

A long waves-short waves model: Darboux transformation and soliton solutions

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Darboux transformation is constructed for a third-order spectral problem. By proper reduction, a Darboux transformation for a long-short wave model is obtained. Furthermore, a closed multi-soliton solution formula is found for this equation.

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I. INTRODUCTION

Construction of solution for nonlinear systems has been a difficult and yet an important problem. During last four decades, important progress has been made and many methods have been developed serving this purpose. For instance, one may try to find solution for a given system by now well-known methods such as inverse scattering transformation,^{1,2} Hirota method,⁸ or Darboux or Bäcklund transformation.⁹

The aim of the this paper is to study a long wave-short wave model, which reads as

$$\begin{aligned} A_t &= 2\sigma(|B|^2)_x, \\ B_t &= iB_{xx} - A_x B + iA^2 B - 2i\sigma B|B|^2, \end{aligned} \quad (1)$$

where $A = A(x, t)$ represents the amplitude of the long wave and $B(x, t)$ the envelope of the short wave. This equation was considered by Newell^{13,14} in terms of the inverse scattering transformation. Chowdhury and Chanda³ took the Weiss-Tabor-Carnevale approach to the Painlevé analysis and considered its integrability and Bäcklund transformation. In Ref. 11, it was shown that (1) is related with a model equation proposed by Yajima and Oikawa through a Muira transformation. For its physical relevance, we refer to the recent paper.⁴ While this is an important model, to our knowledge, not much is known for its solutions. In particular, while one-soliton solution was calculated in the framework of the inverse scattering transformation, a compact formula is not available for the general N -soliton solution.

In this paper, we will take up this problem. We will construct a Darboux transformation for (1). Indeed, Darboux transformation approach has been very successful for construction of soliton solutions for some nonlinear systems.^{5-7,12} We will work with a general 3×3 spectral problem and construct its Darboux transformation. This will be done in Sec. II. Then, in Sec. III, we reduce this Darboux transformation to the particular case we are interested in. In Sec. IV, we will give the N -fold Darboux matrix. In the last section, we present some explicit solutions.

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II. DARBOUX TRANSFORMATION

We begin with the following general spectral problem:

$$\Psi_x = U\Psi, \quad (2a)$$

$$\Psi_t = V\Psi, \quad (2b)$$

where

$$U = \zeta J + P, \quad V = \sum_{i=0}^N \zeta^i V_i \quad (3)$$

with

$$J = \begin{pmatrix} i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -i \end{pmatrix}, \quad P = \begin{pmatrix} 0 & B & iA \\ C & 0 & D \\ iE & F & 0 \end{pmatrix},$$

and ζ is the spectral parameter; A, B, C, D, E, F are field variables depending on x and t and V_i ($i = 1, 2, \dots, N$) are the matrices which could be determined by means of the zero-curvature condition

$$U_t - V_x + [U, V] = 0$$

or

$$[J, V_N] = 0,$$

$$[J, V_{N-1}] + [P, V_N] = 0,$$

$$[J, V_{i-1}] + [P, V_i] - V_{i,x} = 0, \quad i = 1, \dots, N-1,$$

$$P_t - V_{0,x} + [P, V_0] = 0. \quad (4)$$

Generally, Eq. (4) is an evolution equation.

To construct a Darboux transformation for (2), we also consider the associated conjugate equations

$$-\Psi_x^T = \Psi^T U, \quad (5a)$$

$$-\Psi_t^T = \Psi^T V, \quad (5b)$$

where Ψ^T is the 3-row vector.

Now according to Refs. 6 and 10, by combining the elementary Darboux transformation of original spectral problem and that of conjugate spectral problem, we can construct a new Darboux transformation. Indeed, the Darboux matrix for the spectral problem (2) reads as

$$T = I + \frac{\nu_1 - \xi_1}{\zeta - \nu_1} \frac{\Psi_1 \Psi_1^T}{\Psi_1^T \Psi_1}, \quad (6)$$

and the Darboux matrix for the conjugate spectral problem (5) is

$$T^c = I + \frac{\xi_1 - \nu_1}{\zeta - \xi_1} \frac{\Psi_1 \Psi_1^T}{\Psi_1^T \Psi_1}, \quad (7)$$

where Ψ_1 is a special solution of the spectral problem (2) with $\zeta = \xi_1$, Ψ_1^T is a special solution of the conjugate spectral problem (5) at $\zeta = \nu_1$. It is easy to see that $T^c T = T T^c = I$.

III. REDUCTIONS

With the results above, we now turn to Eq. (1). According to Newell,^{13,14} the corresponding spectral problem for (1) is the one (2) but with the specific U and V

reads as

$$\begin{aligned}
 T &= T[2]T[1] \\
 &= \left[I + \frac{\xi_1 - \xi_1^*}{\zeta + \xi_1^*} \frac{T[1]|_{\zeta=-\xi_1} \sigma_2 \Phi_1 \Phi_1^\dagger \sigma_2 T[1]^c|_{\zeta=-\xi_1^*}}{\Phi_1^\dagger \sigma_2 T[1]^c|_{\zeta=-\xi_1^*} T[1]|_{\zeta=-\xi_1} \sigma_2 \Phi_1} \right] \left[I + \frac{\xi_1^* - \xi_1}{\zeta - \xi_1^*} \frac{\Phi_1 \Phi_1^\dagger \sigma_1}{\Phi_1^\dagger \sigma_1 \Phi_1} \right] \\
 &= I - \frac{\Delta_1}{\zeta - \xi_1^*} - \frac{\Delta_2}{\zeta + \xi_1^*},
 \end{aligned} \tag{12}$$

where

$$\Delta_1 + \Delta_2 = \left[\frac{\Phi_1^\dagger \sigma_2 \sigma_1 \sigma_2 \Phi_1}{-\xi_1 + \xi_1^*} \Phi_1 \Phi_1^\dagger \sigma_1 - \frac{\Phi_1^\dagger \sigma_2 \sigma_1 \Phi_1}{\xi_1 + \xi_1^*} \sigma_2 \Phi_1 \Phi_1^\dagger \sigma_1 \right] + \left[\frac{\Phi_1^\dagger \sigma_1 \Phi_1}{\xi_1 - \xi_1^*} \sigma_2 \Phi_1 \Phi_1^\dagger \sigma_2 \sigma_1 - \frac{\Phi_1^\dagger \sigma_1 \sigma_2 \Phi_1}{-\xi_1 - \xi_1^*} \Phi_1 \Phi_1^\dagger \sigma_2 \sigma_1 \right].$$

Then, by direct calculation, one may verify that this Darboux transformation preserves the constraint (10) as well as (11). The associated transformations between field variables are given by

$$\begin{aligned}
 \hat{A} &= A + 2(\Delta_1 + \Delta_2)_{13}, \\
 \hat{B} &= B + i(\Delta_1 + \Delta_2)_{12},
 \end{aligned}$$

where $(\Delta_1 + \Delta_2)_{kl}$ denotes the entry of k th row and l th column for $\Delta_1 + \Delta_2$.

As usual, this particular Darboux transformation may be iterated. However, an alternative approach to this problem may be adopted and a better form exists as we will show in Sec. IV.

IV. THE ITERATED DARBOUX TRANSFORMATION

We consider the fractional form of Darboux matrices (6) and (7) and at first we study the general problem with no reduction involved. Suppose that we are given N solutions Ψ_k at $\zeta = \xi_k$ ($\xi_i \neq \xi_j, i \neq j$) of (2) and N solutions Ψ_k^T at $\zeta = v_k$ ($v_i \neq v_j, i \neq j$) ($\xi_m \neq v_n$) of (5) ($k = 1, 2, \dots, N$), respectively, then composition of Darboux transformations leads to the N -fold Darboux transformation

$$T = T[N]T[N-1] \cdots T[1], \quad T^c = T[1]^c T[2]^c \cdots T[N]^c, \tag{13}$$

where

$$\begin{aligned}
 T[k] &= I + \frac{v_k - \xi_k}{\zeta - v_k} \frac{\Psi_k[k-1] \Psi_k^T[k-1]}{\Psi_k^T[k-1] \Psi_k[k-1]}, \\
 T[k]^c &= I + \frac{\xi_k - v_k}{\zeta - \xi_k} \frac{\Psi_k[k-1] \Psi_k^T[k-1]}{\Psi_k^T[k-1] \Psi_k[k-1]},
 \end{aligned}$$

and

$$\begin{aligned}
 \Psi_k[k-1] &= T[k-1]T[k-2] \cdots T[1]|_{\zeta=\xi_k} \Psi_k, \\
 \Psi_k^T[k-1] &= \Psi_k^T T[1]^c|_{\zeta=v_k} \Psi_k^T.
 \end{aligned}$$

It is easy to see that the rank of A_k is less than or equal to 1. Taking account of the following facts:

$$\det T = \frac{(\zeta - \xi_1)(\zeta - \xi_2) \cdots (\zeta - \xi_N)}{(\zeta - \nu_1)(\zeta - \nu_2) \cdots (\zeta - \nu_N)}, \quad \det T^c = \frac{(\zeta - \nu_1)(\zeta - \nu_2) \cdots (\zeta - \nu_N)}{(\zeta - \xi_1)(\zeta - \xi_2) \cdots (\zeta - \xi_N)},$$

we find that both A_k 's and C_k 's are all rank one matrices. Therefore, we assume

$$A_k = |a_k\rangle\langle b_k|, \quad C_k = |c_k\rangle\langle d_k|,$$

where $|a_k\rangle$ and $|c_k\rangle$ are column vectors, while $\langle b_k|$, $\langle d_k|$ are row vectors. Keeping the identity $TT^c = I$, $T^cT = I$ at $\zeta = \xi_k$, $\zeta = \nu_k$ in mind, we arrive at the following:

$$T|_{\zeta=\xi_k}|c_k\rangle = 0, \quad \langle b_k|T^c|_{\zeta=\nu_k} = 0, \quad (16)$$

$$T^c|_{\zeta=\nu_k}|a_k\rangle = 0, \quad \langle d_k|T|_{\zeta=\xi_k} = 0. \quad (17)$$

Owing to equation (13)–(16), we have the identifications $|c_k\rangle = \Psi_k$ and $\langle b_k| = \Psi_k^T$. Rewriting the equations (16) and (17), we obtain

$$|c_k\rangle = -\sum_{l=1}^N \frac{|a_l\rangle\langle b_l|c_k\rangle}{\xi_k - \nu_l}, \quad \langle b_k| = \sum_{l=1}^N \frac{\langle b_k|c_l\rangle\langle d_l|}{\xi_l - \nu_k}, \quad (18)$$

$$|a_k\rangle = -\sum_{l=1}^N \frac{|c_l\rangle\langle d_l|a_k\rangle}{\nu_k - \xi_l}, \quad \langle d_k| = \sum_{l=1}^N \frac{\langle d_k|a_l\rangle\langle b_l|}{\nu_l - \xi_k}. \quad (19)$$

Introducing the notions

$$D = \left(\frac{\langle b_l|c_k\rangle}{\xi_k - \nu_l} \right)_{lk}, \quad \hat{D} = \left(\frac{\langle d_l|a_k\rangle}{\nu_k - \xi_l} \right)_{lk},$$

which fulfill the relation $D\hat{D} = I$, then in general case we obtain the iterated Darboux matrices

$$T = I - \sum_{k,l=1}^N \frac{1}{\zeta - \nu_k} |c_l\rangle (D^{-1})_{lk} \langle b_k|, \quad (20)$$

$$T^c = I + \sum_{k,l=1}^N \frac{1}{\zeta - \xi_k} |c_l\rangle (D^{-1})_{lk} \langle b_k|. \quad (21)$$

As discussed in the last section, to have a meaningful Darboux transformation for (1), one needs to choose seed solutions properly. Indeed, let μ_k , ($k = 1, 2, \dots, N$)

where $|c_{2k-1}\rangle = \Phi_k$, $|c_{2k}\rangle = \sigma_2 \Phi_k$, $\langle b_{2k-1}| = \Phi_k^\dagger \sigma_1$, $\langle b_{2k}| = \Phi_k^\dagger \sigma_2 \sigma_1$, and

$$D = \begin{pmatrix} \frac{\Phi_1^\dagger \sigma_1 \Phi_1}{\mu_1 - \mu_1^*} & \frac{\Phi_1^\dagger \sigma_1 \sigma_2 \Phi_1}{-\mu_1 - \mu_1^*} & \dots & \frac{\Phi_1^\dagger \sigma_1 \Phi_N}{\mu_N - \mu_1^*} & \frac{\Phi_1^\dagger \sigma_1 \sigma_2 \Phi_N}{-\mu_N - \mu_1^*} \\ \frac{\Phi_1^\dagger \sigma_2 \sigma_1 \Phi_1}{\mu_1 + \mu_1^*} & \frac{\Phi_1^\dagger \sigma_2 \sigma_1 \sigma_2 \Phi_1}{-\mu_1 + \mu_1^*} & \dots & \frac{\Phi_1^\dagger \sigma_2 \sigma_1 \Phi_N}{\mu_N + \mu_1^*} & \frac{\Phi_1^\dagger \sigma_2 \sigma_1 \sigma_2 \Phi_N}{-\mu_N + \mu_1^*} \\ \dots & \dots & \dots & \dots & \dots \\ \frac{\Phi_N^\dagger \sigma_1 \Phi_1}{\mu_1 - \mu_N^*} & \frac{\Phi_N^\dagger \sigma_1 \sigma_2 \Phi_1}{-\mu_1 - \mu_N^*} & \dots & \frac{\Phi_N^\dagger \sigma_1 \Phi_N}{\mu_N - \mu_N^*} & \frac{\Phi_N^\dagger \sigma_1 \sigma_2 \Phi_N}{-\mu_N - \mu_N^*} \\ \frac{\Phi_N^\dagger \sigma_2 \sigma_1 \Phi_1}{\mu_1 + \mu_N^*} & \frac{\Phi_N^\dagger \sigma_2 \sigma_1 \sigma_2 \Phi_1}{-\mu_1 + \mu_N^*} & \dots & \frac{\Phi_N^\dagger \sigma_2 \sigma_1 \Phi_N}{\mu_N + \mu_N^*} & \frac{\Phi_N^\dagger \sigma_2 \sigma_1 \sigma_2 \Phi_N}{-\mu_N + \mu_N^*} \end{pmatrix},$$

that is,

$$\hat{A} = A + 2 \left(\sum_{k,l=1}^{2N} |c_l\rangle (D^{-1})_{lk} \langle b_k| \right)_{13} = A + 2 \sum_{k=1}^{2N} \frac{\det D_k}{\det D}, \quad (23)$$

$$\hat{B} = B + i \left(\sum_{k,l=1}^{2N} |c_l\rangle (D^{-1})_{lk} \langle b_k| \right)_{12} = B + i \sum_{k=1}^{2N} \frac{\det S_k}{\det D}, \quad (24)$$

where D_{2k-1} is the matrix D with $(2k-1)$ th row replaced by the vector

$$\left[\phi_1^{(1)} \phi_3^{(k)*}, \phi_3^{(1)} \phi_3^{(k)*}, \dots, \phi_1^{(N)} \phi_3^{(k)*}, \phi_3^{(N)} \phi_3^{(k)*} \right],$$

D_{2k} is the matrix D with $2k$ th row replaced by the vector

$$\left[\phi_1^{(1)} \phi_1^{(k)*}, \phi_3^{(1)} \phi_1^{(k)*}, \dots, \phi_1^{(N)} \phi_1^{(k)*}, \phi_1^{(N)} \phi_1^{(k)*} \right],$$

S_{2k-1} is the matrix D with $(2k-1)$ th row replaced by the vector

$$\left[-\sigma \phi_1^{(1)} \phi_2^{(k)*}, -\sigma \phi_3^{(1)} \phi_2^{(k)*}, \dots, -\sigma \phi_1^{(N)} \phi_2^{(k)*}, -\sigma \phi_3^{(N)} \phi_2^{(k)*} \right], \quad (25)$$

and S_{2k} is the matrix D with $2k$ th row replaced by the above vector (25).

V. EXPLICIT SOLUTIONS

In what follows, we will construct some solutions for (1) by means of the Darboux transformation constructed above.

(23) and (24) supply us

$$A[1] = A - 4\alpha\beta \frac{2\alpha \operatorname{Im}(\psi_1 \psi_3^*) G - \beta (|\psi_1|^2 - |\psi_3|^2) H}{\alpha^2 G^2 + \beta^2 H^2}, \quad (26)$$

$$B[1] = B - 2\sigma\alpha\beta(\psi_1 - \psi_3)\psi_2^* \frac{\alpha G - i\beta H}{\alpha^2 G^2 + \beta^2 H^2}, \quad (27)$$

where $G = |\psi_1|^2 + |\psi_3|^2 - \sigma|\psi_2|^2$, $H = 2\operatorname{Re}(\psi_1^* \psi_3) - \sigma|\psi_2|^2$, and $(\psi_1, \psi_2, \psi_3)^T$ is a special solution of the linear spectral problems at $\zeta = \mu_1 = \alpha + i\beta$. Depending on the values of the seeds A and B , we have two types of solutions for the system (1).

- $A = B = 0$.

In this case, the solution to the linear spectral problems at $\zeta = \mu_1 = \alpha + \beta i$ is given by

$$\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \exp\left(-\frac{4}{3}\alpha\beta t + i\left[\frac{2}{3}(\alpha^2 - \beta^2)t + \theta_2\right]\right) \times \begin{pmatrix} k_1 \exp(-\beta x + 2\alpha\beta t + i[\alpha x - (\alpha^2 - \beta^2)t + \theta_1]) \\ k_2 \\ k_3 \exp(\beta x + 2\alpha\beta t + i[-\alpha x - (\alpha^2 - \beta^2)t + \theta_3]) \end{pmatrix}, \quad (28)$$

where $k_i \geq 0$, θ_i , ($i = 1, 2, 3$) are arbitrary real numbers. Substituting (28) to (26) and (27) leads to the explicit solutions. By choosing the parameters differently, we obtain the solutions with different behaviors.

(I) Traveling solitary solution

Suppose $k_3 = 0$, we obtain the first kind traveling wave solution

$$A[1] = \frac{-4\alpha\beta^2\sigma k_1^2 k_2^2 \exp(-2\beta x + 4\alpha\beta t)}{\alpha^2 [k_1^2 \exp(-2\beta x + 4\alpha\beta t) - \sigma k_2^2]^2 + \beta^2 k_2^4},$$

$$|B[1]|^2 =$$



FIG. 1. (Color online) (a), (b): Normal breather type ($\mu_1 = 1 + \sqrt{3}i$, $\sigma = -1$, $c_1 = c_3 = 1$, $c_2 = 0$); (c), (d): Y-V-Breather type ($\mu_1 = 1 + \sqrt{3}i$, $\sigma = -1$, $c_1 = c_2 = c_3 = 1$).

if $-x + 2\alpha t = \frac{1}{2\beta} \ln(|\alpha| \sqrt{\alpha^2 + \beta^2 k_1^2 k_2^2})$ and $\alpha\beta t \rightarrow -\infty$, while the second kind traveling solution can be obtained with $x + 2\alpha t = \frac{1}{2\beta} \ln(|\alpha| \sqrt{\alpha^2 + \beta^2 k_2^2 k_3^2})$ and $\alpha\beta t \rightarrow -\infty$.

- $A = 0$, $B = \exp(-2i\sigma t)$.

In present case, the solutions to the linear spectral problems at $\zeta = \mu_1$ reads as

$$\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \exp\left(\frac{2}{3}i\mu_1^2 t\right) \begin{pmatrix} \frac{i}{\mu_1} \exp(-2i\sigma t) & \frac{1}{2}\sigma v_2 \exp(w_1) & -\frac{1}{2}\sigma v_3 \exp(w_2) \\ 1 & \exp(w_1 + 2i\sigma t) & \exp(w_2 + 2i\sigma t) \\ -\frac{i}{\mu_1} \exp(-2i\sigma t) & \frac{1}{2}\sigma v_3 \exp(w_1) & -\frac{1}{2}\sigma v_2 \exp(w_2) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}, \quad (29)$$

where $v_1 = \sqrt{-\mu_1^2 + 2\sigma}$, $v_2 = i\mu_1 + v_1$, $v_3 = -i\mu_1 + v_1$, $w_1 = v_1 x - i\mu_1^2 t$, $w_2 = -v_1 x - i\mu_1^2 t$, and c_k ($k=1, 2, 3$) are arbitrary complex numbers. Substituting (29) into (26), we have two breather type solutions, whose plots are given by Fig. 1 above.

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