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2023 5 22

: ★ , , 25 .

1. (4+5+5+$\boxed{3}$) $X_i, i \geq 1$ μ, $N \sim Pois(\lambda)$ $X_i, i \geq 1$, $\sum_{i=1}^{N} X_i$ Poisson . $\{N(t), t \geq 0\}$ $\lambda > 0$ Poisson ,

(i) T_n Poisson n , $f_{T_n}(\cdot)$,

$$\int_0^t \mathbb{P}(T_{n+1} - T_n > t - s | T_n = s) f_{T_n}(s) \mathrm{d}s;$$

解答: , n s $(s,t]$ $\{N(t) = n | T_n = s\}$,

$$\int_0^t \mathbb{P}(N(t) = n | T_n = s) f_{T_n}(s) \mathrm{d}s = \mathbb{P}(N(t) = n) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}.$$

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(ii) $\mathbb{E}\Big[N(2023) \Big| N(2022) = 2022, N(2024) = 2024\Big]$;

解答: ,

$$\mathbb{E}\Big[N(2023) - N(2022) + 2022 \Big| N(2022) = 2022, N(2024) = 2024\Big]$$

$$= \mathbb{E}\Big[N(2023) - N(2022) \Big| N(2022) = 2022, N(2024) = 2024\Big] + 2022$$

Poisson , 2022 0,

$$\mathbb{E}\Big[N(1) - N(0) \Big| N(0) = 0, N(2) = 2\Big] = \mathbb{E}[N(1)|N(2) = 2] = 1,$$

$N(1)|N(2) = 2 \sim Binomial(2, 1/2)$. $2022 + 1 = 2023$.

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(iii) Poisson $\{N(t), t \geq 0\}$. $G($ $g(\cdot))$, . ; : ...

解答: $\xi = t$, 0, Poisson . , , t , , (, s , $G(t-s))$. Poisson , t ,

$$N_1(t) \sim Poisson\left(\lambda \int_0^t G(t-s)\mathrm{d}s\right) = Poisson\left(\lambda \int_0^t G(s)\mathrm{d}s\right).$$

$$A = \{ \qquad \},$$

$$\begin{aligned} \mathbb{P}(A) &= \int_0^\infty \mathbb{P}(A|\xi = t)g(t)\mathrm{d}t \\ &= \int_0^\infty \mathbb{P}(N_1(t) = 0|\xi = t)g(t)\mathrm{d}t \\ &= \int_0^\infty e^{-\lambda \int_0^t G(s)\mathrm{d}s} g(t)\mathrm{d}t \end{aligned}$$

□

(iv)★ $S(t)$ t , $S(t)$ Poisson ? $\mathbb{E}[S(t)]$.

解答: ! $N(t) = n$ (t), Poisson $(U_i, V_i$), $(V_1, V_2, \cdots, V_n)$,

$$\sum_{i=1}^n \max\{V_i + \xi_i - t, 0\} \stackrel{(d)}{=} \sum_{i=1}^n \max\{U_i + \xi_i - t, 0\} := \sum_{i=1}^n X_i$$

$\xi_i, i = 1, 2, \cdots, n$ G, $S(t)|N(t) = n$ $\sum_{i=1}^n X_i$, $X_i, i \geq 1$ $N(t)$.

$$S(t) \stackrel{(d)}{=} \sum_{i=1}^{N(t)} X_i$$

( ), $S(t)$ Poisson .

, $\mathbb{E}[S(t)] = \mathbb{E}[N(t)]\mathbb{E}[X_1] = \lambda t \mathbb{E}[X_1]$,

$$\begin{aligned}\mathbb{E}[X_1] &= \int_0^\infty \mathbb{P}(\max\{U_1+\xi_1-t,0\}>s)\mathrm{d}s \\ &= \int_0^\infty \mathbb{P}(U_1+\xi_1>t+s)\mathrm{d}s \\ &= \frac{1}{t}\int_0^\infty \left(\int_0^t 1-G(t+s-y)\mathrm{d}y\right)\mathrm{d}s,\end{aligned}$$

$$\mathbb{P}(U_1+\xi_1>t+s) = \frac{1}{t}\int_0^t \mathbb{P}(U_1+\xi_1>t+s|U_1=y)\mathrm{d}y = \frac{1}{t}\int_0^t 1-G(t+s-y)\mathrm{d}y,$$

$$\mathbb{E}[S(t)] = \lambda\int_0^\infty \left(\int_0^t 1-G(t+s-y)\mathrm{d}y\right)\mathrm{d}s.$$

: $\mathbb{E}[X_1]$, , U_1 ξ_1 .
, . □

2. (2+$\boxed{4}$+3+3+3) $\{X_n, n\geq 0\}$ $S=\{1,2,3\}$

$$\begin{pmatrix} 1/2 & 1/4 & 1/4 \\ 0 & 3/4 & 1/4 \\ 0 & 0 & 1 \end{pmatrix}$$

$T_{ij} := \min\{n\geq 0, X_n = j|X_0=i\}$, i j .

(i) $a_k = \mathbb{P}(T_{13}=k), k=0,1,2,\cdots,$ a_{k+1}, a_k ;

解答: $a_0=0, a_1=1/4,$ $k\geq 2,$ X_1

$$\begin{aligned}a_{k+1} &= \frac{1}{2}\mathbb{P}(T_{13}=k+1|X_1=1)+\frac{1}{4}\mathbb{P}(T_{13}=k+1|X_1=2)+\frac{1}{4}\mathbb{P}(T_{13}=k+1|X_1=3) \\ &= \frac{1}{2}\mathbb{P}(T_{13}=k)+\frac{1}{4}\mathbb{P}(T_{23}=k)+0 \\ &= \frac{1}{2}a_k+\frac{1}{4}\left(\frac{3}{4}\right)^{k-1}\frac{1}{4}\end{aligned}$$

2 , 2 1/4
, a_{k+1}, a_k

$$a_{k+1} = \frac{1}{2}a_k + \frac{1}{12}\left(\frac{3}{4}\right)^k, \quad k\geq 1$$

□

(ii)★ (i) T_{13} $\mathbb{E}[T_{13}]$;

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$$g(z) = \sum_{k=0}^{\infty} a_k z^k, \quad |z| \le 1,$$

(i) $g(z)$, z ,

a_k , https://www.cnblogs.com/Tarjan-Zeng/p/15038489.html

解答: , (

) .

$$a_0 = 0, \quad a_k = \frac{1}{4}\left(\frac{3}{4}\right)^{k-1}, \quad k \ge 1.$$

$T_{13} \sim Geom(1/4)$, $\mathbb{E}[T_{13}] = 4$.

: ? $1,2$, 3 ,

$p(1,1) + p(1,2) = p(2,2) = 3/4$, T_{13} $1/4$ Bernoulli

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(iii) (ii) , , 3 2

$\{T_{13} = k\}$, $\mathbb{P}(T_{13} = k)$ $\mathbb{E}[T_{13}]$;

解答: , $k \ge 1$

$$\begin{aligned}\mathbb{P}(T_{13} = k) = \mathbb{P}(X_1 = \cdots = X_{k-1} = 1, X_k = 3)\\ + \sum_{m=1}^{k-1} \mathbb{P}(X_1 = 1, \cdots, X_{m-1} = 1, X_m = 2, \cdots, X_{k-1} = 2, X_k = 3)\\ = \left(\frac{1}{2}\right)^{k-1}\frac{1}{4} + \sum_{m=1}^{k-1}\left(\frac{1}{2}\right)^{m-1}\frac{1}{4}\left(\frac{3}{4}\right)^{k-m-1}\frac{1}{4} = \frac{1}{4}\left(\frac{3}{4}\right)^{k-1},\end{aligned}$$

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(iv) (X_1), $\mathbb{E}[T_{13}]$;

解答: ,

$$\mathbb{E}[T_{13}] = \frac{1}{2}\mathbb{E}[T_{13}|X_1 = 1] + \frac{1}{4}\mathbb{E}[T_{13}|X_1 = 2] + \frac{1}{4}\mathbb{E}[T_{13}|X_1 = 3]$$

$$\mathbb{E}[T_{23}] = \frac{3}{4}\mathbb{E}[T_{23}|X_1 = 2] + \frac{1}{4}\mathbb{E}[T_{23}|X_1 = 3]$$

$\mathbb{E}[T_{23}|X_1 = 3] = \mathbb{E}[T_{13}|X_1 = 3] = 1$, $\mathbb{E}[T_{13}|X_1 = 1] = 1 + \mathbb{E}[T_{13}]$, $\mathbb{E}[T_{13}|X_1 = 2] = 1 + \mathbb{E}[T_{23}]$,$\mathbb{E}[T_{23}|X_1 = 2] = 1 + \mathbb{E}[T_{23}]$, $\mathbb{E}[T_{23}] = 4($ $T_{23} \sim Geom(1/4))$,

$$\mathbb{E}[T_{13}] = \frac{1}{2}(1 + \mathbb{E}[T_{13}]) + \frac{3}{2} \Rightarrow \mathbb{E}[T_{13}] = 4 \qquad +\infty.$$

$+\infty$, $T_{13} < Geom(1/2) + Geom(1/4)$, !

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: $+\infty$, . □

(v) , $i \in S$, $\rho_i := \mathbb{P}($ $i|X_0 = i)$.

: (iv) (v), , .

解答: $\rho_3 = 1$, ρ_2

$$\rho_2 = \frac{3}{4}\mathbb{P}_2(T_2 < \infty | X_1 = 2) + 0 = \frac{3}{4}$$

$$\rho_1 = \frac{1}{2} + \frac{1}{4}\mathbb{P}_1(T_1 < +\infty | X_1 = 2) + 0 = \frac{1}{2}.$$

: $\rho_{xy} := \mathbb{P}_x(T_y < +\infty)$, ,

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