where E and H are electric and magnetic field vectors, and D and B are corresponding electric and magnetic flux densities. The relations between D, B and E, H are called the constitutive relations given by

$$D = \epsilon E, \quad B = \mu H, \tag{5}$$

where ϵ is the permittivity and μ is the permeability. In a vacuum, $c^2 = 1/(\epsilon_0\mu_0)$ with c the velocity of light in vacuum. In the frequency-dependent media,

$$D = \epsilon * E, \quad B = \mu * H, \quad D = \epsilon_0 E + P, \tag{6}$$

where $*$ means the convolution, and P is the electric induced polarization. By eliminating B and D, the resulting wave equation follows:

$$\nabla^2 E - \frac{1}{c^2} E_{tt} = \mu_0 P_{tt}, \tag{7}$$

which describes light propagation in optical fibers. If we assume the local medium response and only the third-order nonlinear effects governed by $\chi^{(3)}$, the induced polarization consists of linear and nonlinear parts $P(r,t) = P_L(r,t) + P_{NL}(r,t)$, with the linear part

$$P_L(r,t) = \epsilon_0 \int_{-\infty}^{\infty} \chi^{(1)}(t - t') \cdot E(r,t')\, dt', \tag{8}$$

and the nonlinear part

$$P_{NL}(r,t) = \epsilon_0 \iiint_{-\infty}^{\infty} \chi^{(3)}(t - t_1, t - t_2, t - t_3) \times E(r,t_1)E(r,t_2)E(r,t_3)\, dt_1 dt_2 dt_3. \tag{9}$$

Here ϵ_0 is the vacuum permittivity and $\chi^{(j)}$ is the jth-order susceptibility. As discussed in [23], the nonlinear response is due to the induced atomic dipole with a response time of the order $1/\Delta\omega$, where $\Delta\omega = |\omega_{ik} - \omega_0|$. ω_{ik} represents the transition frequency from the initial (usually ground) quantum state i into some excited state k, and ω_0 is the central carrier frequency. Since the typical transition frequency from the atomic ground state to the lowest excited state significantly exceeds the usual carrier frequency, $1/\Delta\omega$ is typically less than 1 fs. Therefore, we can assume an instantaneous nonlinear response in the femtosecond regime. Moreover, the nonlinear effects are relatively small in silica fibers, P_{NL} can be treated as a small perturbation. Therefore, we first consider Eq. (7) with $P_{NL} = 0$. Furthermore, we restrict ourselves to the case that the optical pulse maintains its polarization along the optical fiber, and the transverse diffraction term can be neglected. In this case, the electric field can be considered to be one dimensional and expressed as

$$E = \tfrac{1}{2} e_1 [E(z,t) + \text{c.c.}], \tag{10}$$

where e_1 is a unit vector in the direction of the polarization, $E(z,t)$ is the complex-valued function, and c.c. stands for the complex conjugate. Under this case, it is useful to transform Eq. (7) into the frequency domain, which reads

$$\tilde{E}_{zz}(z,\omega) + \epsilon(\omega)\frac{\omega^2}{c^2}\tilde{E}(z,\omega) = 0, \tag{11}$$

where $\tilde{E}(z,\omega)$ is the Fourier transform of $E(z,t)$ defined as

$$\tilde{E}(z,\omega) = \int_{-\infty}^{\infty} E(z,t)e^{i\omega t}\, dt. \tag{12}$$

The frequency-dependent dielectric constant occurring in Eq. (12) is defined as

$$\epsilon(\omega) = 1 + \tilde{\chi}^{(1)}(\omega), \tag{13}$$

where $\tilde{\chi}^{(1)}(\omega)$ is the Fourier transform of $\chi^{(1)}(t)$. Up to now, the consideration is exactly the same as the one for deriving the NLS equation. To derive the NLS equation, the optical field is assumed to be quasimonochromatic, i.e., the pulse spectrum, centered as ω_0, is assumed to have a spectral width $\Delta\omega$ such that $\Delta\omega/\omega_0 \ll 1$. Under this assumption, the NLS equation can be derived to govern the slowly varying envelope of the optical wave packet in weakly nonlinear dispersive media. However, when the width of the optical pulse is of the order of femtoseconds (10^{-15} s), the monochromatic assumption to derive the NLS equation is not valid anymore. We need to construct a suitable fit to the frequency-dependent dielectric constant $\epsilon(\omega)$ in the desired spectral range. More specifically, for the frequency-dependent dielectric constant $\epsilon(\omega) = 1 + \tilde{\chi}^{(1)}(\omega)$, we assume $\tilde{\chi}^{(1)}$ can be approximated by

$$\tilde{\chi}^{(1)} = \tilde{\chi}_0^{(1)} \mp \tilde{\chi}_2^{(1)}\lambda^2, \quad \tilde{\chi}_2^{(1)} > 0. \tag{14}$$

As discussed subsequently, the negative sign represents the focusing media with anomalous group velocity dispersion (GVD), and the positive sign stands for the defocusing media with normal GVD.

Next we proceed to the consideration of the nonlinear effect. Assuming the nonlinear response is instantaneous so that P_{NL} is given by $P_{NL}(z,t) = \epsilon_0 \epsilon_{NL} E(z,t)$ [3], the nonlinear contribution to the dielectric constant is defined as

$$\epsilon_{NL} = \tfrac{3}{4}\chi^{(3)}_{xxxx}|E(z,t)|^2. \tag{15}$$

Therefore, the Helmholtz equation can be modified as

$$\tilde{E}_{zz}(z,\omega) + \tilde{\epsilon}(\omega)\frac{\omega^2}{c^2}\tilde{E}(z,\omega) = 0, \tag{16}$$

where

$$\tilde{\epsilon}(\omega) = 1 + \tilde{\chi}_0^{(1)} \mp \tilde{\chi}_2^{(1)}\lambda^2 + \epsilon_{NL}. \tag{17}$$

In summary, Eq. (16) with Kerr cubic nonlinearity reads

$$\tilde{E}_{zz} + \frac{1 + \tilde{\chi}_0^{(1)}}{c^2}\omega^2\tilde{E} \mp (2\pi)^2\tilde{\chi}_2^{(1)}\tilde{E} + \epsilon_{NL}\frac{\omega^2}{c^2}\tilde{E} = 0. \tag{18}$$

By applying the inverse Fourier transform to Eq. (18), the nonlinear wave equation in physical domain is

$$E_{zz} -$$

Next, we focus on only a right-moving wave packet and assume a multiple scales ansatz

$$E(z,t) = \epsilon E_0(\phi, z_1, z_2, \dots) + \epsilon^2 E_1(\phi, z_1, z_2, \dots) + \cdots, \tag{22}$$

where ϵ is a small parameter, and z_n are the scaled variables defined by

$$\phi = \frac{t - x}{\epsilon}, \quad z_n = \epsilon^n z. \tag{23}$$

Substituting (22) with (23) into (19), we obtain the following partial differential equation for E_0 at the order $O(\epsilon)$:

$$-2\frac{\partial^2 E_0}{\partial \phi \partial z_1} = \pm E_0 + 2\frac{\partial}{\partial \phi}\left(|E_0|^2 \frac{\partial E_0}{\partial \phi}\right). \tag{24}$$

Here the term $E_0^2 E_{0,\phi\phi}$ is ignored but it is validated subsequently. Finally a general complex short-pulse equation can be obtained,

$$q_{xt} \pm q + \tfrac{1}{2}(|q|^2 q_x)_x = 0, \tag{25}$$

by the scale transformations

$$x = \frac{1}{\sqrt{2}}\phi, \quad t = \frac{1}{\sqrt{2}} z_1, \quad q = \sqrt{2} E_0. \tag{26}$$

It is obvious that Eq. (25) with positive sign is the same as Eq. (3), while Eq. (25) with negative sign is equivalent to Eq. (3) by a conversion of time $t \to -t$. Consequently, we derive the CSP equation of both the focusing and defocusing types. We should point out that there are typos in the scaling transformations in [16].

To validate the approximation, we compare the solutions to Maxwell equations with the ones to the CSP equation. As a matter of fact, solitary wave solutions with a few cycles derived directly from the Maxwell equations under the assumption of the Kramers-Kronig relation holds have been investigated in the literature [24–27]. Here we mainly refer to the results in [25] and consider the normalized equation (21) with positive sign. We assume an envelope solitary wave solution is of the form

$$E(z,t) = A(\xi) e^{i\phi(z,t)}, \tag{27}$$

with $\xi = z - vt$, $\phi(z,t) = \kappa(t - vz) + F(\xi)$. Inserting this ansatz into Eq. (21), one obtains the set of equations

$$\begin{aligned} &A_{zz} - A_{tt} - A(\phi_z^2 - \phi_t^2) \\ &\quad - A - (6AA_t^2 + 3A^2 A_{tt} - A^3 \phi_t^2) = 0, \end{aligned} \tag{28}$$

$$2A_z\phi_z + A\phi_{zz} - (2A_t\phi_t + A\phi_{tt}) - (6A^2 A_t \phi_t + A^3 \phi_{tt}) = 0, \tag{29}$$

$$\begin{aligned} &[1 - v^2A^2/(1 - v^2)]AF_{\xi\xi} + 2[1 - 3v^2A^2/(1 - v^2)]A_\xi F_\xi \\ &\quad + 6v\kappa A^2 A_\xi/(1 - v^2) = 0. \end{aligned} \tag{30}$$

By introducing normalized amplitude $a = vA/\sqrt{1 - v^2}$, we obtain

$$F(\xi) = -\frac{\kappa}{2v}\int_{-\infty}^{\xi} \frac{a^2(3 - 2a^2)}{(1 - a^2)} d\xi \tag{31}$$

by integrating Eq. (30) once. Further, inserting $\phi_t = \kappa - vF_\xi$ and $\phi_z = -\kappa v + F_\xi$ into Eq. (28), one obtains a second-order differential equation

$$\begin{aligned} &a_{\xi\xi} - \frac{6aa_\xi^2}{1 - 3a^2} - \frac{\kappa^2 a}{1 - 3a^2}\left(\frac{\lambda^2}{v^2} - \frac{a^2(4(1 - a^2)^2 - a^2)}{4v^2(1 - a^2)^3}\right) \\ &\quad = 0, \end{aligned} \tag{32}$$

where $\lambda^2 = v^2/[\kappa^2(1 - v^2)] - v^2$. Integrating once and requiring $a, a_\xi \to 0$ at $\xi \to \pm\infty$, one arrives at

$$a_\xi = \pm\frac{\kappa\sqrt{1 + \lambda^2}\,a}{v(1 - 3a^2)(1 - a^2)}\sqrt{\left(1 - \frac{3}{2}a^2\right)\left(\frac{1 + 4\lambda^2 + \sqrt{1 - 8\lambda^2}}{4(1 + \lambda^2)} - a^2\right)\left(\frac{1 + 4\lambda^2 - \sqrt{1 - 8\lambda^2}}{4(1 + \lambda^2)} - a^2\right)}. \tag{33}$$

From (33), one can easily show that a localized solution exists with amplitude

$$a_{max} = \frac{1}{2}\sqrt{\frac{1 + 4\lambda^2 + \sqrt{1 - 8\lambda^2}}{1 + \lambda^2}} \tag{34}$$

provided $\lambda^2 \leqslant 1/8$.

As mentioned in [25], in the case of $a_{max} \ll 1$ where the slowly evolving wave field approximation (SEWA) is valid, the solution to Eq. (33) can be written as

$$\frac{9}{2}\sqrt{\lambda^2 - \frac{a^2}{2}} - \cosh^{-1}\left(\frac{\sqrt{2}\lambda}{a}\right) = \pm\frac{\kappa\xi}{v}. \tag{35}$$

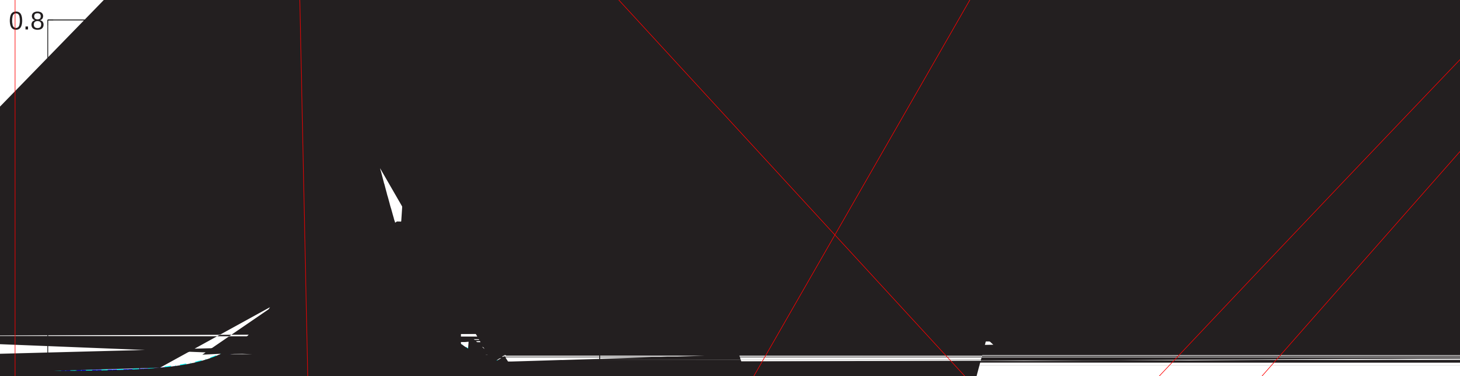
Furthermore, when $\lambda^2 \ll 1/8$ and the first term in Eq. (35) can be neglected, we obtain the one-soliton solution to the NLS equation,

$$a_{NLS} = \sqrt{2}\lambda\,\mathrm{sech}(\kappa\xi/v). \tag{36}$$

Multiplying Eq. (35) by 2 and taking cosh function, we arrive at a localized solution to the higher order nonlinear Schrödinger (HONLS) equation by taking into account dispersions beyond group velocity dispersion (GVD),

$$\begin{aligned} a_{HONLS} &= \frac{2\sqrt{2}\lambda}{\sqrt{9\lambda^2 + \sqrt{81\lambda^2 + 4}\cosh(2\kappa\xi/v)}}, \\ F &= -\frac{3\kappa}{2v}\int_{-\infty}^{\xi} a^2 d\xi. \end{aligned} \tag{37}$$

In Fig. 1, we compare the solutions for Eqs. (21) and (25) with positive sign [16,17,28], and the solution to the NLS equation and the higher order NLS equations (37) and (35) for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where

$$U(q,\rho;\lambda) = \lambda^{-1}\begin{bmatrix} -i\rho & \bar{q}_y \\ q_y & i\rho \end{bmatrix},$$

$$V(q;\lambda) = \frac{i}{4}\lambda\sigma_3 + \frac{i}{2}Q, \quad Q = \begin{bmatrix} 0 & -\bar{q} \\ q & 0 \end{bmatrix},$$

with the overbar representing the complex conjugate and σ_3 being the third Pauli matrix, yields the defocusing CCD equation

$$q_{ys} = \rho q, \tag{45}$$

$$\rho_s - \tfrac{1}{2}(|q|^2)_y = 0. \tag{46}$$

Through the hodograph transformation

$$dx = \rho\, dy + \tfrac{1}{2}|q|^2 ds, \quad dt = -ds,$$

one can obtain the defocusing CSP equation (42). To obtain the soliton equation, we give the following Darboux matrix for the defocusing CSP equation (42) (we omitted the proof here; the interested reader can refer to [28,30–32] for details):

$$\begin{aligned} T &= I + \frac{\bar{\lambda}_1 - \lambda_1}{\lambda - \bar{\lambda}_1}P_1, \quad P_1 = \frac{|y_1\rangle\langle y_1|\sigma_3}{\langle y_1|\sigma_3|y_1\rangle}, \\ \langle y_1| &= |y_1\rangle^{\dagger}, \quad |y_1\rangle = \begin{bmatrix} \phi_1(y,s;\lambda_1) \\ \psi_1(y,s;\lambda_1) \end{bmatrix}, \end{aligned} \tag{47}$$

where $|y_1\rangle$ denotes the special solution for the system (43) and (44) with $\lambda = \lambda_1$, and we can convert system (43) and (44) into a new system:

$$\Phi[1]_y = U(q[1],\rho[1];\lambda)\,\Phi[1], \tag{48}$$

$$\Phi[1]_s = V(q[1];\lambda)\,\Phi[1]. \tag{49}$$

The Bäcklund transformations between $(q[1],\rho[1])$ and (q,ρ) are given through

$$q[1] = q + \frac{(\bar{\lambda}_1 - \lambda_1)\,\bar{\phi}_1\psi_1}{\langle y_1|\sigma_3|y_1\rangle}, \tag{50}$$

$$\rho[1] = \rho - 2\ln_{ys}\left(\frac{\langle y_1|\sigma_3|y_1\rangle}{\bar{\lambda}_1 - \lambda_1}\right), \tag{51}$$

$$|q[1]|^2 = |q|^2 - 4\ln_{ss}\left(\frac{\langle y_1|\sigma_3|y_1\rangle}{\bar{\lambda}_1 - \lambda_1}\right). \tag{52}$$

Furthermore, we have the following N-fold Darboux matrix: The N-fold Darboux matrix can be represented as

$$T_N = I + YM^{-1}D^{-1}Y^{\dagger}\sigma_3, \tag{53}$$

where

$$\begin{aligned} Y &= [|y_1\rangle, |y_2\rangle, \ldots, |y_N\rangle], \\ M &= \left(\frac{\langle y_i|\sigma_3|y_j\rangle}{\bar{\lambda}_i - \lambda_j}\right)_{1\leqslant i,j\leqslant N}, \\ D &= \mathrm{diag}(\lambda - \bar{\lambda}_1, \lambda - \bar{\lambda}_2, \ldots, \lambda - \bar{\lambda}_N), \end{aligned}$$

the vector $|y_i\rangle$ represents the special solution for system (43) and (44) with $\lambda = \lambda_i$, and the Bäcklund transformations for $q[N]$ and $\rho[N]$ are

$$q[N] = q + \frac{\det(\widehat{M})}{\det(M)}, \tag{54}$$

$$\rho[N] = \rho - 2\ln_{ys}[\det(M)], \tag{55}$$

$$|q[N]|^2 = |q|^2 - 4\ln_{ss}[\det(M)], \tag{56}$$

where

$$\widehat{M} = \begin{bmatrix} M & Y_1^{\dagger} \\ -Y_2 & 0 \end{bmatrix},$$

and Y_k represents the kth row of matrix Y. The proof can be given similar to the one in [32], which is omitted here. Instead, we merely comment that the following identities associated with the matrix and determinant are used:

$$\begin{aligned} \alpha^{\dagger}M^{-1}\beta &= \begin{vmatrix} M & \beta \\ -\alpha^{\dagger} & 0 \end{vmatrix} \Big/ |M|, \\ 1 + \alpha^{\dagger}M^{-1}\beta &= \begin{vmatrix} M & \beta \\ -\alpha^{\dagger} & 1 \end{vmatrix} \Big/ |M| = \frac{\det(M + \beta\alpha^{\dagger})}{\det(M)}. \end{aligned}$$

Here M is an $N \times N$ matrix; α, β are the $N \times 1$ matrix. Based on the above N-fold Darboux transformation for the CCD system (40) and (41), we have the solitonic solution formula for the defocusing CSP equation (42) in parametric form,

$$q[N] = q + \frac{\det(\widehat{M})}{\det(M)}, \tag{57}$$

$$x = \int \rho\, dy + \frac{1}{2}\int |q|^2 ds - 2\ln_s[\det(M)], \quad t = -s. \tag{58}$$

IV. ONE- AND MULTI-DARK SOLUTIONS TO THE DEFOCUSING CSP EQUATION

In this section, we derive an explicit expression for the one- and multi-dark-soliton solution to the defocusing CSP equation through formulas (54)–(56) by a limit technique.

A. One-dark-soliton solution

We start with the seed solution

$$\rho[0] = -\frac{\rho}{2}, \quad q[0] = \frac{\rho}{2}e^{i\theta}, \quad \theta = y + \frac{\rho}{2}s, \quad \rho > 0$$

However, the soliton solution obtained above is usually singular. In order to derive the one-dark-soliton solution through the DT method, a limit process $\lambda_1 \to \bar{\lambda}_1$ is needed. To this end, we first pick up one special solution,

$$|y_1\rangle = K L_1 M_1 \begin{bmatrix} 1 \\ \varepsilon_1(\bar{\lambda}_1 - \lambda_1) \end{bmatrix};$$

further, for the sake of convenience, we set

$$\lambda_1 = \rho\cosh(\alpha + i\theta_1) - \kappa, \qquad \xi_1^{\pm} = \rho e^{\pm(\alpha + i\theta_1)} - \kappa,$$

$$\varepsilon_1 = -\frac{e^{-i\theta_1}}{4\rho\sin^2\theta_1},$$

where $\theta_1 \in (0,\pi)$. By taking a limit $\alpha \to 0$, we can obtain

$$\frac{y_1^{\dagger}\sigma_3 y_1}{2(\bar{\lambda}_1 - \lambda_1)} = \frac{e^{2\eta_1}+1}{\rho(e^{-i\theta_1} - e^{i\theta_1})}, \tag{60}$$

where

$$\eta_1 = -\frac{\rho\sin\theta_1}{4}\left(s + \frac{2y}{\rho\cos\theta_1 - \kappa}\right) + a_1.$$

Thus, the single dark soliton can be written as

$$q[1] = \frac{\rho}{2}\left[\frac{1+e^{2(\eta_1 - i\theta_1)}}{1+e^{2\eta_1}}\right]e^{i\phi}$$

$$= \frac{\rho}{4}[(1+e^{-2i\theta_1}) + (e^{-2i\theta_1} - 1)\tanh\eta_1]e^{i\phi}, \tag{61}$$

$$x = -\frac{\kappa}{2}y + \frac{\rho^2}{8}s + \frac{\rho\sin\theta_1 e^{2\eta_1}}{1+e^{2\eta_1}}, \quad t = -s, \tag{62}$$

The nonsingularity condition for the single dark soliton is $\delta[1] \neq 0$ for all $(x,t) \in \mathbb{R}^2$. To analyze the property for the one-soliton solution, we calculate out

$$\delta[1] = \frac{\partial x}{\partial y} = -\frac{e^{4\eta_1} + \left(2 + \frac{2\rho^2\sin^2\theta_1}{\rho\cos\theta_1 - \kappa}\right)e^{2\eta_1} + 1}{2(1+e^{2\eta_1})^2}, \tag{63}$$

thus we can classify this one-dark-soliton solution as follows:

(i) Smooth soliton: when $\kappa - \rho\cos\theta_1 < 0$, or $\kappa - \rho\cos(\theta_1) > 0$ and $\Delta_1 > 0$, where $\Delta_1 = 2\rho^2 - 2\kappa\rho\cos\theta_1 - \rho^2\sin^2\theta_1$, the single-dark-soliton solution is always smooth. An example is illustrated in Fig. 3(a).

(ii) Cuspon soliton: when $\kappa - \rho\cos\theta_1 > 0$ and $\Delta_1 = 0$ then $\delta[1]$ attains zero at only one point, which leads to a cusponed dark soliton as displayed in Fig. 3(b).

(iii) Loop soliton: when $\kappa - \rho\cos\theta_1 > 0$, $\Delta_1 < 0$, then $\delta[1]$ attains two zeros, which leads to a looped dark soliton.

The velocity can be solved with the following relation:

$$x - v_{sp,1}t - c_1 = \frac{(\rho\cos\theta_1 - \kappa)}{\rho\sin\theta_1}\eta_1 + \frac{\rho}{2}\sin\theta_1\tanh(\eta_1).$$

The velocity of the dark soliton is

$$v_{sp,1} = -\frac{2\kappa(\rho\cos\theta_1 - \kappa) + \rho^2}{8},$$

and the initial center is

$$c_1 = \frac{(\kappa - \rho\cos\theta_1)}{\rho\sin(\theta_1)}a_1 + \frac{\rho}{2}\sin\theta_1.$$

The trough of the dark soliton $|q|$ is along the line

$$x - v_{sp,1}t - c_1 = 0,$$

and the depth of the trough is $\frac{1}{2}|\rho(1-\cos\theta_1)|$.

B. Multi-dark-soliton solution

Similar to the process of obtaining the single-dark-soliton solution, starting with the same seed solution, and solving the Lax pair equations (43) and (44) with $(q = q[0], \phi = \phi[0])$ at $\lambda = \lambda_i$, we have

$$|y_i\rangle = K L_i M_i \begin{bmatrix} 1 \\ \varepsilon_i(\bar{\lambda}_i - \lambda_i) \end{bmatrix} \equiv K\begin{bmatrix} \hat{\phi}_i \\ \hat{\psi}_i \end{bmatrix},$$

where

$$L_i = \begin{bmatrix} 1 & 1 \\ \frac{\rho}{\xi_i^{+} + \kappa} & \frac{\rho}{\xi_i^{-} + \kappa} \end{bmatrix}, \quad M_i = \mathrm{diag}(e^{\eta_i^{+}}, e^{\eta_i^{-}}),$$

with

$$\eta_i^{\pm} = \frac{i}{4}(\xi_i^{\pm} - \kappa)\left(s + \frac{2}{\lambda_i}y\right) \pm a_i,$$

$$\xi_i^{\pm} = \lambda_i \pm \sqrt{(\lambda_i + \kappa)^2 - \rho^2},$$

λ_is are appropriate complex parameters and a_is are real parameters. Based on the N-soliton solution (57) and (58) to the defocusing CSP equation, it then follows that

$$q[N] = \frac{\rho}{2}[1 + \hat{Y}_2 M^{-1}\hat{Y}_1^{\dagger}]e^{i\phi} = \frac{\rho}{2}\left[\frac{\det(H)}{\det(M)}\right]e^{i\phi},$$
$$x = -\frac{\kappa}{2}y + \frac{\rho^2}{8}s - 2\ln_s[\det(M)], \quad t = -s, \tag{64}$$

where

$$M = \left(\frac{y_i^{\dagger}\sigma_3 y_j}{2(\bar{\lambda}_i - \lambda_j)}\right)_{1\leqslant i,j\leqslant N}, \quad H = M + Y_1^{\dagger}Y_2,$$

$$\hat{Y}_1 = [\hat{\phi}_1, \hat{\phi}_2, \ldots, \hat{\phi}_N], \quad \hat{Y}_2 = [\hat{\psi}_1, \hat{\psi}_2, \ldots, \hat{\psi}_N].$$

In general, the above N-soliton solution (64) is singular. In order to derive the N-dark soliton solution through the DT method, we need to take a limit process $\lambda_i \to \bar{\lambda}_i$ ($i = 1,2,\ldots,N$) similar to the one in [32]. By a tedious procedure which is omitted here, we finally have the N-dark-soliton solution to the defocusing CSP equation (42) as follows:

$$q = \frac{\rho}{2}\frac{\det(H)}{\det(M)}e^{i\rho[y+(\rho/2)s]}, \tag{65}$$

$$x = -\frac{\kappa}{2}y + \frac{\rho^2}{8}s - 2\ln_s[\det(M)], \quad t = -s, \tag{66}$$

where the entries of the matrices M and H are

$$m_{i,j} = \frac{e^{\eta_i + \eta_j} + \delta_{i,j}}{\rho(e^{-i\theta_i} - e^{i\theta_j})}, \quad h_{i,j} = \frac{e^{(\eta_i - i\theta_i)+(\eta_j - i\theta_j)} + \delta_{i,j}}{\rho(e^{-i\theta_i} - e^{i\theta_j})},$$
$$1 \leqslant i,j \leqslant N, \tag{67}$$

$\delta_{i,j}$ is a Kronecker's δ and

$$\eta_i = -\frac{\rho}{4}\sin\theta_i\left(s + \frac{2y}{\rho\cos\theta_i - \kappa}\right) + a_i, \quad \theta_i \in \mathbb{R}. \tag{68}$$

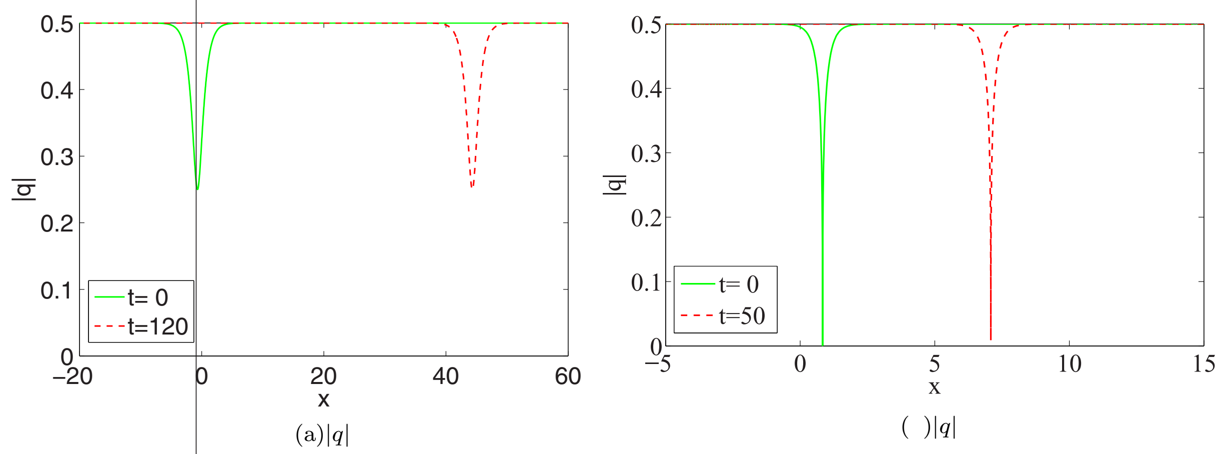


FIG. 3. (a) Smooth dark soliton ($t = 0$: blue solid line; $t = 120$: red dashed line). Parameters $\rho = 1$, $\kappa = 1$, $\phi_1 = 2\pi/3$, $a_1 = 0$. (b) Cuspon dark soliton ($t = 0$: blue solid line; $t = 50$: red dashed line). Parameters $\rho = 1$, $\kappa = \sqrt{2}/2$, $\phi_1 = \pi/2$, $a_1 = 0$.

By taking $N = 2$ in (67), the determinants corresponding to the two-dark-soliton solution can be calculated as

$$|M| = 1 + e^{2\xi_1} + e^{2\xi_2} + a_{12}e^{2(\xi_1+\xi_2)}, \tag{69}$$

$$|H| = 1 + e^{2(\xi_1 - i\phi_1)} + e^{2(\xi_2 - i\phi_2)} + a_{12}e^{2(\xi_1+\xi_2 - i\phi_1 - i\phi_2)}, \tag{70}$$

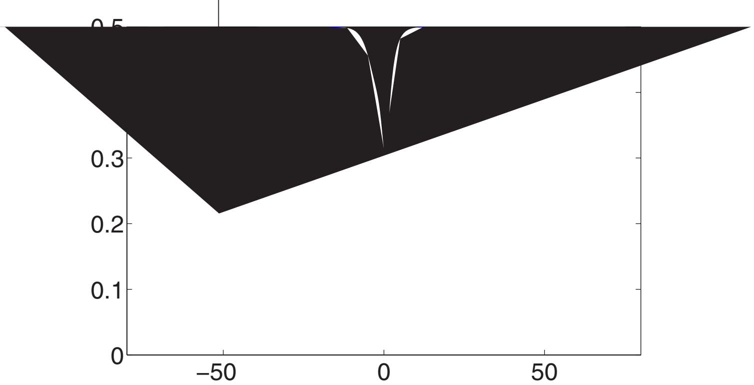
where

$$a_{12} = \frac{\sin^2\left(\frac{\phi_2-\phi_1}{2}\right)}{\sin^2\left(\frac{\phi_2+\phi_1}{2}\right)}. \tag{71}$$

The collision processes between smooth-smooth dark solitons and smooth-cuspon dark solitons are illustrated in Figs. 4(a) and 4(b), respectively. It is seen that the interactions between dark solitons are elastic. Different from the interaction between two smooth bright solitons, which could develop singularity, the interaction between two smooth dark solitons never appears as singularity. When a smoothed dark soliton interacts with a cusponed dark soliton, the singularity of the cusponed dark soliton could vanish as observed in Fig. 4(b).

V. CONCLUSIONS AND DISCUSSIONS

In this paper, we derived the complex short-pulse equation of both focusing and defocusing types from the context of nonlinear optics and found the multi-dark-soliton solution of the defocusing type. Comparing with the classical theory for the SP equation, there are several advantages in using complex representation. First, amplitude and phase are two fundamental characteristics for a wave packet, the information of these two factors is nicely combined into a single complex-valued function. Second, the use of complex representation can allow us to model the propagation of optical pulses in both the focusing and defocusing nonlinear media. Such advantages



can be observed in many analytical results related to the NLS equation and the CSP equation. Therefore, by using a complex representation, we have shown that the focusing CSP equation admits the bright soliton solution [16,17], the breather solution, as well as the rogue wave solution [28], whereas, as shown in the present paper, the defocusing CSP equation has the multi-dark-soliton solution the same as the defocusing NLS equation. It would be a very interesting topic to compare the properties of ultrashort optical pulses experimentally with the theoretical predictions for the CSP equation and the ones for the NLS equations. This, of course, is beyond the scope of the present paper.

The dynamics of the dark soliton has been a hot topic in nonlinear optics. The history for the observation of the dark soliton is even earlier than the bright soliton. In this work, we proposed an integrable equation which admits the multi-dark-soliton solution. Moreover, we provided the dynamics analysis for the single-dark soliton and two-dark soliton in detail. The results would further enrich our understanding of dark solitons in the ultrashort-pulse model.

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