I_1 I_2 $\tilde{u}$ ρ $\tilde{v}$ θ $\tilde{u}^2+1=\rho^2$ $\tilde{v}=\theta$ $\tilde{u}=\sqrt{\rho^1-1}$

$$\begin{aligned} I_1 =& \frac{2\rho^2-1}{\rho^2}(\mathrm{d}\sqrt{\rho^1-1})^2+\rho^2(\mathrm{d}\theta)^2 \\ =& \frac{2\rho^2-1}{\rho^2-1}(\mathrm{d}\rho)^2+\rho^2(\mathrm{d}\theta)^2 \end{aligned}$$

I_2 6.6 □

$$\begin{cases} \rho = \sqrt{u^2+1} \\ \theta = \arctan u + v \end{cases}$$

S_2 $\theta = \arctan u + v$

$$\begin{cases} \sin\theta = \dfrac{u\cos v+\sin v}{\sqrt{u^2+1}} \\ \cos\theta = \dfrac{\cos v - u\sin v}{\sqrt{u^2+1}} \end{cases}$$

S_2

$$\tilde{r}_2(u,v) = (u\cos v+\sin v, \cos v - u\sin v, u)$$

$$\tilde{\sigma}: \begin{cases} u = u \\ v = v \end{cases}$$

- $u-$
- $u=0$ $v-$

6.6

-
-
  6.3
-
  Jacobi
  Jacobi

6.3.5 Gauss

$K_2 = -\frac{1}{(2\rho^2-1)^2}$

$K_1 = -\frac{1}{(2\ ^2+1)^2}$ $K_2 = -\frac{1}{(2\rho^2-1)^2}$ $-\frac{1}{(2\ ^2+1)^2} = -\frac{1}{(2\rho^2-1)^2}$ $\rho = \sqrt{u^2+1}$ $u = \sqrt{\rho^2-1}$

Gauss

$u = \sqrt{\rho^2-1}$ I_1

$$\begin{aligned} I_1 =& 2(\mathrm{d}u)^2 + 2\mathrm{d}u\mathrm{d}v + (u^2+1)(\mathrm{d}v)^2 \\ =& \frac{2\rho^2}{\rho^2-1}(\mathrm{d}\rho)^2 + 2\frac{\rho}{\sqrt{\rho^2-1}}\mathrm{d}\rho\mathrm{d}v + \rho^2(\mathrm{d}v)^2 \end{aligned}$$

v I_1

ρ^2

$$\begin{aligned} I_1 =& \frac{2\rho^2}{\rho^2-1}(\mathrm{d}\rho)^2 + \rho^2\left(2\frac{1}{\rho\sqrt{\rho^2-1}}\mathrm{d}\rho\mathrm{d}v + (\mathrm{d}v)^2\right) \\ =& \frac{2\rho^2-1}{\rho^2-1}(\mathrm{d}\rho)^2 + \rho^2\left(\frac{1}{\rho\sqrt{\rho^2-1}}\mathrm{d}\rho + \mathrm{d}v\right)^2 \end{aligned}$$

$\mathrm{d}\theta = \frac{1}{\rho\sqrt{\rho^2-1}}\mathrm{d}\rho + \mathrm{d}v$

$$\theta = v + \int \frac{1}{\rho\sqrt{\rho^2-1}}\mathrm{d}\rho = v - \arcsin\frac{1}{\rho}$$

$$\arcsin\frac{1}{\rho} = \arcsin\frac{1}{\sqrt{u^2+1}} = \frac{\pi}{2} - \arctan u$$

$\rho = \sqrt{u^2+1}, \theta = \arctan u + v$

Gauss

6.7 $\mathrm{d}\tilde{v} = \frac{\mathrm{d}}{^2+1} + \mathrm{d}v$

$\mathrm{d}\theta = \frac{1}{\rho\sqrt{\rho^2-1}}\mathrm{d}\rho + \mathrm{d}v$

Gauss

例 6.9.

Gauss Gauss

1. $(\cos\theta\cos\phi, \cos\theta\sin\phi, \sin\theta)$

2. $|\vec{r}|^2 = 1$ $\vec{r}\cdot d\vec{r} = 0$ $d\vec{r}\cdot d\vec{r}+\vec{r}\cdot d^2\vec{r} =$ 0 $\vec{n}$ $\vec{r}$ $d\vec{r}\cdot d\vec{r} + \vec{n}\cdot d^2\vec{r} = 0$ $I + II = 0$ $K = 1$

3. $\kappa_{\boldsymbol{r}} = 1 = \kappa$ $\kappa_1 = \kappa_2 = 1$

Gauss 1 Gauss

例 6.10. S $\tilde{S}$

$$\vec{r} = \left(au, bv, \frac{1}{2}(au^2 + bv^2)\right)$$
$$\tilde{\vec{r}} = \left(\tilde{a}\tilde{u}, \tilde{b}\tilde{v}, \frac{1}{2}(\tilde{a}\tilde{u}^2 + \tilde{b}\tilde{v}^2)\right)$$

$ab = \tilde{a}\tilde{b}$ $(a^2, b^2) \neq (\tilde{a}^2, \tilde{b}^2)$ $(a^2, b^2) \neq (\tilde{b}^2, \tilde{a}^2$ $SFRV$ a

$$II_2 : \frac{\tilde{a}}{\sqrt{1+\tilde{u}^2+\tilde{v}^2}}\mathrm{d}\tilde{u}^2 + \frac{\tilde{b}}{\sqrt{1+\tilde{u}^2+\tilde{v}^2}}\mathrm{d}\tilde{v}^2$$

$$K_1 = \frac{1}{ab(1+u^2+v^2)^2}, \quad K_2 = \frac{1}{\tilde{a}\tilde{b}(1+\tilde{u}^2+\tilde{v}^2)^2}$$

$\tilde{u} = u, \tilde{v} = v \qquad K_1 = K_2$

Gauss Gauss

Jacobi

$$\begin{cases} \tilde{u} = \tilde{u}(u,v) \\ \tilde{v} = \tilde{v}(u,v) \end{cases}$$

$$\begin{pmatrix} a^2(1+u^2) & abuv \\ abuv & b^2(1+v^2) \end{pmatrix} = J \begin{pmatrix} \tilde{a}^2(1+\tilde{u}^2) & \tilde{a}\tilde{b}\tilde{u}\tilde{v} \\ \tilde{a}\tilde{b}\tilde{u}\tilde{v} & \tilde{b}^2(1+\tilde{v}^2) \end{pmatrix} J$$

$$J = \begin{pmatrix} \frac{\partial \tilde{}}{\partial} & \frac{\partial \tilde{}}{\partial} \\ \frac{\partial \tilde{}}{\partial} & \frac{\partial \tilde{}}{\partial} \end{pmatrix}$$

Gauss Gauss

$$\frac{1}{\tilde{a}\tilde{b}(1+\tilde{u}^2+\tilde{v}^2)^2} = \frac{1}{ab(1+u^2+v^2)^2}$$

$$\tilde{u}^2 + \tilde{v}^2 = u^2 + v^2$$

Jacobi

$$\begin{cases} \tilde{u}\frac{\partial \tilde{u}}{\partial u} + \tilde{v}\frac{\partial \tilde{v}}{\partial u} = u \\ \tilde{u}\frac{\partial \tilde{u}}{\partial v} + \tilde{v}\frac{\partial \tilde{v}}{\partial v} = v \end{cases}$$

Jacobi

$$\begin{cases} (\frac{\partial \tilde{u}}{\partial u})^2 + \tilde{u}\frac{\partial^2 \tilde{u}}{\partial u^2} + (\frac{\partial \tilde{v}}{\partial u})^2 + \tilde{v}\frac{\partial^2 \tilde{v}}{\partial u^2} = 1 \\ \frac{\partial \tilde{u}}{\partial u}\frac{\partial \tilde{u}}{\partial v} + \tilde{u}\frac{\partial^2 \tilde{u}}{\partial u \partial v} + \tilde{v}\frac{\partial^2 \tilde{v}}{\partial u \partial v} + \frac{\partial \tilde{v}}{\partial u}\frac{\partial \tilde{v}}{\partial v} = 0 \\ (\frac{\partial \tilde{u}}{\partial v})^2 + \tilde{u}\frac{\partial^2 \tilde{u}}{\partial v^2} + (\frac{\partial \tilde{v}}{\partial v})^2 + \tilde{v}\frac{\partial^2 \tilde{v}}{\partial v^2} = 1 \end{cases}$$

$$\tilde{u}=0,\tilde{v}=0$$

$$\begin{cases}(\frac{\partial\tilde{u}}{\partial u})^2+(\frac{\partial\tilde{v}}{\partial u})^2=1\\ \frac{\partial\tilde{u}}{\partial u}\frac{\partial\tilde{u}}{\partial v}+\frac{\partial\tilde{v}}{\partial u}\frac{\partial\tilde{v}}{\partial v}=0\\ (\frac{\partial\tilde{u}}{\partial v})^2+(\frac{\partial\tilde{v}}{\partial v})^2=1\end{cases}$$

$\tilde{u}=0,\tilde{v}=0$ Jacobi

$$J=\begin{pmatrix}\frac{\partial\tilde{}}{\partial} & \frac{\partial\tilde{}}{\partial}\\ \frac{\partial\tilde{}}{\partial} & \frac{\partial\tilde{}}{\partial}\end{pmatrix}=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$$

$\tilde{u}^2+\tilde{v}^2=u^2+v^2$ $\tilde{u}=0,\tilde{v}=0$ $u=0,v=0$

$(0,0)$

$$\begin{aligned}\begin{pmatrix}a^2 & 0\\ 0 & b^2\end{pmatrix}&=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\tilde{a}^2 & 0\\ 0 & \tilde{b}^2\end{pmatrix}\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\\ &=\begin{pmatrix}\tilde{a}^2\cos^2\theta+\tilde{b}^2\sin^2\theta & (\tilde{a}^2-\tilde{b}^2)\sin\theta\cos\theta\\ (\tilde{b}^2-\tilde{a}^2)\sin\theta\cos\theta & \tilde{a}^2\sin^2\theta+\tilde{b}^2\cos^2\theta\end{pmatrix}\end{aligned}\tag{6.3}$$

(6.3)

(6.3)

$(0,0)$

$$\begin{pmatrix}a^2 & 0\\ 0 & b^2\end{pmatrix}=J\begin{pmatrix}\tilde{a}^2 & 0\\ 0 & \tilde{b}^2\end{pmatrix}J$$

$(0,0)$ Jacobi $J^{-1}=J$

$$\begin{pmatrix}a^2 & 0\\ 0 & b^2\end{pmatrix}=J\begin{pmatrix}\tilde{a}^2 & 0\\ 0 & \tilde{b}^2\end{pmatrix}J^{-1}$$

$\begin{pmatrix}a^2 & 0\\ 0 & b^2\end{pmatrix}$ $\begin{pmatrix}\tilde{a}^2 & 0\\ 0 & \tilde{b}^2\end{pmatrix}$

$(a^2,b^2)=(\tilde{a}^2,\tilde{b}^2)$ $(a^2,b^2)=(\tilde{b}^2,\tilde{a}^2)$

Jacobi

Gauss

Gauss Weingarten

Weingarten

Weingarten

Gauss

Gauss Weingarten

Chapter 7

7.1

Gauss

7.1.1

[1]

$$\vec{r}(u,v) = \vec{a}(u) + v\vec{l}(u)$$

$\vec{a}(u)$ $v = 0$ u- v-

$$\vec{r}\ (u,v) = \vec{a}'(u) + v\vec{l}'(u),\ \vec{r}\ (u,v) = \vec{l}(u)$$

[1]

$$\vec{r}_u \times \vec{r}_v = (\vec{a}'(u) + v\vec{l}'(u)) \times \vec{l}(u) \neq \vec{0}$$

$v = 0$ $\vec{a}'(u) \times \vec{l}(u)$

- $\vec{l}(u)$

- $\vec{l}(u)$

$$|\vec{l}(u)| = 1$$

$$\tilde{a}(u) = \vec{a}(u) + \lambda(u)\vec{l}(u)$$

$\lambda(u)$ $\lambda(u)$ $\tilde{a}'(u) \cdot \vec{l}(u) = 0$

$$\tilde{a}'(u) = \vec{a}'(u) + \lambda'(u)\vec{l}(u) + \lambda(u)\vec{l}'(u)$$

$$\tilde{a}'(u) \cdot \vec{l}(u) = \vec{a}'(u) \cdot \vec{l}(u) + \lambda'(u) = 0$$

$\lambda'(u) = -\vec{a}'(u) \cdot \vec{l}(u)$

$$\lambda(u) = -\int_0 \vec{a}'(t) \cdot \vec{l}(t)\mathrm{d}t$$

- $\vec{l}(u)$ $\vec{l}(u)$ $\lambda(u)$

- $\vec{l}(u)$ $\vec{l}(u)$ $\vec{a}(u)$ $\vec{r} = \vec{a}'(u) + v\vec{l}'(u),\ \vec{r}_u = \vec{l}(u)$ (u, v)

-

7.1.2

- $\vec{l}(u)$ $\vec{l}(u) \times \vec{l}'(u) = \vec{0}$

  $\vec{r}(u,v) = \vec{a}(u) + v\vec{l}$ $\vec{l}$ $\vec{l}$

- $\lambda(u)$ $\vec{a}(u) + \lambda(u)\vec{l}(u) = \vec{r}_0$

  $\vec{r}(u,v) = \vec{a} + v\vec{l}(u)$ $\vec{a}$

- $\vec{a}(u)$

  $\vec{r}(u,v) = \vec{a}(u) + v\vec{a}'(u)$[1]

$$\vec{r}_u \times \vec{r}_v = (\vec{a}'(u) + v\vec{a}''(u)) \times \vec{a}'(u) = v\vec{a}''(u) \times \vec{a}'(u)$$

v [2]

定义 7.1. S

命题 7.2. S $\vec{r} = \vec{a}(u) + v\vec{l}(u)$

$$(\vec{a}'(u), \vec{l}(u), \vec{l}'(u)) = 0$$

[1] $v = 0$

[2] v u

.

$$\vec{b}(u,v) = \vec{r}\ \times \vec{r}\ = (\vec{a}'(u) + v\vec{l}'(u)) \times \vec{l}(u)$$

$$\vec{b}(u,v) \qquad\qquad\qquad v$$

$$\frac{\partial \vec{b}}{\partial v} \times \vec{b} = \vec{0}$$

$$\vec{b} \times \underline{\partial \vec{b}}$$

$$\vec{l}(u) \times \vec{l}'(u) \neq \vec{0} \qquad \vec{l}(u) \times \vec{l}'(u) \cdot \vec{a}'(u) = 0$$

$$\vec{a}'(u) = \alpha(u)\vec{l}(u) + \beta(u)\vec{l}'(u)$$

1

$$\vec{b}(u) = \vec{a}(u) + \lambda(u)\vec{l}(u)$$

$$\vec{b}'(u) \equiv \vec{0}$$

$$\vec{b}'(u) = \vec{a}' + \lambda'(u)\vec{l}(u) + \lambda(u)\vec{l}'(u) = (\alpha(u) + \lambda'(u))\vec{l}(u) + (\beta(u) + \lambda(u))\vec{l}'(u) \tag{7.1}$$

$$\begin{cases} \alpha(u) + \lambda'(u) = 0 \\ \beta(u) + \lambda(u) = 0 \end{cases}$$

$\alpha(u) = \beta'(u)$ $\lambda(u) = -\beta(u)$

2 $\vec{b}(u)$ $\vec{b}'(u) \parallel \vec{l}(u)$

(7.1)

$$\begin{cases} \alpha(u) + \lambda'(u) \neq 0 \\ \beta(u) + \lambda(u) = 0 \end{cases}$$

$\alpha(u) \neq \beta'(u)$ $\lambda(u) = -\beta(u)$

$\lambda(u) = -\beta(u)$

□

7.2.2

5.3.1

Weingarten $\mathscr{W}$

$S : \vec{r}(u, v)$ C

$$u = u(t),\ v = v(t)$$

C $\vec{r}(u(t),v(t))$

$$\mathscr{W}\left(\frac{\mathrm{d}\vec{r}(u(t),v(t))}{\mathrm{d}t}\right)=\lambda\frac{\mathrm{d}\vec{r}(u(t),v(t))}{\mathrm{d}t}$$

Weingarten Gauss

$$\mathscr{W}\left(\frac{\mathrm{d}\vec{r}(u(t),v(t))}{\mathrm{d}t}\right)=-\frac{\mathrm{d}\vec{n}(u(t),v(t))}{\mathrm{d}t}$$

Rodriques

定理 7.4. $S:\vec{r}(u,v)$ $C:u=u(t),\ v=v(t)$
S C $\vec{n}(u(t),v(t))$ C C

$$\frac{\mathrm{d}\vec{n}(u(t),v(t))}{\mathrm{d}t}\parallel\frac{\mathrm{d}\vec{r}(u(t),v(t))}{\mathrm{d}t}$$

推论 7.5. S C S C

. C $\vec{r}(t)$ $\vec{n}(t)$

$$\tilde{r}(t,s)=\vec{r}(t)+s\vec{n}(t)$$

$(\vec{a}'(u),\vec{l}(u),\vec{l}'(u))=0$ C

$$(\vec{r}'(t),\vec{n}(t),\vec{n}'(t))=0$$

$$(\vec{r}'(t)\times\vec{n}'(t))\cdot\vec{n}(t)=0$$

$\vec{r}'(t)$ $\vec{n}'(t)$ $\vec{n}(t)$ $\vec{r}'(t)\times\vec{n}'(t)$ $\vec{n}(t)$ $\vec{r}'(t)\times\vec{n}'(t)=\vec{0}$
Rodriques C

□

Rodriques

•

-

例 7.6.

. u- v-

π $\vec{n}$ π

$\vec{r}$ $\vec{r}$

$\vec{r}$ $\vec{r}$ $\vec{r}$ $\vec{r}$

π $\vec{n}$ $\vec{r}$

π

$\vec{n}$ π

$\vec{n}$

□

$(u\cos v, u\sin v, f(u))$

$$\mathrm{I} = (1 + f'^2(u))(\mathrm{d}u)^2 + u^2(\mathrm{d}v)^2$$

$$\mathrm{II} = \frac{f''(u)}{\sqrt{1 + f'^2(u)}}(\mathrm{d}u)^2 + \frac{uf'(u)}{\sqrt{1 + f'^2(u)}}(\mathrm{d}v)^2$$

7.2.3

定理 7.7.

$$I = |\vec{a}'(u) + v\vec{l}'(u)|^2 \mathrm{d}u^2 + 2(\vec{a}'(u) + v\vec{l}'(u)) \cdot \vec{l}(u) \mathrm{d}u \mathrm{d}v + |\vec{l}(u)| \mathrm{d}v^2$$

$$I_{E^2} = \mathrm{d}x^2 + \mathrm{d}y^2 \tag{7.2}$$

$\vec{l}(u)$ $\vec{a}'(u)$

$$I = |\vec{a}'(u) + v\vec{l}'(u)|^2 \mathrm{d}u^2 + \mathrm{d}v^2$$

$\vec{r} = \vec{a}(u) + v\vec{l}$

$$I = |\vec{a}'(u)|^2 \mathrm{d}u^2 + \mathrm{d}v^2$$

(7.2) $\mathrm{d}s = |\vec{a}'(u)| \mathrm{d}u$

$$s(u) = \int |\vec{a}'(u)| \mathrm{d}u$$

$$I = \mathrm{d}s^2 + \mathrm{d}v^2$$

$$\begin{cases} x = s(u) \\ y = v \end{cases}$$

$\vec{a}(u)$ $s(u) = \int |\vec{a}'(u)| \mathrm{d}u$

$x = s(u)$ $\vec{a}(u)$

$\vec{r} = \vec{a} + v\vec{l}(u)$

$$I = v^2 |\vec{l}'(u)|^2 \mathrm{d}u^2 + \mathrm{d}v^2$$

$\mathrm{d}s = v|\vec{l}'(u)| \mathrm{d}u$ $v|\vec{l}'(u)| \mathrm{d}u$ v u

$\mathrm{d}s = |\vec{l}'(u)| \mathrm{d}u$

$$I = v^2 (\mathrm{d}s)^2 + (\mathrm{d}v)^2$$

[1]

$x = v\cos s, y = v\sin s$

$$\mathrm{d}x = \cos s\mathrm{d}v - v\sin s\mathrm{d}s$$
$$\mathrm{d}y = \sin s\mathrm{d}v + v\cos s\mathrm{d}s$$

$$(\mathrm{d}x)^2 + (\mathrm{d}y)^2 = v^2(\mathrm{d}s)^2 + (\mathrm{d}v)^2$$

$$\vec{b}(u) = \vec{a} + \vec{l}(u)$$

$\vec{a}$ 1 $\vec{a}$

$\mathrm{d}s = |\vec{l}'(u)|\mathrm{d}u$ $\vec{b}(u)$

$\vec{b}(u)$

$\vec{a}(s)$

$$\vec{r}(s,t) = \vec{a}(s) + t\vec{a}'(s) = \vec{a}(s) + t\vec{\alpha}(s)$$

Frenet

$$\vec{r}\ = \vec{\alpha}(s) + t\kappa(s)\vec{\beta}(s), \quad \vec{r}\ = \vec{\alpha}(s)$$

$$I = (1 + t^2\kappa^2)\mathrm{d}s^2 + 2\mathrm{d}s\mathrm{d}t + \mathrm{d}t^2$$

τ

$$s = s, t = t$$

s $\kappa(s)$ $\tau(s)$ s

$\kappa(s)$

[1] $v\mathrm{d}s\mathrm{d}v$

H

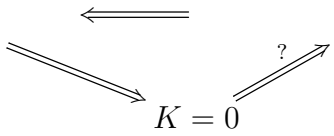
(s,t) (x,y) [1]

v

S S

Gauss [2]

Gauss



命題 7.8. E^3 Gauss S

Gauss

Gauss

Gauss $K=\kappa_1\kappa_2$ $K=0$ $\kappa_2=0$ κ_2

κ_2

[1]

[2] Gauss

v- $\kappa_2 = 0$

v-

$$\vec{r}_{} \times \vec{r}_{} \equiv \vec{0}$$

$$\vec{r}_{} = \Gamma^1_{22}\vec{r}_{} + \Gamma^2_{22}\vec{r}_{} + b_{22}\vec{n}$$

$$\vec{r}_{} \times \vec{r}_{} = \Gamma^1_{22}\vec{r}_{} \times \vec{r}_{} + b_{22}\vec{n} \times \vec{r}_{}$$

$\vec{r}_{} \times \vec{r}_{} \neq \vec{0}$ $\vec{n} \times \vec{r}_{} \neq \vec{0}$ Γ^1_{22}

b_{22}

$\kappa_1 = \frac{}{E}$ $\kappa_2 = \frac{}{}$ $b_{22} = N = 0$

$$\Gamma^1_{22} = \frac{1}{EG - F^2}(G\frac{\partial F}{\partial v} - \frac{G}{2}\frac{\partial G}{\partial u} - \frac{F}{2}\frac{\partial G}{\partial v}) = -\frac{1}{2E}\frac{\partial G}{\partial u}$$

$\kappa_2 = 0$ $\Gamma^1_{22} = 0$ Gauss-Codazzi

(5.17)(5.18)

$\kappa_2 = 0$ $N = 0$

$$-\sqrt{EG}\left((\frac{(\sqrt{E})}{\sqrt{G}}) + (\frac{(\sqrt{G})}{\sqrt{E}})\right) = 0$$

$$\begin{aligned}\frac{\partial L}{\partial v} &= H\frac{\partial E}{\partial v} \\ 0 &= H\frac{\partial G}{\partial u}\end{aligned}$$

$$\Gamma^1_{22} = -\frac{1}{2E}\frac{\partial G}{\partial u}$$

κ_1, κ_2 $H \neq 0$ Codazzi

$\frac{\partial}{\partial} = 0$ $\Gamma^1_{22} = 0$

$\vec{n}$ v- $\vec{n} \equiv \vec{0}$

$$\frac{\partial \vec{n}}{\partial v} = -b_2^1 \vec{r} - b_2^2 \vec{r}$$

$$\begin{pmatrix} b_1^1 & b_1^2 \\ b_2^1 & b_2^2 \end{pmatrix} = \begin{pmatrix} \overline{E} & 0 \\ 0 & - \end{pmatrix}$$

N

$$\vec{n} \cdot \vec{r} = -M = 0, \ \vec{n} \cdot \vec{r} = -N = 0, \ \vec{n} \cdot \vec{n} = 0$$

$\vec{n} \equiv \vec{0}$

7.8

-
- $H \neq 0$

7.8 [15] [16]

7.7

7.7 7.8 7.8 Gauss-Codazzi 7.8

7.3.1

Gauss

Gauss

Gauss

推论 7.9. E^3 S

Gauss

Weingarten

Gauss

7.3 Monge-Ampère Gauss

7.2.3 E^3 Gauss

Monge $(x, y, u(x, y))$

Gauss $u(x, y)$ Monge-Ampère $u \quad u \quad -$ $(u \quad)^2 = 0$ Monge-Ampère $(x, y, u(x, y))$

Monge-Ampère

$$u \quad u \quad -(u \quad)^2 = f(u \ , u \ , u)$$

$$M(u) = u \quad u \quad -(u \quad)^2$$

M

$$\begin{aligned} M(u+t\phi) =&(u+t\phi) \quad (u+t\phi) \quad -((u+t\phi) \quad)^2 \\ =&(u \quad +t\phi \quad)(u \quad +t\phi \quad) - (u \quad +t\phi \quad)^2 \\ =&u \quad u \quad -(u \quad)^2 + t(u \quad \phi \quad -2u \quad \phi \quad +u \quad \phi \quad) + t^2(\phi \quad \phi \quad -(\phi \quad)^2) \end{aligned}$$

$$L(\phi)| \ = \lim_{\to 0} \frac{M(u+t\phi) - M(u)}{t} = u \quad \phi \quad -2u \quad \phi \quad +u \quad \phi$$

$L(\phi)|$ $\begin{pmatrix} u & -u \\ -u & u \end{pmatrix}$ $\begin{vmatrix} u & -u \\ -u & u \end{vmatrix} = u \quad u \quad -(u \quad)^2 =$ $f(u \ , u \ , u)$

1. $f(u \ , u \ , u) > 0$
2. $f(u \ , u \ , u) < 0$
3. $f(u \ , u \ , u) = 0$

Monge-Ampère $u \quad u \quad -(u \quad)^2 = 0$

Monge-Ampère $u \quad = u \quad = u \quad = 0$

$II = 0$

Monge-Ampère

$\begin{pmatrix} u & u \\ u & u \end{pmatrix}$ 1

Gauss

Monge-Ampère

*7.3.1 Monge-Ampère

$\begin{pmatrix} u & u \\ u & u \end{pmatrix}$ $Hess(u)$ $Hess(u)$

0 $\mathbb{R}^2$

$Hess(u)$

$Hess(u)$ 1 $(u \quad , -u$

$$\begin{pmatrix} u & -u \end{pmatrix} \begin{pmatrix} u & u \\ u & u \end{pmatrix} = \begin{pmatrix} 0 & 0 \end{pmatrix}$$

$(u \quad , -u \quad)$ $Hess(u)$ 0 u $-u$

$Hess(u)$ i

j $A_{\ j}$ (A_{11}, A_{12})

(A_{11}, A_{12}) $(x(t), y(t))$

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = A_{11} \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = A_{12} \end{cases}$$

$(x(t), y(t))$ $(x(t), y(t))$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\begin{pmatrix} x(t) & y(t) \end{pmatrix} = \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} A_{11} & A_{12} \end{pmatrix} = \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} u & -u \end{pmatrix}$$

$(x(t), y(t))$

$(x(t), y(t))$

$\begin{pmatrix} A_{11} & A_{12} \end{pmatrix}$ $\frac{\mathrm{d}}{\mathrm{d}}\begin{pmatrix} A_{11} & A_{12} \end{pmatrix}$

$Hess(u)$ $(u \quad , u \quad)$

$\begin{pmatrix} A_{11} & A_{12} \end{pmatrix}$ $\frac{\mathrm{d}}{\mathrm{d}}\begin{pmatrix} A_{11} & A_{12} \end{pmatrix}$ $(u \quad , u \quad)$

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} A_{11} & A_{12} \end{pmatrix}\right)\begin{pmatrix} u \\ u \end{pmatrix} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\begin{pmatrix} A_{11} & A_{12} \end{pmatrix}\begin{pmatrix} u \\ u \end{pmatrix}\right) - \begin{pmatrix} A_{11} & A_{12} \end{pmatrix}\begin{pmatrix} \frac{\mathrm{d}\acute{}}{\mathrm{d}} \\ \frac{\mathrm{d}}{\mathrm{d}} \end{pmatrix}$$

$$\begin{pmatrix} A_{11} & A_{12} \end{pmatrix}\begin{pmatrix} u \\ u \end{pmatrix}$$

$$\frac{\mathrm{d}u}{\mathrm{d}t} = u \quad \frac{\mathrm{d}x}{\mathrm{d}t} + u \quad \frac{\mathrm{d}y}{\mathrm{d}t} = \begin{pmatrix} u & u \end{pmatrix}\begin{pmatrix} A_{11} \\ A_{12} \end{pmatrix}$$

$$\frac{\mathrm{d}u}{\mathrm{d}t} = u \quad \frac{\mathrm{d}x}{\mathrm{d}t} + u \quad \frac{\mathrm{d}y}{\mathrm{d}t} = \begin{pmatrix} u & u \end{pmatrix}\begin{pmatrix} A_{11} \\ A_{12} \end{pmatrix}$$

$$\begin{pmatrix} \frac{\mathrm{d}\acute{}}{\mathrm{d}} \\ \frac{\mathrm{d}}{\mathrm{d}} \end{pmatrix} = \begin{pmatrix} u & u \\ u & u \end{pmatrix}\begin{pmatrix} A_{11} \\ A_{12} \end{pmatrix}$$

$$\begin{pmatrix} u & u \\ u & u \end{pmatrix} = \begin{pmatrix} u & u \\ u & u \end{pmatrix} = \frac{\partial}{\partial y}\begin{pmatrix} u & u \\ u & u \end{pmatrix} = \frac{\partial Hess(u)}{\partial y}$$

$$\begin{pmatrix} A_{11} & A_{12} \end{pmatrix} \begin{pmatrix} \frac{\mathrm{d}}{\mathrm{d}} \\ \frac{\mathrm{d}}{\mathrm{d}} \end{pmatrix}$$

$$= \begin{pmatrix} A_{11} & A_{12} \end{pmatrix} \frac{\partial Hess(u)}{\partial y} \begin{pmatrix} A_{11} \\ A_{12} \end{pmatrix}$$

$$= \begin{pmatrix} A_{11} & A_{12} \end{pmatrix} \frac{\partial}{\partial y} \left(Hess(u) \begin{pmatrix} A_{11} \\ A_{12} \end{pmatrix} \right) - \left(\begin{pmatrix} A_{11} & A_{12} \end{pmatrix} Hess(u) \right) \frac{\partial}{\partial y} \begin{pmatrix} A_{11} \\ A_{12} \end{pmatrix}$$

$Hess(u) \begin{pmatrix} A_{11} \\ A_{12} \end{pmatrix}$ $\begin{pmatrix} A_{11} & A_{12} \end{pmatrix} Hess(u)$ $(u \quad , u \quad)$

$\begin{pmatrix} A_{11} & A_{12} \end{pmatrix}$ $\frac{\mathrm{d}}{\mathrm{d}} \begin{pmatrix} A_{11} & A_{12} \end{pmatrix}$ $\begin{pmatrix} A_{11} & A_{12} \end{pmatrix}$ $\frac{\mathrm{d}}{\mathrm{d}} \begin{pmatrix} A_{11} & A_{12} \end{pmatrix}$

$\begin{pmatrix} A_{11} & A_{12} \end{pmatrix}$ $(x(t), y(t))$

$\mathbb{R}^3$ $(x(t), y(t), u(x(t), y(t)))$

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = A_{11} \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = A_{12} \end{cases}$$

$$\frac{\mathrm{d}u}{\mathrm{d}t} = u \quad \frac{\mathrm{d}x}{\mathrm{d}t} + u \quad \frac{\mathrm{d}y}{\mathrm{d}t} = u \quad A_{11} + u \quad A_{12} = 0$$

$$\frac{\mathrm{d}u}{\mathrm{d}t} = u \quad \frac{\mathrm{d}x}{\mathrm{d}t} + u \quad \frac{\mathrm{d}y}{\mathrm{d}t} = u \quad A_{11} + u \quad A_{12} = 0$$

$u \ , u$ $(x(t), y(t))$ $(\frac{\mathrm{d}}{\mathrm{d}}, \frac{\mathrm{d}}{\mathrm{d}}) = \lambda(x, y)(a, b)$

$$\frac{\mathrm{d}u}{\mathrm{d}t} = u \ \frac{\mathrm{d}x}{\mathrm{d}t} + u \ \frac{\mathrm{d}y}{\mathrm{d}t} = \lambda(au \ + bu \ )$$

$$(\frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t}, \frac{\mathrm{d}u}{\mathrm{d}t}) = \lambda(a, b, au \ + bu \ )$$

$u \ , u$ $(x(t), y(t), u(x(t), y(t)))$

$(x(t), y(t), u(x(t), y(t)))$

xOy

$(x, y, u(x, y))$ $\mathbb{R}^2$

1.

2.

3.

$\mathbb{R}^3$

$u(x,y)$

Monge-Ampère Gauss

$\mathbb{R}^2$

Gauss-Codazzi $\mathbb{R}^2$

E^3 $\mathbb{R}^2$

$Hess(u)$

Monge-Ampère

Monge-Ampère [17]

7.4

*7.4.1

定义 7.10. $\{C_\alpha\}$

C C C_α C C_α

$\{C_\alpha\}$ C

C $\{C_\alpha\}$

C C_α

$F(x,y,\alpha)$ F

C C $(x(t),y(t))$

$(x(t),y(t))$ $\alpha(t)=\alpha(x(t),y(t))$

$$F(x(t),y(t),\alpha(t))=0$$

$F(x,y,\alpha(t))=0$ $(x(s),y(s))$

$$F(x(s),y(s),\alpha(t))\equiv 0$$

s

$$\frac{\partial F}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}s}+\frac{\partial F}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}s}=0$$

$$\frac{\partial F}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t}+\frac{\partial F}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}=0$$

$$\begin{cases} F(x(t),y(t),\alpha(t))=0 & (7.3)\\ \dfrac{\partial F}{\partial x}\dfrac{\mathrm{d}x}{\mathrm{d}t}+\dfrac{\partial F}{\partial y}\dfrac{\mathrm{d}y}{\mathrm{d}t}=0 & (7.4)\end{cases}$$

(7.3)

t

$$\frac{\partial F}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t}+\frac{\partial F}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}+\frac{\partial F}{\partial \alpha}\frac{\mathrm{d}\alpha}{\mathrm{d}t}\equiv 0$$

(7.4) $\frac{\partial}{\partial\alpha}\frac{\mathrm{d}\alpha}{\mathrm{d}}=0$ t α $\frac{\mathrm{d}\alpha}{\mathrm{d}}$

F $\frac{\partial}{\partial\alpha}=0$

$$\begin{cases} F(x,y,\alpha)=0\\ \dfrac{\partial F}{\partial\alpha}(x,y,\alpha)=0\end{cases}$$

∂

α $\Phi(x,y)=0$

$(f(t,\alpha),g(t,\alpha))$ $(f(s,\alpha(s)),g(s,\alpha(s))),$

$$(\frac{\partial f}{\partial s}+\frac{\partial f}{\partial \alpha}\frac{\partial \alpha}{\partial s},\frac{\partial g}{\partial s}+\frac{\partial f}{\partial \alpha}\frac{\partial \alpha}{\partial s})$$

$(\frac{\partial}{\partial},\frac{\partial}{\partial})$ t s $(\frac{\partial}{\partial},\frac{\partial}{\partial})$
$(\frac{\partial}{\partial\alpha},\frac{\partial}{\partial\alpha})$

$$\frac{\partial f}{\partial t}\frac{\partial g}{\partial \alpha}-\frac{\partial g}{\partial t}\frac{\partial f}{\partial \alpha}=0$$

t α α t
$\alpha(t)$ $(x(t,\alpha(t)),y(t,\alpha(t)))$

D

D $\{C_\alpha\}$ D

$\{C_\alpha\}$ D

foliation D
$\{C_\alpha\}$

$\{C_\alpha\}$

D

D

$(s(t,\alpha),g(s,\alpha)(t,\alpha))$

2.4.1

*7.4.2

定义 7.11. $\{S_\alpha\}$
S S S_α S S_α
$\{S_\alpha\}$ S
S $\{S_\alpha\}$

$F(x,y,z,\alpha)$

$$\begin{cases} F(x,y,z,\alpha)=0 \\ \dfrac{\partial F}{\partial \alpha}(x,y,z,\alpha)=0 \end{cases} \tag{7.6}$$

α $\Phi(x,y,z)=0$

$$\begin{cases} F(x,y,z,\alpha)=0 \\ G(x,y,z,\alpha)=0 \end{cases}$$

$$\begin{cases} F(x,y,z,\alpha)=0 \\ G(x,y,z,\alpha)=0 \\ \dfrac{\partial F}{\partial \alpha}(x,y,z,\alpha)=0 \\ \dfrac{\partial G}{\partial \alpha}(x,y,z,\alpha)=0 \end{cases} \tag{7.7}$$

(7.5)

(7.6) α

(x, y, z)

(7.6) (7.7)

$$\begin{cases} F(x,y,z,\alpha) = 0 \\ \dfrac{\partial F}{\partial \alpha}(x,y,z,\alpha) = 0 \\ \dfrac{\partial^2 F}{\partial \alpha^2}(x,y,z,\alpha) = 0 \end{cases}$$

-
-
-

例 7.12.

$$x\cos\alpha + y\sin\alpha - z\sin\alpha = 1$$

.

$$F(x,y,z,\alpha) = x\cos\alpha + y\sin\alpha - z\sin\alpha - 1$$

$$F_\alpha(x,y,z,\alpha) = -x\sin\alpha + y\cos\alpha - z\cos\alpha$$

$$\begin{cases} x\cos\alpha + y\sin\alpha - z\sin\alpha = 1 \\ -x\sin\alpha + y\cos\alpha - z\cos\alpha = 0 \end{cases}$$

$$x^2 + (y - z)^2 = 1$$

□

Chapter 8

2

$\vec{r}(u, v)$

$E(\mathrm{d}u)^2 + 2F\mathrm{d}u\mathrm{d}v + G(\mathrm{d}v)^2$

8.1.1 Darboux

C C

Frenet

$$\vec{e}_1 = \frac{\mathrm{d}\vec{r}(s)}{\mathrm{d}s} = \vec{\alpha}(s)$$
$$\vec{e}_2 = \vec{e}_3 \times \vec{e}_1 = \vec{n}(s) \times \vec{\alpha}(s)$$
$$\vec{e}_3 = \vec{n}(s)$$

Darboux

Frenet $\vec{\alpha}, \vec{e}_2, \vec{n}$ $\vec{\alpha}, \vec{\beta}, \vec{\gamma}$ $\vec{e}_2, \vec{n}$ $\vec{\beta}, \vec{\gamma}$ $\angle(\vec{e}_2, \vec{\beta}) = \angle(\vec{n}, \vec{\gamma})$

$\vec{e}_1$ $\vec{n}$ $90°$ Frenet

8.1.2 Darboux

Darboux

$$\frac{\mathrm{d}\vec{e}_1(s)}{\mathrm{d}s} = \kappa\ \vec{e}_2 + \kappa\ \vec{e}_3 \tag{8.1}$$
$$\frac{\mathrm{d}\vec{e}_2(s)}{\mathrm{d}s} = -\kappa\ \vec{e}_1 + \tau\ \vec{e}_3 \tag{8.2}$$
$$\frac{\mathrm{d}\vec{e}_3(s)}{\mathrm{d}s} = -\kappa\ \vec{e}_1 - \tau\ \vec{e}_2 \tag{8.3}$$

κ κ τ

$$\kappa = \frac{\mathrm{d}\vec{e}_1(s)}{\mathrm{d}s} \cdot \vec{e}_3(s) = \frac{\mathrm{d}\vec{\alpha}(s)}{\mathrm{d}s} \cdot \vec{n}(s)$$

κ C

$$\begin{aligned}
\frac{\mathrm{d}\vec{e}_1(s)}{\mathrm{d}s} &= \qquad\qquad \kappa\ \vec{e}_2 + \kappa\ \vec{e}_3 \\
\frac{\mathrm{d}\vec{e}_2(s)}{\mathrm{d}s} &= -\kappa\ \vec{e}_1 \qquad\quad + \tau\ \vec{e}_3 \\
\frac{\mathrm{d}\vec{e}_3(s)}{\mathrm{d}s} &= -\kappa\ \vec{e}_1 \quad -\tau\ \vec{e}_2
\end{aligned} \tag{8.4}$$

Frenet

$$\begin{aligned}
\vec{\alpha}'(s) &= \qquad\qquad \kappa(s)\vec{\beta}(s) \\
\vec{\beta}'(s) &= -\kappa(s)\vec{\alpha}(s) \qquad\quad + \tau(s)\vec{\gamma}(s) \\
\vec{\gamma}'(s) &= \qquad\quad -\tau(s)\vec{\beta}(s)
\end{aligned}$$

κ S C τ C

κ κ τ

κ τ

$$\begin{aligned}
\kappa &= \frac{\mathrm{d}\vec{e}_1(s)}{\mathrm{d}s} \cdot \vec{e}_2 = \frac{\mathrm{d}^2\vec{r}(s)}{\mathrm{d}s^2} \cdot (\vec{n}(s) \times \vec{r}'(s)) = (\vec{r}'(s), \vec{r}''(s), \vec{n}(s)) \\
\tau &= \frac{\mathrm{d}\vec{e}_2(s)}{\mathrm{d}s} \cdot \vec{e}_3 = \frac{\mathrm{d}(\vec{n}(s) \times \vec{r}'(s))}{\mathrm{d}s} \cdot \vec{n}(s) = (\vec{n}'(s), \vec{r}'(s), \vec{n}(s))
\end{aligned}$$

$\vec{r}(-s)$

$\{-\vec{e}_1, -\vec{e}_2, \vec{e}_3\}$ (8.4)

$$\begin{aligned}
\frac{\mathrm{d} - \vec{e}_1(s)}{\mathrm{d} - s} &= \qquad\qquad -\kappa\ (-\vec{e}_2) + \kappa\ \vec{e}_3 \\
\frac{\mathrm{d} - \vec{e}_2(s)}{\mathrm{d} - s} &= \kappa\ (-\vec{e}_1) \qquad\quad + \tau\ \vec{e}_3 \\
\frac{\mathrm{d}\vec{e}_3(s)}{\mathrm{d} - s} &= -\kappa\ (-\vec{e}_1) - \tau\ (-\vec{e}_2)
\end{aligned}$$

8.2

$$\tau = (\vec{n}'(s), \vec{r}'(s), \vec{n}(s))$$

$$\vec{r}'(s) = \sum_{\gamma=1}^{2} \vec{r}_\gamma \frac{\mathrm{d}u^\gamma(s)}{\mathrm{d}s}$$

$$\frac{\mathrm{d}\vec{n}(s)}{\mathrm{d}s} = \sum_{\alpha=1}^{2} \frac{\partial \vec{n}(s)}{\partial u^\alpha} \frac{\mathrm{d}u^\alpha(s)}{\mathrm{d}s} = \sum_{\alpha=1}^{2}\sum_{\beta=1}^{2} -b_\alpha^\beta \frac{\mathrm{d}u^\alpha}{\mathrm{d}s} \vec{r}_\beta$$

$$\begin{aligned}
\tau =& (\sum_{\alpha=1}^{2}\sum_{\beta=1}^{2} -b_\alpha^\beta \frac{\mathrm{d}u^\alpha}{\mathrm{d}s}\vec{r}_\beta, \sum_{\gamma=1}^{2} \vec{r}_\gamma \frac{\mathrm{d}u^\gamma(s)}{\mathrm{d}s}, \vec{n}) \\
=& \sum_{\beta=1}^{2}\sum_{\gamma=1}^{2}\sum_{\alpha=1}^{2} -b_\alpha^\beta \frac{\mathrm{d}u^\alpha}{\mathrm{d}s}\frac{\mathrm{d}u^\gamma}{\mathrm{d}s}(\vec{r}_\beta, \vec{r}_\gamma, \vec{n}) \\
=& \sum_{\alpha=1}^{2} -b_\alpha^1 \frac{\mathrm{d}u^\alpha}{\mathrm{d}s}\frac{\mathrm{d}u^1}{\mathrm{d}s}(\vec{r}_1, \vec{r}_2, \vec{n}) + \sum_{\alpha=1}^{2} -b_\alpha^2 \frac{\mathrm{d}u^\alpha}{\mathrm{d}s}\frac{\mathrm{d}u^1}{\mathrm{d}s}(\vec{r}_2, \vec{r}_1, \vec{n}) \\
=& \sum_{\alpha=1}^{2} (-b_\alpha^1 \frac{\mathrm{d}u^\alpha}{\mathrm{d}s}\frac{\mathrm{d}u^2}{\mathrm{d}s} + b_\alpha^2 \frac{\mathrm{d}u^\alpha}{\mathrm{d}s}\frac{\mathrm{d}u^1}{\mathrm{d}s})|\vec{r}_1 \times \vec{r}_2| \\
=& \sqrt{g}\left(-b_2^1(\frac{\mathrm{d}u^2}{\mathrm{d}s})^2 + (b_2^2 - b_1^1)\frac{\mathrm{d}u^1}{\mathrm{d}s}\frac{\mathrm{d}u^2}{\mathrm{d}s} + b_1^2(\frac{\mathrm{d}u^1}{\mathrm{d}s})^2\right)
\end{aligned}$$

$$g = \begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix} = g_{11}g_{22} - (g_{12})^2 = \begin{vmatrix} E & F \\ F & G \end{vmatrix}$$

$\frac{\mathrm{d}\ ^1}{\mathrm{d}}, \frac{\mathrm{d}\ ^2}{\mathrm{d}}$

-

-

8.2.1 Weingarten

$$\tau = \sqrt{g}\left(-b_2^1(\frac{\mathrm{d}u^2}{\mathrm{d}s})^2 + (b_2^2 - b_1^1)\frac{\mathrm{d}u^1}{\mathrm{d}s}\frac{\mathrm{d}u^2}{\mathrm{d}s} + b_1^2(\frac{\mathrm{d}u^1}{\mathrm{d}s})^2\right) \tag{8.5}$$

b_α^β Weingarten Weingarten
(8.5)

$$\begin{pmatrix} \frac{\mathrm{d}\ ^1}{\mathrm{d}} & \frac{\mathrm{d}\ ^2}{\mathrm{d}} \end{pmatrix} \begin{pmatrix} b_1^2 & b_2^2 \\ -b_1^1 & -b_2^1 \end{pmatrix} \begin{pmatrix} \frac{\mathrm{d}\ ^1}{\mathrm{d}} \\ \frac{\mathrm{d}\ ^2}{\mathrm{d}} \end{pmatrix}$$

$$\begin{pmatrix} b_1^2 & b_2^2 \\ -b_1^1 & -b_2^1 \end{pmatrix} \begin{pmatrix} \frac{\mathrm{d}\ ^1}{\mathrm{d}} \\ \frac{\mathrm{d}\ ^2}{\mathrm{d}} \end{pmatrix}$$

$$\left| \begin{matrix} \frac{\mathrm{d}\ ^2}{\mathrm{d}} \\ -\frac{\mathrm{d}\ ^1}{\mathrm{d}} \end{matrix} \quad \begin{pmatrix} b_1^2 & b_2^2 \\ -b_1^1 & -b_2^1 \end{pmatrix} \begin{pmatrix} \frac{\mathrm{d}\ ^1}{\mathrm{d}} \\ \frac{\mathrm{d}\ ^2}{\mathrm{d}} \end{pmatrix} \right| = - \left| \begin{matrix} \frac{\mathrm{d}\ ^2}{\mathrm{d}} \\ \frac{\mathrm{d}\ ^1}{\mathrm{d}} \end{matrix} \quad \begin{pmatrix} b_1^2 & b_2^2 \\ b_1^1 & b_2^1 \end{pmatrix} \begin{pmatrix} \frac{\mathrm{d}\ ^1}{\mathrm{d}} \\ \frac{\mathrm{d}\ ^2}{\mathrm{d}} \end{pmatrix} \right| = \left| \begin{matrix} \frac{\mathrm{d}\ ^1}{\mathrm{d}} \\ \frac{\mathrm{d}\ ^2}{\mathrm{d}} \end{matrix} \quad \begin{pmatrix} b_1^1 & b_2^1 \\ b_1^2 & b_2^2 \end{pmatrix} \begin{pmatrix} \frac{\mathrm{d}\ ^1}{\mathrm{d}} \\ \frac{\mathrm{d}\ ^2}{\mathrm{d}} \end{pmatrix} \right|$$

Weingarten

$$\tau = (\vec{n}'(s), \vec{r}'(s), \vec{n}(s))$$

$$\vec{n}'(s) = \frac{\mathrm{d}\vec{n}(s)}{\mathrm{d}s} = \mathscr{G}_*(\frac{\mathrm{d}\vec{r}(s)}{\mathrm{d}s}) = \mathscr{G}_*(\vec{e}_1)$$

Weingarten $\mathscr{W} = -\mathscr{G}_*$ $\vec{r}_1, \vec{r}_2$ $\vec{n}'(s) \times \vec{r}'(s)$

$$\left| -\begin{pmatrix} b_1^1 & b_2^1 \\ b_1^2 & b_2^2 \end{pmatrix} \begin{pmatrix} \frac{\mathrm{d}\ ^1}{\mathrm{d}} \\ \frac{\mathrm{d}\ ^2}{\mathrm{d}} \end{pmatrix} \quad \begin{matrix} \frac{\mathrm{d}\ ^1}{\mathrm{d}} \\ \frac{\mathrm{d}\ ^2}{\mathrm{d}} \end{matrix} \right| \vec{r}_1 \times \vec{r}_2$$

$\vec{n}$

$$\tau = \left| -\begin{pmatrix} b_1^1 & b_2^1 \\ b_1^2 & b_2^2 \end{pmatrix} \begin{pmatrix} \frac{\mathrm{d}\ ^1}{\mathrm{d}} \\ \frac{\mathrm{d}\ ^2}{\mathrm{d}} \end{pmatrix} \quad \begin{matrix} \frac{\mathrm{d}\ ^1}{\mathrm{d}} \\ \frac{\mathrm{d}\ ^2}{\mathrm{d}} \end{matrix} \right| |\vec{r}_1 \times \vec{r}_2| = \sqrt{g} \left| \begin{matrix} \frac{\mathrm{d}\ ^1}{\mathrm{d}} \\ \frac{\mathrm{d}\ ^2}{\mathrm{d}} \end{matrix} \quad \begin{pmatrix} b_1^1 & b_2^1 \\ b_1^2 & b_2^2 \end{pmatrix} \begin{pmatrix} \frac{\mathrm{d}\ ^1}{\mathrm{d}} \\ \frac{\mathrm{d}\ ^2}{\mathrm{d}} \end{pmatrix} \right|$$

5.3.1 (5.13) (5.16)

Weingarten Darboux

(8.3)

$$\frac{\mathrm{d}\vec{e}_3(s)}{\mathrm{d}s} = -\kappa\ \vec{e}_1 - \tau\ \vec{e}_2$$

$$\frac{\mathrm{d}\vec{n}(s)}{\mathrm{d}s} = \mathscr{G}_*(\frac{\mathrm{d}\vec{r}(s)}{\mathrm{d}s}) = \mathscr{G}_*(\vec{e}_1)$$

Weingarten $\mathscr{W} = -\mathscr{G}_*$ (8.3)

$$\mathscr{W}(\vec{e}_1) = \kappa\ \vec{e}_1 + \tau\ \vec{e}_2$$

$$\kappa\ = \mathscr{W}(\vec{e}_1)\cdot\vec{e}_1, \quad \tau\ = \mathscr{W}(\vec{e}_1)\cdot\vec{e}_2$$

Weingarten

8.1.1

[1]

Euler $\vec{d}_1, \vec{d}_2$

$$\vec{e}_1 = \cos\theta\vec{d}_1 + \sin\theta\vec{d}_2 \qquad \vec{e}_2 = -\sin\theta\vec{d}_1 + \cos\theta\vec{d}_2$$

$$\mathscr{W}(\vec{e}_1) = \kappa_1\cos\theta\vec{d}_1 + \kappa_2\sin\theta\vec{d}_2$$

Euler

$$\kappa\ (\theta) = \mathscr{W}(\vec{e}_1)\cdot\vec{e}_1 = \kappa_1\cos^2\theta + \kappa_2\sin^2\theta$$

$$\tau\ (\theta) = \mathscr{W}(\vec{e}_1)\cdot\vec{e}_2 = \frac{\kappa_1 - \kappa_2}{2}\sin 2\theta$$

$\tau\ = \mathscr{W}(\vec{e}_1)\cdot\vec{e}_2$ (8.5)

I

[1] Weingarten $\vec{e}_1, \vec{e}_2$ $-\vec{e}_1, -\vec{e}_2$

$$\mathrm{d}u$$

$$(s \quad \mathrm{d}^{\alpha} \quad {}_{\alpha\beta}\vec{n})$$

8.3

$$\kappa \ = (\vec{r}'(s), \vec{r}''(s), \vec{n}(s))$$

$$\vec{r}'(s) = \sum_{\eta=1}^{2} \vec{r}_\eta \frac{\mathrm{d}u^\eta(s)}{\mathrm{d}s}$$

$$\begin{aligned}
\frac{\mathrm{d}^2\vec{r}}{\mathrm{d}s^2} =& \frac{\mathrm{d}}{\mathrm{d}s}\left(\sum_{\alpha=1}^{2} \vec{r}_\alpha \frac{\mathrm{d}u^\alpha(s)}{\mathrm{d}s}\right) = \sum_{\alpha,\beta=1}^{2} \vec{r}_{\alpha\beta} \frac{\mathrm{d}u^\alpha(s)}{\mathrm{d}s}\frac{\mathrm{d}u^\beta(s)}{\mathrm{d}s} + \sum_{\alpha=1}^{2} \vec{r}_\alpha \frac{\mathrm{d}^2u^\alpha(s)}{\mathrm{d}s^2} \\
=& \sum_{\alpha,\beta=1}^{2} (\sum_{\gamma=1}^{2} \Gamma^\gamma_{\alpha\beta}\vec{r}_\gamma + b_{\alpha\beta}\vec{n}) \frac{\mathrm{d}u^\alpha(s)}{\mathrm{d}s}\frac{\mathrm{d}u^\beta(s)}{\mathrm{d}s} + \sum_{\alpha=1}^{2} \vec{r}_\alpha \frac{\mathrm{d}^2u^\alpha(s)}{\mathrm{d}s^2} \\
=& \sum_{\alpha,\beta=1}^{2} (\sum_{\gamma=1}^{2} \Gamma^\gamma_{\alpha\beta}\vec{r}_\gamma + b_{\alpha\beta}\vec{n}) \frac{\mathrm{d}u^\alpha(s)}{\mathrm{d}s}\frac{\mathrm{d}u^\beta(s)}{\mathrm{d}s} + \sum_{\gamma=1}^{2} \vec{r}_\gamma \frac{\mathrm{d}^2u^\gamma(s)}{\mathrm{d}s^2} \\
=& \sum_{\gamma=1}^{2} (\frac{\mathrm{d}^2u^\gamma(s)}{\mathrm{d}s^2} + \sum_{\alpha,\beta=1}^{2} \Gamma^\gamma_{\alpha\beta} \frac{\mathrm{d}u^\alpha(s)}{\mathrm{d}s}\frac{\mathrm{d}u^\beta(s)}{\mathrm{d}s})\vec{r}_\gamma + \sum_{\alpha,\beta=1}^{2} b_{\alpha\beta} \frac{\mathrm{d}u^\alpha(s)}{\mathrm{d}s}\frac{\mathrm{d}u^\beta(s)}{\mathrm{d}s}\vec{n}
\end{aligned}$$

$$\kappa \ = \sum_{\eta,\gamma=1}^{2} (\frac{\mathrm{d}^2u^\gamma(s)}{}$$

$$u^1 = u^1(s), u^2 = u^2(s)$$

8.3.1

s

t

$$\begin{aligned}
\kappa =& \sqrt{g}\left|\begin{matrix} \frac{\mathrm{d}\ ^1(\ )}{\mathrm{d}} & \frac{\mathrm{d}^2\ ^1(\ )}{\mathrm{d}\ ^2} + \sum_{\alpha,\beta=1}^{2}\Gamma^1_{\alpha\beta}\frac{\mathrm{d}\ ^\alpha(\ )}{\mathrm{d}}\frac{\mathrm{d}\ ^\beta(\ )}{\mathrm{d}} \\ \frac{\mathrm{d}\ ^2(\ )}{\mathrm{d}} & \frac{\mathrm{d}^2\ ^2(\ )}{\mathrm{d}\ ^2} + \sum_{\alpha,\beta=1}^{2}\Gamma^2_{\alpha\beta}\frac{\mathrm{d}\ ^\alpha(\ )}{\mathrm{d}}\frac{\mathrm{d}\ ^\beta(\ )}{\mathrm{d}} \end{matrix}\right| \\
=& \sqrt{g}\left|\begin{matrix} \frac{\mathrm{d}\ ^1(\ )}{\mathrm{d}}\frac{\mathrm{d}}{\mathrm{d}} & \frac{\mathrm{d}^2\ ^1(\ )}{\mathrm{d}\ ^2}(\frac{\mathrm{d}}{\mathrm{d}})^2 + \frac{\mathrm{d}\ ^1(\ )}{\mathrm{d}}\frac{\mathrm{d}^2}{\mathrm{d}\ ^2} + \sum_{\alpha,\beta=1}^{2}\Gamma^1_{\alpha\beta}\frac{\mathrm{d}\ ^\alpha(\ )}{\mathrm{d}}\frac{\mathrm{d}\ ^\beta(\ )}{\mathrm{d}}(\frac{\mathrm{d}}{\mathrm{d}})^2 \\ \frac{\mathrm{d}\ ^2(\ )}{\mathrm{d}}\frac{\mathrm{d}}{\mathrm{d}} & \frac{\mathrm{d}^2\ ^2(\ )}{\mathrm{d}\ ^2}(\frac{\mathrm{d}}{\mathrm{d}})^2 + \frac{\mathrm{d}\ ^2(\ )}{\mathrm{d}}\frac{\mathrm{d}^2}{\mathrm{d}\ ^2} + \sum_{\alpha,\beta=1}^{2}\Gamma^2_{\alpha\beta}\frac{\mathrm{d}\ ^\alpha(\ )}{\mathrm{d}}\frac{\mathrm{d}\ ^\beta(\ )}{\mathrm{d}}(\frac{\mathrm{d}}{\mathrm{d}})^2 \end{matrix}\right| \\
=& \sqrt{g}\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^3\left|\begin{matrix} \frac{\mathrm{d}\ ^1(\ )}{\mathrm{d}} & \frac{\mathrm{d}^2\ ^1(\ )}{\mathrm{d}\ ^2} + \sum_{\alpha,\beta=1}^{2}\Gamma^1_{\alpha\beta}\frac{\mathrm{d}\ ^\alpha(\ )}{\mathrm{d}}\frac{\mathrm{d}\ ^\beta(\ )}{\mathrm{d}} \\ \frac{\mathrm{d}\ ^2(\ )}{\mathrm{d}} & \frac{\mathrm{d}^2\ ^2(\ )}{\mathrm{d}\ ^2} + \sum_{\alpha,\beta=1}^{2}\Gamma^2_{\alpha\beta}\frac{\mathrm{d}\ ^\alpha(\ )}{\mathrm{d}}\frac{\mathrm{d}\ ^\beta(\ )}{\mathrm{d}} \end{matrix}\right| \\
=& \frac{\sqrt{g}}{|\vec{r}'(t)|^3}\left|\begin{matrix} \frac{\mathrm{d}\ ^1(\ )}{\mathrm{d}} & \frac{\mathrm{d}^2\ ^1(\ )}{\mathrm{d}\ ^2} + \sum_{\alpha,\beta=1}^{2}\Gamma^1_{\alpha\beta}\frac{\mathrm{d}\ ^\alpha(\ )}{\mathrm{d}}\frac{\mathrm{d}\ ^\beta(\ )}{\mathrm{d}} \\ \frac{\mathrm{d}\ ^2(\ )}{\mathrm{d}} & \frac{\mathrm{d}^2\ ^2(\ )}{\mathrm{d}\ ^2} + \sum_{\alpha,\beta=1}^{2}\Gamma^2_{\alpha\beta}\frac{\mathrm{d}\ ^\alpha(\ )}{\mathrm{d}}\frac{\mathrm{d}\ ^\beta(\ )}{\mathrm{d}} \end{matrix}\right|
\end{aligned}$$

8.3.2

(8.1)

$$\frac{\mathrm{d}\vec{e}_1(s)}{\mathrm{d}s} = \kappa\ \vec{e}_2 + \kappa\ \vec{e}_3$$

$$\frac{\mathrm{d}\vec{\alpha}}{\mathrm{d}s} = \kappa\vec{\beta} = \kappa\ \vec{e}_2 + \kappa\ \vec{e}_3$$

$$\kappa^2 = \kappa^2 + \kappa^2, \qquad \kappa\ = \kappa\vec{\beta}\cdot\vec{e}_2 = \kappa\cos\theta$$

θ $\vec{\beta}$ $\vec{e}_2$ $\vec{\gamma}$ $\vec{n}$

$\vec{r}''$

- $\vec{n}$
  $\vec{e}_2$
-

例 8.1.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

$-+-=1$ $A=(a,0,0)$ A z

$$\kappa = \sqrt{g}\begin{vmatrix} \frac{\mathrm{d}\ ^1(\ )}{\mathrm{d}} & \frac{\mathrm{d}^2\ ^1(\ )}{\mathrm{d}\ ^2}+\sum_{\alpha,\beta=1}^{2}\Gamma^1_{\alpha\beta}\frac{\mathrm{d}\ ^\alpha(\ )}{\mathrm{d}}\frac{\mathrm{d}\ ^\beta(\ )}{\mathrm{d}} \\ \frac{\mathrm{d}\ ^2(\ )}{\mathrm{d}} & \frac{\mathrm{d}^2\ ^2(\ )}{\mathrm{d}\ ^2}+\sum_{\alpha,\beta=1}^{2}\Gamma^2_{\alpha\beta}\frac{\mathrm{d}\ ^\alpha(\ )}{\mathrm{d}}\frac{\mathrm{d}\ ^\beta(\ )}{\mathrm{d}} \end{vmatrix}$$

$$\kappa = \frac{\sqrt{g}}{|\vec{r}'(t)|^3}\begin{vmatrix} \frac{\mathrm{d}\ ^1(\ )}{\mathrm{d}} & \frac{\mathrm{d}^2\ ^1(\ )}{\mathrm{d}\ ^2}+\sum_{\alpha,\beta=1}^{2}\Gamma^1_{\alpha\beta}\frac{\mathrm{d}\ ^\alpha(\ )}{\mathrm{d}}\frac{\mathrm{d}\ ^\beta(\ )}{\mathrm{d}} \\ \frac{\mathrm{d}\ ^2(\ )}{\mathrm{d}} & \frac{\mathrm{d}^2\ ^2(\ )}{\mathrm{d}\ ^2}+\sum_{\alpha,\beta=1}^{2}\Gamma^2_{\alpha\beta}\frac{\mathrm{d}\ ^\alpha(\ )}{\mathrm{d}}\frac{\mathrm{d}\ ^\beta(\ )}{\mathrm{d}} \end{vmatrix}$$

$(u(t),v(t))$

Christoffel

$\kappa = \kappa\cos\theta$

$$\begin{cases} \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1 \\ \dfrac{x}{a}+\dfrac{y}{b}=1 \end{cases}$$

$t=-$ x,y

$$\vec{r}(t)=(a\frac{1+\sqrt{1-2t^2}}{2},b\frac{1-\sqrt{1-2t^2}}{2},ct)$$

$2t^2 = \sin^2\psi$

$$\vec{r}(\psi) = (a\cos^2\frac{\psi}{2}, b\sin^2\frac{\psi}{2}, c\frac{\sqrt{2}}{2}\sin\psi)$$

$$\vec{r}' = (-\frac{a}{2}\sin\psi, \frac{b}{2}\sin\psi, c\frac{\sqrt{2}}{2}\cos\psi)$$

$$\vec{r}'' = (-\frac{a}{2}\cos\psi, \frac{b}{2}\cos\psi, -c\frac{\sqrt{2}}{2}\cos\psi)$$

$$\kappa(\psi) = \frac{|\vec{r}' \times \vec{r}''|}{|\vec{r}'|^3} = \frac{|(\frac{\sqrt{2}}{4}bc, \frac{\sqrt{2}}{4}ac, 0)|}{(\frac{^2+\ ^2}{4}\sin^2\psi + \frac{^2}{2}\cos^2\psi)^{\frac{3}{2}}}$$

$\psi = 0$

$$\kappa = \frac{\sqrt{a^2+b^2}}{c^2}$$

$\kappa \ = \kappa\cos\theta \quad \theta \qquad \vec{\beta} \qquad \vec{e}_2$

$\vec{e}_2$ $\vec{\alpha}$ $\vec{n}$ $\vec{\gamma}$ $\vec{n}$

$\vec{n} = (1,0,0) \quad \vec{\gamma} = (\frac{}{\sqrt{\ ^2+\ ^2}}, \frac{}{\sqrt{\ ^2+\ ^2}}, 0) \qquad \kappa \ = \frac{}{\ ^2}$

$$\kappa_{\blacktriangledown}(s) = \begin{vmatrix} x'(s) & y'(s) \\ x''(s) & y''(s) \end{vmatrix}$$

$$\begin{aligned} \frac{\mathrm{d}\vec{e}_1(s)}{\mathrm{d}s} &= \qquad\qquad \kappa\ \vec{e}_2 + \kappa\ \vec{e}_3 \\ \frac{\mathrm{d}\vec{e}_2(s)}{\mathrm{d}s} &= -\kappa\ \vec{e}_1 \qquad\quad + \tau\ \vec{e}_3 \\ \frac{\mathrm{d}\vec{e}_3(s)}{\mathrm{d}s} &= -\kappa\ \vec{e}_1 \quad -\tau\ \vec{e}_2 \end{aligned}$$

8.6

8.4

$$\vec{\alpha}'(s) = \kappa_{\boldsymbol{r}}(s)\vec{\beta}(s)$$
$$\vec{\beta}'(s) = -\kappa_{\boldsymbol{r}}(s)\vec{\alpha}(s)$$

Darboux

$$\frac{\mathrm{d}\vec{e}_1(s)}{\mathrm{d}s} = \kappa\ \vec{e}_2 + \kappa\ \vec{e}_3$$
$$\frac{\mathrm{d}\vec{e}_2(s)}{\mathrm{d}s} = -\kappa\ \vec{e}_1 + \tau\ \vec{e}_3$$
$$\frac{\mathrm{d}\vec{e}_3(s)}{\mathrm{d}s} = -\kappa\ \vec{e}_1 - \tau\ \vec{e}_2$$

$\vec{n}$ 3 κ τ

$\kappa_{\boldsymbol{r}}$ κ $\kappa_{\boldsymbol{r}} \equiv 0$

定义 8.2. S $(u(t), v(t))$

$$\kappa = \frac{\sqrt{g}}{|\vec{r}'(t)|^3}\begin{vmatrix} \frac{\mathrm{d}\ ^1(\ )}{\mathrm{d}} & \frac{\mathrm{d}^2\ ^1(\ )}{\mathrm{d}\ ^2} + \sum_{\alpha,\beta=1}^{2}\Gamma^1_{\alpha\beta}\frac{\mathrm{d}\ ^\alpha(\ )}{\mathrm{d}}\frac{\mathrm{d}\ ^\beta(\ )}{\mathrm{d}} \\ \frac{\mathrm{d}\ ^2(\ )}{\mathrm{d}} & \frac{\mathrm{d}^2\ ^2(\ )}{\mathrm{d}\ ^2} + \sum_{\alpha,\beta=1}^{2}\Gamma^2_{\alpha\beta}\frac{\mathrm{d}\ ^\alpha(\ )}{\mathrm{d}}\frac{\mathrm{d}\ ^\beta(\ )}{\mathrm{d}} \end{vmatrix} = 0$$

S pregeodesic

8.4.1

$$\kappa = \frac{\sqrt{g}}{|\vec{r}'(t)|^3}\begin{vmatrix} \frac{\mathrm{d}\ ^1(\ )}{\mathrm{d}} & \frac{\mathrm{d}^2\ ^1(\ )}{\mathrm{d}\ ^2} + \sum_{\alpha,\beta=1}^{2}\Gamma^1_{\alpha\beta}\frac{\mathrm{d}\ ^\alpha(\ )}{\mathrm{d}}\frac{\mathrm{d}\ ^\beta(\ )}{\mathrm{d}} \\ \frac{\mathrm{d}\ ^2(\ )}{\mathrm{d}} & \frac{\mathrm{d}^2\ ^2(\ )}{\mathrm{d}\ ^2} + \sum_{\alpha,\beta=1}^{2}\Gamma^2_{\alpha\beta}\frac{\mathrm{d}\ ^\alpha(\ )}{\mathrm{d}}\frac{\mathrm{d}\ ^\beta(\ )}{\mathrm{d}} \end{vmatrix} = 0$$

$$\kappa = \sqrt{g}\begin{vmatrix} \frac{\mathrm{d}\ ^1(\ )}{\mathrm{d}} & \frac{\mathrm{d}^2\ ^1(\ )}{\mathrm{d}\ ^2} + \sum_{\alpha,\beta=1}^{2}\Gamma^1_{\alpha\beta}\frac{\mathrm{d}\ ^\alpha(\ )}{\mathrm{d}}\frac{\mathrm{d}\ ^\beta(\ )}{\mathrm{d}} \\ \frac{\mathrm{d}\ ^2(\ )}{\mathrm{d}} & \frac{\mathrm{d}^2\ ^2(\ )}{\mathrm{d}\ ^2} + \sum_{\alpha,\beta=1}^{2}\Gamma^2_{\alpha\beta}\frac{\mathrm{d}\ ^\alpha(\ )}{\mathrm{d}}\frac{\mathrm{d}\ ^\beta(\ )}{\mathrm{d}} \end{vmatrix} = 0$$

Darboux

$$\begin{aligned}\frac{\mathrm{d}\vec{e}_1(s)}{\mathrm{d}s} &= \qquad\quad \kappa\ \vec{e}_2 + \kappa\ \vec{e}_3\\ \frac{\mathrm{d}\vec{e}_2(s)}{\mathrm{d}s} &= -\kappa\ \vec{e}_1 \qquad + \tau\ \vec{e}_3\\ \frac{\mathrm{d}\vec{e}_3(s)}{\mathrm{d}s} &= -\kappa\ \vec{e}_1 - \tau\ \vec{e}_2\end{aligned}$$

$\kappa\ = 0$

$$\begin{aligned}\frac{\mathrm{d}\vec{e}_1(s)}{\mathrm{d}s} &= \qquad\quad + \kappa\ \vec{e}_3\\ \frac{\mathrm{d}\vec{e}_2(s)}{\mathrm{d}s} &= \qquad\quad + \tau\ \vec{e}_3\\ \frac{\mathrm{d}\vec{e}_3(s)}{\mathrm{d}s} &= -\kappa\ \vec{e}_1 - \tau\ \vec{e}_2\end{aligned}$$

$$\frac{\mathrm{d}\vec{e}_1(s)}{\mathrm{d}s} = \kappa\ \vec{e}_3$$

$$\frac{\mathrm{d}(\vec{r}_\gamma \frac{\mathrm{d}\ ^\gamma(\ )}{\mathrm{d}})}{\mathrm{d}s} = \kappa\ \vec{n}$$

$$\sum_{\gamma=1}^{2}\left(\frac{\mathrm{d}^2u^\gamma(s)}{\mathrm{d}s^2} + \sum_{\alpha,\beta=1}^{2}\Gamma^\gamma_{\alpha\beta}\frac{\mathrm{d}u^\alpha(s)}{\mathrm{d}s}\frac{\mathrm{d}u^\beta(s)}{\mathrm{d}s}\right)\vec{r}_\gamma + \sum_{\alpha,\beta=1}^{2} b_{\alpha\beta}\frac{\mathrm{d}u^\alpha(s)}{\mathrm{d}s}\frac{\mathrm{d}u^\beta(s)}{\mathrm{d}s}\vec{n} = \kappa\ \vec{n}$$

$\{\vec{r}_1, \vec{r}_2, \vec{n}\}$

$$\begin{cases}\frac{\mathrm{d}^2\ ^\gamma(\ )}{\mathrm{d}\ ^2} + \sum_{\alpha,\beta=1}^{2}\Gamma^\gamma_{\alpha\beta}\frac{\mathrm{d}\ ^\alpha(\ )}{\mathrm{d}}\frac{\mathrm{d}\ ^\beta(\ )}{\mathrm{d}} = 0, \quad \gamma = 1,2\\ \sum_{\alpha,\beta=1}^{2} b_{\alpha\beta}\frac{\mathrm{d}\ ^\alpha(\ )}{\mathrm{d}}\frac{\mathrm{d}\ ^\beta(\ )}{\mathrm{d}} = \kappa\end{cases}$$

$$\sum_{\alpha,\beta=1}^{2} b_{\alpha\beta}\frac{\mathrm{d}u^\alpha(s)}{\mathrm{d}s}\frac{\mathrm{d}u^\beta(s)}{\mathrm{d}s} = \begin{pmatrix}\frac{\mathrm{d}\ ^1}{\mathrm{d}} & \frac{\mathrm{d}\ ^2}{\mathrm{d}}\end{pmatrix}\begin{pmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{pmatrix}\begin{pmatrix}\frac{\mathrm{d}\ ^1}{\mathrm{d}}\\ \frac{\mathrm{d}\ ^2}{\mathrm{d}}\end{pmatrix} = \kappa$$

κ

$$\frac{\mathrm{d}\vec{e}_2(s)}{\mathrm{d}s} = \tau\ \vec{e}_3$$

$$\frac{\mathrm{d}(\vec{n} \times \sum_{\gamma=1}^{2} \vec{r}_\gamma \frac{\mathrm{d}\ ^{\gamma}(\ )}{\mathrm{d}})}{\mathrm{d}s} = \tau\ \vec{n}$$

$$\sum_{\gamma,\eta=1}^{2} \vec{n}_\eta \times \vec{r}_\gamma \frac{\mathrm{d}u^\gamma(s)}{\mathrm{d}s}\frac{\mathrm{d}u^\eta(s)}{\mathrm{d}s} + \sum_{\gamma,\eta=1}^{2} \vec{n} \times \vec{r}_{\gamma\eta} \frac{\mathrm{d}u^\gamma(s)}{\mathrm{d}s}\frac{\mathrm{d}u^\eta(s)}{\mathrm{d}s} + \sum_{\gamma=1}^{2} \vec{n} \times \vec{r}_\gamma \frac{\mathrm{d}^2u^\gamma(s)}{\mathrm{d}s^2} = \tau\ \vec{n}$$

$\vec{n}, \vec{r}_\xi$ $\vec{n}$

$\vec{r}_\xi$

$$\sum_{\gamma,\eta=1}^{2} (\vec{n}, \vec{r}_{\gamma\eta}, \vec{r}_\xi) \frac{\mathrm{d}u^\gamma(s)}{\mathrm{d}s}\frac{\mathrm{d}u^\eta(s)}{\mathrm{d}s} + \sum_{\gamma=1}^{2} (\vec{n}, \vec{r}_\gamma, \vec{r}_\xi) \frac{\mathrm{d}^2u^\gamma(s)}{\mathrm{d}s^2} = 0$$

$$\begin{aligned}
\sum_{\gamma,\eta=1}^{2} (\vec{n}, \vec{r}_{\gamma\eta}, \vec{r}_\xi) \frac{\mathrm{d}u^\gamma(s)}{\mathrm{d}s}\frac{\mathrm{d}u^\eta(s)}{\mathrm{d}s} &= \sum_{\alpha,\beta=1}^{2} (\vec{n}, \vec{r}_{\alpha\beta}, \vec{r}_\xi) \frac{\mathrm{d}u^\alpha(s)}{\mathrm{d}s}\frac{\mathrm{d}u^\beta(s)}{\mathrm{d}s} \\
&= \sum_{\alpha,\beta=1}^{2} (\vec{n}, \sum_{\gamma=1}^{2} \Gamma^\gamma_{\alpha\beta} \vec{r}_\gamma, \vec{r}_\xi) \frac{\mathrm{d}u^\alpha(s)}{\mathrm{d}s}\frac{\mathrm{d}u^\beta(s)}{\mathrm{d}s} \\
&= \sum_{\alpha,\beta=1}^{2} \sum_{\gamma=1}^{2} (\vec{n}, \vec{r}_\gamma, \vec{r}_\xi) \Gamma^\gamma_{\alpha\beta} \frac{\mathrm{d}u^\alpha(s)}{\mathrm{d}s}\frac{\mathrm{d}u^\beta(s)}{\mathrm{d}s}
\end{aligned}$$

$$\sum_{\gamma=1}^{2} (\vec{n}, \vec{r}_\gamma, \vec{r}_\xi) \left(\frac{\mathrm{d}^2u^\gamma(s)}{\mathrm{d}s^2} + \sum_{\alpha,\beta=1}^{2} \Gamma^\gamma_{\alpha\beta} \frac{\mathrm{d}u^\alpha(s)}{\mathrm{d}s}\frac{\mathrm{d}u^\beta(s)}{\mathrm{d}s} \right) = 0, \quad \xi = 1, 2$$

$(\vec{n}, \vec{r}_\gamma, \vec{r}_\xi)$ $\xi \neq \gamma$

$$\frac{\mathrm{d}^2u^\gamma(s)}{\mathrm{d}s^2} + \sum_{\gamma,\eta=1}^{2} \Gamma^\gamma_{\alpha\beta} \frac{\mathrm{d}u^\alpha(s)}{\mathrm{d}s}\frac{\mathrm{d}u^\beta(s)}{\mathrm{d}s} = 0, \quad \gamma = 1, 2 \tag{8.7}$$

$$\tau = 0$$

$\tau = 0$

Weingarten

定义 8.3.

$$\frac{\mathrm{d}^2 u^\gamma(s)}{\mathrm{d}s^2} + \sum_{\gamma,\eta=1}^{2} \Gamma^\gamma_{\alpha\beta} \frac{\mathrm{d}u^\alpha(s)}{\mathrm{d}s} \frac{\mathrm{d}u^\beta(s)}{\mathrm{d}s} = 0, \quad \gamma = 1, 2$$

(8.7)

$$\kappa = \sqrt{g} \begin{vmatrix} \frac{\mathrm{d}\ ^1(\ )}{\mathrm{d}} & \frac{\mathrm{d}^2\ ^1(\ )}{\mathrm{d}\ ^2} + \sum_{\alpha,\beta=1}^{2} \Gamma^1_{\alpha\beta} \frac{\mathrm{d}\ ^\alpha(\ )}{\mathrm{d}} \frac{\mathrm{d}\ ^\beta(\ )}{\mathrm{d}} \\ \frac{\mathrm{d}\ ^2(\ )}{\mathrm{d}} & \frac{\mathrm{d}^2\ ^2(\ )}{\mathrm{d}\ ^2} + \sum_{\alpha,\beta=1}^{2} \Gamma^2_{\alpha\beta} \frac{\mathrm{d}\ ^\alpha(\ )}{\mathrm{d}} \frac{\mathrm{d}\ ^\beta(\ )}{\mathrm{d}} \end{vmatrix} = 0$$

-
-

8.4.2

$\kappa = \sqrt{}$

$\mathrm{d}\ ^2(\ )$

$\mathrm{d}\ ^\beta(\ )$

$4 \sum^2 \quad \Gamma^1 \quad ^\alpha(\ )$

$$\begin{vmatrix} \frac{\mathrm{d}\ ^1(\ )}{\mathrm{d}} & \frac{\mathrm{d}^2\ ^1(\ )}{\mathrm{d}\ ^2} + \sum_{\alpha,\beta=1}^{2} \Gamma^1_{\alpha\beta} \frac{\mathrm{d}\ ^\alpha(\ )}{\mathrm{d}} \frac{\mathrm{d}\ ^\beta(\ )}{\mathrm{d}} \\ \frac{\mathrm{d}\ ^2(\ )}{\mathrm{d}} & \frac{\mathrm{d}^2\ ^2(\ )}{\mathrm{d}\ ^2} + \sum_{\alpha,\beta=1}^{2} \Gamma^2_{\alpha\beta} \frac{\mathrm{d}\ ^\alpha(\ )}{\mathrm{d}} \frac{\mathrm{d}\ ^\beta(\ )}{\mathrm{d}} \end{vmatrix} = 0$$

$$\frac{\mathrm{d}^2 u^\gamma(t)}{\mathrm{d}t^2} + \sum_{\alpha,\beta=1}^{2} \Gamma^\gamma_{\alpha\beta} \frac{\mathrm{d}u^\alpha(t)}{\mathrm{d}t} \frac{\mathrm{d}u^\beta(t)}{\mathrm{d}t} = 0,\ \gamma = 1, 2 \tag{8.8}$$

$\kappa \ = 0$ (8.8)

(8.8)

$$\begin{aligned} \frac{\mathrm{d}^2 \vec{r}}{\mathrm{d}t^2} =& \frac{\mathrm{d}}{\mathrm{d}t} (\sum_{\gamma=1}^{2} \frac{\mathrm{d}u^\gamma}{\mathrm{d}t} \vec{r}_\gamma) \\ =& \sum_{\gamma=1}^{2} (\frac{\mathrm{d}^2 u^\gamma(t)}{\mathrm{d}t^2} + \sum_{\alpha,\beta=1}^{2} \Gamma^\gamma_{\alpha\beta} \frac{\mathrm{d}u^\alpha(t)}{\mathrm{d}t} \frac{\mathrm{d}u^\beta(t)}{\mathrm{d}t}) \vec{r}_\gamma + \sum_{\alpha,\beta=1}^{2} b_{\alpha\beta} \frac{\mathrm{d}u^\alpha(t)}{\mathrm{d}t} \frac{\mathrm{d}u^\beta(t)}{\mathrm{d}t} \vec{n} \\ =& \sum_{\alpha,\beta=1}^{2} b_{\alpha\beta} \frac{\mathrm{d}u^\alpha(t)}{\mathrm{d}t} \frac{\mathrm{d}u^\beta(t)}{\mathrm{d}t} \vec{n} \end{aligned}$$

$$\vec{r}''(t) \cdot \vec{r}'(t) = 0$$

$|\vec{r}'(t)|$

t

$$\begin{vmatrix} \frac{\mathrm{d}\ ^1(\ )}{\mathrm{d}} & \frac{\mathrm{d}^2\ ^1(\ )}{\mathrm{d}\ ^2} + \Gamma^1_{\alpha\beta} \frac{\mathrm{d}\ ^\alpha(\ )}{\mathrm{d}} \frac{\mathrm{d}\ ^\beta(\ )}{\mathrm{d}} \\ \frac{\mathrm{d}\ ^2(\ )}{\mathrm{d}} & \frac{\mathrm{d}^2\ ^2(\ )}{\mathrm{d}\ ^2} + \Gamma^2_{\alpha\beta} \frac{\mathrm{d}\ ^\alpha(\ )}{\mathrm{d}} \frac{\mathrm{d}\ ^\beta(\ )}{\mathrm{d}} \end{vmatrix} = 0$$

$$\Downarrow \quad \Uparrow$$

$$\frac{\mathrm{d}^2 u^\gamma(t)}{\mathrm{d}t^2} + \Gamma^\gamma_{\alpha\beta} \frac{\mathrm{d}u^\alpha(t)}{\mathrm{d}t} \frac{\mathrm{d}u^\beta(t)}{\mathrm{d}t} = 0,\ \gamma = 1, 2$$

(8.8) $|\vec{r}'(t)| \ = \ C$ $s \ =$

$|\vec{r}'(t)|t \ = \ Ct$ $t \ = \ \frac{}{C}$ (8.8)

(8.7)

(8.7) $t = t(s)$

(8.7)

$$\frac{\mathrm{d}^2u^\gamma(t)}{\mathrm{d}t^2}(\frac{\mathrm{d}t}{\mathrm{d}s})^2 + \frac{\mathrm{d}u^\gamma(t)}{\mathrm{d}t}\frac{\mathrm{d}^2t}{\mathrm{d}s^2} + \sum_{\alpha,\beta=1}^{2}\Gamma^\gamma_{\alpha\beta}\frac{\mathrm{d}u^\alpha(t)}{\mathrm{d}t}\frac{\mathrm{d}u^\beta(t)}{\mathrm{d}t}(\frac{\mathrm{d}t}{\mathrm{d}s})^2 = 0, \quad \gamma = 1, 2$$

8.4.3

$$\frac{\mathrm{d}^2u^\gamma(t)}{\mathrm{d}t^2} + \sum_{\alpha,\beta=1}^{2}\Gamma^\gamma_{\alpha\beta}\frac{\mathrm{d}u^\alpha(t)}{\mathrm{d}t}\frac{\mathrm{d}u^\beta(t)}{\mathrm{d}t} = 0, \ \gamma = 1, 2$$

p $\vec{v}$

定理 8.4. S p S p $\vec{v}$
C p p $\vec{v}$

$\vec{v}$
($|\vec{v}|$)

例 8.5. $I = (\mathrm{d}u)^2 + u^2(\mathrm{d}v)^2$

(u, v)

. Christoffel

$$\Gamma^1_{11} = \frac{1}{2}\frac{\partial \log E}{\partial u} = 0, \qquad \Gamma^1_{12} = \Gamma^1_{21} = \frac{1}{2}\frac{\partial \log E}{\partial v} = 0, \qquad \Gamma^1_{22} = -\frac{1}{2E}\frac{\partial G}{\partial u} = -u$$
$$\Gamma^2_{11} = -\frac{1}{2G}\frac{\partial E}{\partial v} = 0, \qquad \Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{2}\frac{\partial \log G}{\partial u} = \frac{1}{u}, \qquad \Gamma^2_{22} = \frac{1}{2}\frac{\partial \log G}{\partial v} = 0$$

$$\begin{cases} \frac{\mathrm{d}^2 u}{\mathrm{d}t^2} - u(\frac{\mathrm{d}v}{\mathrm{d}t})^2 = 0 \\ \frac{\mathrm{d}^2 v}{\mathrm{d}t^2} + \frac{2}{u}\frac{\mathrm{d}u}{\mathrm{d}t}\frac{\mathrm{d}v}{\mathrm{d}t} = 0 \end{cases}$$

$$\frac{\mathrm{d}^2 v}{\mathrm{d}t^2} + \frac{2}{u}\frac{\mathrm{d}u}{\mathrm{d}t}\frac{\mathrm{d}v}{\mathrm{d}t} = 0$$

$f = \frac{\mathrm{d}}{\mathrm{d}}$

$$\frac{\mathrm{d}f}{\mathrm{d}t} + \frac{2}{u}\frac{\mathrm{d}u}{\mathrm{d}t}f = 0$$

$$u^2\frac{\mathrm{d}f}{\mathrm{d}t} + 2u\frac{\mathrm{d}u}{\mathrm{d}t}f = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(u^2 f) = 0$$

$u^2 f = A, \qquad f = \frac{A}{2}$

$$\frac{\mathrm{d}^2 u}{\mathrm{d}t^2} - \frac{A^2}{u^3} = 0$$

$$\begin{cases} \frac{\mathrm{d}u}{\mathrm{d}t} = y \\ \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{A^2}{u^3} \end{cases}$$

$$\frac{\mathrm{d}y}{\mathrm{d}u} = \frac{A^2}{yu^3}$$

$$y = \pm\sqrt{B - \frac{A^2}{u^2}}$$

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \pm\sqrt{B - \frac{A^2}{u^2}}$$

$$\frac{\mathrm{d}u}{\sqrt{B}\sqrt{1 - \frac{A^2}{B^{\ 2}}}} = \pm\mathrm{d}t$$

$\tilde{u} = \frac{\sqrt{B}}{A}$

$$\frac{A}{B}\frac{\tilde{u}\mathrm{d}\tilde{u}}{\sqrt{\tilde{u}^2 - 1}} = \pm\mathrm{d}t$$

$$\frac{A}{B}\sqrt{\tilde{u}^2 - 1} = C \pm t$$

$$\tilde{u}^2 = \frac{B^2(t \pm C)^2}{A^2} + 1$$

$$u^2 = B(t \pm C)^2 + \frac{A^2}{B}$$

t $\tilde{t} = t \pm C$ $\tilde{t}$ t

$$u^2 = Bt^2 + \frac{A^2}{B} \tag{8.9}$$

$\frac{\mathrm{d}}{\mathrm{d}} = f = \frac{A}{^2}$

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{AB}{B^2t^2 + A^2}$$

v

$$v = \int \frac{AB}{B^2t^2 + A^2}\mathrm{d}t = \int \frac{1}{(\frac{B}{A})^2 + 1}\mathrm{d}\frac{Bt}{A} = \arctan(\frac{Bt}{A}) + C \tag{8.10}$$

$$\begin{cases} u = \pm\sqrt{Bt^2 + \dfrac{A^2}{B}} \\ v = \arctan(\dfrac{Bt}{A}) + C \end{cases}$$

t (8.10)

$$\tan(v - C) = \frac{Bt}{A}$$

(8.9)

$$\frac{B}{A^2}u^2 = \tan^2(v - C) + 1 = \sec^2(v - C)$$

$$u\cos(v - C) = \frac{A}{\sqrt{B}}$$

□

8.6

8.4.4

$\kappa \ = \kappa\cos\theta$ θ $\vec{\beta}$ $\vec{e}_2$

命題 8.6. S C
S

- S
-

例 8.7. S

命题 8.8. S p S

p C

. p $\vec{r} = \vec{r}(t)$ S

$$\mathbf{F} = m\frac{\mathrm{d}^2\vec{r}(t)}{\mathrm{d}t^2}$$

m $\mathbf{F}$ $\vec{n}$ $\mathbf{F}$

$$\mathbf{F} = m\frac{\mathrm{d}^2\vec{r}(t)}{\mathrm{d}t^2}$$

t t

$\frac{\mathrm{d}^2\boldsymbol{r}(\ )}{\mathrm{d}\ ^2} = \kappa\beta$ β $\vec{n}$

t $\frac{\mathrm{d}\boldsymbol{r}(\ )}{\mathrm{d}}$

$$\frac{\mathrm{d}^2\vec{r}(t)}{\mathrm{d}t^2} \cdot \frac{\mathrm{d}\vec{r}(t)}{\mathrm{d}t} = \frac{1}{m}\mathbf{F} \cdot \frac{\mathrm{d}\vec{r}(t)}{\mathrm{d}t} = 0$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}\vec{r}(t)}{\mathrm{d}t} \cdot \frac{\mathrm{d}\vec{r}(t)}{\mathrm{d}t}\right) = \frac{\mathrm{d}^2\vec{r}(t)}{\mathrm{d}t^2} \cdot \frac{\mathrm{d}\vec{r}(t)}{\mathrm{d}t} = 0$$

$$\left|\frac{\mathrm{d}\vec{r}(t)}{\mathrm{d}t}\right| =$$

t s $\frac{\mathrm{d}^2\boldsymbol{r}(\ )}{\mathrm{d}\ ^2}$

□

$$\left|\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}}\right| = const$$

8.1: 1415

8.5

1.

2. ;

3.

-

8.5.1

$$\begin{aligned}
\frac{\mathrm{d}\vec{e}_1(s)}{\mathrm{d}s} &= \kappa\ \vec{e}_2 + \kappa\ \vec{e}_3 \\
\frac{\mathrm{d}\vec{e}_2(s)}{\mathrm{d}s} &= -\kappa\ \vec{e}_1 + \tau\ \vec{e}_3 \\
\frac{\mathrm{d}\vec{e}_3(s)}{\mathrm{d}s} &= -\kappa\ \vec{e}_1 - \tau\ \vec{e}_2
\end{aligned}$$

$\kappa\ \equiv 0$

$$\begin{aligned}
\frac{\mathrm{d}\vec{e}_1(s)}{\mathrm{d}s} &= +\kappa\ \vec{e}_3 \\
\frac{\mathrm{d}\vec{e}_2(s)}{\mathrm{d}s} &= +\tau\ \vec{e}_3 \\
\frac{\mathrm{d}\vec{e}_3(s)}{\mathrm{d}s} &= -\kappa\ \vec{e}_1 - \tau\ \vec{e}_2
\end{aligned}$$

$$\frac{\mathrm{d}\vec{e}_1(s)}{\mathrm{d}s} = \kappa\ \vec{e}_3$$

$$\frac{\mathrm{d}(\sum_{\gamma=1}^{2} \vec{r}_\gamma \frac{\mathrm{d}\ \gamma(\ )}{\mathrm{d}})}{\mathrm{d}s} = \kappa\ \vec{n}$$

$$\sum_{\gamma=1}^{2}(\frac{\mathrm{d}^2 u^\gamma(s)}{\mathrm{d}s^2} + \sum_{\alpha,\beta=1}^{2} \Gamma^\gamma_{\alpha\beta} \frac{\mathrm{d}u^\alpha(s)}{\mathrm{d}s}\frac{\mathrm{d}u^\beta(s)}{\mathrm{d}s})\vec{r}_\gamma + \sum_{\alpha,\beta=1}^{2} b_{\alpha\beta}\frac{\mathrm{d}u^\alpha(s)}{\mathrm{d}s}\frac{\mathrm{d}u^\beta(s)}{\mathrm{d}s}\vec{n} = \kappa\ \vec{n}$$

$\{\vec{r}_1, \vec{r}_2, \vec{n}\}$

$$\sum_{\alpha,\beta=1}^{2} b_{\alpha\beta}\frac{\mathrm{d}u^\alpha(s)}{\mathrm{d}s}\frac{\mathrm{d}u^\beta(s)}{\mathrm{d}s}\vec{n} = \kappa\ \vec{n}$$

$$\frac{\mathrm{d}^2 u^\gamma(s)}{\mathrm{d}s^2} + \sum_{\alpha,\beta=1}^{2} \Gamma^\gamma_{\alpha\beta}\frac{\mathrm{d}u^\alpha(s)}{\mathrm{d}s}\frac{\mathrm{d}u^\beta(s)}{\mathrm{d}s} = 0 \quad \gamma = 1, 2$$

$$\frac{\mathrm{d}(\sum_{\gamma=1}^{2}\vec{r}_{\gamma}\frac{\mathrm{d}\ \gamma(\ )}{\mathrm{d}})}{\mathrm{d}s}=\kappa\ \vec{n}$$

$\frac{\mathrm{d}(\sum_{\gamma=1}^{2}\vec{r}_{\gamma}\frac{\mathrm{d}\ \gamma(\ )}{\mathrm{d}})}{\mathrm{d}}$ TS $\vec{n}$ TS

$\frac{\mathrm{d}(\sum_{\gamma=1}^{2}\vec{r}_{\gamma}\frac{\mathrm{d}\ \gamma(\ )}{\mathrm{d}})}{\mathrm{d}}$ TS

$\{\vec{r}_1,\vec{r}_2,\vec{n}\}$ $\vec{r}_1,\vec{r}_2$

$\vec{n}$

TS

$\vec{r}''$

$\vec{r}'$

$\{\vec{r}_1,\vec{r}_2,\vec{n}\}$

$$X(u^1,u^2)=\sum_{\alpha=1}^{2}x^{\alpha}(u^1,u^2)\vec{r}_{\alpha}(u^1,u^2)$$

$C:u^{\alpha}=u^{\alpha}(t)$ S $X(t)$ S $X(u^1,u^2)$ C

$$X(t)=\sum_{\alpha=1}^{2}x^{\alpha}(u^1(t),u^2(t))\vec{r}_{\alpha}(u^1(t),u^2(t))$$

$X(t)$ t

t $\frac{\mathrm{d}\ (\ )}{\mathrm{d}}$ C E^3

S

$\frac{\mathrm{d}\ (\ )}{\mathrm{d}}$

定义 8.9.

$$\frac{\mathrm{D}X(t)}{\mathrm{d}t}=(\frac{\mathrm{d}X(t)}{\mathrm{d}t})^{\top}=\frac{\mathrm{d}X(t)}{\mathrm{d}t}-(\frac{\mathrm{d}X(t)}{\mathrm{d}t}\cdot\vec{n})\vec{n}$$

$X(t)$ C

X S Y p $\frac{\partial}{\partial}$

X Y p X Y

$$D\ X = (\frac{\partial X}{\partial Y})^\top = \frac{\partial X}{\partial Y} - (\frac{\partial X}{\partial Y} \cdot \vec{n})\vec{n}$$

p $C : (u(t), v(t))$ Y p

$$D\ X = \frac{\mathrm{D}X(u(t), v(t))}{\mathrm{d}t}$$

$$\frac{\partial X}{\partial Y} = \frac{\mathrm{d}X(u(t), v(t))}{\mathrm{d}t}$$

8.5.2

$X(t), Y(t)$

C

1. $\frac{\mathrm{D}}{\mathrm{d}}(X(t) + Y(t)) = \frac{\mathrm{D}}{\mathrm{d}}X(t) + \frac{\mathrm{D}}{\mathrm{d}}Y(t)$

2. $\frac{\mathrm{D}}{\mathrm{d}}(f(t)X(t)) = \frac{\mathrm{d}}{\mathrm{d}}X(t) + f(t)\frac{\mathrm{D}}{\mathrm{d}}X(t)$

3. $\frac{\mathrm{d}}{\mathrm{d}}(X(t) \cdot Y(t)) = \frac{\mathrm{D}}{\mathrm{d}}X(t) \cdot Y(t) + X(t) \cdot \frac{\mathrm{D}}{\mathrm{d}}Y(t)$

4. C $t = t(s)$

$$\frac{\mathrm{D}}{\mathrm{d}s}X(t(s)) = (\frac{\mathrm{D}}{\mathrm{d}t}X(t))t'(s) \tag{8.11}$$

$$X(t) = x^\alpha(t)\vec{r}_\alpha(u^1(t), u^2(t))$$

$$\begin{aligned}
\frac{\mathrm{d}X(t)}{\mathrm{d}t} &= \sum_{\alpha=1}^{2} \frac{\mathrm{d}x^\alpha}{\mathrm{d}t}\vec{r}_\alpha + \sum_{\alpha,\beta=1}^{2} x^\alpha \vec{r}_{\alpha\beta}\frac{\mathrm{d}u^\beta}{\mathrm{d}t} \\
&= \sum_{\alpha=1}^{2} \frac{\mathrm{d}x^\alpha}{\mathrm{d}t}\vec{r}_\alpha + \sum_{\alpha,\beta=1}^{2} x^\alpha (\sum_{\gamma=1}^{2} \Gamma^\gamma_{\alpha\beta}\vec{r}_\gamma + b_{\alpha\beta}\vec{n})\frac{\mathrm{d}u^\beta}{\mathrm{d}t} \\
&= \sum_{\alpha=1}^{2} \frac{\mathrm{d}x^\alpha}{\mathrm{d}t}\vec{r}_\alpha + \sum_{\alpha,\beta=1}^{2}\sum_{\gamma=1}^{2} (x^\alpha \Gamma^\gamma_{\alpha\beta}\mathrm{d}u^\beta)\vec{r}_\gamma + \sum_{\alpha,\beta=1}^{2} (x^\alpha b_{\alpha\beta}\frac{\mathrm{d}u^\beta}{\mathrm{d}t})\vec{n} \\
&= \sum_{\gamma=1}^{2} \frac{\mathrm{d}x^\gamma}{\mathrm{d}t}\vec{r}_\gamma + \sum_{\alpha,\beta=1}^{2}\sum_{\gamma=1}^{2} (x^\alpha \Gamma^\gamma_{\alpha\beta}\frac{\mathrm{d}u^\beta}{\mathrm{d}t})\vec{r}_\gamma + \sum_{\alpha,\beta=1}^{2} (x^\alpha b_{\alpha\beta}\frac{\mathrm{d}u^\beta}{\mathrm{d}t})\vec{n} \\
&= \sum_{\gamma=1}^{2} (\frac{\mathrm{d}x^\gamma}{\mathrm{d}t} + \sum_{\alpha,\beta=1}^{2} x^\alpha \Gamma^\gamma_{\alpha\beta}\frac{\mathrm{d}u^\beta}{\mathrm{d}t})\vec{r}_\gamma + \sum_{\alpha,\beta=1}^{2} (x^\alpha b_{\alpha\beta}\frac{\mathrm{d}u^\beta}{\mathrm{d}t})\vec{n}
\end{aligned}$$

$$\frac{\mathrm{D}X(t)}{\mathrm{d}t} = \sum_{\gamma=1}^{2} (\frac{\mathrm{d}x^\gamma}{\mathrm{d}t} + \sum_{\alpha,\beta=1}^{2} x^\alpha \Gamma^\gamma_{\alpha\beta}\frac{\mathrm{d}u^\beta}{\mathrm{d}t})\vec{r}_\gamma = \sum_{\alpha=1}^{2} (\frac{\mathrm{d}x^\alpha}{\mathrm{d}t} + \sum_{\beta,\gamma=1}^{2} x^\beta \Gamma^\alpha_{\beta\gamma}\frac{\mathrm{d}u^\gamma}{\mathrm{d}t})\vec{r}_\alpha$$

8.5.3

定义 8.10. $X(t)$ C

$$\frac{\mathrm{D}X(t)}{\mathrm{d}t} = \vec{0}$$

$X(t)$ C

- (8.11)
- 3 $X(t)$

$$\frac{\mathrm{d}}{\mathrm{d}t}(X(t)\cdot X(t)) = \frac{\mathrm{D}X(t)}{\mathrm{d}t}\cdot X(t) + X(t)\cdot\frac{\mathrm{D}X(t)}{\mathrm{d}t} = 0$$

C $X(t) = x^\alpha(t)\vec{r}_\alpha(u^1(t), u^2(t))$

$$\frac{\mathrm{d}x^\alpha}{\mathrm{d}t} + \sum_{\beta,\gamma=1}^{2} x^\beta \Gamma^\alpha_{\beta\gamma}\frac{\mathrm{d}u^\gamma}{\mathrm{d}t} = 0, \quad \alpha = 1, 2$$

$\Gamma^\alpha_{\beta\gamma}(u^1(t), u^2(t))$ $\frac{\mathrm{d}\ \gamma(\ )}{\mathrm{d}}$ t $x^\alpha(t)$
$t = 0$ ($(u^1(0), u^2(0))$) $(x^1(0), x^2(0))($
$X(0))$ ($X(t))$

例 8.11. $\mathrm{d}u^2 + \mathrm{d}v^2$ $\Gamma^\gamma_{\alpha\beta}$

$$\frac{\mathrm{d}x^\alpha}{\mathrm{d}t} = 0, \quad \alpha = 1, 2$$

8.5.4

1. C
2. $$\frac{\mathrm{d}(\sum_{\gamma=1}^{2}\vec{r}_\gamma\frac{\mathrm{d}\ \gamma(\ )}{\mathrm{d}})}{\mathrm{d}s} = \kappa\ \vec{n}$$
3. $$\frac{\mathrm{D}(\sum_{\gamma=1}^{2}\vec{r}_\gamma\frac{\mathrm{d}\ \gamma(\ )}{\mathrm{d}})}{\mathrm{d}s} = \vec{0}$$

4.

$$\frac{\mathrm{D}(\sum_{\gamma=1}^{2}\vec{r}_{\gamma}\frac{\mathrm{d}^{\ \gamma(\ )}}{\mathrm{d}})}{\mathrm{d}t}=\vec{0}$$

5.

$$\frac{\mathrm{d}^2u^{\gamma}}{\mathrm{d}t^2}+\sum_{\alpha,\beta=1}^{2}\Gamma_{\alpha\beta}^{\gamma}\frac{\mathrm{d}u^{\alpha}}{\mathrm{d}t}\frac{\mathrm{d}u^{\beta}}{\mathrm{d}t}=0,\quad \gamma=1,2$$

定义 8.12. S $C:u^{\alpha}=u^{\alpha}(t)$

$$\frac{\mathrm{D}(\sum_{\gamma=1}^{2}\vec{r}_{\gamma}\frac{\mathrm{d}^{\ \gamma(\ )}}{\mathrm{d}})}{\mathrm{d}t}=\vec{0}$$

(8.8)

-
-
-
- $\tau\ =(\vec{n}'(s),\vec{r}'(s),\vec{n}(s))$

-
-

$$\frac{\mathrm{d}x^\gamma}{\mathrm{d}t} + \sum_{\alpha,\beta=1}^{2} x^\alpha \Gamma^\gamma_{\alpha\beta} \frac{\mathrm{d}u^\beta}{\mathrm{d}t} = 0, \quad \gamma = 1, 2$$

$\frac{\mathrm{d}\ ^\beta}{\mathrm{d}}$ $x^\gamma(t)$

$$\frac{\mathrm{d}^2 u^\gamma}{\mathrm{d}t^2} + \sum_{\alpha,\beta=1}^{2} \Gamma^\gamma_{\alpha\beta} \frac{\mathrm{d}u^\alpha}{\mathrm{d}t} \frac{\mathrm{d}u^\beta}{\mathrm{d}t} = 0, \quad \gamma = 1, 2$$

$u^\alpha(t)$

8.6 Liouville

θ

$$\kappa_{\boldsymbol{r}}(s) = \frac{\mathrm{d}\theta(s)}{\mathrm{d}s}$$

κ

8.6.1 Liouville

定理 8.13. (u, v) S $I = E(\mathrm{d}u)^2 + G(\mathrm{d}v)^2$
$C : u = u(s), v = v(s)$ s C u-
θ

$$\kappa = \frac{\mathrm{d}\theta(s)}{\mathrm{d}s} - \frac{1}{2\sqrt{G}} \frac{\partial \log E}{\partial v} \cos\theta + \frac{1}{2\sqrt{E}} \frac{\partial \log G}{\partial u} \sin\theta \tag{8.12}$$

Liouville

θ C

(8.6) θ Christoffel

$$\frac{\mathrm{d}\vec{e}_1(s)}{\mathrm{d}s} = \kappa\ \vec{e}_2 + \kappa\ \vec{e}_3$$

$$\kappa\ = \frac{\mathrm{d}\vec{e}_1(s)}{\mathrm{d}s} \cdot \vec{e}_2$$

$\vec{r}(s)$ $\vec{r}'(s)$ θ $\vec{e}_2$ $\frac{\mathrm{d}\ _1(\ )}{\mathrm{d}}$ $\vec{r}\ , \vec{r}$

$$\vec{\alpha}_1 = \frac{1}{\sqrt{E}}\vec{r}\ , \quad \vec{\alpha}_2 = \frac{1}{\sqrt{G}}\vec{r}$$

$$\vec{e}_1 = \cos\theta\vec{\alpha}_1 + \sin\theta\vec{\alpha}_2$$

$$\begin{aligned}\frac{\mathrm{d}\vec{e}_1(s)}{\mathrm{d}s} =& \frac{\mathrm{d}}{\mathrm{d}s}\left(\cos\theta\vec{\alpha}_1 + \sin\theta\vec{\alpha}_2\right)\\ =& \frac{\mathrm{d}\theta}{\mathrm{d}s}(-\sin\theta\vec{\alpha}_1 + \cos\theta\vec{\alpha}_2) + \cos\theta\frac{\mathrm{d}\vec{\alpha}_1}{\mathrm{d}s} + \sin\theta\frac{\mathrm{d}\vec{\alpha}_2}{\mathrm{d}s}\end{aligned}$$

$$\vec{e}_2 = -\sin\theta\vec{\alpha}_1 + \cos\theta\vec{\alpha}_2$$

$$\frac{\mathrm{d}\vec{e}_1(s)}{\mathrm{d}s} = \frac{\mathrm{d}\theta}{\mathrm{d}s}\vec{e}_2 + \cos\theta\frac{\mathrm{d}\vec{\alpha}_1}{\mathrm{d}s} + \sin\theta\frac{\mathrm{d}\vec{\alpha}_2}{\mathrm{d}s}$$

$$\begin{aligned}\kappa\ =& \frac{\mathrm{d}^2\vec{r}(s)}{\mathrm{d}s^2} \cdot \vec{e}_2 = \frac{\mathrm{d}\theta}{\mathrm{d}s} + \cos\theta\frac{\mathrm{d}\vec{\alpha}_1}{\mathrm{d}s} \cdot \vec{e}_2 + \sin\theta\frac{\mathrm{d}\vec{\alpha}_2}{\mathrm{d}s} \cdot \vec{e}_2\\ =& \frac{\mathrm{d}\theta}{\mathrm{d}s} + \cos^2\theta\frac{\mathrm{d}\vec{\alpha}_1}{\mathrm{d}s} \cdot \vec{\alpha}_2 - \sin^2\theta\frac{\mathrm{d}\vec{\alpha}_2}{\mathrm{d}s} \cdot \vec{\alpha}_1\\ =& \frac{\mathrm{d}\theta}{\mathrm{d}s} + \frac{\mathrm{d}\vec{\alpha}_1}{\mathrm{d}s} \cdot \vec{\alpha}_2\end{aligned}$$

$\vec{\alpha}_1, \vec{\alpha}_2$

$$\frac{\mathrm{d}\vec{\alpha}_1}{\mathrm{d}s} = \frac{\mathrm{d}(\frac{1}{\sqrt{E}}\vec{r}\ )}{\mathrm{d}s} = \frac{\mathrm{d}\frac{1}{\sqrt{E}}}{\mathrm{d}s}\vec{r}\ + \frac{1}{\sqrt{E}}\left(\frac{\mathrm{d}u}{\mathrm{d}s}\vec{r}\ \ + \frac{\mathrm{d}v}{\mathrm{d}s}\vec{r}\ \ \right)$$

$$\frac{\mathrm{d}\vec{\alpha}_1}{\mathrm{d}s} \cdot \vec{\alpha}_2 = \frac{1}{\sqrt{EG}}\left(\frac{\mathrm{d}u}{\mathrm{d}s}\vec{r}\ \ \cdot \vec{r}\ + \frac{\mathrm{d}v}{\mathrm{d}s}\vec{r}\ \ \cdot \vec{r}\ \right)$$

Christoffel

$\vec{r}\ ,\quad \vec{r}$ [1]

$$\begin{aligned}\vec{r}\ \cdot\vec{r}\ &=\frac{\partial}{\partial u}(\vec{r}\ \cdot\vec{r}\ )-\vec{r}\ \cdot\vec{r}\ \ =-\vec{r}\ \cdot\vec{r}\\ &=-\frac{1}{2}\frac{\partial}{\partial v}(\vec{r}\ \cdot\vec{r}\ )=-\frac{1}{2}\frac{\partial E}{\partial v}\end{aligned}$$

$$\vec{r}\ \cdot\vec{r}\ =\frac{1}{2}\frac{\partial}{\partial u}(\vec{r}\ \cdot\vec{r}\ )=\frac{1}{2}\frac{\partial G}{\partial u}$$

$$\kappa\ =\frac{\mathrm{d}\theta(s)}{\mathrm{d}s}-\frac{1}{2\sqrt{EG}}\left(E\ \frac{\mathrm{d}u}{\mathrm{d}s}-G\ \frac{\mathrm{d}v}{\mathrm{d}s}\right)$$

$$\vec{e}_1=\frac{\mathrm{d}\vec{r}}{\mathrm{d}s}=\vec{r}\ \frac{\mathrm{d}u}{\mathrm{d}s}+\vec{r}\ \frac{\mathrm{d}v}{\mathrm{d}s}=\sqrt{E}\frac{\mathrm{d}u}{\mathrm{d}s}\vec{\alpha}_1+\sqrt{G}\frac{\mathrm{d}v}{\mathrm{d}s}\vec{\alpha}_2$$

$\cos\theta=\sqrt{E}\frac{\mathrm{d}}{\mathrm{d}}, \sin\theta=\sqrt{G}\frac{\mathrm{d}}{\mathrm{d}}$

$$\kappa\ =\frac{\mathrm{d}\theta(s)}{\mathrm{d}s}-\frac{1}{2\sqrt{G}}\frac{\partial\log E}{\partial v}\cos\theta+\frac{1}{2\sqrt{E}}\frac{\partial\log G}{\partial u}\sin\theta$$

Liouville

$$\kappa = \frac{\mathrm{d}\theta(s)}{\mathrm{d}s} - \frac{1}{2\sqrt{G}}\frac{\partial \log E}{\partial v}\cos\theta + \frac{1}{2\sqrt{E}}\frac{\partial \log G}{\partial u}\sin\theta$$

, θ

- $u-$ $\theta \equiv 0$ $\kappa_1 = -\frac{1}{2\sqrt{\ }}\frac{\partial \log E}{\partial}$
- $v-$ $\theta \equiv \frac{\pi}{2}$ $\kappa_2 = \frac{1}{2\sqrt{E}}\frac{\partial \log}{\partial}$

$$\kappa = \frac{\mathrm{d}\theta}{\mathrm{d}s} + \kappa_1\cos\theta + \kappa_2\sin\theta \tag{8.13}$$

(8.13) $\kappa_{\blacktriangleright}(s) = \frac{\mathrm{d}\theta(\ )}{\mathrm{d}}$

- $\kappa_1 = \kappa_2 = 0$
- $\kappa_1 = \kappa_2 = 0$ E u G v

$$I = E(u)(\mathrm{d}u)^2 + G(v)(\mathrm{d}v)^2$$

$\mathrm{d}\tilde{u} = \sqrt{E(u)}\mathrm{d}u, \mathrm{d}\tilde{v} = \sqrt{G(v)}\mathrm{d}v$

$$I = (\mathrm{d}\tilde{u})^2 + (\mathrm{d}\tilde{v})^2$$

(8.13) Euler (3.12)

*8.6.2 Gauss

$$\kappa = \frac{\mathrm{d}\theta(s)}{\mathrm{d}s} - \frac{1}{2\sqrt{EG}}\left(E\,\frac{\mathrm{d}u}{\mathrm{d}s} - G\,\frac{\mathrm{d}v}{\mathrm{d}s}\right)$$

s

$$\kappa = \frac{\mathrm{d}\theta(s) - \frac{1}{2\sqrt{EG}}(E\ \mathrm{d}u - G\ \mathrm{d}v)}{\mathrm{d}s}$$

$$\omega = -\frac{1}{2\sqrt{EG}}(E\ \mathrm{d}u - G\ \mathrm{d}v) = -\frac{(\sqrt{E})}{\sqrt{G}}\mathrm{d}u + \frac{(\sqrt{G})}{\sqrt{E}}\mathrm{d}v$$

$$\kappa = \frac{\mathrm{d}\theta(s)}{\mathrm{d}s} + \frac{\omega}{\mathrm{d}s}$$

$\vec{r}$

$$\kappa = \frac{\omega}{\mathrm{d}s}$$

u- v-

$$\kappa_1 = -\frac{E}{2E\sqrt{G}} = -\frac{(\sqrt{E})}{\sqrt{EG}}, \quad \kappa_2 = \frac{G}{2G\sqrt{E}} = \frac{(\sqrt{G})}{\sqrt{EG}}$$

κ_1 κ_2 Gauss

$$K = -\frac{1}{\sqrt{EG}}\left((\frac{(\sqrt{E})}{\sqrt{G}}) + (\frac{(\sqrt{G})}{\sqrt{E}}) \right)$$

ω Gauss K [1]

$$\mathrm{d}\omega = K\sqrt{EG}\mathrm{d}u \wedge \mathrm{d}v$$

ω Gauss

Gauss-Bonnet

8.6.3

S (u, v) C u- θ

$$\kappa = \frac{\mathrm{d}\theta(s)}{\mathrm{d}s} - \frac{1}{2\sqrt{G}}\frac{\partial \log E}{\partial v}\cos\theta + \frac{1}{2\sqrt{E}}\frac{\partial \log G}{\partial u}\sin\theta$$

[1] ω 1 [18] 5.2

$$\frac{\mathrm{d}\theta(s)}{\mathrm{d}s} = \frac{1}{2\sqrt{G}}\frac{\partial \log E}{\partial v}\cos\theta - \frac{1}{2\sqrt{E}}\frac{\partial \log G}{\partial u}\sin\theta$$

θ s E, G v, u u, v
s u, v s $\cos\theta = \sqrt{E}\frac{\mathrm{d}(\ )}{\mathrm{d}}$
$\sin\theta = \sqrt{G}\frac{\mathrm{d}(\ )}{\mathrm{d}}$

$$\begin{cases} \frac{\mathrm{d}(\ )}{\mathrm{d}} = \frac{1}{\sqrt{E}}\cos\theta \\ \frac{\mathrm{d}(\ )}{\mathrm{d}} = \frac{1}{\sqrt{\ }}\sin\theta \\ \frac{\mathrm{d}\theta(\ )}{\mathrm{d}} = \frac{1}{2\sqrt{\ }}\frac{\partial \log E}{\partial}\cos\theta - \frac{1}{2\sqrt{E}}\frac{\partial \log}{\partial}\sin\theta \end{cases} \tag{8.8}$$

例 8.14. S $\vec{r} = (u\cos v, u\sin v, f(u))$

.

$$I = (1 + (f'(u))^2)(\mathrm{d}u)^2 + u^2(\mathrm{d}v)^2$$

$u-$

$$\begin{cases} \frac{\mathrm{d}(\ )}{\mathrm{d}} = \frac{1}{\sqrt{E}}\cos\theta \\ \frac{\mathrm{d}(\ )}{\mathrm{d}} = \frac{1}{\sqrt{\ }}\sin\theta \\ \frac{\mathrm{d}\theta(\ )}{\mathrm{d}} = \frac{1}{2\sqrt{\ }}\frac{\partial \log E}{\partial}\cos\theta - \frac{1}{2\sqrt{E}}\frac{\partial \log}{\partial}\sin\theta \end{cases}$$

$$\begin{cases} \frac{\mathrm{d}(\ )}{\mathrm{d}} = \frac{1}{\sqrt{1+(\ '(\ ))^2}}\cos\theta \\ \frac{\mathrm{d}(\ )}{\mathrm{d}} = \frac{1}{\ }\sin\theta \\ \frac{\mathrm{d}\theta(\ )}{\mathrm{d}} = -\frac{1}{\sqrt{1+(\ '(\ ))^2}}\sin\theta \end{cases}$$

(u, v) θ s

s s

u, v, θ

u, v

$$\begin{cases} \dfrac{\mathrm{d}v}{\mathrm{d}u} = \dfrac{\sqrt{1+(f'(u))^2}}{u}\tan\theta \\ \dfrac{\mathrm{d}\theta}{\mathrm{d}u} = -\dfrac{1}{u}\tan\theta \end{cases}$$

$$\mathrm{ctg}\theta \mathrm{d}\theta = -\frac{\mathrm{d}u}{u}$$

$$\frac{\cos\theta}{\sin\theta}\mathrm{d}\theta = -\frac{\mathrm{d}u}{u}$$

$$\frac{1}{\sin\theta}\mathrm{d}\sin\theta = -\frac{\mathrm{d}u}{u}$$

$u\sin\theta = c$ c

$$\frac{\mathrm{d}v}{\mathrm{d}u} = \frac{\sqrt{1+(f'(u))^2}}{u}\tan\theta$$

$u\sin\theta = c$ $\sin\theta = -$

$$\tan\theta = \pm\frac{c}{\sqrt{u^2-c^2}}$$

$$\frac{\mathrm{d}v}{\mathrm{d}u} = \pm\frac{c\sqrt{1+(f'(u))^2}}{u\sqrt{u^2-c^2}}$$

$$v(u) = v_0 \pm \int_0 \frac{c\sqrt{1+(f'(t))^2}}{t\sqrt{t^2-c^2}}\mathrm{d}t$$

$|c| \le u_0$ c (u_0, v_0)

- $u_0 = |c|$

  $u_0 \sin\theta_0 = c$ $\sin\theta_0 = \pm 1 \quad \theta_0 = \pm\frac{\pi}{2}$

- $u-$ $c = 0$

  $\frac{\mathrm{d}\theta}{\mathrm{d}} = -\frac{1}{-}\tan\theta$

  $\tan\theta$ $u\sin\theta = c$ $u-$

  c $\sin\theta$

□

$$f'(u) = 0$$

$$I = (\mathrm{d}u)^2 + u^2(\mathrm{d}v)^2$$

$$v(u) = v_0 + \int_{0} \frac{c}{t\sqrt{t^2 - c^2}}\mathrm{d}t = v_0 + \arcsin\frac{c}{u}$$

$$u\sin(v - v_0) = c$$

$$u\sin v\cos v_0 - u\cos v\sin v_0 = c$$

$$\left\{\right.$$

$$\left.\right)$$

$$\frac{\mathrm{d}u}{u} = \cot(c_0 - v)\mathrm{d}v$$

$$u\cos(c_0 - v) = C$$

8.5 Liouville

κ

Liouville (8.12)

$$\begin{cases} \frac{\mathrm{d}(\ )}{\mathrm{d}} = \frac{1}{\sqrt{E}}\cos\theta \\ \frac{\mathrm{d}(\ )}{\mathrm{d}} = \frac{1}{\sqrt{\ }}\sin\theta \\ \frac{\mathrm{d}\theta(\ )}{\mathrm{d}} = \kappa + \frac{1}{2\sqrt{\ }}\frac{\partial \log E}{\partial}\cos\theta - \frac{1}{2\sqrt{E}}\frac{\partial \log}{\partial}\sin\theta \end{cases}$$

θ (2.10)

8.6.4 Liouville

θ

Liouville $\vec{\alpha}_1, \vec{\alpha}_2$ $\vec{r}$, $\vec{r}$

1. X $|X|$
2. $\frac{\mathrm{D}(\ \ )}{\mathrm{d}} = \frac{\mathrm{d}}{\mathrm{d}}X + f\frac{\mathrm{D}}{\mathrm{d}}$

$X = |X|X^0$

X^0 X

Liouville $\kappa\ (X) = \frac{\mathrm{d}}{\mathrm{d}} \cdot X^{\perp}$

X $\kappa\ (X) = 0$

$$\frac{\mathrm{d}\theta(s)}{\mathrm{d}s} = \kappa\ (X) + \frac{1}{2\sqrt{EG}} \left(E\ \frac{\mathrm{d}u}{\mathrm{d}s} - G\ \frac{\mathrm{d}v}{\mathrm{d}s} \right)$$

- θ

  $\kappa\ (X)$

  $\frac{1}{2\sqrt{E}} \left(E\ \frac{\mathrm{d}}{\mathrm{d}} - G\ \frac{\mathrm{d}}{\mathrm{d}} \right)$

- C X

$$\frac{\mathrm{d}\theta(t)}{\mathrm{d}t} = \frac{1}{2\sqrt{EG}} \left(E\ \frac{\mathrm{d}u}{\mathrm{d}t} - G\ \frac{\mathrm{d}v}{\mathrm{d}t} \right)$$

$\frac{\mathrm{d}}{\mathrm{d}} \cdot X^{\perp}$

$\kappa\ (X)$

8.6.5

s $\vec{r}$ θ

X

$$\frac{\mathrm{d}X}{\mathrm{d}s} \cdot X^{\perp} = \frac{\mathrm{d}\theta(s)}{\mathrm{d}s} - \frac{1}{2\sqrt{EG}} \left(E\ \frac{\mathrm{d}u}{\mathrm{d}s} - G\ \frac{\mathrm{d}v}{\mathrm{d}s} \right)$$

$$- \frac{1}{2\sqrt{EG}} \left(E\ \frac{\mathrm{d}u}{\mathrm{d}s} - G\ \frac{\mathrm{d}v}{\mathrm{d}s} \right)$$

Y $\vec{r}$ ϕ

$$\frac{\mathrm{d}Y}{\mathrm{d}s} \cdot Y^{\perp} = \frac{\mathrm{d}\phi(s)}{\mathrm{d}s} - \frac{1}{2\sqrt{EG}} \left(E\ \frac{\mathrm{d}u}{\mathrm{d}s} - G\ \frac{\mathrm{d}v}{\mathrm{d}s} \right)$$

$\psi = \theta - \phi$

$$\frac{\mathrm{d}X}{\mathrm{d}s} \cdot X^{\perp} - \frac{\mathrm{d}Y}{\mathrm{d}s} \cdot Y^{\perp} = \frac{\mathrm{d}\psi(s)}{\mathrm{d}s} \tag{8.14}$$

X $\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}}$ Y

$$\frac{\mathrm{d}X}{\mathrm{d}s} \cdot X^{\perp} = \kappa \ , \quad \frac{\mathrm{d}Y}{\mathrm{d}s} \cdot Y^{\perp} = 0$$

(8.14)

$$\kappa \ = \frac{\mathrm{d}\psi(s)}{\mathrm{d}s}$$

$\kappa_{\boldsymbol{r}} = \frac{\mathrm{d}\theta}{\mathrm{d}}$

8.7

u- v-

8.6.1

- $u-$ $\theta \equiv 0$ $\kappa_{\ 1} = -\frac{1}{2\sqrt{\ }} \frac{\partial \log E}{\partial}$
- $v-$ $\theta \equiv \frac{\pi}{2}$ $\kappa_{\ 2} = \frac{1}{2\sqrt{E}} \frac{\partial \log}{\partial}$

u- v- $\frac{\partial E}{\partial} \equiv \frac{\partial}{\partial} \equiv 0$

Gauss

u- v-

u- (u, v)

8.7.1

1. S C v C

$\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$

2. C $\vec{e}_2$ C $\vec{e}_2$
u

3. Σ Σ C U
(u, v)

$v = const$ $u-$ $u = 0$
$v-$ C $u-$ $v-$ $u-$

$\vec{r}(u, v)$ $F = \vec{r}\ \cdot \vec{r}\ = 0$

$$\frac{\partial F}{\partial u} = \frac{\partial(\vec{r}\ \cdot \vec{r}\ )}{\partial u} = \vec{r}\ \ \cdot \vec{r}\ + \vec{r}\ \cdot \vec{r}$$

$\vec{r}$ $\vec{r}$ Christoffol

$$\frac{\mathrm{d}^2\vec{r}(s)}{\mathrm{d}s^2} = \frac{\mathrm{d}(\sum_{\gamma=1}^{2} \vec{r}_\gamma \frac{\mathrm{d}\ \gamma(\ )}{\mathrm{d}})}{\mathrm{d}s} = \kappa\ \vec{n}$$

u

$$\vec{r}\ \ = \kappa\ \vec{n}$$

$\vec{r}\ \ \cdot \vec{r}\ = 0$

Christoffol

$$\vec{r}\ \ = \sum_{\gamma=1}^{2} \Gamma_{11}^{\gamma} \vec{r}_\gamma + b_{11}\vec{n}$$

$$\vec{r}\ \ \cdot \vec{r}\ = \sum_{\gamma=1}^{2} \Gamma_{11}^{\gamma} g_{\gamma 2}$$

Γ_{11}^{γ}

$$\frac{\mathrm{d}^2 u^\gamma}{\mathrm{d}s^2} + \sum_{\alpha,\beta=1}^{2} \Gamma_{\alpha\beta}^{\gamma} \frac{\mathrm{d}u^\alpha}{\mathrm{d}s} \frac{\mathrm{d}u^\beta}{\mathrm{d}s} = 0, \quad \gamma = 1, 2$$

$s = u \quad v = v_0 \qquad \gamma = 1, 2 \qquad \Gamma^{\gamma}_{11} = 0 \qquad \vec{r} \quad \cdot \vec{r} \ = 0$

$u- \qquad |\vec{r} \ | \qquad 1$

$$\vec{r} \ \cdot \vec{r} \quad = \vec{r} \ \cdot \vec{r} \quad = \frac{1}{2} \frac{\partial (\vec{r} \ \cdot \vec{r} \ )}{\partial v} = 0$$

$\frac{\partial}{\partial} = \vec{r} \quad \cdot \vec{r} \ + \vec{r} \ \cdot \vec{r} \quad = 0 \qquad u = 0 \qquad F(0, v) = 0$

$v \qquad F(u, v) \equiv 0$

$$I = (\mathrm{d}u)^2 + G(u, v)(\mathrm{d}v)^2$$

$u = 0 \qquad v \qquad C$

$$G(0, v) = 1$$

$C : u = 0$

$$\kappa \ _2| \ _{=0} = \frac{1}{2} \frac{\partial \log G}{\partial u} | \ _{=0} = 0$$

$$G \ (0, v) = 0$$

(u, v)

$$I = (\mathrm{d}u)^2 + G(u, v)(\mathrm{d}v)^2$$

$G(u, v)$

$$G(0, v) = 1, \ G \ (0, v) = 0$$

(u, v)

(u, v)

$(u, v) \qquad u = 0 \qquad v- \qquad u-$

命題 8.15. $S \qquad (u, v)$

$$I = (\mathrm{d}u)^2 + G(u, v)(\mathrm{d}v)^2$$

$G(u, v)$

$$G(0, v) = 1, \ G \ (0, v) = 0$$

- u-
- $u = 0$ v- C v-
-

8.7.2

p Σ

$F(s,\theta)$ s $F(0,\theta)$

$$F(s,\theta) = (\vec{r}\ \cdot \vec{r}_\theta) = |\vec{r}\ ||\vec{r}_\theta| \cos\angle(\vec{r}\ , \vec{r}_\theta)$$

$\lim_{\to 0} |\vec{r}\ | = 1$ $\lim_{\to 0} \vec{r}_\theta = \vec{r}_\theta(0,\theta) = \vec{0}$

$$\lim_{\to 0} F(s,\theta) = 0$$

$F(s,\theta) \equiv 0$

$$I = \mathrm{d}s^2 + G(s,\theta)\mathrm{d}\theta^2$$

$$\lim_{\to 0} G(s,\theta) = \lim_{\to 0}(\vec{r}_\theta \cdot \vec{r}_\theta) = 0$$

$G(s,\theta)$ $|\vec{r}_\theta(s,\theta)| =$ s $\lim_{\to 0}(\sqrt{G(s,\theta)}) = 1$

[19]

命題 8.16. S (s,θ)

$$I = (\mathrm{d}s)^2 + G(s,\theta)(\mathrm{d}\theta)^2$$

$G(s,\theta)$

$$\lim_{\to 0} \sqrt{G(s,\theta)} = 0,\ \lim_{\to 0}(\sqrt{G(s,\theta)}) = 1$$

(s,θ)

8.7.3

Gauss

-
- Gauss

$$(u, v)$$

$$I = (\mathrm{d}u)^2 + G(u, v)(\mathrm{d}v)^2$$

$G(u, v)$

$$G(0, v) = 1,\ G\ (0, v) = 0$$

Gauss

$$K = -\frac{1}{\sqrt{EG}}\left((\frac{(\sqrt{E})}{\sqrt{G}}) \ + (\frac{(\sqrt{G})}{\sqrt{E}}) \right) = -\frac{1}{\sqrt{G}}(\sqrt{G})$$

S Gauss K

$$(\sqrt{G}) \quad + K\sqrt{G} = 0$$

$\sqrt{G}$ u

G u v v_0 u

$$(\sqrt{G(u, v_0)}) \quad + K\sqrt{G(u, v_0)} = 0 \tag{8.15}$$

v_0

$$\lambda^2 + K = 0$$

K (8.15)

$$K>0,\ \sqrt{G}=a(v_0)\cos(\sqrt{K}u)+b(v_0)\sin(\sqrt{K}u),$$
$$K=0,\ \sqrt{G}=a(v_0)+b(v_0)u,$$
$$K<0,\ \sqrt{G}=a(v_0)\cosh(\sqrt{-K}u)+b(v_0)\sinh(\sqrt{-K}u),$$

$u=0$

$$G(0,v_0)=1,\ G\ (0,v_0)=0$$

$$K>0,\ \sqrt{G}=\cos(\sqrt{K}u),$$
$$K=0,\ \sqrt{G}=1,$$
$$K<0,\ \sqrt{G}=\cosh(\sqrt{-K}u).$$

$$K>0,\ I=(\mathrm{d}u)^2+\cos^2(\sqrt{K}u)(\mathrm{d}v)^2,$$
$$K=0,\ I=(\mathrm{d}u)^2+(\mathrm{d}v)^2,$$
$$K<0,\ I=(\mathrm{d}u)^2+\cosh^2(\sqrt{-K}u)(\mathrm{d}v)^2.$$

定理 8.17. Gauss

-
- 7.9
-

1

[19]

1

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-
-

a

a [19]

Jacobi

[18] [10]

[19]

-
-

Chapter 9

Laplace

9.1

9.1.1

定义 9.1. $\sigma : S_1 \to S_2$ 3

$p \in S_1$ $\sigma_* : T\ S_1 \to T_{\sigma(\ )}S_2$

$X, Y \in T\ S_1$

$$\angle(\sigma_*\ (X), \sigma_*\ (Y)) = \angle(X, Y)$$

σ S_1 S_2

σ

定理 9.2. S_1 S_2 I_1 I_2 $\sigma : S_1 \to S_2$

$$S_1 \qquad\qquad \lambda$$

$$\lambda^2(u_1,v_1)\begin{pmatrix} E_1(u_1,v_1) & F_1(u_1,v_1) \\ F_1(u_1,v_1) & G_1(u_1,v_1) \end{pmatrix} = J \begin{pmatrix} E_2(u_2,v_2) & F_2(u_2,v_2) \\ F_2(u_2,v_2) & G_2(u_2,v_2) \end{pmatrix} J$$

$$J = \begin{pmatrix} \frac{\partial\ _2}{\partial\ _1} & \frac{\partial\ _2}{\partial\ _1} \\ \frac{\partial\ _2}{\partial\ _1} & \partial\ _2\ _1 \end{pmatrix}$$

σ σ

S_1 S_2 (u, v)

S_1 S_2 $\vec{r}_1(u, v)$ $\vec{r}_2(u, v)$

$$\sigma(\vec{r}_1(u, v)) = \vec{r}_2(u, v)$$

$$\cos\angle(\frac{\partial \vec{r}_1}{\partial u}, \frac{\partial \vec{r}_1}{\partial v}) = \cos\angle(\frac{\partial \vec{r}_2}{\partial u}, \frac{\partial \vec{r}_2}{\partial v})$$

$$\frac{F_1}{\sqrt{E_1 G_1}} = \frac{F_2}{\sqrt{E_2 G_2}}$$

t

$$\cos\angle(\frac{\partial \vec{r}_1}{\partial u}, t\frac{\partial \vec{r}_1}{\partial u} + \frac{\partial \vec{r}_1}{\partial v}) = \cos\angle(\frac{\partial \vec{r}_2}{\partial u}, t\frac{\partial \vec{r}_2}{\partial u} + \frac{\partial \vec{r}_2}{\partial v})$$

$$\frac{tE_1 + F_1}{\sqrt{E_1(t^2E_1 + 2tF_1 + G_1)}} = \frac{tE_2 + F_2}{\sqrt{E_2(t^2E_2 + 2tF_2 + G_2)}}$$

$$\frac{t^2(E_1)^2 + 2tE_1F_1 + (F_1)^2}{t^2(E_1)^2 + 2tE_1F_1 + E_1G_1} = \frac{t^2(E_2)^2 + 2tE_2F_2 + (F_2)^2}{t^2(E_2)^2 + 2tE_2F_2 + E_2G_2}$$

$$\frac{(F_1)^2 - E_1G_1}{t^2(E_1)^2 + 2tE_1F_1 + E_1G_1} = \frac{(F_2)^2 - E_2G_2}{t^2(E_2)^2 + 2tE_2F_2 + E_2G_2}$$

$$\frac{t^2(E_1)^2 + 2tE_1F_1 + E_1G_1}{g_1} = \frac{t^2(E_2)^2 + 2tE_2F_2 + E_2G_2}{g_2}$$

t

$1=6\Delta 3//3$ $_2 + E_1$

õ•Rõs1y=ô•Rñ&!y= g— õ`Ywm ñG,2 2v2cXwP$D

$g_1 = E_1G_1 - (F_1)^2$ $g_2 = E_2G_2 - (F_2)^2$ $f<b>TJ\psi F\psi\psi Td<bTJ\psi F\psi\psi Tf\psi\psi\psi TdTJ\psi F\psi\psi Tf\psi\psi\psi Td<$

$$\frac{f_1}{f_2} = \frac{E_1}{E_2} = \frac{F_1}{F_2} = \frac{G_1}{G_2}$$

Jacobi

推论 9.4. (u, v)

$$I_2 = \lambda(u, v) I_1.$$

.

$$\begin{aligned} E_2(u, v) &= \lambda^2(u, v) E_1(u, v) \\ F_2(u, v) &= \lambda^2(u, v) F_1(u, v) \\ G_2(u, v) &= \lambda^2(u, v) G_1(u, v) \end{aligned}$$

$I_2 = \lambda(u, v) I_1$

S_1 S_2 (u, v) $I_2 = \lambda(u, v) I_1$

S_1 S_2 σ

$$\sigma(\vec{r}_1(u, v)) = \vec{r}_2(u, v)$$

σ (u, v) □

9.1.2

例 9.5. D $\mathbb{C}$ f D $f'(z) \neq 0$ f

$\mathbb{R}^2$ $\mathbb{R}^2$ f

$w = f(z)$ $z = u + iv$ $w = \tilde{u} + i\tilde{v}$ (u, v) $(\tilde{u}, \tilde{v})$ $\mathbb{R}^2$

$E = G = 1, F = 0$ $\tilde{E} = \tilde{G} = 1, \tilde{F} = 0$

f $\begin{cases} \tilde{u}(u,v) \\ \tilde{v}(u,v) \end{cases}$ Jacobi $J = \begin{pmatrix} \frac{\partial \tilde{}}{\partial} & \frac{\partial \tilde{}}{\partial} \\ \frac{\partial \tilde{}}{\partial} & \frac{\partial \tilde{}}{\partial} \end{pmatrix}$

$$J \begin{pmatrix} \tilde{E} & \tilde{F} \\ \tilde{F} & \tilde{G} \end{pmatrix} J^{\top} = JJ^{\top} = \begin{pmatrix} (\frac{\partial \tilde{}}{\partial})^2 + (\frac{\partial \tilde{}}{\partial})^2 & \frac{\partial \tilde{}}{\partial}\frac{\partial \tilde{}}{\partial} + \frac{\partial \tilde{}}{\partial}\frac{\partial \tilde{}}{\partial} \\ \frac{\partial \tilde{}}{\partial}\frac{\partial \tilde{}}{\partial} + \frac{\partial \tilde{}}{\partial}\frac{\partial \tilde{}}{\partial} & (\frac{\partial \tilde{}}{\partial})^2 + (\frac{\partial \tilde{}}{\partial})^2 \end{pmatrix}$$

$f(z)$ Cauch-Riemann

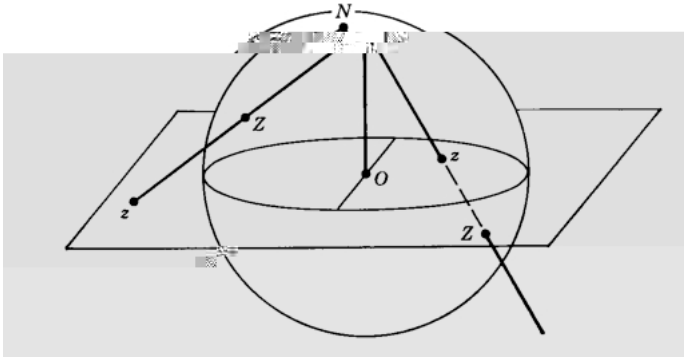
$$\frac{\partial \tilde{u}}{\partial u} = \frac{\partial \tilde{v}}{\partial v}, \quad \frac{\partial \tilde{v}}{\partial u} = -\frac{\partial \tilde{u}}{\partial v}$$

$$\left((\frac{\partial \tilde{u}}{\partial u})^2 + (\frac{\partial \tilde{v}}{\partial u})^2 \right) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = |f'(z)|^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$|f'(z)| \neq 0$ 9.2 f

定义 9.6. D_1 D_2 $\mathbb{C}$ f D_1 D_2 $|f'(z)|$ D_1 f

例 9.7. $x^2 + y^2 + z^2 = 1$ $N = (0, 0, 1)$ $p = (u, v, 0)$ N p p' p' p N



p' [1]

$$x = \frac{2u}{u^2 + v^2 + 1}, \ y = \frac{2v}{u^2 + v^2 + 1}, \ z = \frac{u^2 + v^2 - 1}{u^2 + v^2 + 1}$$

[1] $\varkappa = u + iv$

$$x = \frac{\varkappa + \bar{\varkappa}}{|\varkappa|^2 + 1}, \ y = \frac{1}{\sqrt{-1}} \frac{\varkappa - \bar{\varkappa}}{|\varkappa|^2 + 1}, \ z = \frac{|\varkappa|^2 - 1}{|\varkappa|^2 + 1}$$

$$\vec{r}(\theta,\phi) = (\cos\theta\cos\phi, \cos\theta\sin\phi, \sin\theta)$$

(u,v) N

$$\vec{r}(u,v) = \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right).$$

(u,v)

$$I = \frac{4}{(u^2+v^2+1)^2}(\mathrm{d}u^2+\mathrm{d}v^2)$$

- (u,v) (u,v) (u,v)

-

9.3

例 9.8.

9.4

$I = (\mathrm{d}\theta)^2 + \cos^2\theta(\mathrm{d}\phi)^2$
$\tilde{I} = (\mathrm{d}h)^2 + (\mathrm{d}\psi)^2$
$\mathrm{d}\psi = \cos\theta\mathrm{d}\phi$
$\cos\theta \neq 0$

$$I = \cos^2\theta\left(\frac{1}{\cos^2\theta}(\mathrm{d}\theta)^2 + (\mathrm{d}\phi)^2\right)$$

$\mathrm{d}h = \frac{1}{\cos\theta}\mathrm{d}\theta$

$$h(\theta) = \int_0^\theta \frac{1}{\cos t}\mathrm{d}t = \log\tan(\frac{\theta}{2}+\frac{\pi}{4})$$

$$\sigma : \begin{cases} h = \log\tan(\frac{\theta}{2} + \frac{\pi}{4}) \\ \psi = \phi \end{cases}$$ Mercator

1

2 (u, v)

$$I = \frac{4}{(u^2 + v^2 + 1)^2}(\mathrm{d}u^2 + \mathrm{d}v^2)$$

$$I = \frac{1}{(u^2/4 + v^2/4 + 1)^2}(\mathrm{d}u^2 + \mathrm{d}v^2)$$

$\frac{1}{(\ ^2/4+\ ^2/4+1)^2}\mathrm{d}u\mathrm{d}v$ S

Mercator Mercator

$$\sigma : \begin{cases} h = \log\tan(\frac{\theta}{2} + \frac{\pi}{4}) \\ \psi = \phi \end{cases}$$

- $\lambda(u, v)$ 1
  Jacobi Id

- $\lambda(u, v)$ $c \neq 1$
  (u, v)

- $\lambda(u, v)$

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

$$\frac{2u}{a} + \frac{2v}{b} + \frac{u^2 + v^2 - 1}{c} = u^2 + v^2 + 1$$

$$\frac{1-c}{c}u^2 + \frac{2u}{a} + \frac{1-c}{c}v^2 + \frac{2v}{b} = 1 + \frac{1}{c}$$

9.2

9.2.1

Mercator

9.5

z

S (u, v) $\begin{pmatrix} E & F \\ F & G \end{pmatrix}$ $z = u + iv$

$iz = -v + iu$

[1] $z = u + iv$ i

[1] i S i

$$i_* \begin{pmatrix} \frac{\partial}{\partial} \\ \frac{\partial}{\partial} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial} \\ \frac{\partial}{\partial} \end{pmatrix}$$

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} G & -F \\ -F & E \end{pmatrix}$$

$E = G, F = 0$

$$\lambda^2 \left((\mathrm{d}u)^2 + (\mathrm{d}v)^2 \right)$$

(u, v) $z = u + iv$ i

定义 9.9. S (u, v) $z = u + iv$
(u, v) S

$$\lambda^2 \left((\mathrm{d}u)^2 + (\mathrm{d}v)^2 \right)$$

$(\mathrm{d}u)^2 + (\mathrm{d}v)^2$ 9.4
S S
S

例 9.10. (u, v)

- (u, v)

$$I = \frac{4}{(u^2 + v^2 + 1)^2} (\mathrm{d}u^2 + \mathrm{d}v^2)$$

-

S

Kähler 2- 2
2-

9.2.2

定理 9.11 (Korn-Lichtenstein). S (u, v)

$$\lambda^2(u, v)\left((\mathrm{d}u)^2 + (\mathrm{d}v)^2\right)$$

$\lambda(u, v)$

$$I = E\mathrm{d}x^2 + 2F\mathrm{d}x\mathrm{d}y + G\mathrm{d}y^2$$

$\begin{pmatrix} E & F \\ F & G \end{pmatrix}$ A

P $A = PP$ $\begin{pmatrix} E & F \\ F & G \end{pmatrix}$

T

$$\begin{aligned}\begin{pmatrix} E & F \\ F & G \end{pmatrix} =& T\mathrm{diag}(\lambda_1, \lambda_2)T \\ =& T\mathrm{diag}(\sqrt{\lambda_1}, \sqrt{\lambda_2})\mathrm{diag}(\sqrt{\lambda_1}, \sqrt{\lambda_2})T \\ =& T\mathrm{diag}(\sqrt{\lambda_1}, \sqrt{\lambda_2})(T\mathrm{diag}(\sqrt{\lambda_1}, \sqrt{\lambda_2}))\end{aligned}$$

λ_1, λ_2 $\lambda_1 = \lambda_2$ $\begin{pmatrix} E & F \\ F & G \end{pmatrix}$

$\lambda_1 \neq \lambda_2$

$$T\mathrm{diag}(\sqrt{\lambda_1}, \sqrt{\lambda_2}) = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}$$

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} = \begin{pmatrix} a_1^2 + a_2^2 & a_1b_1 + a_2b_2 \\ a_1b_1 + a_2b_2 & b_1^2 + b_2^2 \end{pmatrix} \tag{9.1}$$

$$\theta_1 = a_1\mathrm{d}x + b_1\mathrm{d}y, \theta_2 = a_2\mathrm{d}x + b_2\mathrm{d}y$$

$$I = \theta_1^2 + \theta_2^2$$

$$\theta_1^2 + \theta_2^2 = (a_1^2 + a_2^2)\mathrm{d}x^2 + 2(a_1b_1 + a_2b_2)\mathrm{d}x\mathrm{d}y + (b_1^2 + b_2^2)\mathrm{d}y^2$$

$$I = E\mathrm{d}x^2 + 2F\mathrm{d}x\mathrm{d}y + G\mathrm{d}y^2 \tag{9.1}$$

$$\phi = \theta_1 + i\theta_2$$

$$I = \phi\bar{\phi} = (\theta_1 + i\theta_2)(\theta_1 - i\theta_2) = \theta_1^2 + \theta_2^2$$

c 1

$$I = c\phi(\bar{c\phi}) = \phi\bar{\phi}$$

ϕ 1

λ z

$$\mathrm{d}z = \lambda\phi$$

$z = u + iv$

$$I = \frac{1}{|\lambda|^2}\mathrm{d}z\mathrm{d}\bar{z} = \frac{1}{|\lambda|^2}((\mathrm{d}u)^2 + (\mathrm{d}v)^2)$$

(u, v)

$\bar{\partial}$

F

[22] 11

Gauss E, F, G u, v

F

推论 9.12. S

例 9.13.

$$\vec{r}(u,v) = (f(u)\cos v, f(u)\sin v, g(u))$$

.

$$I = f^2(u)(\mathrm{d}v)^2 + (f'^2(u) + g'^2(u))(\mathrm{d}u)^2$$

$u = u(x,y), v = v(x,y)$

$$\mathrm{d}u = \frac{\partial u}{\partial x}\mathrm{d}x + \frac{\partial u}{\partial y}\mathrm{d}y, \mathrm{d}v = \frac{\partial v}{\partial x}\mathrm{d}x + \frac{\partial v}{\partial y}\mathrm{d}y$$

u

v

$$\begin{aligned} I =& f^2(u)(\mathrm{d}v)^2 + (f'^2(u) + g'^2(u))(\mathrm{d}u)^2 \\ =& f^2(u)\left((\mathrm{d}v)^2 + \frac{f'^2(u) + g'^2(u)}{f^2(u)}(\mathrm{d}u)^2\right) \\ =& f^2(u)\left((\mathrm{d}v)^2 + (\frac{\sqrt{f'^2(u) + g'^2(u)}}{f(u)}\mathrm{d}u)^2\right) \end{aligned}$$

$$y = \int \frac{\sqrt{f'^2(u) + g'^2(u)}}{f(u)}\mathrm{d}u$$

□

Mercator 9.8

θ, ϕ u, v ψ, h

x, y $f(\theta) = \cos\theta$ $g(\theta) = \sin\theta$ $h = \int \frac{1}{\cos\theta}\mathrm{d}\theta$

Mercator

例 9.14.

9.3

9.3.1

9.5

$|f'(z)|$ 9.5

定理 9.15. (u_1, v_1) S_1 (u_2, v_2) S_2 σ

S_1 S_2 $\begin{cases} u_2(u_1, v_1) \\ v_2(u_1, v_1) \end{cases}$ $z = u_1 + iv_1$

$w = f(z)$ $w = u_2 + iv_2$

1. σ
2. $f(z)$
3. (u_2, v_2)

. (u_1, v_1) S_1

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix}$$

1 2 3 S_2 (u_2, v_2)

$$\begin{pmatrix} E_2 & F_2 \\ F_2 & G_2 \end{pmatrix}$$

σ

$$\lambda^2 \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix} = J \begin{pmatrix} E_2 & F_2 \\ F_2 & G_2 \end{pmatrix} J$$

$$\begin{pmatrix} E_2 & F_2 \\ F_2 & G_2 \end{pmatrix} = J^{-1}\lambda^2 \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix} (J\ \ )^{-1} = \lambda^2 \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix} (J\ \ J)^{-1}$$

$$J\ \ J = \begin{pmatrix} \frac{\partial\ _2}{\partial\ _1} & \frac{\partial\ _2}{\partial\ _1} \\ \frac{\partial\ _2}{\partial\ _1} & \frac{\partial\ _2}{\partial\ _1} \end{pmatrix} \begin{pmatrix} \frac{\partial\ _2}{\partial\ _1} & \frac{\partial\ _2}{\partial\ _1} \\ \frac{\partial\ _2}{\partial\ _1} & \frac{\partial\ _2}{\partial\ _1} \end{pmatrix} = \begin{pmatrix} (\frac{\partial\ _2}{\partial\ _1})^2 + (\frac{\partial\ _2}{\partial\ _1})^2 & \frac{\partial\ _2}{\partial\ _1}\frac{\partial\ _2}{\partial\ _1} + \frac{\partial\ _2}{\partial\ _1}\frac{\partial\ _2}{\partial\ _1} \\ \frac{\partial\ _2}{\partial\ _1}\frac{\partial\ _2}{\partial\ _1} + \frac{\partial\ _2}{\partial\ _1}\frac{\partial\ _2}{\partial\ _1} & (\frac{\partial\ _2}{\partial\ _1})^2 + (\frac{\partial\ _2}{\partial\ _1})^2 \end{pmatrix}$$

$f(z)$ Cauch-Riemann

$$\frac{\partial u_2}{\partial u_1} = \frac{\partial v_2}{\partial v_1}, \quad \frac{\partial v_2}{\partial u_1} = -\frac{\partial u_2}{\partial v_1}$$

$$J\ \ J = \left((\frac{\partial u_2}{\partial u_1})^2 + (\frac{\partial v_2}{\partial u_1})^2 \right) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = |f'(z)|^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

1 3 2 (u_2, v_2) (u_2, v_2)

$$\begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

$$\lambda^2 \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix} = J \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_2 \end{pmatrix} J$$

$$\frac{\lambda_2}{\lambda^2 \lambda_1} JJ \ \ = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$(\frac{1}{\lambda}\sqrt{\frac{\lambda_2}{\lambda_1}}J)(\frac{1}{\lambda}\sqrt{\frac{\lambda_2}{\lambda_1}}J) \ \ = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\frac{1}{\lambda}\sqrt{\frac{\lambda_2}{\lambda_1}}J$ $J = \begin{pmatrix} \frac{\partial\ _2}{\partial\ _1} & \frac{\partial\ _2}{\partial\ _1} \\ \frac{\partial\ _2}{\partial\ _1} & \frac{\partial\ _2}{\partial\ _1} \end{pmatrix} = c \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

Cauchy-Riemann

2 3 1

$$\lambda^2 \begin{pmatrix} E_1(u_1, v_1) & F_1(u_1, v_1) \\ F_1(u_1, v_1) & G_1(u_1, v_1) \end{pmatrix} = J \begin{pmatrix} E_2(u_2, v_2) & F_2(u_2, v_2) \\ F_2(u_2, v_2) & G_2(u_2, v_2) \end{pmatrix} J$$

S_2 (u_2, v_2) $\begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

$$J \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_2 \end{pmatrix} J = \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_2 \end{pmatrix} JJ = \lambda_2 |f'(z)|^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Cauchy-Riemann

(u_1, v_1) S_1 $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix}$

λ^2 $\frac{\lambda_2}{\lambda_1}|f'(z)|^2$ □

(u_1, v_1) (u_2, v_2) S_1 S_2 $z = u_1 + iv_1, w = u_2 + iv_2$ $w = f(z)$ σ

$w = f(z)$ $\Leftrightarrow$ $\frac{}{\sqrt{\det}}$ $\Leftrightarrow \sigma$

$\sqrt{\det J} = |f'(z)|$

9.3.2

(u, v) S (u, v) $z = u + iv$

$$\mathrm{d}z = \mathrm{d}u + i\mathrm{d}v, \quad \mathrm{d}\bar{z} = \mathrm{d}u - i\mathrm{d}v$$

$$\vec{r} = \frac{1}{2}(\vec{r} - i\vec{r}); \quad \vec{r}\text{-} = \frac{1}{2}(\vec{r} + i\vec{r})$$

$\vec{r}$ $\vec{r}\text{-}$ $\mathrm{d}z, \mathrm{d}\bar{z}$

$$\mathrm{d}z(\vec{r}) = 1, \mathrm{d}z(\vec{r}\text{-}) = \mathrm{d}\bar{z}(\vec{r}) = 0, \mathrm{d}\bar{z}(\vec{r}\text{-}) = 1$$

$\{\vec{r}, \vec{r}\text{-}, \vec{n}\}$ ^{1}

$\frac{\partial}{\partial} = \frac{1}{2}(\frac{\partial}{\partial} - i\frac{\partial}{\partial})$ $\frac{\partial}{\partial^-} = \frac{1}{2}(\frac{\partial}{\partial} + i\frac{\partial}{\partial})$ $\vec{r}$ $\vec{r}\text{-}$ S

^{1} $\{\vec{r}_z, \vec{r}_{\bar{z}}\}$ $\vec{n}$

f

$$f_z = \frac{\partial f}{\partial z} = \frac{1}{2}(\frac{\partial f}{\partial u} - i\frac{\partial f}{\partial v}), \quad f_{\bar{z}} = \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial f}{\partial u} + i\frac{\partial f}{\partial v})$$

$$\overline{f_z} = \bar{f}_{\bar{z}}$$

$$f_{z\bar{z}} = \frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}f, \quad f_{\bar{z}z} = \frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z}f$$

$$f_{z\bar{z}} = f_{\bar{z}z} = \frac{1}{4}(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2})f = \frac{1}{4}(f_{uu} + f_{vv})$$

$\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}$ Laplace

Gauss Laplace

$\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}$

Laplace

$$\begin{aligned}
\langle \vec{r}_z, \vec{r}_z \rangle &= \frac{1}{4}(\langle \vec{r}_u, \vec{r}_u \rangle - \langle \vec{r}_v, \vec{r}_v \rangle - 2i\langle \vec{r}_u, \vec{r}_v \rangle) \\
\langle \vec{r}_{\bar{z}}, \vec{r}_{\bar{z}} \rangle &= \frac{1}{4}(\langle \vec{r}_u, \vec{r}_u \rangle - \langle \vec{r}_v, \vec{r}_v \rangle + 2i\langle \vec{r}_u, \vec{r}_v \rangle) \\
\langle \vec{r}_z, \vec{r}_{\bar{z}} \rangle &= \frac{1}{4}(\langle \vec{r}_u, \vec{r}_u \rangle + \langle \vec{r}_v, \vec{r}_v \rangle)
\end{aligned} \tag{9.2}$$

[1]

$$\overline{\langle \vec{r}_z, \vec{r}_z \rangle} = \langle \overline{\vec{r}_{\bar{z}}}, \overline{\vec{r}_{\bar{z}}} \rangle = \langle \vec{r}_{\bar{z}}, \vec{r}_{\bar{z}} \rangle$$

$\langle \vec{r}_z, \vec{r}_z \rangle$ $\langle \vec{r}_{\bar{z}}, \vec{r}_{\bar{z}} \rangle$ $\langle \vec{r}_z, \vec{n} \rangle, \langle \vec{r}_{\bar{z}}, \vec{n} \rangle$

(u, v) S

$z = u + iv$ (u, v)

S $\lambda^2(\mathrm{d}u^2 + \mathrm{d}v^2)$

$$I = \lambda^2 \mathrm{d}z\mathrm{d}\bar{z}$$

[1]

(9.2)

$$\langle \vec{r}\ , \vec{r}\ \rangle = 0 \tag{9.3}$$
$$\langle \vec{r}\ , \vec{r}\text{-}\rangle = \frac{\lambda^2}{2}$$

(9.3) (u, v)

$$\langle \vec{r}\ , \vec{r}\ \rangle = 0, \quad \langle \vec{r}\ , \vec{r}\text{-}\rangle = \frac{\lambda^2}{2}, \quad \langle \vec{r}\ , \vec{n}\rangle = 0, \quad \langle \vec{n}, \vec{n}\rangle = 1 \tag{9.4}$$

$\vec{r}\ \ , \vec{r}\ \text{-}, \vec{n}$ (9.4)

$$\begin{aligned} \langle \vec{r}\ \ , \vec{r}\ \rangle = \langle \vec{r}\ \text{-}, \vec{r}\ \rangle &= 0 \\ \langle \vec{r}\ \ , \vec{r}\text{-}\rangle &= \lambda\lambda \\ \langle \vec{r}\ \ , \vec{n}\rangle &= -\langle \vec{r}\ , \vec{n}\ \rangle \\ \langle \vec{r}\ \ \text{-}, \vec{n}\rangle &= -\langle \vec{r}\text{-}, \vec{n}\ \rangle \\ \langle \vec{n}\ , \vec{n}\rangle &= 0 \end{aligned} \tag{9.5}$$

$\langle \vec{r}\ \ , \vec{n}\rangle = -\langle \vec{n}\ , \vec{r}\ \rangle$

$L = -\langle \vec{r}\ , \vec{n}\ \rangle, \quad M = -\langle \vec{r}\ , \vec{n}\ \rangle = -\langle \vec{r}\ , \vec{n}\ \rangle, \quad N = -\langle \vec{r}\ , \vec{n}\ \rangle$

$$\begin{aligned} -\langle \vec{n}\ , \vec{r}\ \rangle &= -\frac{1}{4}\langle \vec{r}\ - i\vec{r}\ , \vec{n}\ - i\vec{n}\ \rangle \\ &= -\frac{1}{4}\left(\langle \vec{r}\ , \vec{n}\ \rangle - \langle \vec{r}\ , \vec{n}\ \rangle - 2i\langle \vec{r}\ , \vec{n}\ \rangle\right) \\ &= \frac{L - N - 2iM}{4} \end{aligned}$$

$Q = \frac{- \ -2}{4}$

[1]

10.2

$$\langle \vec{r}\ \text{-}, \vec{n}\rangle = \langle \frac{1}{4}(\vec{r}\ \ + \vec{r}\ \ ), \vec{n}\rangle = \frac{1}{4}(L + N) = \frac{\lambda^2}{2}\left(\frac{1}{2}(\frac{L}{\lambda^2} + \frac{N}{\lambda^2})\right) = \frac{\lambda^2}{2}H$$

[1] Q Hopf

H (9.5)

$$\begin{aligned}
&\langle \vec{r}\ _{,}\vec{r}\ \rangle = 0, \qquad \langle \vec{r}\ _{,}\vec{r}_{\text{-}}\rangle = \lambda\lambda\ , \quad \langle \vec{n},\vec{r}\ \ \rangle = Q;\\
&\langle \vec{r}_{\text{-}},\vec{r}\ \rangle = 0, \qquad \langle \vec{r}_{\text{-}},\vec{r}_{\text{-}}\rangle = 0, \qquad \langle \vec{r}_{\text{-}},\vec{n}\rangle = \frac{\lambda^2}{2}H;\\
&\langle \vec{n}\ _{,}\vec{r}\ \rangle = -Q, \quad \langle \vec{n}\ _{,}\vec{r}_{\text{-}}\rangle = \frac{\lambda^2}{2}H, \quad \langle \vec{n}\ _{,}\vec{n}\rangle = 0.
\end{aligned} \tag{9.6}$$

(9.4)

$$\begin{cases}
\vec{r}\ \ = \dfrac{2}{\lambda}\lambda\ \vec{r}\ + Q\vec{n}\\
\vec{r}_{\text{-}} = \dfrac{\lambda^2}{2}H\vec{n}\\
\vec{n}\ \ = -H\vec{r}\ - \dfrac{2}{\lambda^2}Q\vec{r}_{\text{-}}
\end{cases} \tag{9.7}$$

$\vec{r}\ _{\text{-}} = \vec{r}_{\text{-}}$, (9.7) Gauss-Codazzi

$$(\ln\lambda)_{\text{-}} = -\frac{\lambda^2}{4}K \tag{9.8}$$

$$Q_{\text{-}} = \frac{\lambda^2}{2}H \tag{9.9}$$

Gauss-Codazzi

(9.7) Gauss-Codazzi (9.8), (9.9)

$\vec{r}_{\text{-}} = \frac{1}{4}(\vec{r}\ + \vec{r}\ )$ (9.7)

$$\frac{1}{\lambda^2}(\vec{r}\ + \vec{r}\ ) = 2H\vec{n}$$

Laplace

9.4 Laplace

Laplace

Laplace

Laplace

Laplace

9.4.1 Hessian Laplace

Laplace Δ $\mathbb{R}^3$

f $\Delta f = \frac{\partial^2}{\partial\ ^2} + \frac{\partial^2}{\partial\ ^2} + \frac{\partial^2}{\partial\ ^2}$ $\mathbb{R}^2$ f $\Delta f = \frac{\partial^2}{\partial\ ^2} + \frac{\partial^2}{\partial\ ^2}$

$\Delta f = \mathbf{div}(\nabla f)$ Laplace

S (u, v) $\vec{r}(u, v)$ f

f ∇ f f

$\bar{\nabla} f$ S

f

$$f\ \vec{r}\ + f\ \vec{r}$$

$$\mathbf{d}f = f\ \mathbf{d}u + f\ \mathbf{d}v \qquad (u, v)$$

$\{\vec{e}_1, \vec{e}_2\}$

$$\nabla f = \frac{\partial f}{\partial \vec{e}_1}\vec{e}_1 + \frac{\partial f}{\partial \vec{e}_2}\vec{e}_2$$

$\frac{\partial}{\partial\ _1}$ $\frac{\partial}{\partial\ _2}$ f $\vec{e}_1$ $\vec{e}_2$

f_1, f_2

$$\nabla f = f_1\vec{e}_1 + f_2\vec{e}_2$$

$$\{\vec{e}_1, \vec{e}_2\}$$

∇f S ^{1}

$$D_{\vec{1}}(\nabla f) = \frac{\partial f_1}{\partial \vec{e}_1}\vec{e}_1 + f_1 D_{\vec{1}}\vec{e}_1 + \frac{\partial f_2}{\partial \vec{e}_1}\vec{e}_2 + f_2 D_{\vec{1}}\vec{e}_2$$
$$D_{\vec{2}}(\nabla f) = \frac{\partial f_1}{\partial \vec{e}_2}\vec{e}_1 + f_1 D_{\vec{2}}\vec{e}_1 + \frac{\partial f_2}{\partial \vec{e}_2}\vec{e}_2 + f_2 D_{\vec{2}}\vec{e}_2$$

$$D_{\vec{1}}(\nabla f) = f_{11}\vec{e}_1 + f_{21}\vec{e}_2$$
$$D_{\vec{2}}(\nabla f) = f_{12}\vec{e}_1 + f_{22}\vec{e}_2$$

$\{\vec{e}_1, \vec{e}_2\}$ $D_{\vec{1}}\vec{e}_1, D_{\vec{2}}\vec{e}_1$ $\vec{e}_2$ $D_{\vec{1}}\vec{e}_2, D_{\vec{2}}\vec{e}_2$ $\vec{e}_1$

$$f_{11} = \frac{\partial f_1}{\partial \vec{e}_1} + f_2\langle D_{\vec{1}}\vec{e}_2, \vec{e}_1\rangle; \quad f_{12} = \frac{\partial f_1}{\partial \vec{e}_2} + f_2\langle D_{\vec{2}}\vec{e}_2, \vec{e}_1\rangle$$
$$f_{21} = \frac{\partial f_2}{\partial \vec{e}_1} + f_1\langle D_{\vec{1}}\vec{e}_1, \vec{e}_2\rangle; \quad f_{22} = \frac{\partial f_2}{}$$

D

Hessian

f Hessian f Laplace

$$\Delta f = \mathrm{Tr} Hess(f) = f_{11} + f_{22}$$

Hessian

Hessian Laplace Laplace

Laplace

Laplace Laplace

Laplace

$\Delta f = 0$

例 9.16. $\mathbb{R}^2$ f Hessian Laplace

$\frac{\partial}{\partial}, \frac{\partial}{\partial}$

$x, y-$ $\frac{\partial}{\partial}, \frac{\partial}{\partial}$

$$\nabla f = f \ \frac{\partial}{\partial x} + f \ \frac{\partial}{\partial y}$$

D

$$D\_(\nabla f) = f \ \ \frac{\partial}{\partial x} + f \ \ \frac{\partial}{\partial y}$$
$$D\_(\nabla f) = f \ \ \frac{\partial}{\partial x} + f \ \ \frac{\partial}{\partial y}$$

$$Hess(f) = \begin{pmatrix} f & f \\ f & f \end{pmatrix}$$

Hessian

Monge-Ampère Hessian

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

Laplace

f Hessian f

9.4.2 Laplace

例 9.17. S $I=\lambda^2((\mathrm{d}u)^2+(\mathrm{d}v)^2)$ Laplace

$\frac{\partial}{\partial},\frac{\partial}{\partial}$ $u,v-$ $\vec{r}\ ,\vec{r}$

$\frac{\partial}{\partial},\frac{\partial}{\partial}$ $\vec{e}_1=\frac{1}{\lambda}\frac{\partial}{\partial}\quad\vec{e}_2=\frac{1}{\lambda}\frac{\partial}{\partial}$

$$\nabla f=\frac{1}{\lambda^2}\left(f\ \frac{\partial}{\partial u}+f\ \frac{\partial}{\partial v}\right)$$

$$\begin{aligned}D_{\ 1}\nabla f=&\frac{1}{\lambda}D_-\left(\frac{f}{\lambda^2}\frac{\partial}{\partial u}+\frac{f}{\lambda^2}\frac{\partial}{\partial v}\right)\\=&\frac{1}{\lambda}\left((\frac{\partial}{\partial u}\frac{f}{\lambda^2})\frac{\partial}{\partial u}+\frac{f}{\lambda^2}D_-\frac{\partial}{\partial u}+(\frac{\partial}{\partial u}\frac{f}{\lambda^2})\frac{\partial}{\partial v}+\frac{f}{\lambda^2}D_-\frac{\partial}{\partial u}\right)\\=&\frac{1}{\lambda}\left((\frac{f}{\lambda^2}-\frac{2f\ \lambda}{\lambda^3})\frac{\partial}{\partial u}+\frac{f}{\lambda^2}D_-\frac{\partial}{\partial u}+(\frac{f}{\lambda^2}-\frac{2f\ \lambda}{\lambda^3})\frac{\partial}{\partial v}+\frac{f}{\lambda^2}D_-\frac{\partial}{\partial u}\right)\end{aligned}$$

$$\begin{aligned}D_-\frac{\partial}{\partial u}=&\Gamma^1_{11}\frac{\partial}{\partial u}+\Gamma^2_{11}\frac{\partial}{\partial v}\\D_-\frac{\partial}{\partial u}=&\Gamma^1_{21}\frac{\partial}{\partial u}+\Gamma^2_{21}\frac{\partial}{\partial v}\end{aligned}$$

Christoffel

$$\Gamma^1_{11}=\frac{\partial}{\partial u}\log\lambda,\quad\Gamma^2_{11}=-\frac{\partial}{\partial v}\log\lambda,\quad\Gamma^1_{21}=\frac{\partial}{\partial v}\log\lambda,\quad\Gamma^2_{21}=\frac{\partial}{\partial u}\log\lambda$$

$$\begin{aligned}D_{\ 1}\nabla f=&\frac{1}{\lambda}\left((\frac{f}{\lambda^2}-\frac{2f\ \lambda}{\lambda^3})\frac{\partial}{\partial u}+\frac{f}{\lambda^2}D_-\frac{\partial}{\partial u}+(\frac{f}{\lambda^2}-\frac{2f\ \lambda}{\lambda^3})\frac{\partial}{\partial v}+\frac{f}{\lambda^2}D_-\frac{\partial}{\partial u}\right)\\=&\frac{1}{\lambda}\left((\frac{f}{\lambda^2}-\frac{f\ \lambda}{\lambda^3}+\frac{f\ \lambda}{\lambda^3})\frac{\partial}{\partial u}+(\frac{f}{\lambda^2}-\frac{f\ \lambda}{\lambda^3}-\frac{f\ \lambda}{\lambda^3})\frac{\partial}{\partial v}\right)\\=&(\frac{f}{\lambda^2}-\frac{f\ \lambda}{\lambda^3}+\frac{f\ \lambda}{\lambda^3})\vec{e}_1+(\frac{f}{\lambda^2}-\frac{f\ \lambda}{\lambda^3}-\frac{f\ \lambda}{\lambda^3})\vec{e}_2\end{aligned}$$

$$D_{\ 2}\nabla f=(\frac{f}{\lambda^2}-\frac{f\ \lambda}{\lambda^3}-\frac{f\ \lambda}{\lambda^3})\vec{e}_1+(\frac{f}{\lambda^2}+\frac{f\ \lambda}{\lambda^3}-\frac{f\ \lambda}{\lambda^3})\vec{e}_2$$

$f_{11} = \frac{}{\lambda^2} - \frac{\lambda}{\lambda^3} + \frac{\lambda}{\lambda^3}, \quad f_{22} = \frac{}{\lambda^2} + \frac{\lambda}{\lambda^3} - \frac{\lambda}{\lambda^3}$

$$\Delta\ f = \frac{1}{\lambda^2}(f \quad + f \ )$$

9.3.2 (u, v) $z = u+iv$

$$f\text{-} = f\text{-} \ = \frac{1}{4}(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2})f = \frac{1}{4}(f \quad + f \ )$$

$$\Delta\ f = 4\lambda^2 f\text{-} = 4\lambda^2 f\text{-}$$

Laplace

$$\Delta \ = \frac{4}{\lambda^2}\frac{\partial^2}{\partial\bar{z}\partial z} = \frac{4}{\lambda^2}\frac{\partial^2}{\partial z\partial\bar{z}} \tag{9.11}$$

1

命題 9.18. f S (u, v) $z = u + iv$
$\frac{\partial}{\partial} = \frac{1}{2}(\frac{\partial}{\partial} - i\frac{\partial}{\partial})$ z

命題 9.19. (u, v) S $z = u + iv$ ψ z
Reψ Imψ $\ln|\psi|^2$ S

. Reψ Imψ Cauchy-Riemann

$\ln|\psi|^2$

$$\Delta \ \ln|\psi|^2 = \frac{4}{\lambda^2}\frac{\partial^2}{\partial z\partial\bar{z}}\ln(\psi\bar{\psi}) = \frac{4}{\lambda^2}\frac{\partial^2}{\partial z\partial\bar{z}}(\ln\psi + \ln\bar{\psi}) = \frac{4}{\lambda^2}\left(\frac{\partial}{\partial z}(\frac{\partial}{\partial\bar{z}}\ln\psi) + \frac{\partial}{\partial\bar{z}}(\frac{\partial}{\partial z}\overline{\ln\psi})\right)$$

ψ $\ln\psi$ $\frac{\partial}{\partial\text{-}}\ln\psi = 0$ $\frac{\partial}{\partial}\overline{\ln\psi} = \overline{\frac{\partial}{\partial\text{-}}\ln\psi}$
$\Delta \ \ln|\psi|^2 = 0$ □

1

9.4.3 Hessian Laplace

Hessian Laplace Hessian

f S X, Y S

$D^2 f$ f

$$D^2 f(X,Y) = \frac{\partial^2 f}{\partial Y \partial X} - \frac{\partial f}{\partial (D \quad X)}$$

a^{j1}

. (9.13) (9.12)

(9.13)

$$\frac{1}{\sqrt{g}}\sum_{=1}^{2}\frac{\partial}{\partial u}\left(\sum_{=1}^{2}g\ \ \sqrt{g}\frac{\partial f}{\partial u}\right)=\sum_{,=1}^{2}\left(g\ \ \frac{\partial^2 f}{\partial u\ \partial u}+(\frac{\partial g}{\partial u}+\frac{g}{2}\frac{\partial \ln g}{}\right.$$

H $\vec{n}$ $\vec{r}$ $\Delta\ \vec{r}$
$(\Delta\ x, \Delta\ y, \Delta\ z)$

. (9.10) f_{11}, f_{22} $\langle \vec{e}_1, \vec{e}_2 \rangle \equiv 0,$

$$\begin{aligned} f_{11} &= \frac{\partial f_1}{\partial \vec{e}_1} - f_2 \langle D_{\ 1}\vec{e}_1, \vec{e}_2 \rangle \\ f_{22} &= \frac{\partial f_2}{\partial \vec{e}_2} - f_1 \langle D_{\ 2}\vec{e}_2, \vec{e}_1 \rangle \end{aligned} \tag{9.16}$$

$\vec{r}$ 3.2.6 $\frac{\partial \boldsymbol{r}}{\partial} = \vec{v}$
$\vec{v}$ $\frac{\partial \boldsymbol{r}}{\partial\ _1} = \vec{e}_1, \frac{\partial \boldsymbol{r}}{\partial\ _2} = \vec{e}_2$ $\vec{r}$ (9.16) f $\vec{r}$
x, y, z (9.16)

$$\begin{aligned} r_{11} &= \frac{\partial \vec{r}_1}{\partial \vec{e}_1} - \langle D_{\ 1}\vec{e}_1, \vec{e}_2 \rangle \vec{r}_2 = \frac{\partial \vec{e}_1}{\partial \vec{e}_1} - \langle D_{\ 1}\vec{e}_1, \vec{e}_2 \rangle \vec{e}_2 \\ r_{22} &= \frac{\partial \vec{r}_2}{\partial \vec{e}_2} - \langle D_{\ 2}\vec{e}_2, \vec{e}_1 \rangle \vec{r}_1 = \frac{\partial \vec{e}_2}{\partial \vec{e}_2} - \langle D_{\ 2}\vec{e}_2, \vec{e}_1 \rangle \vec{e}_1 \end{aligned}$$

$D_{\ 1}\vec{e}_1$ $\vec{e}_2$ $D_{\ 2}\vec{e}_2$ $\vec{e}_1$

$$D_{\ 1}\vec{e}_1 = \langle D_{\ 1}\vec{e}_1, \vec{e}_2 \rangle \vec{e}_2, \quad D_{\ 2}\vec{e}_2 = \langle D_{\ 2}\vec{e}_2, \vec{e}_1 \rangle \vec{e}_1$$

$\vec{r}_{11}, \vec{r}_{22}$

$$\begin{aligned} r_{11} &= \frac{\partial \vec{e}_1}{\partial \vec{e}_1} - D_{\ 1}\vec{e}_1 \\ r_{22} &= \frac{\partial \vec{e}_2}{\partial \vec{e}_2} - D_{\ 2}\vec{e}_2 \end{aligned}$$

$$\vec{r}_{11} = \langle \frac{\partial \vec{e}_1}{\partial \vec{e}_1}, \vec{n} \rangle \vec{n}, \quad \vec{r}_{22} = \langle \frac{\partial \vec{e}_2}{\partial \vec{e}_2}, \vec{n} \rangle \vec{n}$$

$$\langle \frac{\partial \vec{e}_1}{\partial \vec{e}_1}, \vec{n} \rangle = -\langle \frac{\partial \vec{n}}{\partial \vec{e}_1}, \vec{e}_1 \rangle = b_{11} = L, \quad \langle \frac{\partial \vec{e}_2}{\partial \vec{e}_2}, \vec{n} \rangle = -\langle \frac{\partial \vec{n}}{\partial \vec{e}_2}, \vec{e}_2 \rangle = b_{22} = N$$

$$2H = b_{11} + b_{22} = L + N \qquad \Delta\ \vec{r} = \vec{r}_{11} + \vec{r}_{22} = 2H\vec{n} \qquad \square$$

$$\Delta = \frac{4}{\lambda^2}\frac{\partial^2}{\partial^- \partial} = \frac{4}{\lambda^2}\frac{\partial^2}{\partial\ \partial^-} \qquad (9.15)$$

9.3.2 (9.7) $\vec{r}\ \text{-} = \frac{\lambda^2}{2} H\vec{n}$

Laplace

Laplace

Laplace

Laplace Cauchy-Riemann

Sobolev

Kähler

Laplace

Cauchy-Riemann

[22]

Newlander-Nirenberg

Newlander-Nirenberg

K

Laplace Laplace

Laplace Hessian

Laplace [23]

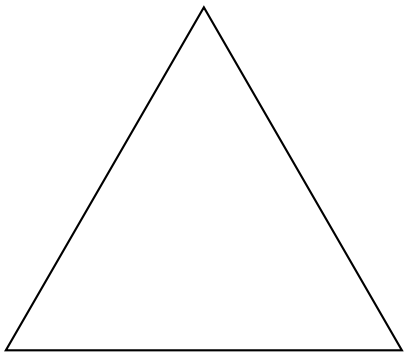
Laplace

Laplace

Laplace

, ,

Gauss



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Laplace

Laplace

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- Laplace

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Monge-Ampère

Monge-Ampère

$$f \quad (1+f^2) - 2f \quad f \; f \; + f \quad (1+f^2) = 0$$

Laplace

Chapter 10

Laplace

Laplace

10.1

Σ Monge $\Sigma : \vec{r}(x,y) = (x.y, f(x,y))$

$$2H = \frac{1}{(\sqrt{1+f^2+f^2})^3}\left(f\ \ (1+f^2) - 2f\ \ f\ f\ + f\ \ (1+f^2)\right)$$

Σ

$$f\ \ (1+f^2) - 2f\ \ f\ f\ + f\ \ (1+f^2) = 0 \tag{10.1}$$

Monge

Laplace

10.1.1 Weierstrass

Σ E^3 Σ (u,v)

$$I=\lambda^2(\mathrm{d}u^2+\mathrm{d}v^2)$$

$z=u+iv$ $\vec{r}=(x_1,x_2,x_3)$ Σ 9.3.2
$\{\frac{\partial \boldsymbol{r}}{\partial},\frac{\partial \boldsymbol{r}}{\partial^-},\vec{n}\}$ (u,v)

$$\langle\frac{\partial\vec{r}}{\partial z},\frac{\partial\vec{r}}{\partial z}\rangle=0 \tag{10.2}$$

$\langle\ ,\ \rangle$ ( ) [1]

$$2\frac{\partial\vec{r}}{\partial z}=\frac{\partial\vec{r}}{\partial u}-i\frac{\partial\vec{r}}{\partial v}=\left(\frac{\partial x_1}{\partial u}-i\frac{\partial x_1}{\partial v},\frac{\partial x_2}{\partial u}-i\frac{\partial x_2}{\partial v},\frac{\partial x_3}{\partial u}-i\frac{\partial x_3}{\partial v}\right)\triangleq(\varphi_1,\varphi_2,\varphi_3)$$

(10.2)

$$\varphi_1^2+\varphi_2^2+\varphi_3^2=0$$

$\frac{\partial \boldsymbol{r}}{\partial}$ [2]

$$(\varphi_1+i\varphi_2)(\varphi_1-i\varphi_2)=-\varphi_3^2 \tag{10.3}$$

Σ 9.21 Σ $\vec{r}=(x,y,f(x,y))$

$$\Delta_\Sigma\vec{r}=(\Delta_\Sigma x,\Delta_\Sigma y,\Delta_\Sigma f(x,y))=\vec{0} \tag{10.4}$$

(10.4) x_1,x_2,x_3 Σ Laplace $\Delta_\Sigma=\frac{1}{\lambda^2}(\frac{\partial^2}{\partial\ ^2}+\frac{\partial^2}{\partial\ ^2})$
9.18 $\varphi\ =\frac{\partial}{\partial}-i\frac{\partial}{\partial}$ $k=1,2,3$ $z=u+iv$

$$\Phi=\varphi_1-i\varphi_2,\quad \Psi=\frac{\varphi_3}{\varphi_1-i\varphi_2}$$

---

[1] $\frac{\partial\vec{r}}{\partial z}$ 2

[2] $\varphi\ =i\sqrt{\varphi_1^2+\varphi_2^2}$

Φ Ψ (10.3)

$$\varphi_1 + i\varphi_2 = -\Phi\Psi$$

$$\begin{aligned}\varphi_1 =& \frac{1}{2}\Phi(1-\Psi^2)\\ \varphi_2 =& \frac{i}{2}\Phi(1+\Psi^2)\\ \varphi_3 =& \Phi\Psi\end{aligned} \tag{10.5}$$

$\varphi_1, \varphi_2, \varphi_3$ $\varphi = \frac{\partial}{\partial} -$
$i\frac{\partial}{\partial}$

$$\begin{aligned}x_1 =& \mathrm{Re}\frac{1}{2}\int \Phi(1-\Psi^2)\mathrm{d}z\\ x_2 =& \mathrm{Re}\frac{i}{2}\int \Phi(1+\Psi^2)\mathrm{d}z\\ x_3 =& \mathrm{Re}\int \Phi\Psi\mathrm{d}z\end{aligned}$$

Weierstrass Weierstrass

$\varphi_1, \varphi_2, \varphi_3$

1. Ψ

2. Weierstrass

Weierstrass Φ Ψ
$\varphi_1^2 + \varphi_2^2 + \varphi_3^2 = 0$ $(\varphi_1, \varphi_2, \varphi_3)$
x_1, x_2, x_3 $\mathbb{R}^3$

Weierstrass

例 10.1. $\Phi = 1$ $\Psi = 0$ $\mathbb{C}$

例 10.2. $\Phi = 1$ $\Psi = \frac{1}{}$ $\mathbb{C}\backslash\{0\}$ $\Phi = -ie$ $\Psi = e^{-}$ $\mathbb{C}$

例 10.3. $\Phi = i$ $\Psi = \frac{1}{}$ $\mathbb{C}\backslash\{0\}$ $\Phi = e$ $\Psi = e^{-}$ $\mathbb{C}$

Weierstrass Weierstrass

[18] §10.2 [24] §10.2.4

$$\varphi_1^2+\varphi_2^2+\varphi_3^2=0$$

φ Weierstrass Weierstrass

$$\frac{\partial\vec{r}}{\partial z}=\frac{1}{2}(\frac{\partial\vec{r}}{\partial u}-i\frac{\partial\vec{r}}{\partial v}),\quad \frac{\partial\vec{r}}{\partial\bar{z}}=\frac{1}{2}(\frac{\partial\vec{r}}{\partial u}+i\frac{\partial\vec{r}}{\partial v})$$

$$\frac{\partial\vec{r}}{\partial z}\times\frac{\partial\vec{r}}{\partial\bar{z}}=\frac{1}{4}(\frac{\partial\vec{r}}{\partial u}-i\frac{\partial\vec{r}}{\partial v})\times(\frac{\partial\vec{r}}{\partial u}+i\frac{\partial\vec{r}}{\partial v})=\frac{i}{2}\vec{r}\ \times\vec{r}\ =\frac{i}{2}|\vec{r}\ \times\vec{r}\ |\vec{n}=\frac{i}{2}\lambda^2\vec{n}$$

$\vec{n}$ Σ (9.4)

$$\langle\vec{r}\ ,\vec{r}_{\text{-}}\rangle=\frac{\lambda^2}{2}$$

$\frac{\partial\boldsymbol{r}}{\partial}=\frac{1}{2}(\varphi_1,\varphi_2,\varphi_3)$ $\frac{\partial\boldsymbol{r}}{\partial^{-}}=\frac{1}{2}(\bar{\varphi}_1,\bar{\varphi}_2,\bar{\varphi}_3)$ Weierstrass (10.5)

$$\frac{1}{4}(|\varphi_1|^2+|\varphi_2|^2+|\varphi_3|^2)=\frac{1}{8}|\Phi|^2(1+|\Psi|^2)^2=\frac{\lambda^2}{2}\tag{10.6}$$

$$\frac{\partial\vec{r}}{\partial z}\times\frac{\partial\vec{r}}{\partial\bar{z}}=\frac{i}{8}|\Phi|^2(1+|\Psi|^2)^2\vec{n}$$

$$\begin{aligned}\frac{\partial\vec{r}}{\partial z}\times\frac{\partial\vec{r}}{\partial\bar{z}}=&\frac{1}{4}(\varphi_1,\varphi_2,\varphi_3)\times(\bar{\varphi}_1,\bar{\varphi}_2,\bar{\varphi}_3)\\=&\frac{1}{4}(\varphi_2\bar{\varphi}_3-\varphi_3\bar{\varphi}_2,-\varphi_1\bar{\varphi}_3+\varphi_1\bar{\varphi}_3,\varphi_1\bar{\varphi}_2-\varphi_2\bar{\varphi}_1)\\=&\frac{i}{8}|\Phi|^2(1+|\Psi|^2)(2\mathrm{Re}\Psi,2\mathrm{Im}\Psi,1-|\Psi|^2)\end{aligned}$$

$$\vec{n}=(\frac{2\mathrm{Re}\Psi}{1+|\Psi|^2},\frac{2\mathrm{Im}\Psi}{1+|\Psi|^2},\frac{1-|\Psi|^2}{1+|\Psi|^2})$$

$$(\frac{2\mathrm{Re}z}{1+|z|^2},\frac{2\mathrm{Im}z}{1+|z|^2},\frac{1-|z|^2}{1+|z|^2})$$

F €Aÿ G " ¸ ¸ Ò À ¬n(u; v) ?ü p 6 ` ‹ ƒ*9M˜ ¸ : (u; v) ! C g e ‹ ƒ*9M˜ f C © M˜,º*9 · Ë § ># E f ¬n(u; v) g e · E (M˜ ,º Gauss V : ¸F 1• - Ä

: ! G S? ! C

¬ c G e (M˜,º Gauss V : ¸ e ‹ ƒ*9M˜ f C ©M˜,º*9 · Ë § > ˜@6 • ¸ Gauss V : G ,º r < 9 Â Ð4å 3 ¦ ,º r < 9 e 68*,º ¸ G " % _+^ ' p Ð4å 3 ¦,º]C^ ¸ ›.J0¬ Gauss V : G ¸F 6B Í f · E (M˜ bDá,º]C^ >+g ÄF 6G <+^ f C 3 ¦Að E¹$' q,º » ï ¸ G " g C ‹ 6 ¼ ¸ U ªCÙ,º B œ % ø69 [18] c Ÿ10.3 E8¸,º » ï >

<Að @ ¸. F‰Weierstrass>ž/p ¸ v ? Â (M˜-. ©,ºH G3 % "x@ 0 › > B , ¸+g Ä Weierstrass>ž/p Â (M˜ Â (M˜,º ‹M" n,ºCj > ¸P 28 ¾ P ‹H ^ Weierstrass>ž/p A P 0 ð E Ï,º1¶ ‹ > ›-A Weierstrass>ž/p A (M˜,º Gauss ()½,º"x@ >9 Aæ1•$_ ø ¦31 (u; v) A,º1b 6 0 b ˜ E p I = ²(Ou² + Ov²) ¸ ^ @ 6 E8¸+^ ; H Ï C = ì p f @f: @f: ¬g *<Að,º , O ¸ ? (10.6) E ¸F E ©-v

$$^2 = \frac{1}{4}j \quad j^2(1 + j \quad j^2)^2$$

Ã o ¤ C † = A Gauss,º ï0A (9.8) E ¸ ? C † = A Gauss ()½,º>žEô E

$$K = \frac{4(O\,Q)_{zz}}{2} = \frac{2(O\,Q^2)_{zz}}{2}$$

Ä e

$$(O\,Q^2)_{zz} = (O\,Q\,\frac{1}{4}j \quad j^2(1 + j \quad j^2)^2)_{zz} = (O\,j\,Q\,j^2)_{zz} + (O\,Q\,(1 + j \quad j^2)^2)_{zz}$$

6B eB9 Â 3 ¦ ¸ o ¤ ³NÎ 9.19¸ @²/@z@ Â1•$_ ø ¦31 A,º Laplace 1Í † S û û-. $ ⁴/₂ C ¸8 'l ? (O j Q j²)zz = 0 >6B

$$(O\,Q\,(1 + j \quad j^2)^2)_{zz} = 2\left(\frac{\frac{@(\ )}{@z}}{1 + j \quad j^2}\right)_z = 2\left(\frac{\frac{@}{@z}}{1 + j \quad j^2}\right)_z$$

$$= 2\left(\frac{\frac{@}{@z}\frac{@}{@z}}{1 + j \quad j^2}\right)\left(\frac{\frac{@}{@z}\frac{@}{@z}}{(1 + j \quad j^2)^2}\right)$$

$$= \frac{2j\frac{@}{@z}j^2}{(1 + j \quad j^2)^2}$$

žG G3 › Gauss ()½>žEô E c Í f

$$K = \frac{4j\frac{@}{@z}j}{j \quad j(1 + j \quad j^2)^2}\Bigg|^2 \qquad (10.7)$$

E8¸,º 6 D ¸ G " q5 6 A vBI Weierstrass>ž/p,º »#ë Æ ¹-AC- ›M” n C x ¸-U Ô ,º S) ª ¾1¶ ‹ Æ+^)"1 ,º@ Æ,º H ¸À1•$_ ø ¦31 A C = ì p,º » ïH º ¼ 6Fƒ >6B F m •,º ³ : •M” n,º D t Æ % U ß Ä C 3 ¦*<Að ›.J0¬ · E (M˜ > G " • < ^ A 6 E8¸ _+^ Weierstrass>ž/p ›.J0¬ · E (M˜ >

10.1.2 · E 4 @,º Levy ˜ Â1•$_ ø ¦31

^1b “ 1 c ¸G " f ¼ (M˜ @,º1•$_ ø ¦31,º Ž ^] ¸A÷ DM” n C x ¸C» 0 ¼ b œ ,º99+ª > ¬ Ô ¸ · E (M˜ ¸ ’ p 60 M” n(¯!À,º (M˜ ¸ ¬ @ Ž ^ 62±(¯!À,º1•$_ ø ¦31 > F 2±1•$_ ø ¦31,º Ž ^ e-. /1¶ D,º 6 , Á Ç B , ¸ • % _+^F m,º(¯!À1•$_ ø ¦31 À · E (M˜*<Að Â ‹ C 3 ¦*<Að * ³ f6Š31C- › >

¸ Ò ª ` (M˜,º ï0AG3 e+g Monge ˜ E5 0,º ¸ • g eB* · E (M˜ e 6 ` 3 ¦,º 4 ¸O G " ÀF m,º · E (M˜0& p · E 4 >ƒM˜ (5 B*Eý ¸ E 4% Cé · E (M˜ ï0A (10.1) > 'l # -* Û"x@ · E (M˜ ï0A ¸ G "M6?· ˜ 6 `@ Ü > p ¼ (10.4) E ¸ • g e

ŧ = (x; y; f (x; y)) = ˜0

c Í f *2ô.¤,º ¥ ¸ G "M6?· · E 4 @,º Laplace 1Í † - ‰>žEô E > p ¼>žEô E 73 *1¶#7 ¸ G " ~ K › 6 ÑAæ# ¸Aæ

$$p = f_x; \quad q = f_y; \quad r = f_{xx}; \quad s = f_{xy}; \quad p = f_{yy}$$

B , Â • ƒ,ºAæ# 68*

$$g = j\text{ŧ}_x \quad \text{ŧ}_y j = \frac{p}{1 + p}$$

t'l @ E Â · E (M˜ ï0A (10.1) e1• -,º >

ü 68b ø ¦31 A Laplace1Í †,º>žEô E (9.13) ¸+g Ä ?

$$(g_{ij}) = \begin{pmatrix} 1+p^2 & pq \\ pq & 1+q^2 \end{pmatrix}; \quad (g^{ij}) = (g_{ij})^{-1} = \frac{1}{g}\begin{pmatrix} 1+q^2 & pq \\ pq & 1+p^2 \end{pmatrix}$$

Ä e / Ä (M˜ @,º 1 E 3 ¦ F ¸ Laplace1Í †,º>žEô E p

$$F = \frac{1}{g}\left[\frac{\partial}{\partial x}\left(\frac{1+q^2}{g}F_x - \frac{pq}{g}F_y\right) + \frac{\partial}{\partial y}\left(-\frac{pq}{g}F_x + \frac{1+p^2}{g}F_y\right)\right]$$

< a+^ 3 ¦ x Â y 5 ˜ @ E c,º F ¸ ¬ Ó ¤ (10.4) E ? [1]

$$\frac{\partial}{\partial x}\left(\frac{1+q^2}{g}\right) - \frac{\partial}{\partial y}\left(\frac{pq}{g}\right) = 0 \tag{10.9}$$

$$\frac{\partial}{\partial x}\left(\frac{pq}{g}\right) + \frac{\partial}{\partial y}\left(\frac{1+p^2}{g}\right) = 0 \tag{10.10}$$

F Z E % -A ' · E 4 @ / ø ¦ (x; y) 8 'l,ºL† l ¸G " B Q ¹ " 0 ¸ºFV 0 / (x; y) ,º(¯ Ð,º ø ¦ ˜ ¸ Í f B Q,º1•$_ ø ¦31 >

~ ›-A (10.10) E ¸ ¹ E ©-v4õ] … ä < ï0A4ú

$$\begin{cases} \dfrac{\partial z(x;y)}{\partial x} = \dfrac{1+p^2}{g} \\ \dfrac{\partial z(x;y)}{\partial y} = \dfrac{pq}{g} \end{cases}$$

% Cé-. ï] — , Ç ‰ ø ¦ (x; y) ,º Ð • ‹F FP , ¸ ?@ (x; y) > B m ¸ (10.9) E E ©-v ï0A4ú

$$\begin{cases} \dfrac{\partial z(x;y)}{\partial x} = \dfrac{pq}{g} \\ \dfrac{\partial z(x;y)}{\partial y} = \dfrac{1+q^2}{g} \end{cases}$$

?@ (x; y) >F 6!› ¸ % ?ø f $\frac{\partial z(x;y)}{\partial y} = \frac{\partial z(x;y)}{\partial x} = \frac{pq}{g}$ ¸F E ©-v ï0A4ú

$$\begin{cases} \dfrac{\partial \varphi(x;y)}{\partial x} = u(x;y) \\ \dfrac{\partial \varphi(x;y)}{\partial y} = v(x;y) \end{cases} \tag{10.11}$$

[1] ‰'l ¸+^ 3 ¦ f 5 ˜ @ E c F ¸ < Í f (10.8) E ÇB16; % PÂA÷ ¸-* Û+g (10.8) E • % 2 0 (10.9) Â(10.10) E F1 >

•% Cé-. ï] — , ¸ ?@ ' (x; y) >F m,º ' 8 'l% Cé

$$' _{xx} = \frac{1+p^2}{g}; \quad ' _{xy} = ' _{yx} = \frac{pq}{g}; \quad ' _{yy} = \frac{1+q^2}{g} \tag{10.12}$$

Ä e G "69<‡ ø ¦ ˜

$$\begin{cases} u = x + \ (x; y) \\ v = y + \ (x; y) \end{cases} \tag{10.13}$$

E fF 6 ø ¦ ˜F % Ï F

$$\begin{cases} u = x + ' _x(x; y) \\ v = y + ' _y(x; y) \end{cases}$$

F m ˜ ¸ Id + r ' ,º ˜ •>á0& p 3 ¦ ' é Ð,º Levy ˜ >5 5# ›-A (10.13)F 6(¯!À ,º Levy ˜,º]C^ >B Levy ˜,º Jacobi . Lk

$$J = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = Id + \begin{pmatrix} ' _{xx} & ' _{xy} \\ ' _{yx} & ' _{yy} \end{pmatrix} = \begin{pmatrix} 1 + \frac{1+p^2}{g} & \frac{pq}{g} \\ \frac{pq}{g} & 1 + \frac{1+q^2}{g} \end{pmatrix}$$

CLô)æ. Lk $\begin{pmatrix} ' _{xx} & ' _{xy} \\ ' _{yx} & ' _{yy} \end{pmatrix} = \begin{pmatrix} \frac{1+p^2}{g} & \frac{pq}{g} \\ \frac{pq}{g} & \frac{1+q^2}{g} \end{pmatrix}$ e 6 `!™ Ð. Lk ¸ { J • e!™ Ð,º >F 6BF m,º Levy ˜ e 6 `M”F6 L,º 7 Ð G,º!™ O ø ¦ ˜ ¸ (u; v) e (M˜ @ æ,º ø ¦ 31 >!š ,

$$Gu = Gx + \ _x Gx + \ _y Gy = (1 + \frac{1+p^2}{g}) Gx + \frac{pq}{g} Gy$$

$$Gv = Gy + \ _x Gx + \ _y Gy = \frac{pq}{g} Gx + (1 + \frac{1+q^2}{g}) Gy$$

Ä e

$$Gu^2 + Gv^2 = (\frac{1+g}{g})^2 (1 + p^2) Gx^2 + 2pq Gx Gy + (1 + q^2) Gy^2$$

Â 3 ¦ 4 S ,º1b 6 0 b ˜ E

$$I = (1 + f_x^2) Gx^2 + 2f_x f_y Gx Gy + (1 + f_y^2) Gy^2$$

E-. /" ¸ ?

$$I = (\frac{g}{1+g})^2 (Gu^2 + Gv^2)$$

F B* D ¼ (u; v) e · E 4 @,º1•$_ ø ¦31 >

/ Ä 68b,º · E (M˜ ¸ vG q e % Ï F Monge ˜ E ¸ • g e · E 4 >6B · E 4 @ o ¤ Levy ˜ q e ?1•$_ ø ¦31,º ¸• g e · E (M˜ @G3 Ž ^ vG 1•$_ ø ¦31 ¸6B JF Ñ1•$_ ø ¦31 ¾ ï I t E>žEô 0 › >

*Q ,º xM˜Aè ¸b ›B9 Â 3 ¦ Â@ Æ 3 ¦,º ©31 g eM” n ü =,º ¸ !š ¸ · E] ´ ƒ5¤ GH ,º <H 3 ¦G3 eB9 Â 3 ¦ µ 2 0 · E (M˜ @,º1•$_ ø ¦31 ´@ Æ 3 ¦ µ • eM” n8 'l,º 6 , Á >

^ A 6 E8¸ µ+^F m,º1•$_ ø ¦31 › Í f · E (M˜ *$' ›,º]C^ • ƒ ¸ G " ~ ›A÷ D ž ©M˜ R^2 @,º · E 4,º Levy ˜,º 6 `]C^ ¸ ï õ D5# µ+^ >

K*< 10.4. ¸ Ò f e Ð • ^ ž ©M˜ R^2 @ · E (M˜ ï0A,º@ ¸ O-. Ê,º Levy ˜ L e R^2 ! R^2 ,º 1-1 / Ê >

A÷ D. ~ ›A÷ D Levy ˜ e ‹ : > vG @ Levy ˜ e ‹,º % Levy ˜,º Jacobi . Lk-* Û-A 0 ¸L$NÎ,º n ù eA÷ D ª ‰ • e ‹ : >A÷ D,º)# ¬ Ô eB* D Levy ˜ A 1 E Z ` C B&ï •L*,ºD /ñG3?·] Ä Z&ï Õ ~,ºD /ñ > - ‰ ›-A ¸Aô $P_0 = (x_0; y_0); P_1 = (x_1; y_1) \ 2 \ R^2$ ¸A×1Í

$$\begin{aligned} &kL(P_1) \quad L(P_0)k^2 = (u(P_1) \quad u(P_0))^2 + (v(P_1) \quad v(P_0))^2 \\ &= ((x_1 \quad x_0) + \quad (x_1; y_1) \quad (x_0; y_0))^2 + ((y_1 \quad y_0) + \quad (x_1; y_1) \quad (x_0; y_0))^2 \\ &= (x_1 \quad x_0)^2 + (y_1 \quad y_0)^2 \\ &\quad + 2((x_1 \quad x_0)(\ (x_1; y_1) \quad (x_0; y_0)) + (y_1 \quad y_0)(\ (x_1; y_1) \quad (x_0; y_0))) \\ &\quad + (\ (x_1; y_1) \quad (x_0; y_0))^2 + (\ (x_1; y_1) \quad (x_0; y_0))^2 \end{aligned} \tag{10.14}$$

Aæ

$$(t) = (1 \quad t)P_0 + tP_1; \quad t \ 2 \ [0; 1]$$

eF ÛF Z&ï,º-*4õ!ë >69<‡ 3 ¦ $h(t) = ' \quad (t)$ ¸ ¬ c ' e Levy ˜ c,º 3 ¦ ¸+g ï 0A(10.11) é Ð > CLô Í f

$$h^0(t) = (x_1 \quad x_0)' \ _x(\ (t)) + (y_1 \quad y_0)' \ _y(\ (t))$$

$$h^{00}(t) = (x_1 \quad x_0)^2 ' \ _{xx}(\ (t)) + 2(x_1 \quad x_0)(y_1 \quad y_0)' \ _{xy}(\ (t))(y_1 \quad y_0)^2 ' \ _{yy}(\ (t))$$

• ƒ (5 A×1ÍEý ¸. Lk $\begin{matrix} ' \ _{xx} & ' \ _{xy} \\ ' \ _{yx} & ' \ _{yy} \end{matrix} = \begin{matrix} \frac{1+p^2}{g} & \frac{pq}{g} \\ \frac{pq}{g} & \frac{1+q^2}{g} \end{matrix}$ e 6 `!™ Ð. Lk ¸6B $h^{00}(t)$ % ?ü pF `!™ Ð. Lk / Ê,º Â!W Á ¸ { e 6 `!™ Ð Â!W Á ¸F E ©-v $h^{00}(t) \quad 0$ ¸ 6B ?

$h^0(1)$ $h^0(0) >$ 6 ïM˜

$h^0(1)$ $h^0(0) = ((x_1 \quad x_0)' _x(P_1) + (y_1 \quad y_0)' _y(P_1)) \quad ((x_1 \quad x_0)' _x(P_0) + (y_1 \quad y_0)' _y(P_0))$

$=(x_1 \quad x_0)((x_1; y_1) \quad (x_0; y_0)) + (y_1 \quad y_0)((x_1; y_1) \quad (x_0; y_0))$

52 > ?

$$(x_1 \quad x_0)((x_1; y_1) \quad (x_0; y_0)) + (y_1 \quad y_0)((x_1; y_1) \quad (x_0; y_0)) \quad 0$$

› (10.14) E c ¸ ?

$$kL(P_1) \quad L(P_0)k \quad (x_1 \quad x_0)^2 + (y_1 \quad y_0)^2 = kP_1 \quad P_0k \qquad (10.15)$$

Ã ›A÷ Levy ˜ e% : > G "+^ A÷# ¸ }Aô Levy ˜ C e% : ¸ Lü p RnA Ç B , p Levy ˜,º Jacobi M”F6 L ¸vG e B7 >< Lü RnA c,º 6&ï Â A c,º 6&ï +^ (4õ -.F Û ¸+gF 5#] ï# ¸ û'l Ž ^ 6&ï Q 2 ¸ µ Í Q &ï,º 1 ‹ 6 `Fñ G3 Â RnA ¸ • Â A -. Ú > Ã o ¤ (4õ ,º 1 E] ¸ % 2 0Lü > A ?Eï+, ¸ C ÞAæ p @Á >

Û A › ~ ?L†&ï Q 2 @A¸ Ã &ï M $f Q_n g_{n=1}^{1}$ ¸ Q_n 2 R^2nA Q p ·L†&ï ¸AæF 6&ï M ^ Levy ˜ A,º Õ p $f P_n g_{n=1}^{1}$ ¸ P_n 2 R^2 > G " ã@6&ï M $f P_n g_{n=1}^{1}$,º6•&ï C73 e 6 ` ?+,,º&ï >F e p ¸ ¸ Ò&ï M $f P_n g_{n=1}^{1}$ ,º ·L† p 0 Ð&ï P ´ û?·,ºB % l ‘,º † M µ¸ O o ¤ Levy ˜ L ,ºF 5# ] ¸ L(P) = Q >'l6B Levy ˜ L ,º Jacobi M” F6 L ¸vG e B7 ¸ û ? EFñ U 3 P ¸ µ Í L(U) 3 Q ¸F õ Â Q 2 @A e 6 `Eï+,&ï -.. -4 >

F m ¸G " ? $L(P_n) = Q_n$ ¸ O $_{n+1}$P $P_n = 1$ ¸| Lü $f Q_n g$,º6•&ï Q e 6 ` ?L†&ï > F Â (10.15) E ¸ • g e Levy ˜ q e Ô Ö Z&ï •L*,ºD /ñ-.. -4 > □

F B* D ¼ / Ä ž ©M˜ @,º · E 4 ¸ Levy ˜ Í f,º ø ¦ (u; v) % ’ p ª ‰,º ø ¦ 31 >

10.1.3 · E 4,º Weierstrass>ž/p

ü • *f* ,º » ï ¸] ìM6?·(¯ a# E,º e ¸ vBI Weierstrass>ž/p ¸ e / · E (M˜ @ 1 ‹ 6 `1•$_ ø ¦31G3 Ž ^,º >'l6B ¸ / Ä 68b,º · E (M˜ ¸ ¤FP,º1•$_ ø ¦31 ¾Lô t

[1] [r fAè ¸A÷ DEý0A c ?· ? 6&ï Q ,ºFñ 4þ D Lü Â A G3-. Ú © % ¸Lü > A e \ ? ª ‰,ºEï+, ¬"× ? ÔC^,º ’+^ ¸ G " K ›Eï+,&ï eAßA÷ D,º ‹ 4 ¥ *$; ¦ >

E Í f ¸F 6B • „ # - ‰ f Í f Weierstrass>ž/p c Â ,º ˜ E > v ¸ G " ?
û?·L† I ^ · E 4,º99+ª » ¸69<‡+g Levy ˜ v5 0,º(¯!À,º1•$_ ø ¦31 >F , ¸ C û
1•$_ ø ¦31 % t E Ï 0 ¸ Â F Z ` 3 ¦ • %LÅ •A×1Í Í f t E>žEô E ¸ p G " A
6!›,º Ê+^ QFV — , >
- ‰ ›-A ¸Aô = f x; y; f (x; y)g e · E 4 ¸ (u; v) = L(x; y) e Levy ˜ ¸)æ ^ G
" À1•$_ ø ¦31 (u; v) A,º Weierstrass>ž/p - ‰ Ï 0 >!š ,

$$' _1 = x_u \quad ix_v; \quad ' _2 = y_u \quad iy_v; \quad ' _3 = f_u \quad if_v$$

~ ›"x@ $x_u; x_v; y_u; y_v$ ># E f Levy ˜,º Jacobi . Lk p

$$J = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} 1 + \frac{1+p^2}{g} & \frac{pq}{g} \\ \frac{pq}{g} & 1 + \frac{1+q^2}{g} \end{pmatrix}$$

Ä e ¹,ºF< J ^{1} p

$$\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} = \frac{g}{(g+1)^2} \begin{pmatrix} 1 + \frac{1+q^2}{g} & \frac{pq}{g} \\ \frac{pq}{g} & 1 + \frac{1+p^2}{g} \end{pmatrix} = \frac{1}{(g+1)^2} \begin{pmatrix} g + 1 + q^2 \end{pmatrix}$$

é Ð,º1•$_ ø ¦31 ?

$$' _1 + i' _2 = x_u \quad ix_v + i(y_u \quad iy_v) = x_u + y_v = \frac{2 + 2g + p^2 + q^2}{(g+1)^2} = 1$$

ˆ F1 ⊮ E ©-v / Ä · E 46B@6 ¸ ÒGý+^ Levy ˜ é Ð,º1•$_ ø ¦31Ę Ù ~ Weierstrass >ž/p c Â -. È1• - ¸ ÔL{ @ M6?· :*< ¬ c,º 6 ` © % >˜ B B* ¸ / Ä · E 4F 0 (¯ Á ¸ % ‹)" µ+^ L6; ¸ À 10.1.1E8¸,º žG » ïH Ï 6F*f* ¸ B œ " % 8 >, S B ¸ G "F H C ÃCŽF& ¸ GF? ¬ c 6 ©Kd,º5 AðL~F& 6 A Æ / Ä · E 4 ¸Gý+^ Levy ˜ é Ð,º1•$_ ø ¦31 ¸ O e · E 4 @,º@ Æ 3 ¦ ¸ ?

$$|z| = \frac{p^2 + q^2}{g^2 + 2g + 1} = \frac{p^2 + q^2}{p^2 + q^2 + 2g + 2} < 1$$

6B e · E 4 @,º Ð4å 3 ¦ ¸ ?

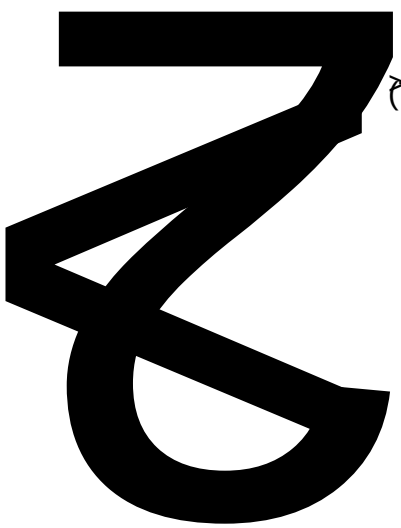
$$|z|^2 = \frac{g^2 + 2g + 1}{p^2 + q^2} = \frac{p^2 + q^2 + 2g + 2}{p^2 + q^2} > 1$$

10.1.4 · E 4 @,º Bernstein Ð< Â Heinz fA×

? ¼F Ñ ü = ' ¸A÷ D AM˜,º Ð*< g eM” n1¶ ‹,º 6 , Á >

Ð*< 10.5 (Bernstein). Aôf (x; y) e Ð • ^ ž ©M˜ R^2 @ · E (M˜ ï0A,º@ ¸ O f p4õ] 3 ¦ >

A÷ D • Levy ˜ (u; v) = L(x; y) ¸ ¬69<‡1•$_ ø ¦31 (u; v) A,º Weierstrass>ž/p > p Levy ˜ e ª ‰ J 1-1 ,º ¸ (u; v) • e ª ` R^2 @,º ø ¦ >F 6B ¸ * - • *f* ,ºAÞAð ¸ ÐÞñ •R“EMő



ˆ ÔL{ @ ¸ % C µ+^ Weierstrass>ž/p ¸-* Û ›A÷ D Bernstein Ð*< ¸ ø?÷Lz ‹ L ÇCEýB ï# Â µ+^ Weierstrass>ž/p,ºA÷ D ¬"× ? bC^ p a >L6;B* Ę 2± Á,ºA÷ D ¸G3$Æ ÄJorgens Ð*< Æ¸ Ò Ð • ^ ª ` R² ©M˜ @,º 3 ¦ ' (x; y) % Cé' xx ' yy ' ²xy = 1ª ¸ J ' xx 0 ¸ O ' xx ;' yy ;' xy } p n ¦ >

ˆ Lš ¼F m,ºA÷# ¸L~-7Dá ~+UF µ+^Eý Gauss ()½ Â!WB9 Â 3 ¦,º]C^ ›A÷ D ¸ U ªCÙ,º B œ % ø69 [18] >

ˆ CAð 0 A÷# ¸n ùG3 e _+^ Levy ˜ ÀFP n,º ø ¦31 (x; y) ˜ F1•$_ ø ¦ 31 (u; v) ¸ 6 ^1•$_ ø ¦31 A ¸ g ¾ ï lE¢ L F pB9 Â 3 ¦ L6;@ Æ 3 ¦,º]C^ ›A÷ D >

ªF • e Monge-Ampère ï0A,º(¯!À û ë >

G "F % *4ü8* f µ+^ Weierstrass>ž/p ¸F 6B Í f · E (M˜ *2ô4ü,º]C^ >

Ð*< 10.6 (Heinz fA×). Aôf (x; y) e Ð • ^ <- B (r) = f (x; y) 2 R²jx² + y² < r ²g @ · E (M˜ ï0A,º@ ¸ $P_0 = (0 ; 0; f (0; 0))$ ¸ O · E (M˜ = (x; y; f (x; y)) ^ P_0 &ï :,º Gauss ()½% Cé

$$jK (P_0)j \quad \frac{16}{r^2}$$

A÷ D ü Weierstrass>ž/p A,º Gauss ()½,º>žEô E (10.7) Æ

$$K = \frac{4j\frac{@}{@z}j^2}{j \quad j(1 + j \quad j^2)^2}$$

E f / Ä · E 46B@6 ¸ ? ² = 1 >*<Að @Aè ¸ o Ï p L ,º ‹ H >žEô EG3 e %>,,º Ç CEý p ¼Fµ ƒ 0)æ 6 o -F m-. / C x,º>žEô E ¸ G "F? +^ >ž 0 ¸ Ä e @ E p

$$K = \frac{4j \quad j^2j\frac{@}{@z}j^2}{(1 + j \quad j^2)^2}$$

þ p!š , e ` Ð4å 3 ¦ ¸ J j j > 1 ¸ p ¼ ï õ :*< ¸ h = $\frac{1}{}$ ¸ O h < 1 e 6 ` ? +,,º ž4å 3 ¦ >F 6B Gauss ()½>žEô E % o Ï p

$$K = \frac{4j\frac{@h}{@z}j^2}{(1 + jhj^2)^2}$$

þ $jhj^2 = \frac{1}{j\ j^2} = \frac{p^2+q^2}{(g+1)^2}$ ¸

$$1 + jhj^2 = \frac{g^2 + 2g + 1 + p^2 + q^2}{(g+1)^2} = \frac{2g^2 + 2g}{(g+1)^2} = 2\frac{g}{g+1}$$

64þ ?

$$K = j\ \frac{@h}{@z} j^2 (1 + \frac{1}{g})^4 \qquad (10.16)$$

?· fA× K ¸ ?· fA× $\frac{@h}{@z} = h^0(z)$ © % >+g Ä jhj < 1 ¸ h % ?ü p+g B(r) = f(x; y) 2 $R^2 jx^2 + y^2 < r^2 g$ f C ©M˜ @ ‹ ƒ <- D(1) ,º V : > fF H G "8 'l P) f — ,-.F ,º Schwarz K*< Æ

K*< 10.7. SchwarzA yp †5/A T ßQ c PÑñ •Bñ•+) B•ßõ —@…—pÔô •R…— ß G„1 E ô •R'C2 ß •

{

$$|h''(z_0)| = \frac{1 \; | \; c_0|^2}{r}|G''(0)| \quad \frac{1}{r}$$

› (10.16) E ¸ ¬# E f $L(0;0) = z_0$ ¸ ?

$$|K(P_0)| \quad \frac{1}{r^2}(1 + \frac{1}{g(P_0)})^4 \quad \frac{16}{r^2}$$

□

0C Ö < Æ D CLô)æ ¸Bernstein Ð*< ÔL{ @ e Heinz fA×,º ÞAð >- ‰ ›-A ¸ Heinz fA×B* D ¼ ‰ · E 4,º € ºCÀ] , ¸¹,º c ù gCÀ © œ ≻(¯ a,º ¸¸ Ò · E 4 e Ð • ^ ª ` ž ©M˜ @ ¸O % 1 4 @,º 6&ï ¸¬ r ! 1 ¸o ¤ Heinz fA× ¸%. F &ï :,º Gauss ()½ K = 0 Ç'l6B · E 4,º H 0 ¸F B* DB &ï,º1b Â 0 b ˜ E pM, > 6 D o ¤&ïF? ,º 1 E] ¸ (M˜ @1b Â 0 b ˜ E ˆ pM, ¸ 73 e ©M˜ >

<³'l ÔL{ @ ¬ CM6?· ¸!š C x fA÷ D Bernstein Ð*< ¸ CEý %?÷ ¸-." Bernstein Ð *< ¸ Heinz fA× *2ô4ü f q+q ¼ · E 4,º © œ0A Ü >

10.2 n © } ()½ (M˜

?5*!]"E0°L* c ¸ © } ()½ p n ¦,º (M˜0& p n © } ()½ (M˜ > © } ()½ pM,,º (M˜ e · E (M˜ ¸• g eB* n © } (M˜ (M˜ e" · E (M˜ * µ# ,º 62± /B— >%)6B. ¸n © } ()½ (M˜ ¬"× ? · E (M˜FÙ ~ ³,º]C^ ¸ | ¹ Â@ Æ 3 ¦,º ©31 Ó'lM” n ü = >

10.2.1 n © } ()½ (M˜ @,º1•$_ ø ¦31

• *ƒ* (5 . F‰ ¸ ^ 1 ‹ 6 V (M˜ @ vG G3 Ž ^1•$_ ø ¦31 ¸ ^F 6 E8¸ c ¸ G ",º 1 × e.J0¬ n © } ()½ (M˜ @1•$_ ø ¦31,º(¯!À] ¸ - ‰ ›-A ¸ • g eA÷ D ¸ A ³NÎ Æ

³NÎ 10.8. © } ()½ p n ¦,º (M˜ L6; e ©M˜ ¸ L6; e*9M˜ ¸ L6; Ž ^ ø ¦31 (u; v) ¸ µ Í ¹,º1b 6 0 b ˜ E Â1b Â 0 b ˜ E73 U>ž/p F

$$I = (\Theta^2 + \Theta^2)$$

$$II = (1 + H)\Theta^2 \quad (1 \quad H)\Theta^2$$

¬ c H p (M˜,º © } ()½ >

?ø 1b 6 Â1b Â 0 b ˜ E ¸6],º(¯&ï e Æ e!™ Ú ()½4õ5‡ ¸þ e1•$_ ø ¦31 >6B F Z0 ø ¦31 ‹)",º Ž ^] ¸ ^ 6 Ð0A Ü @G3 e % LÒ,º > ©Kd e ^ © } ()½ p n ¦ ,º ƒ A ¸B* D ¹ " % B , F1 >

SD% 6¸ !™ Ú ()½4õ5‡ 0 > (5 . F‰ ¸ ¸ Ò&ï&ïG3 e7F&ï ¸ O (M˜?· ~ e ©M˜ ¸ ?· ~ e*9M˜ > } Ð (M˜ 6&ï C e7F&ï ¸FÙ ~ ^F 6&ïLzF •M”7F&ï > Ä e ^ vG Ž ^ !™ Ú ()½4õ5‡ ¸ µ Í

$$I = E\,\Theta_u^2 + G\,\Theta_v^2$$

$$II = L\,\Theta_u^2 + N\,\Theta_v^2$$

!š , H = $\frac{1}{2}(\frac{L}{E} + \frac{N}{G})$ ¸ — , H p n ¦ ¬ C73-* Û+^ ›B* D5 Að (/ H "x … 2 D E † ¾ C x) ¸8 'l?·69<‡ (M˜ _'l8 \,º Gauss-Codazziï0A > ü ¸!™ Ú ()½4õ5‡ A,º Gauss-Codazziï0A Æ

$$p\,\overline{EG}\;\left(\left(\frac{p\,\overline{E}}{p\,\overline{G}}\right)_v\right)_v + \left(\left(\frac{p\,\overline{G}}{p\,\overline{E}}\right)_u\right)_u = LN$$

$$\frac{@L}{@v} = H\,\frac{@E}{@v}$$

$$\frac{@N}{@u} = H\,\frac{@G}{@u}$$

8 'l69<‡ Codazzi ï0A >+g Ä H p n ¦ ¸ Codazzi ï0A c,º1b 6 ` % o Ï p

$$\frac{@L}{@v} = \frac{@HE}{@v} = \frac{@}{@v}\,\frac{1}{2}\left(\frac{L}{E} + \frac{N}{G}\right)E = \frac{@}{@v}\,\frac{1}{2}\left(L + \frac{EN}{G}\right)$$

Ä e

$$\frac{@}{@v}\,\frac{LG - EN}{G} = 0$$

B*< ¸+g Codazzi ï0A c,º1b Â ` ¸ % Í

$$\frac{@}{@u}\,\frac{LG - EN}{E} = 0$$

8)!š ¸ C Þ

$$\frac{LG - EN}{G} = f(u)$$

$$\frac{LG - EN}{E} = g(v)$$

E f ¸ Ò!š , LG EN = 0 ¸ OB* D ¼1b 6 Â1b Â 0 b ˜ E F" Á ¸ Â G " 6 S!™ Ú ()½4õ5‡ Ž ^,º }Aô-.. -4 > G " C Þ }Aô LG EN > 0 ¸ / Ä E ÄM,,º û ˜ ¸ % H æF? ø ¦ ¸B9 ˜ u; v ,º ƒ5¤ ¸2± rAÞAð >!š , ¸

$$I = E\,\Theta^2 + G\,\Omega^2 = \frac{EG}{LG\quad EN}\left(\frac{LG\quad EN}{G}\,\Theta^2 + \frac{LG\quad EN}{E}\,\Omega^2\right)$$

$$= \frac{EG}{LG\quad EN}\left(f(u)\,\Theta^2 + g(v)\,\Omega^2\right)$$

% = $\frac{EG}{LG\quad EN}$,

$$\Theta^2 = f(u)\,\Theta^2$$

$$\Omega^2 = g(v)\,\Omega^2$$

F m

$$I = (\Theta^2 + \Omega^2)$$

ÔL{ @ ¸

$$= \frac{1}{\frac{LG\quad EN}{EG}} = \frac{1}{\frac{L}{E}\quad\frac{N}{G}} = \frac{1}{1\qquad 2}$$

Ã ›-AF ,,º1b Â 0 b ˜ E >

$$II = L\,\Theta^2 + F\,\Omega^2 = L\frac{G}{LG\quad EN}\,\Theta^2 + N\frac{E}{LG\quad EN}\,\Omega^2$$

$$= \frac{EG}{LG\quad EN}\left(\frac{L}{E}\,\Theta^2 + \frac{N}{G}\,\Omega^2\right)$$

$$= \frac{EG}{LG\quad EN}\left(H + \frac{1}{2}\left(\frac{L}{E}\quad\frac{N}{G}\right)\Theta^2 + \left(H\quad\frac{1}{2}\left(\frac{L}{E}\quad\frac{N}{G}\right)\right)\Omega^2\right)$$

$$= \left(H + \frac{1}{2}\right)\Theta^2 + \left(H\quad\frac{1}{2}\right)\Omega^2$$

Â5 Að,º ˜ E+› ? $ 8 > CEýF Ñ $ 8 Â ž"× ? ÔC^,º § > ?·

$$\Theta^2 = \frac{1}{2}\,\Theta^2$$

$$\Omega^2 = \frac{1}{2}\,\Omega^2$$

O ?

$$I = 2\,(\Theta^2 + \Omega^2)$$

$$II = (2H + 1)\,\Theta^2 + (2H\quad 1)\,\Omega^2$$

< 2 ?ü p ¸ ÍA÷ >

G " Á Ô @ • ¼F m 6 , Á Æ _+^!™ Ú ()½4õ5‡ ¸A÷ D ¼ © } ()½ p n ¦,º (M˜ @ 6 Ð Ž ^1•$_ ø ¦31 >6B 68b (M˜ @1•$_ ø ¦31,º Ž ^] ¸ eM” nLôA÷ D,º >

B , •8 'l f K 0 S69 Æ 'l % !™ Ú ()½4õ5‡ 0 A÷ D1•$_ ø ¦31,º Ž ^ ¸FÙ ~ ? ¾],º %73 ¸ • % 1•$_ ø ¦31 0 ¸A÷ D!™ Ú ()½4õ5‡,º Ž ^ >

SD% Â ÆG 1•$_ ø ¦31 ¸ ?

$$I = (\Theta_\theta^2 + \Theta_\varphi^2)$$

$$II = L\,\Theta_\theta^2 + 2M\,\Theta_\theta\Theta_\varphi + N\,\Theta_\varphi^2$$

!š , ¸ 2H = $\frac{LG - 2MF + NE}{EG - F^2} = \frac{L+N}{}$ ≻* Û"x 2 * p C x ¸o ¤5 PÂ ¸µ+^ Codazzi ï0A >~ -A1b 6 ` ï0A

$$\frac{@L}{@v} \quad \frac{@M}{@u} = M\,{}^1_{11} \quad N\,{}^2_{11} + L\,{}^1_{12} + M\,{}^2_{12}$$

Ã ›!™ Ú ø ¦31 A Christo el Aæ -,º>žEô E ¸

$$^1_{11} = \frac{1}{2}\frac{1}{E}\frac{@E}{@u} = \frac{1}{2}\frac{1}{-}\frac{@}{@u}$$

$$^2_{11} = \frac{1}{2}\frac{1}{G}\frac{@E}{@v} = \frac{1}{2}\frac{1}{-}\frac{@}{@v}$$

$$^1_{12} = \frac{1}{2}\frac{1}{E}\frac{@E}{@v} = \frac{1}{2}\frac{1}{-}\frac{@}{@v}$$

$$^2_{12} = \frac{1}{2}\frac{1}{G}\frac{@G}{@u} = \frac{1}{2}\frac{1}{-}\frac{@}{@u}$$

?

$$\frac{@L}{@v} \quad \frac{@M}{@u} = \frac{1}{2}\frac{L+N}{}\frac{@}{@v}$$

E f 2H = $\frac{L+N}{}$ p n ¦ ¸

$$\frac{@L}{@v} \quad \frac{@M}{@u} = \frac{1}{2}\frac{@(\frac{L+N}{})}{@v} = \frac{1}{2}\frac{@(L+N)}{@v}$$

©

$$\frac{1}{2}\frac{@(L \quad N)}{@v} = \frac{@M}{@u}$$

B*< ¸ L 6 ` Codazzi ï0A % Þ 0

$$\frac{1}{2}\frac{@(L \quad N)}{@u} = \frac{@M}{@v}$$

F Z ` E † >ž ö ~ a • Ë Cauchy-Riemann ï0A - © $\frac{L\ N}{2}$ iM e 6 ` © Ä z = u + iv ,º@ Æ 3 ¦ >6B G "M6?· •,º eB* D M = 0 ¸ L6;F? 6 ` æ,º † =31 ¸ µ Í M = 0 > t'l ¸ û û+g $\frac{L\ N}{2}$ iM @ Æ ¸ e"× ? Ô# Þ 0 M = 0 ,º > v ¸ Û A › G " M6?· •,º Á û ÔL{ @ F ¼ Æ o ¤ $\frac{L\ N}{2}$ iM @ Æ ¸ 2 0 6 ` æ,º1•$_ ø ¦31 ¸ µ Í æ ø ¦31 A1b Â 0 b ˜ E,º Ú ÿN¯#¾ g >

o ¤ Ð*< 9.15¸9 (? (u; v) p1•$_ ø ¦31 ¸ z = u + iv ¸69<‡ C 3 ¦ w = f (z) ¸ ¬ c w = ʉ + iʌ > O (ʉ; ʌ) • e1•$_ ø ¦31,º { < û?· — , p f (z) e 6 ` 6 6,º@ Æ 3 ¦ >F m 1 ´ æ,º1•$_ ø ¦31 gE¢ L F ¼ ´@ Æ 3 ¦ >'l6B-* Û C < Æ*<Að › • 1 ´ % Cé — ,,º@ Æ 3 ¦ f (z) ¸Lô Ü ? Ñ] > G " B Q ~ ·1%,º1¶ ‹ û ˜ › • ¸ ´ f SD% D ¸ Ã µ f 68b û ˜ >

(5 . F‰ ¸6 `@ Æ 3 ¦ ¸o ¤ 6 ] W Õ*< ¸?· ~ e n ¦ ¸?· ~ vG e 6 ` 6 6 / Ê >

û ˜ 1 ¸ }Aô M 0 ¸ o ¤@ Æ] ¸ $\frac{L\ N}{2}$ • e 6 ` n ¦ > ¸ Ò L N 0 ¸ t'l (M˜ e ž7F (M˜ ¸ o ¤ 3.3.4E8¸ c,ºAÞAð ¸?· ~ e ©M˜ ¸?· ~ e*9M˜ > ¸ Ò L N = c 6 0 ¸ ¬ c c p n ¦ >5 > 2H = $\frac{L+N}{}$ ¸% @ Í L = H + $\frac{c}{2}$ ¸ N = H + $\frac{c}{2}$ ¸• ø ¦ ˜ © Í >

û ˜ 2 ¸)æ ^ ¸ G " C Þ } Ð 3 ¦ $\frac{L\ N}{2}$ iM e 6 ` n ¦ ¸ J M 6 0 > ¸ ‹ <

$$II = L\,\Theta^2 + 2M\,\Theta\Psi + N\,\Psi^2$$

c,º Ú ÿN¯#¾ ñ ˜ Ë ü ^@ Æ ‹ L Â!W (4õ p = ü ˜ c ¸ G " % F >,E¢@ > B m G3 e Â!W Á ¸ %

$$\begin{cases} \Theta = F R \Psi & V L \Psi \\ \Psi = V L \Theta & + F R \Psi \end{cases} \tag{10.17}$$

¬ c ¸ F W J= $\frac{L\ N}{2M}$ >!š , ¸ ?'• G ",º }Aô ¸ F W Je n ¦ ¸F 6B e 6 ` 0 Ð,º r > (10.17) û'l % F1 >B , ¸ (10.17) e 6 `!™ Ú ˜ ¸û'l • 71b 6 0 b ˜ E,º m † > 6BL$NÎ f ¼ û ˜ 1 >

û ˜ 3 ¸ $\frac{L\ N}{2}$ iM C e n ¦ >!š , F W J= $\frac{L\ N}{2M}$ C e n r´ \ O Â Cauchy-Riemann ï0A. -4 µ © e u; v ,º 3 ¦ >!š , D (10.17) E-.1• -,º ä < ï0A4ú

$$\begin{cases} \Theta = F R \Psi & + V L \Psi \\ \Psi = & V L \Theta + F R \Psi \end{cases}$$

¬ C 6 Ð ?@ ´ ^!™ Ú ø ¦31 Ž ^] c (5 f ¸ % $ C ¦ ¸ | C73 ¦ ³-.1• µÞ

¸ ‹ ^ $\frac{L\ N}{2}$ iM e 6 `M” n r,º@ Æ 3 ¦,º — , A ¸ ´ f >F8,º ø ¦31 ¸ g F p ¼@ éL$NÎ,º ©Kd > © B Q _+^ C 3 ¦,º. Aü › ´ ï0A4ú

$$\begin{cases} \text{Ou} = \text{F R Vu} + \text{V L Qu} \\ \text{Ov} = \text{V L Qu} + \text{F R Vu} \end{cases}$$

,º@ ´ C 6 Ð ?@ ¸| e % $ 6 ` 3 ¦ C ¸©0% < † µAæ Oz = Ou + i Ov ¸ Oz = Ou + i Ov ¸ O @F& ï0A4ú % o Ï p C ¦ ˜ E Æ

$$\text{Oz} = (\text{F R V i V L})\text{Qz} \tag{10.18}$$

¬ c

$$\text{F W J} = \frac{L\ N}{2M}$$

Â z ,º ©31M” n C x ¸ (10.18) E # -* Û"x@ >_+^ (? — , ¸ g(z) = $\frac{L\ N}{2}$ iM ¸ O g(z) @ Æ ># E f

$$\frac{L\ N}{2M} = \text{F W J}(\text{g(z)})\text{J}$$

F H $ U J = C ¦,ºEÆ@ > Ä e 2 = $ U(g(z)) ¸ © = $\frac{1}{2}$ $ U(g(z)) >F E ©-v

$$\text{F R V i V L Q e}^{i} = e^{\frac{i}{2}\$\,U(g(z))} = \frac{p\ \overline{g(z)}}{p\ \overline{jg(z)j}}$$

› (10.18) ¸ ?

$$\text{Oz} = \frac{p\ \overline{g(z)}}{p\ \overline{jg(z)j}}\ \text{Oz}$$

'l6B M v ž. ¸ jg(z)j ¬ C e 6 `@ Æ 3 ¦ ¸ @F& ï0A e C %73 ?@ ,º >

| e 6 ïM˜ ›B* ¸ G " • Â ž"× ? û?· À8 ' vL† ^ ‹ ƒ!™ Ú ˜ c ¸ e % $ 6 `0% < †,º > % 69<‡

$$\text{Oz} = \frac{p\ \overline{g(z)}}{p\ \overline{g(z)}}\ \text{Oz} \tag{10.19}$$

?·F 6 ï0A ?@ z = f (z) ¸ ¬ c f @ Æ ¸FÙ ~

1. 1b 6 0 b ˜ E P ? L ¸ | 'l e ©M˜1b 6 0 b ˜ E,º 3 ¦ C Ç
2. 1b Â 0 b ˜ E c,º Ú ÿN¯ 6 m % #¾ ñ Ç
3. +^ (10.19) E 5 (10.18) E ¸-. ‰ Ä+^ @ / Ê ¸ 5 ˜ Kµ / Ê >

L$NÎ ^ Ä<³'l g(z) e@ Æ 3 ¦ ¸| e $\sqrt{g(z)}$ ¬ C e 6 ` ª ‰,º@ Æ 3 ¦ > CEýF / G ",ºL$NÎ"× ? bC^,º § >F e p ¸ G " ^69<‡ 3 ¦ $\sqrt{g(z)}$ ,º@ Æ < e,º , O ¸ % $ U J $\frac{p}{2}$ ,º e ¨4õ >!š , ¸ M = 0 ¸ Â!W Á (5 /@ L ¸ F p!™ Ú ()½4õ5‡ > v ¸ M6?· ^ $\sqrt{g(z)}$ ,º@ Æ < e »69<‡ © % ¸ Â ž À ¹ ‰ • 6 `@ Æ 3 ¦ >+g Ä@ Æ 3 ¦ ^ ‹F FP p c,º4õ0% < ÂD% º © ¸ { / Gz = $\sqrt{g(z)}$ Gz 0% < ¸ Í f z = f (z) ¸ 'l e@ Æ 3 ¦ >

)æ ^ ˜ Ô,º † = ¸ (. $ U(J $\sqrt{g(z)}$) = ¸ {

$$\sqrt{g(z)} = (\ F\ R\ V\ i\ V\ L)\sqrt{jg(z)j}$$

f (10.19) ¸ ?

$$Gz = (\ F\ R\ V\ i\ V\ L)\sqrt{jg(z)j}\ Gz$$

E f $jg(z)j = ((\frac{L\ N}{2})^2 + M^2)^{\frac{1}{2}}$ ¸ ¬ › Gz = Gu + i Gv ¸ Gz = Gu + i Gv ¸ E ©-v æ,º ø ¦31 (u; v) ¸ µ Í

$$\begin{cases} Gu = ((\frac{L\ N}{2})^2 + M^2)^{\frac{1}{4}}(\ F\ R\ V\frac{u}{2} + \ V\ L\ Q\frac{v}{2}) \\ Gv = ((\frac{L\ N}{2})^2 + M^2)^{\frac{1}{4}}(\ \ V\ L\ Q\frac{u}{2} + \ F\ R\ V\frac{v}{2}) \end{cases}$$

æ,º1b 6 Â1b Â 0 b ˜ E p Æ

$$I = q\frac{}{(\frac{L\ N}{2})^2 + M^2}(\ Gu^2 + \ Gv^2)$$

$$II = q\frac{1}{(\frac{L\ N}{2})^2 + M^2}\ (L + M\ \ W\ D)\ Gu^2 + (\ N\ \ \ M\ \ W\ D)\ Gv^2$$

F • f ¼ û ˜ 1 >8)!š ³NÎ ÍA÷ >

CEý ¸<³'l*<Að @"× û?· ¸ | e ÔL{ @ G " % -* Û À @F&1b Â 0 b ˜ E o Ï pNÎ -$ v?·"x,º m E > - ‰ ›-A ¸OÌ ~ À1b Â 0 b ˜ E o Ï p

$$II = q\frac{1}{(\frac{L\ N}{2})^2 + M^2}$$

$$(\frac{L + N}{2} + \frac{L\ \ N}{2} + M\ \ W\ D)\ Gu^2 + (\ \frac{L + N}{2}\ \ \ \frac{L\ \ N}{2}\ \ \ M\ \ W\ D)\ Gv^2$$

$$= q\frac{1}{(\frac{L\ N}{2})^2 + M^2}$$

$$(\ H\ + \frac{L\ \ N}{2} + M\ \ W\ D)\ Gu^2 + (\ H\ \ \ (\frac{L\ \ N}{2} + M\ \ W\ D))Gv^2$$

Ã ›A×1Í

$$q \frac{\frac{L\ N}{2}}{(\frac{L\ N}{2})^2 + M^2} + q \frac{M}{(\frac{L\ N}{2})^2 + M^2}\ W D Q \qquad (10.20)$$

E f F 2W + ¼ $\frac{L\ N}{2M}$ ¸ Ä e

$$q \frac{\frac{L\ N}{2}}{(\frac{L\ N}{2})^2 + M^2} = F R 2 V ; \quad q \frac{M}{(\frac{L\ N}{2})^2 + M^2} = V L 2 Q$$

64þ (10.20) E % Ï p

$$F R 2 V + V L 2 Q W D Q = 2 F R^2 V \ 1 + 2 V L Q F R V W D Q$$
$$= 2 F R^2 V \ 1 + 2 V L^2 Q = 1$$

64þ1b Â 0 b ˜ E

$$I I = (q \frac{}{(\frac{L\ N}{2})^2 + M^2} H + 1) \Theta^2 + (q \frac{}{(\frac{L\ N}{2})^2 + M^2} H \ 1) \Theta^2$$

1¶6B@6 • ¸ SD% Â,º n ù ^ Ä _+^@ Æ 3 ¦,º]C^ ¸ › ´ ä < ï0A4ú

$$\begin{cases} \Theta = F R V \Theta + V L Q \Theta \\ \Theta = V L Q \Theta + F R V \Theta \end{cases}$$

,º ¢ §,º0% < † $((\frac{L\ N}{2})^2 + M^2)^{\frac{1}{4}}$ >-. ‰ Ä+^ @ / Ê 5 ˜ Kµ / Ê >

G " % -A f ¸<³'l SD% ÂF F " SD% 6 C x ´ 6 ïM˜ bDá1•$_ ø ¦31,º Ž ^],º A÷ D gM" n C x ¸ 6 ïM˜ } Ð1•$_ ø ¦31 Ž ^ D ¸ Þ 2,ºEý0A •M" n C x µ | e ‰)æ 0 ¼@ Æ 3 ¦*<Að ^ (M˜ @ Ê+^,º µ#] >

10.2.2 Hopf ä <

^ @ 68¸ ³NÎ 10.8,º :*< c ¸ 6 ©Kd,º f ï ^ Ä g(z) = $\frac{L\ N}{2}$ iM e 6 `@ Æ 3 ¦ > ¬ ÔF `@ Æ 3 ¦] ì ¬ CL,+U > G " ^1b 9.3.2E8¸ C = ì p,ºF Þ ï0A c K › ¼ C 3 ¦ Q = h n$_z$; r$_z$i = L N 2iM

C 3 ¦ Q /.J0¬0°L* c (M˜,º]C^ ? ·],º d ß ¸ G " ? û?·$' › ØAÞ >'l6B ¸OÌ ~M˜ j,º 6 `L$NÎ g e Q Â † =31,ºF? ? © ¸ e vG Ð •,º > p!š ¸ G " K › Â!W ä <

$$= Q(\mathcal{G}z)^2$$

0& p (M˜ S ,º Hopf ä < > Û A › G "B* D ¹ Â vG 1•$_ ø ¦31,ºF? © Æ ¸ Ò w = x + iy e 6 ` D z = u + iv Ð G 68*,º vG 1•$_ ø ¦31 ¸Aô z = z(w) e " •L*,º ø ¦ ˜ ¸ o ¤ Ð*< 9.15¸ z e w ,º@ Æ 3 ¦ ¸ Ä e

$$h\,\mathcal{n}_w;\mathcal{r}_w i\,(\mathcal{G}w)^2 = h\,\mathcal{n}_z;\mathcal{r}_z i\,(\frac{\mathcal{G}z}{\mathcal{G}w})^2(\mathcal{G}w)^2 = h\,\mathcal{n}_z;\mathcal{r}_z i\,(\mathcal{G}z)^2$$

Hopf ä < 0 ¸ % ¾ ï I fA÷ D ¸ A ` ³NÎ >

³NÎ 10.9. (M˜ e ž7F,º ‰ J û ‰ ¹,º Hopf ä < 0 >

A÷ D. ^ (5 ³1•$_ ø ¦31,º *f* A ¸7F&ï E ©-v L = N ¸ M = 0 ¸6BF ¦ ¦1• - Ä Q = 0 > □

³NÎ 10.10. (M˜ S ,º © } ()½ e n ¦ ‰ J û ‰ ¹,º Hopf ä < e ž4å Â!W ä < >

A÷ D. # E f © } ()½ H e Ô 3 ¦ ¸ { H p n ¦,º { < û?· — , e $H_z = 0$ > Ã o ¤ C ˜ E,º Codazzi ï0A (9.9), !š , $Q_z = 0$ ¸ Q e@ Æ 3 ¦ ¸ 6B = $Q(\mathcal{G}z)^2$ e ž4å Â!W ä < > □

+g Ä!š , Q e@ Æ 3 ¦ ¸ Ã5 > ³NÎ 9.19¸ CLô Í f

ÞAð 10.11. Aô S e E^3 c,º n © } ()½ (M˜ ¸ ^M"7F&ï : ¸ ?

$$_S\ Oj\mathbf{Q}j^2 = 0$$

© $Oj\mathbf{Q}j^2$ e (M˜ @,ºB9 Â 3 ¦ >

+g Ä ^1•$_ ø ¦31 A

$$K = \frac{1}{4}(LN \quad M^2)$$

$$H = \frac{1}{2^2}(L + N)$$

6B

$$jQj^2 = \frac{(L \quad N)^2 + 4M^2}{16} = \frac{4}{4}(H^2 \quad K)$$

Ä e

$$\text{s } Q(K\Phi^2 \quad K) = \text{ s } Q(\frac{4}{4} jQj^2) = \text{ s } O\frac{4}{4}Q + \text{ s } OjQj^2 = 4 \text{ s } O Q$$

Ã# E f C † = A,º Gauss ï0A (9.8) ¸ B , _+^ Laplace 1Í † ^ C † = A,º ˜ E (9.11) ¸ t'l % Í f

ÞAð 10.12. AôS e E^3 c,º n © } ()½ (M˜ ¸ ^M"7F&ï : ¸ ?

$$\text{s } Q(K\Phi^2 \quad K) = 4K$$

M6?·# E,º e ¸ ÞAð,º) Z1% Â † =31,ºF? e ©,º ¸ © µ ^M"1•$_ ø ¦31 A • F1 >

*10.2.3 n © } ()½ (M˜,º e 6]

b E8¸,º 6 D ¸ G " ›A÷ D n © } ()½ (M˜,º e 6] Ð*< Æ

Ð*< 10.13 (Hopf). Aô e E^3 cL#,º n © } ()½ (M˜ ¸ ¸ Ò ,º Å r [1] pM, ¸O 6 Ð e*9M˜ >

A÷ D. OÌ ~ ¸ o ¤1b ? 1 ,º » ï ¸ ¸ Ò e ž7F,ºB ¸FÙ ~ ¹?· ~ e*9M˜ ¸?· ~ e ©M˜ > 6B — , c (5 . F‰ (M˜ ,º Å r pM, ¸FÙ ~ g 73 e*9M˜ >F m,ºB ¸?·A÷ D e*9M˜ ¸ ?·B* D ž7F > Ã o ¤ ³NÎ 10.9¸F E ©-v G "M6?·B* D @,º Hopf ä < 0 >

A÷ D @,º Hopf ä < 0 ,º)# " E¹(¯ a ¸M6?·+^ f C 3 ¦ c$' q,º Ð*< ¸ Riemann ‹ r L Ð*<´ • ! Poincare-Koebe ‹ r L Ð*< µ> Riemann ‹ r L Ð*< e = Æ1 E 6 ` ‹F FP,ºU 2M˜G3 Â ? ` = üU 2M˜ Æ ‹ ƒ*9M˜ ¸C ©M˜ ¸ ‹ ƒ <- ¸c,º 6 ` § ˜1• - > 1 ‹ 6 ` •% (M˜G3 % ?ü pU 2M˜ ¸6B ,º Å r pM, ¸ 73 § ˜1• - Ä ‹ ƒ *9M˜[2] >

Aæ ‹ ƒ*9M˜ p S^2 ¸ Â S^2 •L*,º § ˜1• - V : p ¸ o ¤ Ð*< 9.15¸F m,º ^ :8 ,º1•$_ ø ¦31 A e 6 `@ Æ 3 ¦ ¸ E ©-v À @,º ž4å ä < ˜ E ¸ V p [3] $S^2$ @,º ž4å ä < ˜ E >6B e n © } ()½ (M˜ ¸ ¬ @,º Hopf ä < (5 e 6 ` ž4å ä < ˜ E ¸

---

[1] F e 6 ` ‡ C H >

[2] F ,,º ‹ ƒ*9M˜ e ` »;ª,º (M˜ ¸ ¬ C 6 Ð?· t5¤ ^ E^3 c > B , ¸ vBI § ˜1• - ¸ g e Ž ^ @ / Ê >

[3] [r fAè ¸ eFPEý V : < ä < ˜ E ÿ >

!š V : f S^2 @ ¸ • e S^2 @,º ž4å ä < ˜ E ¸ C Þ Ó'lAæ ’ ># E ¸F ` , O,º ¬M” S^2 @,º Hopf ä < ¸ G "?· ¬ @ ÜB* D S^2 @,º Â!W ä < ˜ E pM, >

G "-U!™?·A÷ D,º e ¸ S^2 @ v ?,º ž4å Â!W ä < ˜ E } pM, > o ¤1b “ 1 c @ V :,º » ï ¸Aô $_1$: S^2nf N g ! R^2 Â $_1$: S^2nf Sg ! R^2 < a p M · Â • · 0 ,º*9 · Ë § ¸!š ,Aæ Z6; / Ê,º1•$_ ø ¦31 < a p z = u + iv Â w = x iy ¸ ¹ "G3 e S^2 ,º v G 1•$_ ø ¦31 ¸ CLôPÂA÷ Z6; •L*,º ˜ ©31 p z = $\frac{1}{w}$ >

}Aô e S^2 @ 1 ‹ 6 ` ž4å Â!W ä < E ¸ ^1•$_ ø ¦31 z A ¸ = $P_1(z)(\overset{\circ}{z})^2$ Ç ^ 1•$_ ø ¦31 w A ¸ = $P_2(w)(\overset{\circ}{w})^2$ > J P_1 Â P_2 < a e z Â w ,º ž4å 3 ¦ > _+^ w = $\frac{1}{z}$?

$$P_1(z)(\overset{\circ}{z})^2 = P_2(w)(\overset{\circ}{w})^2 = P_2(\frac{1}{z})\frac{1}{z^2}(\overset{\circ}{z})^2$$

6B

$$P_1(z) = \frac{P_2(\frac{1}{z})}{z^2}$$

F 6!›

$$\underset{z!1}{O}\ LP_1(z) = P_2(0)\ \underset{z!1}{O}\ L\frac{1}{z^2}P = 0$$

+g Liouville Ð*< ¸ $P_1(z)$ 0 ¸ 6B ž4å Â!W ä < E 0 > ä < E e S^2 @ 6 `(¯!À ,º ž4å Â!W ä < E ¸ • û'l pM, > □

Lš ¼ Hopf Ð*< L ¸F ? Alexsandrov Ð*< • e F& n © } ()½ (M˜,º e 6],º ¸6B J — , * ó ´ ¸ CM6?· Å r pM,,ºL† l >

Ð*< 10.14 (Alexsandrov).Aô p E^3 c,º , › ´ ©"× ?8 Ú&ï µ3]8* (M˜ ¸ J © } ()½ p n ¦ ¸ O e*9M˜ >

F 6 Ð*<,ºA÷ D ¸M6?·*<$; (M˜,º Laplace 1Í † D L *!]"E0°L* E^3 @FP n,º Laplace 1Í † •L*,º ©31 ¸F 1Í" E^1 C x ¸ G "F H C Ã ‹ 6 ¸ U ªCÙ,º B œ % ø 69[18] >

b1 # Aæ

bAè •,º 6 D 6G <.J0¬,º e (M˜ (M˜,º ª ‰]C^ ¸ ¬ c b1 .J0¬,º e (M˜,º L ^ ª ‰]C^ ><³'l G "F 3Aè •,ºFqEÇ e L ^ vG f »;ª ª ‰ ¸ %F ¬ C E ©-v L ^ ‹]C^ g CH ?· > 6 ïM˜ ¸ · E (M˜ ^)æ Ô c ?-v-* Û,º(Ÿ*<7 ¥ ¸ ¸5 ÐEï+, ˜ F,º

6Û,¸# ´M˜0% · E ¸ Plateau L$NÎ µÇ 6 ïM˜ ¸ · E (M˜ • % +^ ^.J0¬ *Q 5*U 2#w ˜,º »;ª]C^ c >

68b6B@6 ¸ · E (M˜ Â n © } ()½ (M˜ ¸ Ç L *Q 5*,º · E †#w ˜ Â n © } ()½ † #w ˜ ¸ q?·,º.J0¬ - eM”4õ] … ä < ï0A ¸8 'l e ?M” nQ ,ºLô Ü ¸ G "F m 6L S 1•,ºB40A ¬ CF8 >#¿)DF 2±L$NÎ > • v $1 ÖF 61 ¸ o b Õ e ¸ / Ä ?5*!] E0°L* c,º · E (M˜ Â n © } ()½ (M˜ ¸Lš ¼ n ù,ºM”4õ] … ä < ï0A ï# L ¸+g Ä ¦ ³ e (M˜ ¸ % _+^1•$_ ø ¦31 Laplace

Chapter 11

(4õ Â (M˜‚º »;ª ª ‰ ‹F?Aè

p ¼AßB40A‚º M s %73 µ# ¸ G "F? » ï‚ºLô Ü (5 s %73‚ºL*ƒ* „ ¸ À q?·1ý {G3 t ^ ¼ (4õ Â (M˜‚º vG *<Að @ >

œ –Eý *ƒ*M˜‚º » ï D ¸ B œ " ¬ Ô ÊB 73 U U f Æ ä < ‹‚º vG *<Að ¸ C73B* Â ä0% < ¸ ä < ï0A Â ž1• - ¸ p.¤ ÔF M6?· 6 Đ‚º ‹-*?ø ¸ •?·+^ f 6 Ñ4õ] ¦‚º - Ç | e $ aM” n ?L† >

0¬ ¬ Õ ¸• ¾ ï I*<@ ¸vG @‚º ‹ ¸q ˆ e1¶ ‹‚º >vG ä < ‹*<Að ¸ e 0.v Â ›L > 64þ‚º-$‚º ¸ Ó'l e ª ‰ ä < ‹ ¸F *ƒ* e G "‚º UEæ]#- >

ª ‰ ä < ‹‚º.J0¬ SD% ¸¶ e •!ë ¸Â vG ä < ‹ ¶ ¶]-. º ã >G " B Q U ß F 6G < CKµ‚º1ý { ¸73 U1× c0ÛB¯ ¸5] ì ‹/p 6 A ª ‰ ä < ‹.J0¬‚º /B— Â ï# >

11.1 ©M˜ @1¶ ‹L# (4õ‚º ª ‰]C^

G " ~ K ›9 ¨ ¸ + Æ

Đ • 11.1. ¸ Ò ƒ(s) e E² @‚º 6 —F 5# % ä (4õ ¸ a s b ¸ ¬ J ? ƒ(a) = ƒ(b) ¸ ƒ⁰(a) = ƒ⁰(b) ¸ O0& ¹ p F 5# % äL# (4õ >

Đ • 11.2. ¸ Ò ƒ(s) e E² @‚º 6 —L# (4õ ¸ a s b ¸¬ J / Ä 1 E‚º a $s_1 < s_2$ b G3 ? ƒ(s_1) ƃ ƒ(s_2) ¸ O0& ¹ p 1¶ ‹L# (4õ >

11.1.1 1• ž C1• E

©M˜ (4õ,º1• žL$NÎ e Æ ^5 Ð žKµ,º v ?1¶ ‹L# (4õ c ¸ ö ~ m,º (4õ v * F,º M˜0% 6] ËF e 6 `M” n 67,ºL$NÎ ¸ B7€ ð (5 5 01Š ~ e < ¸ |1b 6 ` [r,ºA÷ D e+g Weierstrass ^ 1870 ª5 0,º (1• žL$NÎ,º ¼ (Â :0 C B,ºA÷ D ï# ¸ % ø 69[26-27]) ¸ © ?

Ð*< 11.3. Aô ©M˜ @,º1¶ ‹L# (4õ C ,ºKµ Ü p L ¸ C v * F,º (M˜,ºM˜0% p A ¸ O ?

$$L^2 \quad 4\,A \quad 0$$

¬ c1• - F1 ‰ J û ‰ C e 6 ` < >

G " ^F H 5 +g Hurwitz ^ 1901 ª5 0,ºA÷ D [1] ¸ q?·+^ f,º - e Green ¢ E ¸ A,º

³NÎ 11.4 (Wirtinger C1• E). ¸ Ò f 2 C^1 $[0;2\,]$ ¸FÙ ~

$$\int_0^{2} [f'(t)]^2 \, dt \qquad \int_0^{2} [f(t) \quad f]^2 \, dt$$

F H f = $\frac{1}{2}\int_0^{2} f(t)\,dt$ e f ,º © } r > J1• - F1 ‰ J û ‰ f (t) = a F R ¥ V b V L Q c ¸ a; b; cG3 e n ¦ >

1ý { vL† ¸U AÚ Wirtinger C1• E (F1 >›-A Ð*<,º - ‰A÷ D ÆAô ^ ©M˜-*@ † = 31 A (4õ C ,º ø ¦ ï0A p x(s); y(s) ¸ s p]Kµ ø ¦ ¸ J s 2 [0; L] > • ø ¦ ˜ t = $\frac{2}{L}$s ¸ Ä e

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 \left(\frac{ds}{dt}\right)^2 = \frac{L^2}{4\ ^2}$$

F 6B

$$\int_0^{2} \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 \, dt = \frac{L^2}{2}$$

AôC v * F,º p p D ¸ ¬M˜0%Aæ pA > o ¤ Green ¢ E

$$\int_C x\,dy = \iint_D \frac{\partial x}{\partial x}\,dx\,dy = A$$

[1]Hurwitz ¬"× ?-* Û µ+^ Wirtinger C1• E ¸6B e+^ ¼ * 0 b,º Fourier ‹ 6 E Â Parsevel Ð*<

6 ïM˜ ¸

$$\int_C x\,dy = \int_0^2 x(t)\frac{dy}{dt}\,dt = \int_0^2 (x(t) - x)\frac{dy}{dt}\,dt + x\int_0^2 \frac{dy}{dt}\,dt =$$

4 11.1: =ð

F Í/p G "L# (4õ E¢ = = ^ 0 E • A e 6 ` ‡ C H ¸ Â (4õ - ‰,º ˜(ì © >-*?ø @-AC- › ¸F 6 Ð*< t'l e F1 ,º > G " j7G c %)B—F m 6 { 4 Æ

ˆ ¸ Ò C e < ¸ O i = 1 L6; 1 ¸!™CU é Ä (4õ C ø ¦ L,º ï G >

ˆ ¸ Ò C e1¶ ‹L# (4õ ¸ O C % •% f ˜ p < [1] ¸ J ^ ˜,ºEý0A c ¸ E¢ = = i • eF 5# f L,º > | i e 6 ` ª ¦ ¸ 73 7 C ¸ { 1 E1¶ ‹L# (4õ,º E¢ = = } p 1 >

'l6B ¸+g Ä %73 P 0)æ ¸ A2± r <ÀOÏ ˜(ì,º (4õ ¸)?· [rA÷ DF 6 Á ÔF ")B—,º ?· &Lô >

Û A › ¸ G " 5 Hopf ,º E¢ = = Ð*<,º S1•A÷ D ï# > ¬ 6 n ù » ï e ºFV (4õ ,º \ V : ¸6 ïM˜+^ \,ºE¢@ › Ý l (4õ = GH E¢,º { Ü ¸ 6 ïM˜ ¸µ Í01 Þ Õ&ï 0 \,º4þ&ï ¸ L6;01 Þ = G Õ&ï,º \,ºC-&ï ¸ \,ºE¢@ G3 p ¸FÙ ~ q,ºE¢@ p 2 > 1¶ õC-?÷ ¸ G " } Ð (4õ C ,º Ð GG3 eF< ,J¾!™ G ¸ • g e = GEÆ@ q ‰ e Ô Ö,º > - ‰ ›-A Æ

1b 6!› ¸ ºFV \ V : >OÌ ~M6?· ^ (4õ C @ 6 ` >F8,ºC-&ï >+g ÄL# (4õ e3]8* ,º ¸ û'l % ´ f (4õ @,º 6&ï ¸ µ Í ª ` (4õ Â ž:s ^F &ï =4õ,º 6 Ý > ˜@6 • ¸F — =4õ D ª ` (4õ C ,º Ú&ï ? J ? 6 ` > C g 68b] ¸ }AôF — =4õ p x Eª ¸ =&ï p Õ&ï O > F 6&ï O pC-&ïH æ µ+^]Kµ ø ¦ L (4õ ¸ Í f ŧ(s); s 2 [0; L] > 4 = f (s1; s2)j0 s1 s2 L g p ?@ ˜ p ¸ Ð • \ V : : 4 ! S[1] ¸ A

$$(s_1; s_2) = \begin{cases} \dfrac{\text{ŧ}(s_2) \quad \text{ŧ}(s_2)}{j\text{ŧ}(s_2) \quad \text{ŧ}(s_2)j} & 0 < s_2 \quad s_1 < L; \\ \text{ŧ}^0(s)(= \ \sim(s)) & s_1 = s_2 = s; \\ \text{ŧ}^0(0)(= \ \sim(0)) & s_1 = 0; s_2 = L: \end{cases}$$

t'l ^ 4 »G e •% ,º ¸ J ^ 4 ,ºEï+, @F 5# >M6?·# E,º e ¸ Ð • \ V : , ¸ G "+^ f ¼1¶ ‹L# (4õ"× ?8 Ú&ïF 6]C^ ¸ \ O ^ 0 < s 2 s1 < L F 6 û ˜ c <" %73 pM, >

[1] F ¬ Ô e B \,º 0 b)#

1b Â!› ¸FPEý \ V : ¸ K › \,º E¢ = = >)æ ^69<‡ ?@ ˜ p 4 @ (0; 0) &ï f (L; L) &ï,º žG F 5#D% º > ¬ c ¸"õ-*@ ?@ ˜ ÒEï,ºD% ºAæ p $_0$ ¸"õ Z —-*@ Eï,ºD% ºAæ p $_1$ > CLôAô) ¸ cL* ? 631 MF 5#,ºD% º $_t$; 0 t 1 ¸ $_0$ f $_1$ >

Ã69<‡ 3 ¦ ^ $_t$ @,ºL† l ¸Aæ $_t$ = j $_t$ ¸O $_t$: I $_t$! S^1 Ç H I $_t$ % ?ü p R @,º 6!ëL# pL* >D K*< 2.10 2± r ¸% fF 5# % ä 3 ¦ $_t$: I $_t$! R µ Í $_t$ = (F R $_t$ V V L $_t$)Q> Ð •"õD% º ,º Ü p D($_t$) = $\frac{t j_{(L;L)} \quad t j_{(0;0)}}{2}$ >-*?ø @-A ¸ D © p-. Ê \,º E¢ = = >

6 ïM˜ ¸ CAðF? FÙ —D% º ¸ (0; 0) &ï Â (L; L) &ï : (4õ,º =4õG3 e-. B,º ¸ • g e \ L6; =4õ ï G@ ,º L û'l e 2 ,º ª ¦ C ¸F E ©-v V : D ,º r ûN± e ª ¦ > 6 ïM˜ ¸D% º $_t$ 3 ¦ G3 eF 5# L,º ¸ D($_t$) © Ä t • ûN± eF 5#,º >52 > Z6; ¸ D ,º r û p n ¦ >

1b ?!› ¸+^ \ V :>ž/p (4õ = G {@ ,º L ¸¬+^ \,ºE¢@ Ý l = G {@ ,º L >#

E f $_0$ g e G " ^ K*< 2.10 c v ºFV,º 3 ¦ ¸ 6B D($_0$) g e1¶ ‹L# (4õ C ,º E¢ = = i(C) > Û A › G " Ã ›.J0¬ D($_1$) ¸F 6!› 6 p ©Kd > $_1$ % < p1 -* Â"j ©,º ZG < ¸ < 669<‡ > ^1 -*,º4õ!ë @ ¸ G "M6?·69 ,º e \ V : (0 ; s) ¸ ¹>ž/p,º e O &ï 0 ,º ‹ ƒ = GH L6; \ > s = 0 ,º S , q (0 ; 0) = ~(0) ¸ = G {@ p 0 Çs = L ,º4þ 5 , q (0 ; L) = ~(0) ¸ = G {@ p > UL* ¸ \,º4þ&ïG3 ^ =4õ x Eª,º @ Ý | Þ ¸ 6B = G {@ ,º Ô ÖH û'l p >

2± r f ¸ ^1 -*,º4õ!ë @ ¸69 \ V : (s; L) ¸ ¹>ž/p,º e4þ&ï p O ,º ‹ ƒ = GH L6; \ > B m+g Ä!š , \,ºC-&ï žG3:s ^ =4õ x Eª,º @ Ý ¸ E ©-v GH C-&ï ©018) Õ&ï D ¸ GH ^ x Eª A ÝE¢ Þ >F E ©-vEý0A c = GEÆ@ Ó'l e Ô Ö,º ¸ • p > 6B"õ-v $_1$ ¸ D($_1$) = 1 >

52 > D e 6 ` n ¦ ¸ G ". F‰ D($_0$) = 1 • F1 ¸ { (4õ C ,º E¢ = = p 1 > (4õ Ð GN° ,J¾CU G ,AÞAð Â ž-. B > Hopf ,º E¢ = = Ð*< gA÷ D ³ ¼ >

Hopf E¢ = = Ð*<F ? L 60 A÷# ¸ 'l ^ \ V :,º | ì A ¸ | e ºFV ³ \ V : : 4 ! S^1 D ¸+g Ä 4 e 6 ` ‹F FP p ¸ o ¤ V : } Ð*< (M6?· B \1• ¸ + ’ p 0 .v ¸ ø?÷ [11] ¸ Ð*< 5.3) ¸ % ºFVF 5# 3 ¦ $(s_1; s_2)$ ¸ µ Í

$$(s_1; s_2) = (\text{F R}(s_1; s_2);\ \text{V L}(s_1; s_2)):$$

Ã (s) = (s; s) ¸ o ¤ 3 ¦ Â ,ºF? ¸ g e% Cé (2.9) ,ºF 5# % ä 3 ¦ > B , +g Ä $(s_1; s_2)$ eF 5#,º ¸

$$(\text{L})\ (0) = (\text{L}; \text{L})\ (0; 0) = (\text{L}; \text{L})\ (0; \text{L})) + (\text{L})\ (0; \text{L})\ (0; 0))$$

Ÿ A,ºAÞAð Â • *f* \ V : ï# e 6 m,º >

ü1b 6G < ©M˜ (4õ » ï c,º (2.11) E ¸ ?

$$(b)\ (a) = \int_a^b k_r(s)\, ds$$

6B

$$\int_a^b k_r(s)\, ds = 2$$

©1¶ ‹L# (4õ,º q-. / ()½ p 2 ¸ ¬ c!™CU - é Ä ø ¦F? ,º Ð G >

Hopf E¢ = = Ð*< % Þ µ f <!ë •% ,º1¶ ‹L# (4õ @ ¸}Aô C e Ð G p!™,º ¸< !ë •% ,º1¶ ‹L# (4õ ÷ : [0; L] ! E^2 ¸ ¡ p <!ë :,º L@ ¸ O ?

Ð*< 11.7 (Þ µ ¼,º Hopf's Umlaufsatz).

$$\int_0^L {}_r(s)\, ds + \sum_i {}_i = 2$$

B Ð*< % ?ü p D5# Gauss-Bonnet ¢ E,º(¯ Á ¸ G "F H g C5 0 - ‰A÷ D ¼ > U ªCÙ,º B œ % ø69 [24] > G ") = 0,º eF H µ+^ L@ ¡ e8 'l,º ¸ p (4õ ? Ð G ¸ "õ-v Ð G ¸ ^ <!ë :EÆ@ ,º LH g eB &ï :,º L@ >

* P © Ä ©M˜ (4õ,º]C^ ¬A÷ D ¸B- ø69 [24,28] >

11.2 (M˜,º »;ª ª ‰]C^

11.2.1 Gauss-Bonnet¢ E,º vG S Ø

FPEý • *f* ,º œ – % -A 0 ¸ Gauss5 Ï Ð*< ÔL{ @5 G " 6 e ¼ »;ª ‹.J0¬,º] L ÆJ0¬ - ?1b 6 0 b ˜ E,º 6 V óB— (M˜ ¸6B C e t ^ E^3 c,º 6 V - ‰ (M˜ > ˜

B B* ¸ G "69<‡,º vBI (M˜ ¸ ¬ Ô e A M 6 `!™ Ð,º Â!W ä < ˜ E Æ

$$\mathfrak{G}^2 = E(\mathfrak{H})^2 + 2F\,\mathfrak{H}\,\mathfrak{V} + G(\mathfrak{V})^2$$

•0& pB óB— (M˜ @,º ÜH ˜ E > s1× ¹ ¬M"-U Ô %?÷,º 6 V (M˜ ¸ | Ó'l % A×1Í ¬ @ (4õ,ºKµ Ü ¸ o@ ¸ p ,ºM˜0%1• ‹H >" ¸B* ¸ Z — (4õ ^ Ú&ï (u; v) :,º = GH < a p ($\mathfrak{H}$; $\mathfrak{V}$) ¸ (u; v) ¸ O ¬ o@ ,º • \ % >ž/p ¸ A

$$F\,R\,\forall\ p\ \frac{E\,\mathfrak{H}\,u + F(\mathfrak{H}\,v + \mathfrak{V}\,u) + G\,\mathfrak{V}\,v}{E(\mathfrak{H})^2 + 2F\,\mathfrak{H}\,\mathfrak{V} + G(\mathfrak{V})^2}\ p\ \frac{}{E(u)^2 + 2F\,u\,v + G(v)^2}$$

(M˜Kµ Ü ¸ p M˜0% •2±,ºA×1Í g C ÃCŽF& > q • ¸ ^ óB—,º (M˜ @ 'l ?Cé U f ,º ‹ œ % .J0¬ >

)æ ^ G " ›.J0¬ »;ª ‹ c#¿ @ Ü,º 6 `L$NÎ > (5 . F‰ ¸ A Z ` Á Ô

ˆ ©M˜ @ ?@ ˜ ? ` »@ Â p 180 Ç

ˆ (M˜ @,º#• f4õ-. ‰ Ä ©M˜ @,º-*4õ ¸ % ºFV (M˜ @,º#• f ?@ ˜ >

8 'l,ºL$NÎ Æ (M˜ @#• f ?@ ˜,º »@ Â e P G Ë G " 61¶ ‹,º ‹ ƒ*9M˜ › 1 ´ SD% > ‹ ƒ*9M˜,º Gauss ()½ p 1 ¸ e n ()½ (M˜ > ¬ @,º#• f4õG3 e] <,º 6G < > CLôA× 1Í 0 ‹ ƒ*9M˜ @#• f ?@ ˜,º »@ Â] Ä 180 >

F 65 Ò Â K 0 ,º!]"E0°L* ? ¾],º p a > %)R#• ¸ »@ Â D (M˜,º »;ª ‹ H Gauss ()½-. © > ÔL{ @ ¸F g e vBI,º Gauss-Bonnet ¢ E Æ

Ð*< 11.8. (4õ C e ? G (M˜ S @,º 6 —+g n !ë •% (4õ4ú F,º1¶ ‹L# (4õ ¸ ¹ v ; * ,º p D e (M˜ S ,º 6 ` ‹F FP p ¸ O

$$\mathfrak{G}_g + \int_C \mathfrak{G}_D K\,G = 2\sum_{i=1}^{n} \mathfrak{I}_i \qquad (11.2)$$

¬ c $_g$ e (4õ C ,º#• f ()½ ¸ K e (M˜,º Gauss ()½ ¸ G eM˜0% y ¸ $_i$ >ž/p (4õ C ^@ &ï s_i :,º L@ >

¾ ï I-A 0 ¸ ‰ Gauss-Bonnet ¢ E+^ Ä © œ,º!]"E0°L* ¸ g Í f ¼ Ð*< 11.7¸ Þ µ ¼,º Hopf E¢ = = Ð*< >

G "1¶F& Gauss-Bonnet ¢ E,ºA÷ D SD% > pF H #¿ f ¼ (4õ •L*,º o@ ¸8 'l P69<‡ (4õ = GH ,ºE¢@ ¸ • g e#• f ()½,º Liouville ¢ E > p ¼1¶ ‹C-?÷ ¸ C Þ } Ð ^ ‹F FP p @ (5 ³!™ Ú ø ¦31 ¸ Ä e Æ

$$_g = \frac{G(s)}{\mathfrak{G}}\ p\ \frac{1}{2\,\overline{EG}}\ E_v\frac{\mathfrak{H}}{\mathfrak{G}}\ G_u\frac{\mathfrak{V}}{\mathfrak{G}}$$

'Õ ¿ ä0% <,º B œ 6-r g % -A 0 ¸]8* @ ¸ ?· / ¢ E) Z1% B , © Ä1¶ ‹L# (4õF >,0% < ¸ H C G(s) G G g e)1%,º 2 n i=1 i ¸6B

I C 2 p 1 EG E v G G G u G G G = I C 2 p 1 EG (E v G G u G)

/ Ä <!ë •% L# (4õ ¸ Ó'l % µ+^ Green ¢ E ¸ <1b Â2± (4õ0% <E¢ L p v * p ,º ÂH 0% < Æ

ZZ 1 2 D (p G u EG)u + (p E v EG)v G G

E f0°L* (M˜,ºM˜0% y ÔL{ @ p G = p EG G G ¸ @ E © p

ZZ D p 1 EG (p E v G)v + (p G u E)u ! G

6 D ¸+g Ä!™ Ú ø ¦31 A

K = p 1 EG (p E v G)v + (p G u E)u !

01N¯ D © e 1%,º RR D K G >

F m,º SD%] ï G e"× ?L$NÎ,º ¸- ‰ 'C- › ¸¾ ! g PF} f ¸ A • ƒ"× ?F} fEý ,ºL$NÎ ÆG ",ºA÷ D SD% ÓCŒ Ä Liouville ¢ E ¸ Liouville ¢ E Ž ^,º ƒ e ¼!™ Ú ø ¦ 31 >| e C?· Aæ ¸!™ Ú ø ¦31,º Ž ^ e vG ,º ¸ 6 `!™ Ú ø ¦31 ¸¬ C 6 Ð % ?¼- ª `L# (4õ v * F,º p ¸• g e @F&*<) û ˜ A,ºA÷ D SD% 73F8+^ Ä vG ,º ?@ ˜ ´:s ^!™ Ú ø ¦31 Ž ^,ºFñ » µ>F • e p ö ~ G "F 6 E8¸,º =NÎ p Gauss-Bonnet ¢ E,º vG Ø0¬ >FÙ0¬1 ¸ ‹A÷ D ^ ‹F FP p @ Gauss-Bonnet ¢ E e ª ‰ F1 ,º Ë

? Ñ B œ PB* ÆLiouville ¢ E ÓCŒ!™ Ú ø ¦31 ¸ G " % ˜ 6 ` C ÓCŒ Ä!™ Ú ø ¦ 31,º ¢ E > CEýF m,º ¢ EM6?·+^ f#q Þ = ì# F 6Q 4Ý,º - ¸ ¬ C ^ G "F bAè • ØAÞ,º99 * (ø?÷[18] ¸1b 5.5 E8¸ µ>

•Aî • ? B œ PB* ÆFÙ •"× ©31 ¸ % ~L† l ^ vG F >,AÞAð ¸ Ã À" ` EFñ » ,º5 Ò 2 ÛC- › C g % ¼ >

F mAè ‰'l C731ÍKO ¸6B JF m,º SD% ¦ ¦ e ª ‰ ä < ‹ 6 n ù,º •D% Ç e c L* ?M" n P,º4ü8¸M6?·>› { ¸ A 6 E8¸ ¸ G " õ Ó'•F 6 SD% 5 Gauss-Bonnet ¢ E,º Â ªA÷ D >

11.2.2 ‹F FP p @,º Gauss-Bonnet¢ E,ºA÷ D SD%

F H ,º ‹F FP p ¸ =,º e (M˜ @,º 6(} B7 D Â5* 6 <- ,º p ¸1¶ ‹,º*<@ g
e p cL*"× ? 6 Š > ‡ @-A ‹F FP p e © ,º ¸ | e ‹ @ C 6 Ð ¸ p ^ Ü
H ,º E • A ¸F ` ‹F FP p % ¾ ] ¸ 8) Ä # +^ 6 ` ª ‰,º!™ Ú ø ¦31?¼- >
¾8 'l ¸ ¸ Ò (4õ C v *,º p D C73 ; a ^ (M˜ S ,º 6 ` - ?!™ Ú ø ¦31,º †
=Fñ » ¸ G " % +^ <!ë •% ,º (4õ < p D < ¨ F ?L† ` ‹F FP,º E p ¸ µ Í

ˆ " 6 ` E p G3:s ^ S ,º ` - ?!™ Ú ø ¦31,º † = c Ç

ˆ " 6 ` E p ,ºEï+, 'l p 6 —1¶ ‹L# (4õ ¸ B , ¸ % ?·"x 1 E Z `-.Fñ,º E p
G3 ? 6!ë •% ,ºEï+, Â žH > >

Ä e (11.2) E ^" ` E p @ F1 > G " <" ` E p @,º ¢ E • Â ¸ B Q 64þ % Í f
ª ` p @,º Gauss-Bonnet ¢ E > | p ¼AðF&1¶ ‹ ¸ G " 69<‡ Z ` E p D_1 Â D_2
4ú >,º û ˜ ¸ * P ¦H p 4ú >,º û ë ¸ % 2± rAðA÷ >
AæC$_1$ Â C_2 < a p D_1 Â D_2 ,ºEï+, ¸ ? •% (4õ!ë C = C_1 \ C_2 ' p ¢ §,ºEï+, >
+g Ä D_1 Â D_2 :8 :s ^ `!™ Ú ø ¦ † =Fñ » ¸ B ,# E f K Â ÂF? ,º ø ¦31
© ¸ Ä e ?

$$\mathrm{I}_{C_1}\ {}_{g}\mathrm{G}+\mathrm{ZZ}_{D_1}\ \mathrm{K}\ \mathrm{G} = 2\ {}^{n_1}_{i=1}\ {}_{i}$$

$$\mathrm{I}_{C_2}\ {}_{g}\mathrm{G}+\mathrm{ZZ}_{D_2}\ \mathrm{K}\ \mathrm{G} = 2\ {}^{n_2}_{j=1}\ {}_{j}$$

OÌ ~ ¸ D_1 Â D_2 ?-. B,º Ð G ¸ Ä e (4õ!ë C ^ Z `0% < E c,º Ð G-. > < @F& Z `
E † ' Â ¸ ^ C @ © Ä#• f ()½ 0% <,º ZG <1\ --. ¸ ¦ ³-. ë#¾ ¸ Ä e ?

$$\mathrm{I}_{\tilde{C}}\ {}_{g}\mathrm{G}+\mathrm{ZZ}_{D_1[\ D_2}\ \mathrm{K}\ \mathrm{G} = 4\ \left({}^{n_1}_{i=1}\ {}_{i} + {}^{n_2}_{j=1}\ {}_{j} \right) \qquad (11.3)$$

!š , ¸ $\tilde{C}$ = (C_1/ C) [(C_2/ C) e 6 —1¶ ‹L# (4õ ¸ p p D_1 [D_2 ,ºEï+, ¸ J D_1 [D_2 ^
$\tilde{C}$ @ vB' 2,º Ð G ¸ Â C_1/ C C_2/ C Õ ?,º Ð G e 68*,º >
Û A ›H &ï ©# (11.3) E,º)1% > C ÞAô p; q < a p (4õ!ë C ,º1%&ï ¸ JL† l Ä C ¸
C_1 ,º Ð G p = G q, ^ p &ï,º L@ p $_1$ ¸ q &ï,º L@ p $_2$ ÇC_2 ,º Ð G q = G p, ^
p &ï,º L@ p $_2$ ¸ q &ï,º L@ p $_1$ >

p ¼AÞAð * ? >ž] ¸ G " o ¤ e \ p æ,º (4õ Ć ,º@ &ï ¸ < p &ï Â q &ï,º û ë ˆ p 0 C B,º2± Á > G " ~ ¬ c 60 p ÁF >,B* D > ^ p &ï ¸ }Aô $_1 + _2 =$ ¸!š , (4õ Ć ^ p &ïF 5# % ä ¸ p &ïM”@ &ï Ç ^ q &ï ¸ }Aô $_2 + _1$ ¸!š , ¸ æ,º L@ p

$$\sim = (\ _2)\ (\ _1) = +(\ _2 + \ _1)$$

Ä e

$$4\ (\sum_{i=1}^{n_1} {}_i + \sum_{j=1}^{n_2} {}_j) = 4\ (\sum_{i=3}^{n_1} {}_i + \sum_{j=3}^{n_2} {}_j)\ (\ _1 + \ _2)\ (\ _2 + \ _1)$$

$$= 4\ (\sum_{i=3}^{n_1} {}_i + \sum_{j=3}^{n_2} {}_j)\ (\ + \sim)$$

$$= 2\ (\sim + \sum_{i=3}^{n_1} {}_i + \sum_{j=3}^{n_2} {}_j)$$

6B ÷ $_3$; ; $_{n_1}$; $_3$; ; $_{n_2}$ ¦ ³ g e æ,ºL# (4õ Ć ,º žG L@ >© Gauss-Bonnet ¢ E ^ p D_1 [D_2 @ • F1 [1] > L ?0 û ˜ ¸ p; q }M”@ &ï =p; q } p@ &ï = p p@ &ï q M” @ &ïEý0AM- B >

2± Þ ¸ % A÷ D ^ 1 E ‹F FP p @,º Gauss-Bonnet ¢ EG3 F1 >

)æ ^ G " % › 1Š • *ƒ* 0,º (M˜ @#• f ?@ ˜,º »@ Â e P GF 6L$NÎ Æ!š , ,º#• f ()½ žG pM, ¸ Ä e

$$\int\int_D K\ G = 2\sum_{i=1}^{3} {}_i$$

©

$$\sum_{i=1}^{3} (\ _i) = + \int\int_D K\ G$$

%?÷

ˆ K=0 ¸ »@ Â1• Ä 180

ˆ K<0 ¸ »@ Â E Ä 180

ˆ K>0 ¸ »@ Â] Ä 180

-." Liouville ¢ E ¸ ‹F FP p @,º Gauss-Bonnet ¢ E ^ 6 Ð0A Ü @ % AÚ p e ª ‰ ,º5 Að ¸ e / EFñ » F1 ,º Liouville ¢ E0% < • D ¸ Ã 2 Û6B ›,º Ç | ÔL{ @F m,º ’ Ó'l e ^ vBI,º (M˜(} @,º ¸ bC^F e vG ,º5 Að > B , ¸ ‹F FPF 6 — ,,ºL† l ¸ µ ÍF 65 Að # Ê+^ fM” ‹F FP,º (M˜ ¸ • g e Å r ²] Ä1• Ä 1 ,º (M˜ >

[1] ÔL{ @ ¸F 5# % ä&ï,º L@ } pM, ¸ • g eF 5# % ä&ï % ?ü p(¯!À,º@ &ï ¸ % C • p < ¸5 6A÷ D

[2] ‡ ¸ + ¸ % ø69 [11] >

A 6 E8¸ c ¸ G "0·.j ‹F FP,ºL† l ¸ ^ Â ª,º 6 V % Ð G3]8* Eï (M˜ @ 01 ª ‰,º Gauss-Bonnet Ð*< >

*11.2.3 (M˜,º Ð G

G " ^ 3.1.2E8¸ 5 Eý ø ¦ (M˜(} ƒ(u; v) ,º Ð G ¸ ©5 Ð# GH ƒu ƒv ¸ L6; Eý › ¸ ƒv ƒu >

F 68¸ 6 ¸ G ".J0¬,º /B— C Ã e (M˜(} ¸6B e Â ª,º (M˜ >1¶ ‹ f*<@ ¸g e À 631 M,º (M˜(}2ÎCjC- › ¸ Í f vBI,º Â5*#w ˜ ¹ > t ` 61¶ ‹,º Á † ¸ ‹ ƒ*9M˜ ¸ ¹ e "× ? Ô# +^ ‹ 6,º ø ¦ ï0A › F&,º ¸ © õ µ+^?¼- 99 * 6 ],º*9 · Ë § ¸ •M6?· • M · : 6 `*9 · Ë § ¸ • g e8) G+^ Z ` (M˜(} ›?¼- ª ` ‹ ƒ*9M˜ >

(M˜ ¸ ’ p 6 ` ª ‰#w ˜,º Ð G ¸ % Z `1• -@ Ü ›-A > ¸ Ò (M˜(}2Î >,º @ Ü-A ¸ vBI (M˜,º Ð G ¸ e = 631 M,º ø ¦ L,º (M˜(} f (uᵢ ; vᵢ)g ¸ ^H ,º&ï : ¸ } ? Jacobi . Lk,º>, M E

$$\begin{matrix} \frac{@ψ}{@ψ} & \frac{@ψ}{@ɣ} \\ \frac{@ɣ}{@ψ} & \frac{@ɣ}{@ɣ} \end{matrix} > 0$$

‰'l ¸ p G " /#w ˜,ºB#@6 ¬ C'Õ ¿ ¸ v G " B Q 6 ` *-*?ø,º@ Ü › K › Ð G Æ1¶ ‹ ›B* ¸g e p (M˜,º" 6 `&ï ´ •% f µ = Ð 6 `!™ Ý Â 6 `CU Ý ¸F 1• - Ä ^ (M˜ @F 5# fF? 6 ` ‹ ƒ# GH ´ % g p : :M”M,,º# GH µÞ

¬M” v ?,º (M˜G3 % ´ fF m,º ø ¦ L (M˜(} f (uᵢ ; vᵢ)g ›?¼- ¸ L6; : :M”M,,º # GH p ¸" ¸B*] ì'Õ ¿,º Möbius \ ¸ g # p < » L >73 ´ fF m Ð G,º (M˜ 0& p % Ð G (M˜ ¸ •0& p C % Ð G (M˜ > % Ð G (M˜ } % ? Z0 C B,º Ð G % F? >

M6?·NÓ L# E,º e ¸ ‹F FP,º#öyp Vqy=ö nð ´ •'“U!õg “Ð7 - E8¸ 5 Eñ— D jC- “a&%6 g ôÃ ð G %R ¸"

11.2.4 % Ð G3]8 EïL# (M˜ @,º Gauss-Bonnet¢ E1¶

)æ ^ ¸ G " B Q73 ^ Â ª,º (M˜ @ 01 Gauss-Bonnet Ð*< ¸F 6B73 U À ª V (M˜,º ‡G3 ; aF › >

Ð*< 11.9 (Gauss-BonnetÐ*<). 9 S e % Ð G3]8* EïL# (M˜ ¸ O ?

$$\int\int_S K \, G = 2 \quad (S)$$

F H (S) =,º e (M˜ S ,º!] ÿ/p] ¦ ¸ e 6 ` ‡ C H ¸ J ? (S) = 2(1 g) ¸ ¬ c g e (M˜,º Å r > Û A › G "1¶F& ª ‰ Gauss-Bonnet Ð*<,ºA÷ D SD% Æ 6 ïM˜ ¸ Ð *<,ºA÷ D6å Ð e?·+^ f ‹F FP p ,º Gauss-Bonnet ¢ E > 6 ïM˜ ¸ pF 6 ¢ E þ ^ ‹F FP p F1 ¸6B ª V (M˜ þ C ‹F FP ¸-*?ø @-A ¸ % À ª V (M˜>÷ F ‹F FP ,º E • ¸'l D À" 6 E • @,º Gauss-Bonnet ¢ E (11.2) 2 ÛC- › >

/ Ä Gauss-Bonnet ¢ E,º 2 Û ¸G " (5 ? ¼ :*<5 PÂ ¸F H ¬ C" • ƒ C x ` P > ©Kd,ºL$NÎ g F ¼ ¸ ‹ ƒ73Q ~ f Œ < ª V (M˜ p ‹F FP,º E • ¸ µ Í • D 2 Û,ºA× 1ÍH >á Ý I ^73 :*<,º99 * » >

˜ B ¸ G " B Q ^73Eô f Œ < p ‹F FP E •,º 0.v @ ¸ Œ <CÀ1¶ ‹CÀ ³ >8 'l) f+^ 61¶ ‹,º ?@ ˜ ›F >, Œ < >4Ü Ð (M˜ S @,º (Eï ?@ ˜ ´ @ A ½$; ¦ ,1¶0& p ? @ ˜ µ p ? ` C B,º&ï ¸ ÂF ÛF ?&ï,º ? — •% (4õ º F,º1¶ ‹L# (4õ ¸ * 0,º ‹F FP p > G " ÀF ? ` C B,º&ï0& p ?@ ˜,ºN¬&ï ¸ ? — •% (4õ0& p ?@ ˜,ºEï Ç ? —Eï > ^ 6C-0& p ?@ ˜,ºEï+, ÇEï+, v * F,º ‹F FP p 0& p ?@ ˜,ºM˜ ¸ ¬ cLš ñEï,ºG <0& p ?@ ˜,º »G >

1 E (M˜ S ,º ?@ Œ < e S ,º 6 ?@ ˜ f T_lg ¸% Cé

1. / S ,º 1 E 6&ï P ¸ P û'l:s ^ 6 ` ?@ ˜ @ ¸ J P C:s ^ ?@ ˜,ºEï+, @ , ¸ P ” Ä e 6,º ` ?@ ˜ Ç

2. ‰P :s ^ ` ?@ ˜ $T_1$ ,ºEï @ ¸ J C e $T_1$ ,ºN¬&ï , ¸ O?· ~ P C Ã ” Ä ¬ ? @ ˜,º 1 ‹Eï ¸?· ~:s ^ L e 6 6 ` ?@ ˜ T_2 ,º 6 —Eï @ ¸ J C e T_2 ,ºN¬&ï Ç

3. / Ä 1 ‹ ?@ ˜,º 1 ‹ 6 `N¬&ï P ¸ P pN¬&ï,º ?@ ˜ ? ?L† ` ¸ B ,F Ñ P p ¢ §N¬&ï,º ?@ ˜H ¸ 1 E Z ` ?@ ˜G38) P ? 6 ` ¢ §Eï >

E f ¸ Ò>á Œ <,º (M˜ e3]8* Eï,º ¸FÙ ~1b 2 ` — , g F ¼ Æ ‰ P :s ^ ` ?@ ˜ T_1 ,ºEï @ ¸ J C e T_1 ,ºN¬&ï , ¸ O P 6 Ð •:s ^ L e 6 6 ` ?@ ˜ T_2 ,º 6 —Eï @ ¸ J C e T_2 ,ºN¬&ï >

@F& / 1 E (M˜ ?@ Œ <,ºL† Ð ¸ % ˜ F *1¶#7,º1• - F& [1] ¸ © ^

$$S = \left[\; T_l \right.$$

,º *f* A ¸ / 1 ‹ l 6 l^0 ¸ T_l Â T_{l^0} ¸ û ? A ?0 û ëH ,º 60 û ë F1

1. T_l \ T_{l^0} = ; Ç
2. T_l Â T_{l^0} ? J ? 6 ` ¢ §N¬&ï Ç
3. T_l Â T_{l^0} ? 6 — ¢ §Eï ´ ; aEï,º Z `1%&ï µ

B m,º ¸ / Ä3]8* Eï (M˜ ¸F M6?· Ö @ — , Æ 1 ‹ 6 —Eï }:s ^ Z ` C B,º ?@ ˜ c >

?L† ?@ Œ <,º Ž ^] e 6 `M” n C x,ºL$NÎ ¸M6?·+^ f$' ›,ºU 2 ‹,º. Aü ¸U ªCÙ,º B œ % ø69 [24] c,º1b 12 1 >)æ ^ G " C ÞU AÚ3]8* Eï % Ð G,º (M˜ S @ ( 5 Ž ^ ?L†,º ?@ Œ < f $T_l$ $g^f_{l=1}$ ¸AæF ` Œ < c žG N¬&ï,º ¦-$ p v ¸ žG Eï,º ¦-$ p e ¸ 6B žG M˜,º ¦-$ t'l p f >

+g Ä" 6 ` ?@ ˜G3 e ‹F FP,º ¸ G " % ^ T_l @ µ+^ Gauss-Bonnet ¢ E (11.2) ¸ ?

$$\mathop{\mathrm{ZZ}}_{T_l} K \; G + \mathop{\mathrm{I}}_{@T_l} g \; G = 2 \quad (\; l_1 + \; l_2 + \; l_3)$$

F H $_l$ (= 1; 2; 3) e ?@ ˜ T_l ,º L@ >Aæ $_l$ = $_l$ p / Ê,º »@ ¸ @ E o Ï p

$$\mathop{\mathrm{ZZ}}_{T_l} K \; G + \mathop{\mathrm{I}}_{@T_l} g \; G = 2 \quad 3 \; + (\; l_1 + \; l_2 + \; l_3)$$

< f ` ?@ ˜ @,º1• E-. Ö ># E f" —EïG3:s ^-.Fñ,º Z ` ?@ ˜ c ¸ B ,+g Ä (M˜ % Ð G ¸ (4õ0% <,º ï G-. ¸ žG G3 ë#¾ ¸ { 1% p

$$\mathop{X^f}_{l=1} \mathop{\mathrm{ZZ}}_{T_l} K \; G + \mathop{X^f}_{l=1} \mathop{\mathrm{I}}_{@T_l} g \; G = \mathop{\mathrm{ZZ}}_{S} K \; G$$

[1] ÔL{ @ Z0 F&,º 0 &ï e / ¬,º

)1% p

$$2f \quad 3f + \sum_{I=1}^{f} (I_1 + I_2 + I_3)$$

H &ï ›-A »@ Â $\sum_{I=1}^{f}(I_1 + I_2 + I_3)$ ¸ G " ? C B,ºN¬&ïH æ <4ú >+g Ä Eï ¸ J (M˜ •% ¸" `N¬&ï ž *,º »@ Â û'l e 2 ¸ !š q,º »@ Â û p 2 v > Ä e ?

$$K\ G = 2\ f \quad 3\ f + 2\ v \iint_S$$

F H "× ? < 2 f 3 f > ¬ B2±N¯,º Õ ^ Ä ¸ / Ä Eï (M˜ ¸" —EïG3 ^ J û ^ Z ` ?@ ˜ c ¸6B" ` ?@ ˜ • 0 Ð ? ? —Eï ¸ {M˜ ¦ f ÂEï ¦ e ? ¸ A ©31

$$\frac{3f}{2} = e$$

6B ?

$$K\ G = 2\ (f \quad e + v) \iint_S$$

‡ @ % A÷ D f e + v D ?@ Œ <,ºF? © ² ¸ J ?

$$f \quad e + v = (S)$$

64þ Â F ¼ Ð*<,ºA÷ D > / Ä3]8* \Eï,º % Ð G (M˜ ¸ • ?-.F ,º ˜ E ¸B16; % 8 >, Þ 2 ¸ L6; ø69 [18] >

Gauss-Bonnet Ð*<"ÕFP ¼ ‹ Â ‡ ¸ Â5Ä f ‹)æ ¼ ‹ Â ‡ •L*,º-. È l4Ü © 31 ¸ % AÚ p e b0 ä < ‹ c 6$' q ¸ 6 N5Ä,º Ð*< >

b1 # Aæ

(4õ Â (M˜,º » ^ ª ‰ ‹ ¸ FqEÇ @Aè ¸3]D ^#• f4õ » ï • D e 6Cj =,º ¸ p ¹+^ f,º - q?· g e#• f ()½ Â#• f4õ > | e / Ä b0 ,º (4õ Â (M˜Að6B@6 ¸ » ï,ºH ?·0A Ü @ ›-A ¸F 61 » ï t'l"× ?1b “ 1 Â1b wH ?· ¸ v 1J6; ÀF 61 t ^ ¼Aè •,º 6 D >

FÙ ~6å Ð P ?B16;+Ç ¸ 'l –?ÿ Í ¹ CH ?· ¸p ö ~?· Ï ^Aè • c ˜ Ë pH ?· L 6; CH ?· ¸G3?· Ó ¤B16; v ^,ºLl!ë ›-A > 3 » 68bQ W,º b0 ä < ‹ e CAè ¾F Ñ

[1] ÔL{ @+P8) % Gý+^ PEï ˜ Œ < ¸ CL† Ä ?@ ˜ >

[2] ‰'l ¸ C+^ ‡,º ï# Ü % ¸-* Û?ø @F&1• E,º 1% ¸0% < $\iint_S$ K G t'l D ?@ Œ < © >

ª ‰ ‹ » ï‚º ¸6B J ÔL{ @.J0¬+U ä < ‹B40A ¶ ¶ • CAè ¾F Ñ vBI‚º S1• »
ï ¸FV F ¼ Á Ô @‚º-(p Ç6B œ+U "8 ' ¸(5 +è Ä Ê :0 ï0A = (4õ ¸© õ • † c#¿
¼ ¸ •"× ? ù S8 œ > | e D5# ª ‰ ä < ‹ œ –‚º@ Ü-A ¸F 61 þ 0 }‚ºH ?· ¸
p / Ä ‹6B@6 ¸vG q e1¶ ‹‚º ¸+P8) e © ≻U!™ ? - r‚º » ïG3 e ª ‰‚º ¸< ›
.J0¬+U ä < ‹ vAÞAð‚ºL$NÎ ¸ Ç L0 .J c-U!™?·.J0¬‚ºL$NÎ ¸ 6 ÐG3 e ª ‰‚º >

y>— • A ¸ G ` ð?ÿ Í ^ ‰ ³ vG ‹ ´ 0 b n ä < … ä < ï# p n ù ¸Lz Ö 6
Ñ C 3 ¦‚º ï# µ‚º 0.v @ ¸F e ? û?· ¼@ ª ‰ ä < ‹ D vG ä < ‹.J0¬ SD%
‚º $ 8 ≻Ŧ ` $ 8M” n‚º] ¸1¶ ‹ fAè ¸-.E¹ D vG ä < ‹ ¸ª ‰ ä < ‹ „ % B*
"× ? ö ~ 0 Ð‚º •D% ¸ C BL$NÎGý+^‚º ï# %]-. º ã >

B ‚ ¸ © õ e 61¶ ‹‚º ©M˜ (4õ‚º ª ‰]C^ ¸.J0¬‚ºLô Ü •F ^ vG ä < ‹ • @ >
!š G " ÀF 61 t5¤ ^ • †M– D‚ºG < ¸6B C 6 Ñ 3 L • † 6 m < a < ™ ^ (4õ Â
(M˜‚º » ï ‰ c ñ >F 6 ïM˜ e p ¼ * Ö0· 0+g L ^ f »;ª ¸+g vG f ª ‰‚ºFqEÇ q4õ ¸
6 ïM˜ • e+g I fLô ¸ û ù] ì 6 6 g.J0¬ (4õ‚º ª ‰]C^ P Ý+U+…Lô ù*< >

• g eB* ¸G " ÀF 61 t ^-$ ƒF 6 ƒ5¤ ¸e B Q8 œ‚º B œ73 U / ª ‰ ä < ‹
L P L G ? 6 Ð‚º ƒ-ñ ¸ © õ e ¬ c‚º ¾ P4ü8¸ ¬"× ? Â ž :$; • • S"!"× ? ©31 >
p ¸ Ò C •F ïM˜‚º.J0¬ ¸8 'l+^ C @ Ç6B < ›?· •F ïM˜‚º.J0¬ ¸ !™ • P ^.J0¬+U
ä < ‹ c IL ›.J0¬#w ˜‚º ª ‰ ‹ ‡]C^ >

6 D ¸F 61 eF? ¼ 6 Ñ-. /H ?· ¸ JLô Ü C1Í(¯ aQ ‚º (4õ ª ‰ ‹ Â (M˜
»;ª ª ‰ ‹‚º.J0¬99 Á ¸] ì % ^ • † [18] c ´ f * P‚º ª ‰ ‹ » ï >

1b ?G <# Aæ

1b ?G < » ï ^ ª ` • † c,º f ƒ" E¹(¯!À ¸(¯ a e1b w1 ,º » ï ¸.¤ ÔC» 0 ¼ 68b b
0 ä < ‹,º99+ª ¸ Â.J0¬+ULl!ë,º C ‹ ¸U 2 ‹ ¸ †#w ˜ ‹ ? ¾],ºH >
¾ P B œ8 'l P ?+ÇL$ ÆF 3Aè •,º C • C e E ä < ‹ ›L F ~ Ë 'l e ›L ¸þ
‹ û 5 F ~$' ›,º » ï ˜ Ë
G) Z ` ïM˜ › 1Š p ö ~ 6 Ð?· Ö @F Ñ » ï Æ
¬ 6 ¸ vBI ›L ¸FÙ ~ ‰'lOÌ?· g e?· ‰ ³ 0.v ¸F • e G " ƒ ZG < v?· Â F,º
1 × > | e ¸F 6!›Aè ¸/ Ä ›L 6B@6 ¸* ©Kd,º ^ Ä?· = D • D ƒF ,º ï G >"× ?
ï G,º ‰ 0.v ¸FÙ e E œ+U ¸c œ+U,º œ – ï E >/ Ä] œ+U6B@6 ¸e?· D,³8 ')Eô f
6 ` D m,º-$ = D ¸ ÃF >, œ –,º ¿ È >F 6G < » ï ¸ '+^ g e ‹/p5] ì < ›?· œ
–,º ‹] ¸ P e ` ö ~ m † ¸ d ß] ì ´ f8 '?· à Ñ,º ï G >
¬ Â ¸ G ` ð C ` F Õ,º ‹ œ –5 ¼ ›-A ¸F › f.J0¬+ULl!ë ¸ œ –U 2 ‹ ¸
†#w ˜ ¸ C ‹,º , O ¸FÙ Ñ Ð*< ¸<³'lCo Ñ Ñ"J % -A ø ¸ | e q e U?ÿ*<@ ,º C$'
q ¸• C ` ˜ ¸L6; • C f&«#q f µ+^F Ñ Ð*< > P ª • D ¸‰ G5 œ+U " • ¾ b0 ä <
‹,º , O ¸ ª)æ.J0¬+U ä < ‹ c GFÙ Ñ*<@ ,º ¬ C ` ³,º Ð*< ¸ ^ b0 ä < ‹
cG373 ´ f ¹ ",º Õ Á ¸" ¸ Newlander-Nirenberg Ð*< ¸ Bernstein Ð*<1• >6B J ¸ 0
b @ •5 µ ¼ (M˜ @F Ñ Õ Á Ð*<,ºA÷ D SD% > ˜ `@ ÜB* ¸ ¸ ÒB1 b0 , g73 Û@ f
F Ñ » ï ¸ g1Í œ,º C e(¯ a D,³ ¸FÙ ~ f.J0¬+ULl!ë!™ E œ – ‹,ºB • P e Á € Õ
C,º ~ Ò >F C73 CB* e G œ –5 ¼ c,º 6]F• ô ¸ G8 'l • B Q ? Ä œ – ‹,º B
œ73Fµ ƒ > • g eB* ¸Lƒ „ ƒ €0A,º œ –Lô Ü ¸Â Ô Ö 6 ÑE¹Q Ll,º » ï ¬ C. -4 ¸64þ
-$ =G3 eLƒ „ ž ž U ´ b0 f.J0¬+ULl!ë µ,º œ – F b >

DAæ

4ë?ø ª1ýAè • ¸ ¾ D t < p ¼ ? `Ll!ë Æ

1. µ+^4õ] ï0A ›.J0¬ (M˜ Â (4õ,º vG L ^ ‹ ¸ ¬ ^1b 6G < 6 D 01 ¼ Gauss Ð*< ¸ ‰ 6FP G »;ª ‹,º]L Ç

2. µ+^M”4õ] ï0A ›.J0¬ (M˜,º vG L ^ ‹ Â vG »;ª ‹ ¸ q?·Lü c Ä »;ª ‹ >

3. ‹/p ¼ ¸ ‹ µ+^ :0 - ›.J0¬ (4õ Â (M˜,º ª ‰ ä < ‹ Ę 6G < » ï Â ‡ ¸ @ Æ 3 ¦,º ©31 ü = ¸Nº/p ¼)æ ,º ä < ‹ e 6L 52 >],º œ0 ¸¹,º.J0¬ /B— ¸ .J0¬ ï# ¸G3 Â ® ä < ‹ ?-v bC^,º p a >

G ",ºOÌ?·-$ = eAß] ì8çCo 6 G,º ,L* Â2ô Ñ g73 Â (4õ Â (M˜,º vG *<Að ¸ ¬ ^F 6Eý0A c Â ä0% < Â ä < ï0A ^ ‹ c,º Ê+^ > !š ¸ G " ÀH ù6•'\ Ä (4õ Â (M˜,º vG *<Að ¸ • g e1b 6 Â1b ÂG < > CAð e / Ä ¦ œ I PM” ‹ ï G ¸F eM” ¦ œ I P ¸F Ñ. Aü ^ 6 Ð0A Ü »G3 e U+^,º ¸ ‰'l ¸F H ,º U+^ =,º e • D,º œ – Â 0 .J c ¸ % 1b 6 Â1b ÂG < p 0.v ¸F 6!› f œ – Â ‹-. ©. Aü >

'l6B ¸/ Ä ¦ œ I PM” ‹ ï G ¸L eM” ¦ œ I P,º B œ ¸G " C B QF 3Aè •5] ìFV F 6 `KO?ÿ ¸ g e ® ä < ‹ g e vG *<Að > / Ä b › ? Ä Á ‹ ï G0 .J ,º ¦ œ I P B œ ´<³'l G "Aô ÐF 3Aè •,º q?· M ¬M”F 2± B œ µ C B Q "+›B1 ¼ Aè • • D ¸)æ vBI,º ä < ‹ ¸ M” g e À8 ' ('Õ4ù Â ,º ä0% < ¸ n ä < ï0A ¸ … ä < ï0A+^ f @ Æ ‹ ,+i 0,º | ì c › ¸F 6BB% p ä < ‹Lš ¼ < Æ -,º Ê +^ • L ¸ g e 6L "! (¯8¨,º œ0 >

v ¸^ ’6;73 Ñ"j ©G3 w < ?L†,º û ë A ¸Ó'l ÿ Ñ p • ¸æ Ï ¼1b ?G < ¸SB S!› Ø0¬ ª ‰ ä < ‹ > • e)>ž D ¸ ^ ® ä < ‹,º99+ª » ¸ (5 +U ¼ vG ‹ G ª ‰ ‹Eý$WF 6)æB— > ª ‰ ä < ‹ ¸ *ƒ* e ä < ‹,º n ù v ^ > © õ e / Ä ¦ œ

I P cM” ‹ ï G ¸ L eM” ¦ œ I P,º B œ ¸ ª ‰ ä < ‹ c,º :0 @ éL$NÎ,º ï# M”
n2ô Ï ¸ ` ðAÚ p ? ¾],º%’ Ñ+^ ›@ é ¬ œ0 *<Að L6; e ÔL{ c,ºL$NÎ >
/ Ä ‰1Í œ – ‹,º ¦ œ I P œ+U ¸F g * C#¾B* > ª ‰ ä < ‹ Â ‡ œ ¸ ¦
‹ ¸ C < Æ,º$' q6Š31 eM” n r Í G " ñ$' › Ø0¬,º >
v ¸ G " B Q ¸ ^ Â ³ ƒ ZG < » ï D ¸] ì 6 ³F e8ç Ñ ,L*#…?þ 6 A1b ?
G < » ï > •Aî)æ ^"× ?-* Û,º+^ : ¸ |B* C Ð D P ? E Ï • L,º I- >

Lz ‹

Lz ‹ A

>, UF Þ n ä < ï0A,ºF 6!›"x@

^ 1.2.4E8¸ c ¸ G " (5 1Í 0 ¼>, U,ºEžF‰ > ÔL{ @ ¸ G " % Û-v À>, UF >,,º ¬
¥ ¸(¯ a e 6 ©Kd,º , q,º@ Ü ¸ @ FU Ü •"x@ 0 › > - ‰ ›-A ¸+g Ä (5 . F‰

$$\frac{G}{\text{Œ}} = \frac{h}{r^2}$$

$$cr = \frac{1}{1 + a\ F\ R\ V}$$

¬ c c = $\frac{M}{h^2}$ J 0 a 1 > Ä e ?

$$\frac{G}{\text{Œ}} = \frac{M^2}{h^3}(1 + a\ F\ R)^2$$

</ñ H ?

$$\frac{G}{(1 + a\ F\ R)^2} = \frac{M^2}{h^3}\ \text{Œ}$$

)æ ^69<‡0% <

$$Z \frac{1}{(1 + a\ F\ R)^2}\ G$$

¹,º"x@ e ? 0 Ð W E,º ¸ © u = W Ð¸Q@F&0% < p

$$Z \frac{1}{(1 + a\frac{1\ u^2}{1+u^2})^2} \frac{2}{1 + u^2}\ \text{Œ} = Z \frac{2(1 + u^2)}{(1 + a + (1\ a)u^2)^2}\ \text{Œ} \qquad (A.1)$$

M6?· © Äa ,º r < û ëAÞAð > G " ~ ›-A1¶ ‹,º û ˜ ¸ a = 1 ¸ • g e / Ê>, UEžF‰ p Ñ(Ÿ4õ,º û ë >!š ,0% < p

$$\int \frac{1+u^2}{2}\,du = \frac{u}{2} + \frac{u^3}{6} + D$$

Ä e G " q % H æF? >F8,º t ¸ µ Í

$$\frac{M^2}{h^3}t = \frac{u}{2} + \frac{u^3}{6} \qquad \text{(A.2)}$$

©

$$u^3 + 3u - \frac{6M^2t}{h^3} = 0$$

F 6B ¸ _+^ ?!W ï0A,º"x o ¢ E ¸ G " % Í f @F& ?!W ï0A,º Ô o p

$$u = \sqrt[3]{\frac{3M^2t}{h^3} + \sqrt{\frac{9M^4t^2}{h^6} + 1}} + \sqrt[3]{\frac{3M^2t}{h^3} - \sqrt{\frac{9M^4t^2}{h^6} + 1}}$$

64þ ?

$$= 2\ D\ U\ F\ W\ D\ Q\left(\frac{1}{h}\sqrt[3]{3M^2t + \sqrt{9M^4t^2 + h^6}} + \sqrt[3]{3M^2t - \sqrt{9M^4t^2 + h^6}}\right)$$

p ¼"x 2 ¸ g eA×1Í@ FU Ü ï õ ¸ $\sqrt[3]{3M^2t + \sqrt{9M^4t^2 + h^6}} + 3M^2t = \sqrt[3]{3M^2t - \sqrt{9M^4t^2 + h^6}}$ > CLô-A 0 ; •L* ? ¸ A ©31

1. $= h^2$
2. $(\ +\ )(\ ^2\ +\ ^2) = 2\sqrt{9M^4t^2 + h^6}$
3. $(\ )(\ ^2 +\ +\ ^2) = 6M^2t$
4. $(\ )^3 + 3h^2(\ )\ \ 6M^2t = 0$
5. $^0(t) = \sqrt{\frac{M^2}{9M^4t^2+h^6}}\ (t)$ ¸ $^0(t) = -\sqrt{\frac{M^2}{9M^4t^2+h^6}}\ (t)$

ÔL{ @ ƒ ? ` ©31 % Þ 01b 4 ` ©31 ´ bC^ @ g e ?!W ï0A o,º Cardano ¢ E,º Þ 2 Eý0A µÞ ^F m,ºAæ# A

$$= 2\ D\ U\ F\ W\ D\ Q\left(\frac{1}{h}\right)$$

Ä e

$$\frac{G}{\mathfrak{G}} = \frac{2h}{(\quad)^2 + h^2}(\ {}^0(t) \quad {}^0(t)) = \text{þ}\frac{2M^2h}{9M^4t^2 + h^6}\frac{+}{(\quad)^2 + h^2}$$

Ã+g ©313 ¸ ?

$$\text{þ}\frac{+}{9M^4t^2 + h^6} = \frac{2}{{}^2 \quad + \quad {}^2} = \frac{2}{(\quad)^2 + h^2}$$

64þ@ FU Ü,º>žEô E % Ï p

$$\frac{G}{\mathfrak{G}} = \frac{4M^2h}{((\quad)^2 + h^2)^2}$$

9 0 < a < 1 ¸0% <,º"x@ -. / P &Lô ¸ | Ó'l e % ',º ¸ G "F H g C ÃCŽF&
¼ > ˜@6 • ¸>, U,ºF Þ(ì 7 e Â ž.¤ Ð,º ¸+P8) % +^ S1• 3 ¦>žEô 0 › >
)æ ^ G " ›-A L 6 `L$NÎ ¸^ 1.2.4E8¸ c ¸"x@ >, UEžF‰ ¸e À>, UF Þ ï0AE¢ L
p r Â n ä < ï0AF >,"x@ ,º > G "8 'l). F‰ ¸-* Û"x@ ï0A (1.4)

$$\frac{\mathfrak{G}^2 r}{\mathfrak{G}^2} \quad \frac{h^2}{r^3} + \frac{M}{r^2} = 0$$

e \ %>, > ' p t p8 H ,ºM"4õ] ï0A ¸8 'l C ï I@ > ¸ Ò r p n ¦ ¸ t'l Í f
EžF/ e < > ¸ Ò r ¬M" n ¦ ¸ O % Eý › À r -A '8 H ¸ ?

$$\frac{G\frac{\mathfrak{G}}{\mathfrak{G}}}{\mathfrak{G}} \quad \frac{h^2}{r^3} + \frac{M}{r^2} = 0$$

Ä e

$$\frac{G\frac{1}{\frac{\mathfrak{G}}{\mathfrak{G}}}}{\mathfrak{G}}\frac{\mathfrak{G}}{\mathfrak{G}} \quad \frac{h^2}{r^3} + \frac{M}{r^2} = 0$$

$\frac{\mathfrak{G}}{\mathfrak{G}} = f(r)$ ¸ O ?

$$\frac{1}{f^3(r)}\frac{\mathfrak{G}}{\mathfrak{G}} \quad \frac{h^2}{r^3} + \frac{M}{r^2} = 0$$

</ñ H ?

$$\frac{\mathfrak{G}}{f^3} = \frac{h^2}{r^3} \quad \frac{M}{r^2} \quad \mathfrak{G}$$

Ä e

$$\frac{1}{2}\frac{1}{f^2} = \frac{M}{r} \quad \frac{1}{2}\frac{h^2}{r^2} + \frac{C}{2}$$

% Í

$$\frac{\mathfrak{G}}{\mathfrak{G}} = \text{q}\frac{1}{2\frac{M}{r} \quad \frac{h^2}{r^2} + C}$$

F 6B

$$t = \int \frac{1}{\sqrt{2\frac{M}{r} \quad \frac{h^2}{r^2} + C}} \, dr = \int \frac{r}{\sqrt{Cr^2 + 2Mr \quad h^2}} \, dr \qquad (A.3)$$

)æ ^ G " ›"x@)1%,º0% < > < û ëAÞAð ¸ ‰ C = 0 , ¸

$$\int \frac{r}{\sqrt{2Mr \quad h^2}} \, dr = \frac{r}{M} \sqrt{2Mr \quad h^2} - \int \frac{\sqrt{2Mr \quad h^2}}{M} \, dr$$

$$= \frac{r\sqrt{2Mr \quad h^2}}{M} - \int \frac{\sqrt{2Mr \quad h^2}}{M} \, dr$$

$$= \frac{r\sqrt{2Mr \quad h^2}}{M} - \frac{(\sqrt{2Mr \quad h^2})^3}{3M^2} + D$$

$$= \frac{h^2\sqrt{2Mr \quad h^2}}{2M^2} + \frac{(\sqrt{2Mr \quad h^2})^3}{6M^2} + D$$

E f G " q e % H æF? S , q ¸µ Í D = 0

ÔL{ @ ¸ fF H (5 M” n D t ¼ ¸-* Û"x@ r © Ä t ,º ä < ï0A Â ~"x@ r © Ä ,º ï0A 64þ e1• -,º > ÔL{ @ ¸ (A.1) E c a = 1 ¸ Â (A.3) E c C = 0 ¸G3 / Ê>, UEžF/ p Ñ(Ÿ4õ,º(¯!À û ˜ Ç6B J (A.4) E c,º x = p 2Mr h² ¸ < r = h² M 1 1+ F R V › D ¸ ?

$$x = r \frac{1}{2h^2 \frac{1}{1+ F R V} h^2} = h r \frac{1 \quad F R V}{1+ F R V} = h \; W \frac{D}{2} Q$$

!š , ¸ (A.4) E % o Ï p

$$t = \frac{h^3}{2M^2} W \frac{D}{2} Q - \frac{h^3}{6M^2} W^3 \frac{D}{2} Q$$

Â(A.2) E Â ž 68* ¸ 64þ,º5 Ò • û'l-. B > a 6 1 Â C 6 0 , Z0 "x@ ï E,º1• -] ¸ U ªCÙ,º B œ % 8 >,PÂA÷ >

‰'l ¸<³'l G " ‰ ù ËH ¸ Á ~ g73)R f Z0 "x@ D% º 64þ,º5 ÒG3 e-. B,º Ç | eE¢ ˜ F r Â ,º ï0AF >,"x@ ¸ Ó'l?·" -* Û"x@ r Â t ,º ï0A?·1¶ ‹ Í P ¸F e] ì _+^ · † =A×1Í(¯ aM6?·# E,º f ï >

!š L ¸" E¹ ð Aì,º f ï ^ Ä ¸ G ". F‰>, UF ÞEžF/ e 6 ` c < ¸ ¸ Ò?·"x c <,º žKµ ¸FÙ ~ g P 0)æ vBI,º c <0% < ´ % ø69 [29] ¸1b ¡1 ¸1b 5 8¸ µ>'l6B ¸Lš ¼ c < žKµF ` M” vG ,º ‹H L ¸>, UF Þ v#¿ ,º :0 ©Kd(Ÿ*<H ¸žG3 e % +^ S1• 3 ¦>žEô 0 ›,º >" E¹-*?ø,º 6 ` Õ ^ Ä ¸Aæ R p 6 ` Â y,º ?*< 3 ¦ ¸ ' p 6 ` 6 y PN¯ E ¸ / Ä ¸ A ˜ E,º0% <

$$Z \quad R(x; p \overline{\;' (x)}) \; Q x$$

¸ Ò PN¯ E ' (x) ,º!W ¦ p 3 L6; 4 ¸FÙ ~ g e c <0% < ¸ # +^ S1• 3 ¦>ž/p Ç ¸ Ò P N¯ E' (x) ,º!W ¦ p 1 L6; 2 ¸FÙ ~ o ¤ (? ¦ œ < Æ,º. Aü ¸ Õ 3 ¦ 6 Ð % t E f ´ f > ¦ ¦#¿ f>, UF Þ,º0% < ¸G3 e D 60 % t E0% 0,º [1] ¸+P8)

$$\frac{G}{G} = \frac{4M^2 h}{\left(p_3 \overline{3M^2 t + p \overline{9M^4 t^2 + h^6}} + p_3 \overline{3M^2 t \quad p \overline{9M^4 t^2 + h^6}}\right)^2 + h^2}$$

F m-AC- › C x Í/ñBg,º 3 ¦ ¸ ¹,º Õ 3 ¦ •G3 % +^ S1• 3 ¦ t E Ï 0 Æ

$$= 2 \; D \; U \; F \frac{1}{h} W \; D \; Q \; q_3 \overline{3M^2 t + p \overline{9M^4 t^2 + h^6}} + q_3 \overline{3M^2 t \quad p \overline{9M^4 t^2 + h^6}}$$

-. ‰ ÄB* vBI,º _ ‰F Þ ´ ‰'l e ^*<),º(ì ë A µ Â Â ž ž % +^ S1•,º 3 ¦>ž/p 0 › >F m,º5 Að ¸ CAð e / 17 L4à,º ð ¸F e 21 ÔL{,º)æ ð ¸G3 e " M= ò,º >

[1] ¸ Ò C e û ? 6NÍ ˆ U Â 6NÍ>, U ¸69<‡ ¬ _ ‰,º § ¸>, UF Þ c • P 0)æ c <0% < >

Lz ‹ A >, UF Þ n ä < ï0A,ºF 6!›"x@

Lz ‹ B

>, M E"x 2,º 68b :*<

/>, M EF >,"x 2F 1Í ^ … ä < ï0A ¸ ‹ < Æ c µ+^,ºM” nNC4w ¸ G " ¾ ? û?·$' › 1

69 >, M E"x 2,º ï# > G "F e 3 Ll. Lk Q(s) = $\begin{pmatrix} q_{11}(s) & q_{12}(s) & q_{13}(s) \\ q_{21}(s) & q_{22}(s) & q_{33}(s) \\ q_{31}(s) & q_{32}(s) & q_{33}(s) \end{pmatrix}$ p ÁB*

D >AæQ$_{ij}$ p q$_{ij}$,º ¦ • † E ¸ O

$$\frac{\partial Q(s)j}{\partial s} = \frac{G}{\partial s}\left(\ 1\right)^{(i_1 i_2 i_3)} q_{1i_1} q_{2i_2} q_{3i_3}$$

$$= \sum \left(\left(\ 1\right)^{(i_1 i_2 i_3)} \frac{\partial q_{1i_1}}{\partial s} q_{2i_2} q_{3i_3} + \left(\ 1\right)^{(i_1 i_2 i_3)} q_{1i_1} \frac{\partial q_{2i_2}}{\partial s} q_{3i_3} + \left(\ 1\right)^{(i_1 i_2 i_3)} q_{1i_1} q_{2i_2} \frac{\partial q_{3i_3}}{\partial s}\right)$$

$$= \sum_{i;j} \frac{\partial q_{ij}}{\partial s} Q_{ij}$$

¾ ï I ?ÿ ¸F m,º :*< ï# / 1 ELl ¦,º>, M E } F1 >

F m,º 68b ¢ E ‰'l • % d ß G " * ? ~ f@ é (4õ-. ©,ºL$NÎ > ›-A Frenet = ì Ž ^] c © Ä = ì p 7) •],ºAÞAð >!š , ¸. Lk Q(s) % CéFrenet F Þ ¢ E ¸ % À (2.7) E › @M˜,º ¢ E ¸5 5#A×1Í ?

$$\frac{\partial Q(s)j}{\partial s} = \sum_{i;j} \frac{\partial q_{ij}}{\partial s} Q_{ij} = \sum_{i;j} \left(\sum_{k=1}^{3} a_{ik} q_{kj}\right) Q_{ij} = \sum_{i;j;k} a_{ik} q_{kj} Q_{ij}$$

$$= \sum_{i;k} a_{ik} \left(\sum_{j=1}^{3} q_{kj} Q_{ij}\right) = \sum_{i;k} a_{ik}\, jQ(s)j_{i}^{k} = \sum_{i=1}^{3} a_{ii}\, jQ(s)j = W_{i}^{k} jQ(s)j$$

B m %+g W U 0 2 0 jQ(s)j p n ¦ ¸F 6B o ¤ jQ(0)j = 1 Í 0 = ì p ˆ p) •31 >

Lz ‹ C

H5¤ |4õ,º1• ,] Â 6FUL*ƒ*4õ

G " ^Lz ‹H v#¿ f,º |4õ ¸G3 e H5¤,º |4õ ¸ ¸ A 4 Æ

4 C.1: H5¤ |4õ

ø ¦ ï0A p

$$\begin{cases} x = a(\quad V\ L)Q \\ y = a(\ F\ R\ V\ 1) \end{cases}$$

G " ~ ›A÷ D |4õ,º [r1• ,] ¸ © < E*9 t5¤ ^]4õ @,º 1 E 6&ï ¸ E*9 A:s f ËG ,º ,L*G3 e 6 m,º > ‰'l ¸+^ S1•,º ï# • % B* D ¸ CEý p G " (5 . F‰ ¼ | 4õ ø ¦ ï0A ¸ v % p _+^ ä0% <-* ÛA×1Í > C ÞAô S *ƒ*5¤ ^ $_0$ ¸ O o ¤73H ¾ ˆ ¸ :,ºFU Ü] E p $\overline{2ag(\ F\ R_0V\quad F\ R)V}$> B ,Aô t Â ,º ©31 p (t) ¸ O t , q,ºFU Ü p

$$a\frac{G}{dt}(1\quad F\ R;VV\ L)Q$$

Ä e

$$a\frac{G}{dt}p\ \overline{2\quad 2\ F\ R\ V}\ \forall\ p\ \overline{2ag(\ F\ R_0V\quad F\ R)V}$$

</ñ H ?

$$G = s\ \overline{\frac{a}{g}\frac{1\quad F\ R\ V}{F\ R_0V\quad F\ R\ V}}\ G$$

E*9 A:s f ËG ,º ,L*

$$T = Z_0^T G = r\ \frac{a}{g}Z_0^r\ \overline{\frac{1\quad F\ R\ V}{F\ R_0V\quad F\ R\ V}}\ G$$

E f

$$Z_0^r\ \overline{\frac{1\quad F\ R\ V}{F\ R_0V\quad F\ R\ V}}\ G = Z_0^r\ \overline{\frac{2}{F\ R_0V\quad F\ R\ V}}\ V\ L\ Q\ G$$

$$= 2Z_0^r\ \overline{\frac{2}{F\ R_0V\quad F\ R\ V}}\ G\ F\ R\ V$$

$$= 2Z_0^s\ \overline{\frac{2}{1+\ F\ R_0V\quad 2\ F\ R_2V}}\ G\ F\ R\ V$$

$$= 2Z_0^q\ \frac{1}{F\ R_2V\quad F\ R_2V}\ G\ F\ R\ V$$

$$=2Z_0^{F\ R_2V}\ q\ \frac{1}{F\ R_2V\quad u^2}\ Q\ u$$

$$=2\ D\ U\ F\ V\ L\ Q\ \frac{u}{F\ R_2V}\ \Big|_0^{F\ R_2V} =$$

© T = q $\frac{a}{g}$ ¸Â E*9,º S ƒ5¤ © >˜@6 • ¸E*9F Þ,º 6 ` ž U 0 Ð p 4 q $\frac{a}{g}$ >‰'l ¸ -* Û+^ H5¤ |4õ ›A× ,´ ¦ µ¬ C)æ Ô ¸ 'lM6?· À ¹ oFV F ‹ |,º m † >5 > 2.4.1E8¸ c (. ,º5 Að Æ |4õ,º$F n´ ~ µ4õ Ó'l e |4õ > G " À 4 2.4.1H5¤ ¸Aß *-v E*9,º4ü4õ ^ Z — |4õL* | Þ ¸ O E*9,ºEžF/ Â ž:s ^ H5¤,º |4õ @ ¸ g Í f [r1• ,,º ‹ | ¸ ¸ A 4 Æ

[r fAè ¸5 Ð $_0$ 2 (0;] ¸ ?· H5¤ |4õ ËG 2 [$_0$; + $_0$] ,º 6!ë ¸ g % Ô)æ1• ,] > B m ¸ / ÄL† l4ü4õ | Þ,º H5¤ |4õG < ¸ • CM6?· Â ª f lFV 0 › >

+g ÄAè • # t/p Þ 4 ¸ / [r BNÇJÕ | U ªCÙ,º B œ ¸ % ø69. „ @,º ½1 Æ https://zhuanlan.zhihu.com/p/370400808 ¸ ? * ³,º ‹/p >

)æ ^ G " ›69<‡ L 6 `L$NÎ ¸ • g e 6FULƒ4õ,ºL$NÎ ¸1¶ ‹ fAè ¸ g e A &ï f B &ï ´ ‰'l ¸U AÚ A &ï,ºQ Ü] Ä B &ï,ºQ Ü µ E*9 M•!˜(ì 7 ¸ ^H Ñ,º ’+^ A"õ +g A f B (4õ,º (4õF Þ ¸ ö ~ m,º (4õ µ Í fEô4þ&ï,º ,L* 6.# >

1Š ~ t'l C eF Û AB ,º-*4õ!ë > p G " g ? 6 `)æ F,º(¯ Á ¸69<‡ 4 C ¸E*9"õ "j ©-*4õ!ë AB F Þ ¸ A &ï fEô B &ï,º ,L* ęq 0-] ´ p 6-* : ÄM•!˜(ì 7 µ6B "õ H5¤ |4õF Þ,º ,L* G " (5 "x@ Eý ¸ p 2 $\frac{\frac{jAB\,j}{2}}{g} = \sqrt{\frac{2\,jAB\,j}{g}}$ ¸ D t ,L* *.# > Û A › ¸ G " _+^ <# [1] ›B* D 6FU (4õ g e5 Eý AB Z&ï,º H5¤ |4õ [2] >

C ÞAô A &ï,º † = p (0; 0) ¸ B &ï,º † = p (b; h) >}Aô (4õ p 3 ¦ f (x) ,º 4 ¸ © f (0) = 0 ; f (b) = h >2± r Ä • ƒ H5¤ |4õ1• ,],ºA×1Í ¸ o ¤73H ¾ ˆ ¸ x :,ºFU Ü] E p $\sqrt{2gf(x)}$ > 6B A:s ,L* p

$$T(f) = \frac{\mathrm{d}s}{v} = \int_0^b \frac{\sqrt{1+(f'(x))^2}\,\mathrm{d}x}{\sqrt{2gf(x)}} = \frac{1}{\sqrt{2g}}\int_0^b \frac{\sqrt{1+(f'(x))^2}}{f(x)}\,\mathrm{d}x$$

} ¸ ^ f : f · E r ¸FÙ ~ / Ä 1 E,º t Â 3 ¦ g ¸ ¬ c g(0) = g(b) = 0 ¸G3 ? T(f + tg) T(f) ¸ 6B û'l ?

$$\left.\frac{\mathrm{d}T(f+tg)}{\mathrm{d}t}\right|_{t=0} = 0$$

/ 1 E% CéL† l — ,,ºF 5# % ä 3 ¦ g G3 F1 ># E f>á0% 3 ¦,º ˜ E" E¹ C x ¸ p ¼

[1] G "G3. F‰(‘NµGý+^ <# A÷ D ¼ 6FULƒ4õ g e |4õ ¸ | e C µ+^ <# F m-. /Q 4Ý,º - ¸+^ S 1•,º ï# • 6 m % A÷ D Æ Johann Bernoulli g e < AB 1 -* ï G @ L† < x ¸2±" D •,º Î : µ Í5 ¼ ,L* 6.# ¸ Í f 6FULƒ4õ @ 1 E 6&ï @ =4õ Â1 -* ï G v F,º@ Ü,º!™ \ DB &ï A:s,ºQ Ü,º © ï o,º" e n ¦ ¸6B ?F m]C^,º (4õ 73 e H5¤,º |4õ >

[2] [r fAè ¸ 6FULƒ4õ ÊB e H5¤ |4õ,º 6G < ¸ © ø ¦ 2 [0; $_0$] ¸ ¬ c $_0$ e [0; 2] •L*,º 6.¤ Ð r >

1¶ LAæ - ¸ G " ¸ ,Aæ

$$F(f; f') = \frac{\sqrt{1 + (f'(x))^2}}{f(x)}$$

Ä e

$$T(f) = \rho\frac{1}{\sqrt{2g}} \int_0^b F(f; f')\,dx$$

6B

$$T(f + tg) = \rho\frac{1}{\sqrt{2g}} \int_0^b F(f + tg; f' + tg')\,dx$$

B ,

$$\frac{d}{dt} F(f + tg; f' + tg') \Big|_{t=0} = \frac{\partial F}{\partial f} g + \frac{\partial F}{\partial f'} g'$$

F 6B

$$\frac{dT(f + tg)}{dt} \Big|_{t=0} = \rho\frac{1}{\sqrt{2g}} \int_0^b \left(\frac{\partial F}{\partial f} g + \frac{\partial F}{\partial f'} g' \right) dx = 0$$

Ã# E f g(0) = g(b) = 0 ¸ ?

$$\int_0^b \frac{\partial F}{\partial f'} g' \, dx = \int_0^b \frac{\partial F}{\partial f'} dg(x) = \frac{\partial F}{\partial f'} g(x) \Big|_0^b - \int_0^b g(x) \frac{d}{dx}\left(\frac{\partial F}{\partial f'}\right) dx$$

f @ E ¸ ?

$$\int_0^b \left(\frac{\partial F}{\partial f} - \frac{d}{dx}\left(\frac{\partial F}{\partial f'}\right) \right) g(x)\,dx = 0$$

+g ÄF 6 E † / 1 E,º g(x) G3 F1 ¸ û'l ?

$$\frac{\partial F}{\partial f} - \frac{d}{dx}\left(\frac{\partial F}{\partial f'}\right) = 0 \qquad \text{(C.1)}$$

© vBI,º © Ä f ,º Euler-Lagrange ï0A >)æ ^ G " B QF 6!› L1¶ Euler-Lagrange ï 0A ¸# E f

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial f}\frac{df}{dx} + \frac{\partial F}{\partial f'}\frac{df'}{dx}$$

<F 6 ©31 E › (C.1) E ¸ % Í

$$\frac{\partial F}{\partial x} - \frac{\partial F}{\partial f} f' - \left(\frac{d}{dx}\left(\frac{\partial F}{\partial f'}\right) f' + \frac{d}{dx}\left(\frac{\partial F}{\partial f'}\right) f'' \right) = 0$$

©

$$\frac{d}{dx}\left(F - \frac{\partial F}{\partial f'} f'\right) = 0$$

Í f Euler-Lagrange ï0A,º 64þ ˜ E Æ

$$F - f' \frac{\partial F}{\partial f'} = C$$

)æ ^ 6 ¸ G " À F ,º - ‰ ˜ E › ¸ ?

$$\frac{\sqrt{1 + (f')^2}}{f} - f' \frac{(f')^2}{f\sqrt{1 + (f')^2}} = C$$

L1¶ % Í

$$f(1 + (f')^2) = C$$

©

$$f' = \sqrt{\frac{C}{f} - 1}$$

</ñ H ?

$$\frac{\sqrt{f}}{\sqrt{C - f}}\,dy = dx$$

u = $\frac{f}{C}$ ¸ ?

$$C\sqrt{\frac{u}{1-u}}\,du = dx$$

Ã u = V²L Q Ä e ?

$$C\frac{V L Q}{F R V}2V L Q F R V\,G = C2\,V^2 L Q\,G = C(1 + F R2V)\,G = dx$$

0% < % Í

$$C\left(\frac{1}{2}V\,2Q + \frac{1}{2}V\,2Q\right) = x + c$$

B ¸ ?

$$y = f(x) = Cu = C\,V^2 L Q = \frac{C}{2}(1 + F R2V)$$

6Š1 % Í ø ¦ ï0A

$$\begin{cases} x = \frac{C}{2}(2\theta + V\,2Q) + c \\ y = \frac{C}{2}(1 + F R2V) \end{cases}$$

E f f (0) = 0 ¸ %@ Í c = 0 > a = $\frac{C}{2}$ ¸ = 2 ¸ O f ¼ = ü,º |4õ ï0A Æ

$$\begin{cases} x = a(\quad V\ L)Q \\ y = a(1 \quad F\ R)V \end{cases}$$

o ¤ f (b) = h ¸ % F 6!›.¤ Ð a % Cé

$$F\ R\ V\frac{\left(\sqrt{2ah \quad h^2} + b\right)}{a} = \frac{a \quad h}{a} \tag{C.2}$$

© |4õ Â ž.¤ Ð e 6 > ¸ A 4

Lz ‹ D

(M˜,º1b ? 0 b ˜ E

^ 3.38¸ c ¸ G "FPEý.J0¬ (M˜ @ (4õ,º e (0A Ü ¸F 6B K ›1b Â 0 b ˜ E › V (M˜,º e (0A Ü >'l6B ¸2±" Ä (4õ û ˜ ¸ 6-* Û V (M˜ e (0A Ü,º ª e (M˜ ‹ *f* # GH ɲ E¢ Þ,º ! ˜ >+g!š ¸)æ ^ G " B QA×1Í ⊕ɲ = ɲu ⊕ + ɲv ⊗ ,º WKµ ¸ Ç L ⊕ɲ ⊕ɲ >

Ð • D.1.

$$III = \oplus\eta \quad \oplus\eta$$

0& p (M˜,º1b ? 0 b ˜ E >

+g Ä ⊕ɲ = ɲu ⊕ + ɲv ⊗ ¸ G "M6?· ~ ›A×1Í ɲu ; ɲv ># E f ɲ e ‹ *f* GH ¸ ¬ 1 ‹ ï G,º 2 ¦G3 D8 Dá ¸-* ¸ ⊕ɲ û'l:s ^ = ©M˜ c > < a % < ɲu ; ɲv Ï p ɾu Â ɾv ,º4õ]4ú > >

$$ɲ_u = Aɾ_u + Bɾ_v$$

$$ɲ_v = Cɾ_u + Dɾ_v$$

L$NÎ › ¼ ¸ ¬ c,º A; B; C; D 31 ¦ < a e P G Ë < @F& o Ï p. Lk ˜ E

$$\begin{matrix} ɲ_u \\ ɲ_v \end{matrix} = \begin{matrix} A & B \\ C & D \end{matrix} \begin{matrix} ɾ_u \\ ɾ_v \end{matrix}$$

1• E ZEï B , D ɾu ɾv • »0%

$$\begin{matrix} ɲ_u \\ ɲ_v \end{matrix} \quad ɾ_u \; ɾ_v = \begin{matrix} A & B \\ C & D \end{matrix} \begin{matrix} ɾ_u \\ ɾ_v \end{matrix} \quad ɾ_u \; ɾ_v$$

+g Ä

$$\text{ñ}_u \ \text{ŕ}_u = \ L; \ \text{ñ}_u \ \text{ŕ}_v = \text{ñ}_v \ \text{ŕ}_u = \ M; \ \text{ñ}_v \ \text{ŕ}_v = \ N$$

6B Í f

$$\begin{matrix} A & B \\ C & D \end{matrix}^{!} = \begin{matrix} L & M \\ M & N \end{matrix}^{!} \begin{matrix} E & F \\ F & G \end{matrix}^{!\ 1}$$

F g e Weingarten V : W ,º. Lk W >˜@6 • ¸1b Â 0 b ˜ E Â1b ? 0 b ˜ E e % -. È é Ð,º >

)æ ^ G " ›-A1b ? 0 b ˜ E,º - ‰>žEô E >Aæ P = ñ_u ñ_u ¸ Q = ñ_u ñ_v ¸ R = ñ_u ñ_u ¸ ?

$$III = P(\text{Ĝ}_u)^2 + 2Q\,\text{Ĝ}_u\,\text{Ĝ}_v + R(\text{Ĝ}_v)^2$$

J

$$\begin{matrix} P & Q \\ Q & R \end{matrix}^{!} = \begin{matrix} \text{ñ}_u \\ \text{ñ}_v \end{matrix}^{!} \ \text{ñ}_u \ \text{ñ}_v = \begin{matrix} A & B \\ C & D \end{matrix}^{!} \begin{matrix} \text{ŕ}_u \\ \text{ŕ}_v \end{matrix}^{!} \ \text{ŕ}_u \ \text{ŕ}_v \ \begin{matrix} A & B \\ C & D \end{matrix}^{!\ T}$$

$$= \begin{matrix} L & M \\ M & N \end{matrix}^{!} \begin{matrix} E & F \\ F & G \end{matrix}^{!\ 1} \begin{matrix} L & M \\ M & N \end{matrix}^{!}$$

%?÷1b ? 0 b ˜ EA×1ÍC- ›F " 1b Â 0 b ˜ ETñ' ¸F • e ¹ b ›FqEÇ @" 1b Â 0 b ˜ E *1¶ ‹ ¸ | ª>á0& p1b ? 0 b ˜ E,º Õ >

)æ ^ G " ›-AF 6!› ØAÞ1b 6 ¸1b Â ¸1b ? 0 b ˜ E •L*,º ©6Š >

³NÎ D.2. (M˜ S @,º ? ` 0 b ˜ E% Cé ©31 E

$$III \quad 2HII + KI \quad 0 \qquad (D.1)$$

¬ c ¸ H ¸ K e (M˜,º © } ()½ ÂQ å ()½ >

68b û ë A-* Û › : ˜ E,º31 ¦ ¸ H; K ,º>žEô EPÂ1Í PM" n C x ¸| e# E f I; II; III e ä < ˜ E ¸8 'l Â ø ¦F? © >6B H ¸ K +g Weingarten V :,º(¯ · r é Ð ¸ • D ø ¦F? © > v G " ?·F? 6 ` 6(¯!À,º ø ¦31 ›PÂA÷ © % ¸ ©!™ Ú ()½4õ5‡ >F H g C ‹ 6 ¼ >

!š L ¸^ 5 ¼ Weingarten V : • D ¸G ". F‰ W (Ĝ) = Ĝn >Ã o ¤ Weingarten V : © Ä »0% e8 j,º ¸ ?

$$I = \text{Ĝ} \ \text{Ĝ}$$

$$II = W(\text{Ĝ}) \ \text{Ĝ}$$

$$III = W(\text{Ĝ}) \ W(\text{Ĝ}) = W^2(\text{Ĝ}) \ \text{Ĝ}$$

Ä e (D.1) E1• - Ä

$$W^2 \quad 2HW + \quad , G = 0$$

©Weingarten V :. Lk,º(‾ · ï0A jW , ĵG= 0 >F e / @F& ³NÎ ¸ • g e ? ` 0 b ˜
E •L* ©31 ¸ * x!W,º*<@ >

Lz ‹ E

73H ï# A÷ D (4õ Frenet = ì p Â (M˜8 'l = ì p,º e 6]

G " < a ^ 2.2.2E8¸ Â 4.4.2E8¸ cB* D ¼ (4õ @ Frenet = ì p Â (M˜ @8 'l = ì p,º e 6] > Ó ¤ eF Z0 û ë A = ì p v% Cé,º ï0AG3 e4õ] ï0A ¸6B4õ] ï0A% Cé S r,º @ û'l e e 6,º >F m,º ï# p&þ ÓCŒ Ä ï0A,º4õ] ÇM”4õ],º ï0A ¸ 68b û ë A e # A÷@ ,º Ž ^] Â e 6],º > C Þ K › L 60 Ê+^99 * * µ,º ï# ¸ © vBI,º73H ï# ¸ ›.J0¬ = ì p,º e 6] >

›-A (4õ @ Frenet = ì p,º e 6] > µ+^ A÷# ¸} Ð (4õ ₹1(s) Â ₹2(s) ?-. B,º ()½ Â V)½ ¸ © ¹ ",º Frenet = ì p f ~1; ~1; ~1g Â f ~2; ~2; ~2g % Cé B m,º ï0A ¸ J Z 4ú = ì ^ s = 0 :-. B >

OÌ ~ Ð •73H # 3

$$f(s) = j\tilde{\gamma}_1(s) \quad \gamma_2(s)j^2 + j\tilde{\gamma}_1(s) \quad \tilde{\gamma}_2(s)j^2 + j\gamma_1(s) \quad \gamma_2(s)j^2$$

+g — , %. f (0) = 0 >

¬!WA×1Í73H # 3,º 2 ¦

$$\frac{1}{2}\frac{\mathfrak{G}(s)}{\mathfrak{G}} = (s)(\tilde{\gamma}_1(s) \quad \tilde{\gamma}_2(s)) \quad (\gamma_1(s) \quad \gamma_2(s))$$

$$(s)(\gamma_1(s) \quad \gamma_2(s)) \quad (\tilde{\gamma}_1(s) \quad \tilde{\gamma}_2(s)) + (s)(\gamma_1(s) \quad \gamma_2(s)) \quad (\tilde{\gamma}_1(s) \quad \tilde{\gamma}_2(s))$$

$$(s)(\tilde{\gamma}_1(s) \quad \tilde{\gamma}_2(s)) \quad (\gamma_1(s) \quad \gamma_2(s)) = 0$$

6B % Í f (s) 0 ¸F 6B Z (4õ,º Frenet = ì Â žH > > % -A f ¸73H ï# Â ž C ÓCŒ@ ,º Ž ^] ¸ * CM6?· ~; ˜;~ e ‹ *f* !™ Ú,º >

Û A › ›-A (M˜ @8 'l = ì p,º e 6] ¸% / Ä Z `8 'l = ì f ŧ$^{(1)}$; ŧ$^{(2)}$g ¸f ŧ$^{(1)}$; ŧ$^{(2)}$g K ›73H 3 ¦

$$f \quad (u^1; u^2) = (\text{ŧ}^{(1)} \quad \text{ŧ}^{(2)}) \quad (\text{ŧ}^{(1)} \quad \text{ŧ}^{(2)})$$

$$f \quad (u^1; u^2) = (\text{ŧ}^{(1)} \quad \text{ŧ}^{(2)}) \quad (\text{ŧ}^{(1)} \quad \text{ŧ}^{(2)})$$

$$f\ (u^1; u^2) = j\text{ŧ}^{(1)} \quad \text{ŧ}^{(2)} j^2$$

/F Ñ 3 ¦"x … 2 ¸ % Í

$$\begin{cases} \dfrac{@f}{@u} = \displaystyle\sum_{=1}^{X^2} (\quad f \quad + \quad f \quad) + b \quad f \quad + b \quad f \\ \dfrac{@f}{@u} = \displaystyle\sum_{=1}^{X^2} b\, f \quad + \displaystyle\sum_{=1}^{X^2} f \quad + b \quad f \\ \dfrac{@f}{@u} = 2 \displaystyle\sum_{=1}^{X^2} b\, f \end{cases}$$

E fF 6 ï0A Â1b 4.4 E8¸ ¸A÷ D (M˜ Ž ^] c,º ï0A4ú (4.22) ¬ Â8* > o ¤ … ä < *<Að ¸ï0A4ú ? e 6,ºM,@ ¸ !š : `73H # 3 ˆ pM, ¸e 6 ] ÍA÷ > fF H e C e U?ÿ ? Ñ } ` Ë • g eB* Â Frenet = ì,ºL$NÎ C B ¸^A÷ D8 'l = ì e 6],ºEý0A c Ó'lM6 ?·+^ f … ä <@ ,º e 6] > ¾ P B œ8 'l P ?+ÇL$ Æ FÙ p ö ~ C 6 6 -* Û+^ … ä < ï 0A,º e 6] ¸6B5 F ~] 6 ` > † Ë

.¤ Ô ¸ ? @F& Ô# ºFV9 ¨ `)"1 ,º# 3 ›A÷ D e 6 ] ¸ Ô ”+q<ý$1Cé ¸ P!š 6 t ¸ "! E • ¸ D73H ï# ,º q -. $+PF > ] ìB ) 6 A ¸ ¸ Ò % ´ f 6 Ñ >F8,º!™ 3 ¦ a $(u^1; u^2)$ ¸ a $(u^1; u^2)$ ¸ a$(u^1; u^2)$ ’ p31 ¦ ¸ ºFV ‹)",º 6 `73H # 3

$$F(u^1; u^2) = \sum_{;}^{X} a \quad f \quad + \sum_{=1}^{X^2} a\, f \quad + af$$

73 U µ Í / = 1 ; 2 ¸ $\frac{@F}{@u}$ 0 G3 F1 ,ºB ¸FÙ g % |7g ï0A,º4õ]]C^ ¸-* Û Í f = ì p,º e 6] >F *f* e73H ï# ÊB ?,º m † > ‰'l1J6;-$ *f* ¬ `) 0 >F8,º ºFV ï E ¸ •AîF m,º73H ï# ¬ CF8+^ Ä8 'l = ì p,º e 6]F 6L$NÎ ¸ U ªCÙ,º B œ % $' › S69 A >

q • ¸ ¸ Ò73 ºFV 0 >F8,º73H # 3 ¸ O73H ï# ^A÷ DM”4õ] … ä < ï0A@ ,º e 6] , •M” n ? ~ ¸ Ê+^M˜ * µ >

Lz ‹ F

Â y 6!W ä < E0% < †,º Ž ^]

Ô Â y 6!W ä < E

G " (5 . F‰ ¸ 6 ïM˜ ¸ 6!W ä < E a(x; y) Œ + b(x; y) Œ ,º0% < † Ž ^]1• - Ä ^ (M˜ @ 0 (¯!À ø ¦31,º Ž ^] ¸ ^ ä < ‹ c ?-vH ?·,º Ê+^ Ç 6 ïM˜ ¸ 6!W ä < E a(x; y) Œ + b(x; y) Œ ,º0% < † Ž ^] ¸ /*<@ n ä < ï0A

$$a(x;y)\,\frac{\partial}{\partial x} + b(x;y)\,\frac{\partial}{\partial y} = 0 \qquad \text{(F.1)}$$

•8) ©H ?· >F e p ¸ Ò Ž ^0% < † (x; y) ¸ 3 ¦ ' (x; y) µ Í

$$G(x;y) = (x;y)\left(a(x;y)\,\frac{\partial}{\partial x} + b(x;y)\,\frac{\partial}{\partial y}\right) \qquad \text{(F.2)}$$

FÙ ~ ¸ï0A (F.1) g1• - Ä G = 0 ¸) ZEï B ,0% < % Í ' (x; y) = C > ÃFPEýF 6LÆ E ï0A ¸ % Í f@ y(x) >

52 @ Z&ï ¸ G " ? û?· Â 0% < † Ž ^],ºA÷ D ><³'l 1 ‹ 6 b n ä < • † ´" ¸ [12] µG3 P#¿ fF 6G < » ï ¸'l6BL† Ä n ä < • †8 Dá+g I fLô,ºFqEÇ ¸ n ä <B4 0A 68b › C Aè ¾F Ñ » ï ÇB ,+g Ä • †1ý {L† l ¸¶ ¶ • ¬"× ?>žF&,º(¯ a$; ¦ >µ Í œ+U ¸ ; "B1 b0 ,,º1J6; ¸M" n & ¸+P8) Ý+U ¼KµEô 20 • ª,ºB%@ > v ¸ G " Û A › - ‰A÷ D0% < †,º Ž ^] ¸ • g e Ð*< 5.7 ¸ ¬*<$; ¹ Â"x@ n ä < ï0A 4õ] … ä < ï0A,º ©31 > A v ?AÞAðU AÚG3 ^ 6!W ä < E,º Ð • cF >, >

^5 0 [rA÷ D • *ƒ* ¸ G " ~5 0 60 -A r!™.¤,º 8 'l,º SD% ¸F • e ¾ P S œ

6; ï IL- ›,ºB% p > (F.2) E1• - Ä

$$\begin{cases} \frac{@w}{@x} = (x;y)a(x;y) \\ \frac{@w}{@y} = (x;y)b(x;y) \end{cases} \tag{F.3}$$

M6?· ~ À (x; y) "x 0 › > ›-A … ä < ï0A4ú,º-. ï] — , Æ1b 6 E © Ä y "x 2 ¸1b Â E © Ä x "x 2 ¸ ?

$$\frac{@}{@y}a(x;y) + (x;y)\frac{@a}{@y} = \frac{@}{@x}b(x;y) + (x;y)\frac{@b}{@x}$$

©"x@ ¸ A © Ä (x; y) ,º 6Ll4õ] … ä < ï0A Æ

$$b(x;y)\frac{@}{@x}\ a(x;y)\frac{@}{@y} + \frac{@b}{@x}\ \frac{@a}{@y}\ (x;y) = 0 \tag{F.4}$$

6 "x@ 0 (x; y) ¸FÙ ~ Ã o ¤-. ï] — , ¸ ï0A (F.3) c,º ' • g % "x@ 0 › ´" ¸ FPEýD% º0% < µ› ‹"x@ … ä < ï0A F.4 Ë8 'l e L p n ä < ï0A > • » Ð,º H ˜ Æ

$$\begin{cases} = ' (x;y) \\ = (x;y) \end{cases}$$

µ Í

$$\begin{matrix} \frac{@'}{@x} & \frac{@}{@x} \\ \frac{@'}{@y} & \frac{@}{@y} \end{matrix} \ 6 = 0$$

+g Ä %F< ¸ { x; y % @ p ; ,º 3 ¦ > ˜(;) = (x(;); y(;)) ¸ Ä e (x; y) = ˜((x; y); (x; y)) > o ¤K4 E# O ?

$$(b\frac{@'}{@x}\ a\frac{@'}{@y})\frac{@˜}{@} + (b\frac{@}{@x}\ a\frac{@}{@y})\frac{@˜}{@} + \frac{@b}{@x}\ \frac{@a}{@y}\ ˜(;) = 0$$

G "B 4 f >F8 ' ¸ µ Í

$$b\frac{@'}{@x}\ a\frac{@'}{@y} = 0 \tag{F.5}$$

6 73 ´ fF m,º ' ¸ ÂF? Ð,º (x; y) ´ C e 6 ¸ ?· µ Í Jacobi M”F6 L © % µ¸FÙ ~ › ˜ ,º ï0A c ¸ g P Í f

$$(b\frac{@}{@x}\ a\frac{@}{@y})\frac{@˜}{@} + \frac{@b}{@x}\ \frac{@a}{@y}\ ˜(;) = 0$$

η $\lambda(x,y)$

(F.5) (F.4)

(F.1) $\varphi(x,y) =$

h $\varphi(x,y)$ (F.5)

$b(x,y)\mathrm{d}y + a(x,y)\mathrm{d}x$

$$a(x,y)\mathrm{d}x + b(x,y)\mathrm{d}y = 0$$

[12] 5

$a(x,y)\mathrm{d}x + b(x,y)\mathrm{d}y$

5.2.2

定义 F.1. D

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x,y) \tag{F.6}$$

D G $\varphi(x,y)$ x y

$\varphi(x,y)$ F.6 $y = y(x)$ $\varphi(x,y) = C$

F.6 G C $\varphi(x,y)$

定理 F.2. $p \in D$ p $G \subset D$ F.6 G

. [12], 4

x_0 I y_0 J $y_1 \in J$ x_0

(F.6) $y = t(x, y_1), \quad x \in I$

$$\frac{\partial t(x, y_1)}{\partial x} = f(x, t(x, y_1))$$

y_1

$$t(x_0, y_1) = y_1 \tag{F.7}$$

$t(x, y_1)$ y_1

(F.7) y_1

$$\frac{\partial t(x_0, y_1)}{\partial y_1} = 1 \neq 0$$

$\frac{\partial(\ ,\ _1)}{\partial\ _1}$ x x_0 $I' \subset I$

$$\frac{\partial t(x, y_1)}{\partial y_1} \neq 0, \quad \forall (x, y_1) \in I' \times J$$

$(x_1, y_1) \in I' \times J$ $I' \times J$ $\mathbb{R}^2$

$$\begin{cases} x = s(x_1, y_1) \equiv x_1 \\ y = t(x_1, y_1) \end{cases}$$

Jacobi

$$\begin{vmatrix} \frac{\partial}{\partial\ _1} & \frac{\partial}{\partial\ _1} \\ \frac{\partial}{\partial\ _1} & \frac{\partial}{\partial\ _1} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ f(x_1, t(x_1, y_1)) & \frac{\partial(\ _1,\ _1)}{\partial\ _1} \end{vmatrix} = \frac{\partial t(x_1, y_1)}{\partial y_1} \neq 0$$

$$\begin{cases} x_1 = \psi(x, y) \\ y_1 = \varphi(x, y) \end{cases}$$

$$\begin{cases} \psi(s(x_1, y_1), t(x_1, y_1)) = x_1 \\ \varphi(s(x_1, y_1), t(x_1, y_1)) = y_1 \end{cases}$$

$$\varphi(x, t(x, y_1)) = y_1$$

y_1 C (F.6) $y = t(x, C)$

$$\varphi(x, t(x, C)) = C$$

$\varphi(x, y)$ □

(F.6)

引理 F.3. $\varphi(x, y)$ $\frac{\mathrm{d}}{\mathrm{d}} = f(x, y)$ G

$$\frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial y} f = 0 \tag{F.8}$$

G

. $(x_0, y_0) \in G$ $\frac{\mathrm{d}}{\mathrm{d}} = f(x, y)$ $y = y(x; x_0, y_0)$ $\varphi(x, y(x; x_0, y_0)) \equiv 0$ $y = y(x; x_0, y_0)$ (x_0, y_0) $\Gamma_{(\ 0,\ 0)}$ (F.8)

(x_0, y_0) (F.8) G □

(F.6) $\varphi(x, y)$ $p \in G$ p $G_0 \subset G$ $\frac{\partial \varphi}{\partial} \neq 0$ $\varphi(x, y(x)) = C$ $y_C(x)$

$$\varphi(x, y_C(x)) \equiv C$$

x

$$\frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial y} \frac{\mathrm{d} y_C}{\mathrm{d} x} = 0$$

$\frac{\partial\varphi}{\partial}$ (F.8)

$$-f(x,y)+\frac{\mathrm{d}y_C}{\mathrm{d}x}=0$$

$y_C(x)$ (F.6)

$a(x,y)\mathrm{d}x+b(x,y)\mathrm{d}y$

$b(x,y)$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{a(x,y)}{b(x,y)}$$

$f(x,y)=-\frac{(\ ,\ )}{(\ ,\ )}$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=f(x,y)$$

$\varphi(x,y)$ $\varphi(x,y)$ $\varphi(x,y)$

$$\begin{aligned}\mathrm{d}\varphi=&\frac{\partial\varphi}{\partial x}\mathrm{d}x+\frac{\partial\varphi}{\partial y}\mathrm{d}y=-\frac{f(x,y)}{\frac{\partial(\ 1,\ 1)}{\partial\ 1}}\mathrm{d}x+\frac{1}{\frac{\partial(\ 1,\ 1)}{\partial\ 1}}\mathrm{d}y\\=&\frac{a(x,y)}{b(x,y)\frac{\partial(\ 1,\ 1)}{\partial\ 1}}\mathrm{d}x+\frac{1}{\frac{\partial(\ 1,\ 1)}{\partial\ 1}}\mathrm{d}y\\=&b(x,y)\frac{\partial t(x_1,y_1)}{\partial y_1}(a(x,y)\mathrm{d}x+b(x,y)\mathrm{d}y)\end{aligned}$$

$\lambda(x,y)=b(x,y)\frac{\partial(\ 1,\ 1)}{\partial\ 1}$ 5.7

[30]

9.2.2

$$\phi = p(x,y)\mathrm{d}x + q(x,y)\mathrm{d}y$$

$p(x,y), q(x,y)$ x, y

λ z

$$\mathrm{d}z = \lambda\phi$$

Korn-Lichtenstein Korn-Lichtenstein

Gauss $p(x,y), q(x,y)$

x y [1]

$f(z,w)$ z w

命題 F.4. $f : \mathbb{C}\times\mathbb{C} \to \mathbb{C}$ $(z^0, w^0) \in \mathbb{C}\times\mathbb{C}$

$$\frac{\mathrm{d}w}{\mathrm{d}z} = f(z,w) \tag{F.9}$$

$z^0 \in \mathbb{C}$ $w = w(z)$ $w(z^0) = w^0$

. f, z, w $f(z,w) = f_1(z,w) + if_2(z,w)$ $z = z_1 + iz_2$ $w = w_1 + iw_2$ (F.9)

$$\begin{cases} \dfrac{\partial w_1}{\partial z_1} = \dfrac{\partial w_2}{\partial z_2} = f_1(z_1, z_2, w_1, w_2) \\ \dfrac{\partial w_2}{\partial z_1} = -\dfrac{\partial w_1}{\partial z_2} = f_2(z_1, z_2, w_1, w_2) \end{cases} \tag{F.10}$$

$f_{11} = f_{22} = f_1, \quad f_{12} = -f_{21} = -f_2$ (F.10)

$$\frac{\partial w}{\partial z_j} = f_{,j}(z_1, z_2, w_1, w_2), \quad 1 \le i, j \le 2$$

[1] y x x

$\frac{\partial^2 \quad _i}{\partial \quad \partial} = \frac{\partial^2 \quad _i}{\partial \quad \partial}$

$$\frac{\partial f_{\ j}}{{}_2\,\partial z^{+}} - \frac{\partial f}{\partial z_j} +$$

$$f_1$$

$$\frac{\partial f_1}{\partial z_2} +$$

$$\frac{\partial f_2}{\partial z_2} -$$

[redacted]

5.7

[10]

§1.5

G

$$h(x) = g(f(x))$$

$$\frac{\mathrm{d}h(x)}{\mathrm{d}x} = \frac{\mathrm{d}g}{\mathrm{d}f}\frac{\mathrm{d}f}{\mathrm{d}x}$$

$$\begin{cases} f_1(x_1, x_2, \cdots, x\ ) \\ f_2(x_1, x_2, \cdots, x\ ) \\ \quad \vdots \\ f\ \ (x_1, x_2, \cdots, x\ ) \end{cases}$$

Jacobi $(\frac{\partial\ _i}{\partial})_{(\ ,j)}$ m $g_1, g_2, \cdots, g$

$h\ (x_1, x_2, \cdots, x\ )$

$=g\ (f_1(x_1, x_2, \cdots, x\ ), f_2(x_1, x_2, \cdots, x\ ), \cdots, f\ \ (x_1, x_2, \cdots, x\ ))$,

$i = 1, \cdots l$

$$(\frac{\partial h}{\partial x_j}) = \sum_{=1} (\frac{\partial g}{\partial f})(\frac{\partial f}{\partial x_j})$$

Jacobi Jacobi

Jacobi

- $(f_1, f_2, \cdots, f\ ) : \mathbb{R}\ \to \mathbb{R}$
  Jacobi $(\frac{\partial\ _{\prime}}{\partial})_{(\ ,j)}$
- M N σ M
  $x^1, \cdots, x$ N $y^1, \cdots, y$ $\sigma_* : TM \to TN$
  $\{\frac{\partial}{\partial\ ^{\prime}}\}_{=1}$ $\{\frac{\partial}{\partial}\}_{j=1}$ $\tilde{\sigma}$ Jacobi $\tilde{\sigma}$ σ

Jacobi

定理 G.1. $\sigma_1 : S_1 \to S_2$ $\sigma_2 : S_2 \to S_3$ $\tau = \sigma_2 \circ \sigma_1 :$
$S_1 \to S_3$ $p \in S_1$

$\tau_{*,\ } = \sigma_{2*,\sigma_1(\ )} \circ \sigma_{1*,}$

H

$$I = v^2|\vec{l'}(u)|^2\mathrm{d}u^2 + \mathrm{d}v^2$$

$\mathrm{d}s = |\vec{l'}(u)|\mathrm{d}u$

$$I = v^2(\mathrm{d}s)^2 + (\mathrm{d}v)^2$$

$$\begin{cases} (\frac{\partial f}{\partial s})^2 + (\frac{\partial g}{\partial s})^2 = v^2 \\ \frac{\partial f}{\partial s}\frac{\partial f}{\partial v} + \frac{\partial g}{\partial s}\frac{\partial g}{\partial v} = 0 \\ (\frac{\partial f}{\partial v})^2 + (\frac{\partial g}{\partial v})^2 = 1 \end{cases}$$

$$\begin{cases} (\frac{1}{v}\frac{\partial f}{\partial s})^2 + (\frac{1}{v}\frac{\partial g}{\partial s})^2 = 1 \\ (\frac{1}{v}\frac{\partial f}{\partial s})\frac{\partial f}{\partial v} + (\frac{1}{v}\frac{\partial g}{\partial s})\frac{\partial g}{\partial v} = 0 \\ (\frac{\partial f}{\partial v})^2 + (\frac{\partial g}{\partial v})^2 = 1 \end{cases}$$

$$\begin{cases} \frac{1}{v}\frac{\partial f}{\partial s} = \cos\theta(s,v) \\ \frac{\partial f}{\partial v} = \sin\theta(s,v) \\ \frac{1}{v}\frac{\partial g}{\partial s} = -\sin\theta(s,v) \\ \frac{\partial g}{\partial v} = \cos\theta(s,v) \end{cases}$$

$$\begin{cases} \frac{\partial f}{\partial s} = v\cos\theta(s,v) \\ \frac{\partial f}{\partial v} = \sin\theta(s,v) \\ \frac{\partial g}{\partial s} = -v\sin\theta(s,v) \\ \frac{\partial g}{\partial v} = \cos\theta(s,v) \end{cases} \tag{H.1}$$

$$\frac{\partial^2}{\partial\,\partial} = \frac{\partial^2}{\partial\,\partial} \quad \frac{\partial^2}{\partial\,\partial} = \frac{\partial^2}{\partial\,\partial} \tag{H.1}$$

$$\begin{cases} \cos\theta = v\sin\theta\frac{\partial\theta}{\partial v} + \cos\theta\frac{\partial\theta}{\partial s} \\ -\sin\theta = v\cos\theta\frac{\partial\theta}{\partial v} - \sin\theta\frac{\partial\theta}{\partial s} \end{cases}$$

$\frac{\partial\theta}{\partial}, \frac{\partial\theta}{\partial}$

$$\frac{\partial\theta}{\partial v} = 0, \quad \frac{\partial\theta}{\partial s} = 1$$

$\theta = s$ (H.1),

$$\begin{cases} f = v\sin s \\ g = v\cos s \end{cases}$$

$$I = v^2(\mathrm{d}s)^2 + (\mathrm{d}v)^2$$

$$I_{E^2} = \mathrm{d}x^2 + \mathrm{d}y^2,$$

I

8.2 $\tau = (\vec{r}'(s), \vec{n}(s), \vec{n}'(s))$ τ

$$\tau = \sqrt{g} \begin{pmatrix} \frac{\mathrm{d}\ ^1}{\mathrm{d}} & \frac{\mathrm{d}\ ^2}{\mathrm{d}} \end{pmatrix} \begin{pmatrix} b_1^2 & b_2^2 \\ -b_1^1 & -b_2^1 \end{pmatrix} \begin{pmatrix} \frac{\mathrm{d}\ ^1}{\mathrm{d}} \\ \frac{\mathrm{d}\ ^2}{\mathrm{d}} \end{pmatrix}$$

Weingarten

$$\tau = \mathscr{W}(\vec{e}_1) \cdot \vec{e}_2$$

τ

$$\vec{e}_1 = (\frac{\mathrm{d}u^1}{\mathrm{d}s}, \frac{\mathrm{d}u^2}{\mathrm{d}s}) \begin{pmatrix} \vec{r}_1 \\ \vec{r}_2 \end{pmatrix}$$

$$\mathscr{W}(\vec{e}_1) = (\frac{\mathrm{d}u^1}{\mathrm{d}s}, \frac{\mathrm{d}u^2}{\mathrm{d}s}) \begin{pmatrix} b_1^1 & b_1^2 \\ b_2^1 & b_2^2 \end{pmatrix} \begin{pmatrix} \vec{r}_1 \\ \vec{r}_2 \end{pmatrix}$$

$$\vec{e}_2 = \begin{pmatrix} \frac{\mathrm{d}\ ^1}{\mathrm{d}} & \frac{\mathrm{d}\ ^2}{\mathrm{d}} \end{pmatrix} \begin{pmatrix} \vec{n} \times \vec{r}_1 \\ \vec{n} \times \vec{r}_2 \end{pmatrix}$$

$$
\begin{aligned}
\mathscr{W}(\vec{e}_1)\cdot\vec{e}_2 &= \begin{pmatrix} \frac{\mathrm{d}\ ^1}{\mathrm{d}} & \frac{\mathrm{d}\ ^2}{\mathrm{d}} \end{pmatrix} \begin{pmatrix} b_1^1 & b_1^2 \\ b_2^1 & b_2^2 \end{pmatrix} \begin{pmatrix} \vec{r}_1 \\ \vec{r}_2 \end{pmatrix} \begin{pmatrix} \vec{n}\times\vec{r}_1 & \vec{n}\times\vec{r}_2 \end{pmatrix} \begin{pmatrix} \frac{\mathrm{d}\ ^1}{\mathrm{d}} \\ \frac{\mathrm{d}\ ^2}{\mathrm{d}} \end{pmatrix} \\
&= \begin{pmatrix} \frac{\mathrm{d}\ ^1}{\mathrm{d}} & \frac{\mathrm{d}\ ^2}{\mathrm{d}} \end{pmatrix} \begin{pmatrix} b_1^1 & b_1^2 \\ b_2^1 & b_2^2 \end{pmatrix} \begin{pmatrix} 0 & \vec{n}\times\vec{r}_2\cdot\vec{r}_1 \\ \vec{n}\times\vec{r}_1\cdot\vec{r}_2 & 0 \end{pmatrix} \begin{pmatrix} \frac{\mathrm{d}\ ^1}{\mathrm{d}} \\ \frac{\mathrm{d}\ ^2}{\mathrm{d}} \end{pmatrix} \\
&= \sqrt{g} \begin{pmatrix} \frac{\mathrm{d}\ ^1}{\mathrm{d}} & \frac{\mathrm{d}\ ^2}{\mathrm{d}} \end{pmatrix} \begin{pmatrix} b_1^1 & b_1^2 \\ b_2^1 & b_2^2 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\mathrm{d}\ ^1}{\mathrm{d}} \\ \frac{\mathrm{d}\ ^2}{\mathrm{d}} \end{pmatrix} \\
&= \sqrt{g} \begin{pmatrix} \frac{\mathrm{d}\ ^1}{\mathrm{d}} & \frac{\mathrm{d}\ ^2}{\mathrm{d}} \end{pmatrix} \begin{pmatrix} b_1^2 & -b_1^1 \\ b_2^2 & -b_2^1 \end{pmatrix} \begin{pmatrix} \frac{\mathrm{d}\ ^1}{\mathrm{d}} \\ \frac{\mathrm{d}\ ^2}{\mathrm{d}} \end{pmatrix}
\end{aligned}
$$

F† ò`lÛñ — ô—t3•Rl G G— ó#9p " w @ 3Ÿr — ôñT3•R#Đ p

J

Liouville

8.6.1 $\kappa = \frac{\mathrm{d}\,{}_1(\,)}{\mathrm{d}} \cdot \vec{e}_2$

Liouville

$$\kappa = \frac{\mathrm{d}\theta(s)}{\mathrm{d}s} - \frac{1}{2\sqrt{G}}\frac{\partial \log E}{\partial v}\cos\theta + \frac{1}{2\sqrt{E}}\frac{\partial \log G}{\partial u}\sin\theta$$

(u, v)

$$\kappa = \sqrt{g}\begin{vmatrix} \frac{\mathrm{d}\,{}^1(\,)}{\mathrm{d}} & \frac{\mathrm{d}^2\,{}^1(\,)}{\mathrm{d}\,{}^2} + \sum_{\alpha,\beta=1}^{2}\Gamma^1_{\alpha\beta}\frac{\mathrm{d}\,{}^\alpha(\,)}{\mathrm{d}}\frac{\mathrm{d}\,{}^\beta(\,)}{\mathrm{d}} \\ \frac{\mathrm{d}\,{}^2(\,)}{\mathrm{d}} & \frac{\mathrm{d}^2\,{}^2(\,)}{\mathrm{d}\,{}^2} + \sum_{\alpha,\beta=1}^{2}\Gamma^2_{\alpha\beta}\frac{\mathrm{d}\,{}^\alpha(\,)}{\mathrm{d}}\frac{\mathrm{d}\,{}^\beta(\,)}{\mathrm{d}} \end{vmatrix}$$

Liouville

Liouville Liouville

$\cos\theta, \sin\theta$ θ

$$\cos\theta = \sqrt{E}\frac{\mathrm{d}u}{\mathrm{d}s}, \sin\theta = \sqrt{G}\frac{\mathrm{d}v}{\mathrm{d}s}$$

$$\begin{aligned} -\sin\theta\frac{\mathrm{d}\theta}{\mathrm{d}s} &= \frac{\mathrm{d}\sqrt{E}}{\mathrm{d}s}\frac{\mathrm{d}u}{\mathrm{d}s} + \sqrt{E}\frac{\mathrm{d}^2u}{\mathrm{d}s^2} \\ \cos\theta\frac{\mathrm{d}\theta}{\mathrm{d}s} &= \frac{\mathrm{d}\sqrt{G}}{\mathrm{d}s}\frac{\mathrm{d}v}{\mathrm{d}s} + \sqrt{G}\frac{\mathrm{d}^2v}{\mathrm{d}s^2} \end{aligned}$$

$-\sin\theta \qquad \cos\theta$

$$\frac{\mathrm{d}\theta}{\mathrm{d}s}=\begin{vmatrix}\sqrt{E}\frac{\mathrm{d}}{\mathrm{d}} & \frac{\mathrm{d}\sqrt{E}}{\mathrm{d}}\frac{\mathrm{d}}{\mathrm{d}}+\sqrt{E}\frac{\mathrm{d}^2}{\mathrm{d}^{\,2}}\\ \sqrt{G}\frac{\mathrm{d}}{\mathrm{d}} & \frac{\mathrm{d}\sqrt{G}}{\mathrm{d}}\frac{\mathrm{d}}{\mathrm{d}}+\sqrt{G}\frac{\mathrm{d}^2}{\mathrm{d}^{\,2}}\end{vmatrix}$$

$$-\frac{1}{2\sqrt{EG}}\left(E\ \frac{\mathrm{d}u}{\mathrm{d}s}-G\ \frac{\mathrm{d}v}{\mathrm{d}s}\right)=-\begin{vmatrix}\sqrt{E}\frac{\mathrm{d}}{\mathrm{d}} & \frac{1}{2\sqrt{E}}(\log G)\\ \sqrt{G}\frac{\mathrm{d}}{\mathrm{d}} & \frac{1}{2\sqrt{G}}(\log E)\end{vmatrix}$$

$$\begin{aligned}\kappa\ =&\begin{vmatrix}\sqrt{E}\frac{\mathrm{d}}{\mathrm{d}} & \frac{\mathrm{d}\sqrt{E}}{\mathrm{d}}\frac{\mathrm{d}}{\mathrm{d}}+\sqrt{E}\frac{\mathrm{d}^2}{\mathrm{d}^{\,2}}-\frac{1}{2\sqrt{E}}(\log G)\\ \sqrt{G}\frac{\mathrm{d}}{\mathrm{d}} & \frac{\mathrm{d}\sqrt{G}}{\mathrm{d}}\frac{\mathrm{d}}{\mathrm{d}}+\sqrt{G}\frac{\mathrm{d}^2}{\mathrm{d}^{\,2}}-\frac{1}{2\sqrt{G}}(\log E)\end{vmatrix}\\ =&\sqrt{EG}\begin{vmatrix}\frac{\mathrm{d}}{\mathrm{d}} & \frac{\mathrm{d}^2}{\mathrm{d}^{\,2}}+\frac{1}{2}(\log E)\ \frac{\mathrm{d}}{\mathrm{d}}-\frac{1}{2E}(\log G)\\ \frac{\mathrm{d}}{\mathrm{d}} & \frac{\mathrm{d}^2}{\mathrm{d}^{\,2}}+\frac{1}{2}(\log G)\ \frac{\mathrm{d}}{\mathrm{d}}-\frac{1}{2G}(\log E)\end{vmatrix}\end{aligned}$$

s

$$E(\frac{\mathrm{d}u}{\mathrm{d}s})^2+G(\frac{\mathrm{d}v}{\mathrm{d}s})^2=1$$

$$\begin{aligned}(\log E)\ =&(\log E)\ \frac{\mathrm{d}u}{\mathrm{d}s}+(\log E)\ \frac{\mathrm{d}v}{\mathrm{d}s},\\ (\log G)\ =&(\log G)\ \frac{\mathrm{d}u}{\mathrm{d}s}+(\log G)\ \frac{\mathrm{d}v}{\mathrm{d}s}.\end{aligned}$$

$\kappa\ =\sqrt{EG}$

$$\begin{vmatrix}\frac{\mathrm{d}}{\mathrm{d}} & \frac{\mathrm{d}^2}{\mathrm{d}^{\,2}}+\frac{1}{2}(\log E)\ (\frac{\mathrm{d}}{\mathrm{d}})^2+\frac{1}{2}(\log E)\ \frac{\mathrm{d}}{\mathrm{d}}\frac{\mathrm{d}}{\mathrm{d}}-\frac{1}{2}(\log G)\ (\frac{\mathrm{d}}{\mathrm{d}})^2-\frac{1}{2}\frac{E}{\sqrt{E}}(\frac{\mathrm{d}}{\mathrm{d}})^2\\ \frac{\mathrm{d}}{\mathrm{d}} & \frac{\mathrm{d}^2}{\mathrm{d}^{\,2}}+\frac{1}{2}(\log G)\ (\frac{\mathrm{d}}{\mathrm{d}})^2+\frac{1}{2}(\log G)\ \frac{\mathrm{d}}{\mathrm{d}}\frac{\mathrm{d}}{\mathrm{d}}-\frac{1}{2}(\log E)\ (\frac{\mathrm{d}}{\mathrm{d}})^2-\frac{1}{2}\frac{E}{\sqrt{G}}(\frac{\mathrm{d}}{\mathrm{d}})^2\end{vmatrix}$$

Christoffel

$$\begin{aligned}\Gamma^1_{11}=&\frac{1}{2}\frac{\partial\log E}{\partial u}, \qquad \Gamma^1_{12}=\Gamma^1_{21}=\frac{1}{2}\frac{\partial\log E}{\partial v}, \qquad \Gamma^1_{22}=-\frac{1}{2E}\frac{\partial G}{\partial u}\\ \Gamma^2_{11}=&-\frac{1}{2G}\frac{\partial E}{\partial v}, \qquad \Gamma^2_{12}=\Gamma^2_{21}=\frac{1}{2}\frac{\partial\log G}{\partial u}, \qquad \Gamma^2_{22}=\frac{1}{2}\frac{\partial\log G}{\partial v}\end{aligned}$$

$$
\begin{aligned}
\kappa \; =&\sqrt{EG}\begin{vmatrix} \frac{\mathrm{d}}{\mathrm{d}} & \frac{\mathrm{d}^2}{\mathrm{d}^{\,2}} + \Gamma^1_{11}(\frac{\mathrm{d}}{\mathrm{d}})^2 + \Gamma^1_{12}\frac{\mathrm{d}}{\mathrm{d}}\frac{\mathrm{d}}{\mathrm{d}} - \Gamma^2_{12}(\frac{\mathrm{d}}{\mathrm{d}})^2 + \Gamma^1_{22}(\frac{\mathrm{d}}{\mathrm{d}})^2 \\ \frac{\mathrm{d}}{\mathrm{d}} & \frac{\mathrm{d}^2}{\mathrm{d}^{\,2}} + \Gamma^2_{22}(\frac{\mathrm{d}}{\mathrm{d}})^2 + \Gamma^2_{12}\frac{\mathrm{d}}{\mathrm{d}}\frac{\mathrm{d}}{\mathrm{d}} - \Gamma^1_{12}(\frac{\mathrm{d}}{\mathrm{d}})^2 + \Gamma^2_{11}(\frac{\mathrm{d}}{\mathrm{d}})^2 \end{vmatrix} \\
=&\sqrt{EG}\begin{vmatrix} \frac{\mathrm{d}}{\mathrm{d}} & \frac{\mathrm{d}^2}{\mathrm{d}^{\,2}} + \Gamma^1_{11}(\frac{\mathrm{d}}{\mathrm{d}})^2 + 2\Gamma^1_{12}\frac{\mathrm{d}}{\mathrm{d}}\frac{\mathrm{d}}{\mathrm{d}} + \Gamma^1_{22}(\frac{\mathrm{d}}{\mathrm{d}})^2 \\ \frac{\mathrm{d}}{\mathrm{d}} & \frac{\mathrm{d}^2}{\mathrm{d}^{\,2}} + \Gamma^2_{22}(\frac{\mathrm{d}}{\mathrm{d}})^2 + 2\Gamma^2_{12}\frac{\mathrm{d}}{\mathrm{d}}\frac{\mathrm{d}}{\mathrm{d}} + \Gamma^2_{11}(\frac{\mathrm{d}}{\mathrm{d}})^2 \end{vmatrix} \\
&-\sqrt{EG}\begin{vmatrix} \frac{\mathrm{d}}{\mathrm{d}} & \Gamma^1_{12}\frac{\mathrm{d}}{\mathrm{d}}\frac{\mathrm{d}}{\mathrm{d}} + \Gamma^2_{12}(\frac{\mathrm{d}}{\mathrm{d}})^2 \\ \frac{\mathrm{d}}{\mathrm{d}} & \Gamma^2_{12}\frac{\mathrm{d}}{\mathrm{d}}\frac{\mathrm{d}}{\mathrm{d}} + \Gamma^1_{12}(\frac{\mathrm{d}}{\mathrm{d}})^2 \end{vmatrix} \\
=&\sqrt{EG}\begin{vmatrix} \frac{\mathrm{d}}{\mathrm{d}} & \frac{\mathrm{d}^2}{\mathrm{d}^{\,2}} + \Gamma^1_{11}(\frac{\mathrm{d}}{\mathrm{d}})^2 + 2\Gamma^1_{12}\frac{\mathrm{d}}{\mathrm{d}}\frac{\mathrm{d}}{\mathrm{d}} + \Gamma^1_{22}(\frac{\mathrm{d}}{\mathrm{d}})^2 \\ \frac{\mathrm{d}}{\mathrm{d}} & \frac{\mathrm{d}^2}{\mathrm{d}^{\,2}} + \Gamma^2_{22}(\frac{\mathrm{d}}{\mathrm{d}})^2 + 2\Gamma^2_{12}\frac{\mathrm{d}}{\mathrm{d}}\frac{\mathrm{d}}{\mathrm{d}} + \Gamma^2_{11}(\frac{\mathrm{d}}{\mathrm{d}})^2 \end{vmatrix}
\end{aligned}
$$

K

$2n$

n Kähler

$2n$ M J $J^2 = -\mathrm{Ido}$

X, Y M [,]

Nijenhuis

https://www.zhihu.com/question/38290928

Nijenhuis

$2n$ M

i

Newlander-Nirenberg

定理.

[31] [32]

(M, J) h

X, Y $h(X, Y) = h(JX, JY)$ h Hermitian (M, J, h)

Hermitian Hermitian

Hermitian Hermitian 2- $\Omega(X, Y) \triangleq h(JX, Y)$

2- Ω h Kähler Kähler

Kähler https://mathoverflow.net/questions/186139/

1. Hermitian

$$h(X, JX) = h(JX, J^2X) = -h(JX, X) = -h(X, JX)$$

$h(X, JX) = 0$ X JX 2

$90°$

$\langle , \rangle$ X

[1]

$\langle , \rangle$ Hermitian $h(,)$ $\langle , \rangle$

[1] $\vec{e}_1$ $\vec{e}_2$ $J\vec{e}_1 = \vec{e}_2$ $J\vec{e}_2 = -\vec{e}_1$ J

J $X = a\vec{e}_1 + b\vec{e}_2$ $J(JX) = J(a\vec{e}_2 - b\vec{e}_1) = -a\vec{e}_1 - b\vec{e}_2 = -X$

2. Newlander-Nirenberg

$$z = u + iv \qquad J\frac{\partial}{\partial} = \frac{\partial}{\partial}^{1}$$

3. 2- Hermitian h Ω Kähler

4. Kähler $J\frac{\partial}{\partial} = \frac{\partial}{\partial}$ $\frac{\partial}{\partial}$ $\frac{\partial}{\partial}$ $h(\frac{\partial}{\partial}, \frac{\partial}{\partial}) = h(J\frac{\partial}{\partial}, J\frac{\partial}{\partial}) = h(\frac{\partial}{\partial}, \frac{\partial}{\partial})$ $I = \lambda^2(\mathrm{d}u^2 + \mathrm{d}v^2)$ (u, v) $z = u + iv$ Kähler Kähler

Newlander-Nirenberg Korn-Lichtenstein Newlander-Nirenberg

J $Y = c\vec{e}_1 + d\vec{e}_2$

$$\langle JX, JY\rangle = \langle a\vec{e}_2 - b\vec{e}_1, c\vec{e}_2 - d\vec{e}_1\rangle = ac + bd = \langle X, Y\rangle$$

Hermitian

[1] $\vec{e}_1, \vec{e}_2$ $\frac{\partial}{\partial u} = \lambda\vec{e}_1, \frac{\partial}{\partial v} = \lambda\vec{e}_2$ $\frac{\partial}{\partial u} = \lambda\vec{e}_1, \frac{\partial}{\partial v} = \mu\vec{e}_2$ J

L

Bernstein Weierstrass

10.1.4 Bernstein

定理 (Bernstein). $f(x,y)$ $\mathbb{R}^2$ f

Weierstrass Weierstrass

Nitsche

. $(x,y,f(x,y))$ (10.13) Levy Levy

1-1

$$F(u,v) = (x - \xi(x,y)) - i\,(y - \eta(x,y)) = (2x - u) - i(2y - v).$$

$F(u,v)$ $w = u + iv$

$$\alpha(u,v) = 2x - u$$
$$\beta(u,v) = -\,(2y - v)$$

(u,v) Cauchy-Riemann x,y u,v

Levy Jacobi

$$J = \begin{pmatrix} u & u \\ v & v \end{pmatrix} = \begin{pmatrix} 1+\varphi & \varphi \\ \varphi & 1+\varphi \end{pmatrix} \tag{L.1}$$

$$J^{-1} = \begin{pmatrix} x & x \\ y & y \end{pmatrix} = \frac{1}{\det J} \begin{pmatrix} 1+\varphi & -\varphi \\ -\varphi & 1+\varphi \end{pmatrix}$$

(10.12)

$$\det J = 1+\varphi \quad +\varphi \quad +\varphi \quad \varphi \quad -\varphi^2 \quad = 2+\varphi \quad +\varphi$$

$$\begin{aligned} \alpha \quad &= 2x \quad -1 = \frac{2}{\det J}(1+\varphi \quad) - 1 = \frac{\varphi \quad -\varphi}{\det J} \\ \beta \quad &= -2y \quad +1 = -\frac{2}{\det J}(1+\varphi \quad) + 1 = \frac{\varphi \quad -\varphi}{\det J} \\ \alpha \quad &= 2x \quad = -\frac{2\varphi}{\det J} \\ \beta \quad &= -2y \quad = \frac{2\varphi}{\det J} \end{aligned}$$

$\alpha \quad = \beta \ , \quad \alpha \quad = -\beta$ Cauchy-Riemann

$F(w)$

$$F'(w) = \alpha \quad + i\beta \quad = \frac{\varphi \quad -\varphi \quad +2i\varphi}{\det J}$$

$$|F'(w)|^2 = \frac{(\varphi \quad -\varphi \quad)^2 + 4\varphi^2}{(2+\varphi \quad +\varphi \quad)^2} = \frac{(\frac{^2- \ ^2}{\quad})^2 + 4(-)^2}{(\frac{^2 \ +2+ \ ^2+ \ ^2}{\quad})^2} = \left(\frac{p^2+q^2}{2g+2+p^2+q^2}\right)^2 < 1$$

$F'(w)$ Liouville $F'(w)$

$$1 - |F'| = \frac{2g+2}{2g+2+p^2+q^2} = \frac{2g+2}{g^2+2g+1} = \frac{2}{g+1}$$

g

$$\det J = \frac{2g+2+p^2+q^2}{g} = \frac{(g+1)^2}{g}$$

φ $F'(w)\det J$ $\varphi \quad -\varphi$

$\det J$ $\varphi \quad +\varphi$

Nitsche Jörgens

定理. $\varphi(x,y)$ Monge-Ampère

$$\varphi \quad \varphi \quad -\varphi^2 \equiv 1$$

$\varphi \quad > 0$ $\varphi \quad , \varphi \quad , \varphi$ $\varphi(x,y)$ x y

$F(w)$ $F'(w)$ Levy

Weierstrass Φ

$$F'(w) = 2x \quad -1 - i2y$$

$$\Phi = x \quad -y \quad -i(x \quad +y \ )$$

x y

Bernstein

10.1.4

Heinz Bernstein

[33] [34]

[1] , . [M]. . : , 2019.

[2] . [M]. . : , 2015.

[3] ABBENA E, SALAMON S, GRAY A. Modern Differential Geometry of Curves and Surfaces with Mathematica[M]. Third ed. Chapman and Hall/CRC, 2006.

[4] . Galilei [M]. : , 2021.

[5] , . [M]. . : , 2001.

[6] , . [M]. : , 2003.

[7] , , . : [M]. . : , 2016.

[8] . : , [M]. . : , 2016.

[9] . [M]. . : , 2001.

[10] . [M]. . : , 2017.

[11] . [M]. . : , 1997.

[12] , . [M]. . : , 2022.

[13] TESCHL G. Graduate studies in mathematics (140): Ordinary Differential Equations and Dynamical Systems[M]. American Mathematical Society, 2012.

[14] SPIVAK M. A Comprehensive Introduction to Differential Geometry: Vol. 3[M]. Third ed. Houston: Publish or Perish, 1999.

[15] KLINGENBERG W. Graduate texts in mathematics: Vol. 51 A Course in Differential Geometry[M]. First ed. Springer, 2013.

[16] KUHNEL W. Student mathematical library: Vol. 77 Differential Geometry: Curves-Surfaces-Manifolds[M]. Third ed. American Mathematical Society, 2015.

[17] FOOTE R L. Differential geometry of real Monge-Ampère foliations [J]. Mathematische Zeitschrift, 1987, 194: 331-350.

[18] , . [M]. . : , 2021.

[19] , , . : [M]. : , 2014.

[20] MOROIANU A. London mathematical society student texts (69): Lectures on Kähler Geometry[M]. Cambridge University Press, 2010.

[21] HUYBRECHTS D. Universitext (utx): Complex Geometry-An Introduction[M]. Springer, 2005.

[22] VAROLIN D. Graduate studies in mathematics (125): Riemann Surfaces by Way of Complex Analytic Geometry[M]. American Mathematical Society, 2011.

[23] LI P. Cambridge studies in advanced mathematics (134): Geometric Analysis[M]. Cambridge University Press, 2012.

[24] . [M]. . : , 2021.

[25] COLDING T H, MINICOZZI W P. Graduate studies in mathematics(121): A Course in Minimal Surfaces[M]. American Mathematical Society, 2011.

[26] SIEGEL A. A Historical Review of the Isoperimetric Theorem in 2-D, and its place in Elementary Plane Geometry[Z]. : Courant Institute of mathematical Sciences New York University, 2003.

[27] HEHL A. The Isoperimetric Inequality[Z]. : Eberhard-Karls Universitaet Tuebingen, 2013.

[28] TAPP K. Undergraduate texts in mathematics(utm): Differential Geometry of Curves and Surfaces[M]. Springer, 2016.

[29] . [M]. 8 . : , 2023.

[30] IBRAGIMOV N H. Elementary Lie Group Analysis and Ordinary Differential Equations[M]. Wiley, 1999.

[31] NIRENBERG L. Cbms regional conference series in mathematics(17): Lectures on Linear Partial Differential Equations[M]. American Mathematical Society, 1973.

[32] DEMAILLY J P. Complex Analytic and Differential Geometry[M]. 2012.

[33] NITSCHE J C C. Lectures on Minimal Surfaces: Vol. 1[M]. Cambridge University Press, 2011.

[34] KAWAKAMI Y. A Survey on Bernstein-type Theorems for Entire Graphical Surfaces [EB/OL]. 2024[2025-08-07]. https://arxiv.org/pdf/2403.14199v2.