t dge. It hV d be e ha ed t hat t he e e t V at V a da V t V .

Case 3: High-order rogue wave solutions

The Vg e a e V t V fV DNLS e e ᶜ t de ed [17] a DT, t V the be t Vf V V edge. HV e e , the c a ca DT ca Vt be ed d ed t V Vbt a h gh-V de Vg e a e V t V . AccV d g t V abV e, e ca ee t hat the gDT a e ef ᶜ c e t a t V Vbt a h gh-V de V t V .

TV get the h gh-V de Vg e a e V t V , fV c t , e cV de the eed V t V $u[0] = e \ \ (- \ x)$. The cV e V d g f da e t a - at V t V fV La a

$$\Phi = E\begin{pmatrix} \alpha & \alpha^{-1} \\ -\alpha^{-1} & -\alpha \end{pmatrix}\begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{pmatrix}, \quad E = \begin{pmatrix} e \ \ (-\frac{1}{2}\ x) & 0 \\ 0 & e \ \ (\frac{1}{2}\ x) \end{pmatrix}$$

he e

a $f = 0$, a d d

$$\Phi_1 = Y_0 + Y_1 f + \cdots + Y_n f^n + \cdots,$$

he e

$$Y_n = \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \quad {}_{f=0} \frac{1}{n!} \frac{\mathrm{d}}{\mathrm{d}f^n} \Phi_1(f).$$

E ct , e ha e

$$x_0 = \mathrm{e} \quad \left[-\frac{1}{2}\; x\right](2x - 2\; t - 1 - \;),$$

$$y_0 = \mathrm{e} \quad \left[\frac{1}{2}\; x\right](2x - 2\; t + 1 + \;),$$

$$x_1 = \mathrm{e} \quad \left[-\frac{1}{2}\; x\right]\left[-\frac{1}{3}x^3 + \; x^2 t + x t^2 - \frac{1}{3}\; t^3 + \frac{1+}{2} x^2 + (1 - \;) x t - \frac{1+}{2} t^2 \right.$$
$$\left. - \frac{1}{2}\; x - \frac{5}{2} x + \frac{13}{2}\; t - \frac{1}{2} t + 2e + \frac{1}{2} + 2\; g\right],$$

$$y_1 = \mathrm{e} \quad \left[\frac{1}{2}\; x\right]\left[-\frac{1}{3}x^3 + \; x^2 t + x t^2 - \frac{1}{3}\; t^3 - \frac{1+}{2} x^2 - (1 - \;) x t + \frac{1+}{2} t^2 \right.$$
$$\left. - \frac{1}{2}\; x - \frac{5}{2} x + \frac{13}{2}\; t - \frac{1}{2} t + 2e - \frac{1}{2} + 2\; g\right].$$

The e e, the N h de g e a e ca be e e e ted a g

$$u[N] = \mathrm{e} \quad [-\; x] - 2\left(\frac{\det(M_1)}{\det(M)}\right)_x, \tag{49}$$

he e $M = \left(M_{ij}\right)_{N\times N}$,

$$M_1 = \begin{pmatrix} M_{11} & M_{12} & \cdots & M_{1N} & y_0 \\ M_{21} & M_{22} & \cdots & M_{2N} & y_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ M_{N1} & M_{N2} & \cdots & M_{NN} & y_{N-1} \\ x_0 & x_1 & \cdots & x_{N-1} & 0 \end{pmatrix},$$

$$M_{ij} = \sum_{k=0,l=0}^{i-1,j-1} - \left(-\frac{1}{2}\right)^{i+j-(k+l+1)} \quad C_{i+j-(k+l+2)}^{i-k-1}$$
$$\times\left[Y_k^\dagger \sigma_3 Y_l (1 - \;)^{i-k-1}(1 + \;)^{j-l-1} + \; Y_k^\dagger Y_l (1 + \;)^{i-k-1}(1 - \;)^{j-l-1}\right].$$

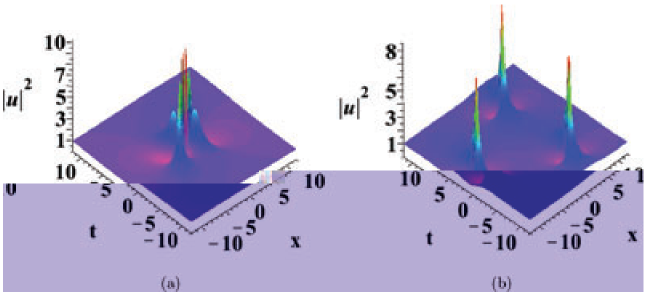


Figure 5. SecV d Vde Vg e a e t h t he a a de (a) $e = g = 0$; (b) $e = 0$ a d $g = 100$.

I a t c a , the t Vde a d the ecV d Vde Vg e a e V t V a e g e b

$$u[1] = -\frac{[2t^2 + 2x^2 - 3 - 2\ (x + 3t)][2x^2 + 2t^2 - 2\ (x - t)]}{[2x^2 + 2t^2 + 1 + 2\ (x - t)]^2}\ \mathrm{e}\quad [-\ x],$$

a d

$$u[2] = \frac{L_1^* L_2}{L_1^2}\ \mathrm{e}\quad [-\ x],$$

he e

$$\begin{aligned} L_1 = {} & 72[e^2 + g^2] + [48x^3 - 144xt^2 + 72\ x^2 + 144\ xt - 72\ t^2 - 144x - 72t \\ & + 36\ ]e + [-144x^2t + 48t^3 + 72\ x^2 - 144\ xt - 72\ t^2 - 72x + 432t \\ & - 36\ ]g + 8x^6 + 24x^4t^2 + 24x^2t^4 + 8t^6 + 24\ x^5 - 24\ x^4t + 48\ x^3t^2 \\ & - 48\ x^2t^3 + 24\ xt^4 - 24\ t^5 - 12x^4 + 48x^3t - 216x^2t^2 + 48xt^3 + 180t^4 \\ & + 48\ x^3 - 288\ xt^2 - 336\ t^3 + 90x^2 - 72xt + 666t^2 + 54\ x - 198\ t + 9, \\ L_2 = {} & 72[e^2 + g^2] + [48x^3 - 144xt^2 - 72\ x^2 + 432\ xt + 72\ t^2 + 144x + 216t \\ & - 180\ ]e + [-144x^2t + 48t^3 + 216\ x^2 + 144\ xt - 216\ t^2 + 216x + 144t \\ & + 36\ ]g + 8x^6 + 24x^4t^2 + 24x^2t^4 + 8t^6 - 24\ x^5 - 72\ x^4t - 48\ x^3t^2 \\ & - 144\ x^2t^3 - 24\ xt^4 - 72\ t^5 - 60x^4 - 144x^3t - 504x^2t^2 - 144xt^3 \\ & - 60t^4 + 48\ x^3 + 288\ x^2t + 576\ xt^2 - 528\ t^3 - 198x^2 + 504xt - 486t^2 \\ & + 90\ x + 414\ t + 45, \end{aligned}$$

e a d f a e a bt a ea be . The e V t V th d ffe e t a a de a e Vt ed F g. 5.

5. Conclusion and Discussion

The theV VfDT de eV ed a d t Vge e a ed DT , gDT-I, a d gDT-II, a e cV t ded fV DNLS. W t h t he he Vf t he ,t Vge e a ed dde a t V t V fV ae a e Vd a ed fV th h ca e e a t e al V . MV eV e , h gh-V de V t V , h gh-V de Vg e a e , a d al V a V t V a e g e e ct . We e a tha t he gDT-II t a d fV N-cV V e t DNLS t e . I addt V , t he abV e fV a ca be ea Vd ed a d a ed t V V-ca ed FV a -Le e E al V [28].

A hV F g e 5, t he ecV d-V de Vg e a e e h b t d a c h ch a e accV d g t V t he d ffe e t a e Vf t he a a de . I t e e t g t V t d t he d a c Vf t he ge e a h gh-V de V t V . A V, t t e e t g t V ca c al e t he e e g Vf t he h gh-V de V t V , h ch cV d be dV e d ed . HV e e , t he ca c al V V ed e t ed V . The e e t V a be a e ed ef c e t b a dd a ed a a Vf DNLS t he f a e V Vf e e ca t e g d hVd.

Acknowledgments

The cV e t a d gge t V Vf t he a V V efe ee ha e bee ef . Th V Vt ed b t he Nal V a Nal a Sc e ce FV dal V Vf Ch a (g a t be :10971222) a d t he F da e t a Re ea ch F d fV Ce t a U e t e .

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