(2023–2024–1)

6858>]TJ(10.909 T0f 2 Tr 0.4710<083c>]TJ(a)1.955 Tf 108.66 0 5<9781420b0/F1 10.909 Tf 108.66 0 [(3ac11d404434>-7<2cba>0.909 T0f 2 Tr 0.4728.010)]TJ /a)1.955 12f 108.66 0 83690 X

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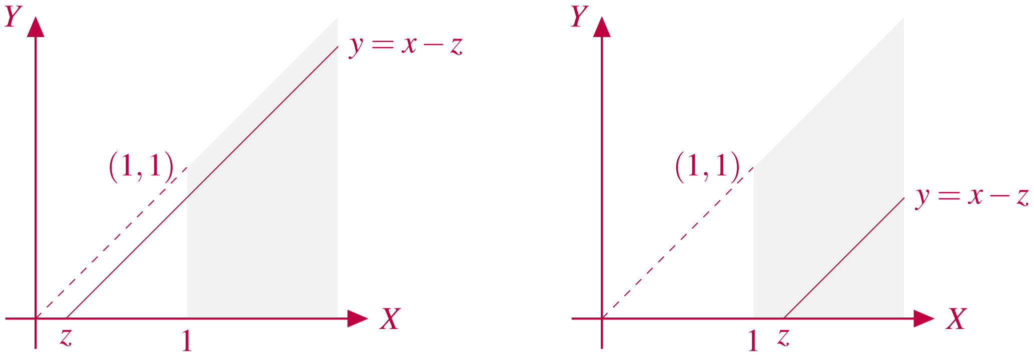

(1) $0<z<1$,

$$\mathbb{P}(Y \geq X-z)=\int_1^{\infty}\left(\int_{x-z}^{x} x^{-3}\mathrm{d}y\right)\mathrm{d}x=\frac{z}{2};$$

(2) $z \geq 1$,

$$\mathbb{P}(Y \geq X-z)=1-\int_z^{\infty}\left(\int_0^{x-z} x^{-3}\mathrm{d}y\right)\mathrm{d}x=1-\frac{1}{2z},$$

Z

$$f_Z(z)=\begin{cases} 0.5, & 0<z<1; \\ 0.5z^{-2}, & z\geq 1; \\ 0, & . \end{cases}$$

: , $(X-Y,Y)$, ,

...., ,

.

2. (50) $X_1, X_2, \cdots$, ,

$S=\{0,1\}$, Bernoulli .

(i) $X_1, X_2, \cdots$ n $i_1, i_2, \cdots, i_n \in S$

$$\mathbb{P}(X_n=i_n|X_1=i_1,X_2=i_2,\cdots,X_{n-1}=i_{n-1})=\mathbb{P}(X_n=i_n)$$

; 15

$X_1, X_2, \cdots$ n, $X_1, X_2, \cdots, X_n$,

$$\mathbb{P}(X_1=i_1,\cdots,X_{n-1}=i_{n-1},X_n=i_n)=\mathbb{P}(X_1=i_1)\cdots\mathbb{P}(X_{n-1}=i_{n-1})\mathbb{P}(X_n=i_n)$$

.

$$=\mathbb{P}(X_1=i_1)\mathbb{P}(X_2=i_2|X_1=i_1)\cdots\mathbb{P}(X_n=i_n|X_1=i_1,\cdots,X_{n-1}=i_{n-1}).$$

, $X_1, X_2, \cdots$ n

$$\begin{aligned}\mathbb{P}(X_1=i_1)\cdots\mathbb{P}(X_{n-1}=i_{n-1})\mathbb{P}(X_n=i_n)\\ =\mathbb{P}(X_1=i_1)\mathbb{P}(X_2=i_2|X_1=i_1)\cdots\mathbb{P}(X_n=i_n|X_1=i_1,\cdots,X_{n-1}=i_{n-1})\end{aligned}$$

,

$$\mathbb{P}(X_n=i_n|X_1=i_1,X_2=i_2,\cdots,X_{n-1}=i_{n-1})=\mathbb{P}(X_n=i_n)$$

n $i_1,i_2,\cdots,i_n\in S$.

(ii) , , . , " ", 3 , $3,4,5$. $n+1$, $k\in\mathbb{N}$? ;

: $i=2,3,\cdots$, , (i).

. . . . 20

$i=2,3,\cdots$,

$$X_i=\begin{cases}1, & i \quad ;\\ 0, & .\end{cases}$$

$A=\{X_1=x_1,\cdots,X_{n-1}=i_{n-1}\}, B=\{\ n-1\ \}$,

$$\begin{aligned}\mathbb{P}(X_n=1|A)&=\mathbb{P}(X_n=1|A,B)\mathbb{P}(B|A)+\mathbb{P}(X_n=1|A,B^c)\mathbb{P}(B^c|A)\\&=(1/2)\mathbb{P}(B|A)+(1/2)\mathbb{P}(B^c|A)=1/2\end{aligned}$$

$$\mathbb{P}(X_n=1)=\mathbb{P}(X_n=1|B)\mathbb{P}(B)+\mathbb{P}(X_n=1|B^c)\mathbb{P}(B^c)=1/2;$$

$$\mathbb{P}(X_n=1|A)=\mathbb{P}(X_n=1), \qquad \mathbb{P}(X_n=0|A)=\mathbb{P}(X_n=0),$$

(i) , .

, $B(1,1/2)$. , $n+1$, $k\in\mathbb{N}$

$$b(k;n,1/2)=C_n^k(1/2)^n, k=0,1,2,\cdots,n;$$

(iii) (ii) , $1+\xi$, $\xi\sim Poisson(\lambda)$, (Y) 10

(ii) Poisson , $Y\sim Poisson(\lambda/2)$.

$$\frac{e^x-e^{-x}}{2}=\frac{1}{2}\left(\sum_{k=0}^{\infty}\frac{x^k}{k!}-\sum_{k=0}^{\infty}\frac{(-x)^k}{k!}\right)=\sum_{k=1,3,5,\cdots}\frac{x^k}{k!},$$

Y

$$\sum_{k=1,3,5,\cdots}^{\infty}\frac{(\lambda/2)^k}{k!}e^{-\lambda/2}=\frac{1}{2}(1-e^{-\lambda});$$

: , ,

$g(s) := \mathbb{E}[s^Y] = \exp((\lambda/2)(s-1)), s \in \mathbb{R},$ Y

$$\mathbb{P}(Y \qquad\qquad) = \frac{1}{2}(g(1) - g(-1)) = \frac{1}{2}(1 - e^{-\lambda}).$$

(iv) (iii) , $\xi - Y$ Y ? 5

Poisson , . ,

.

3. (50) $\mathbb{R}$

$$sign(x) = \begin{cases} 1, & x > 0; \\ 0, & x = 0; \\ -1, & x < 0. \end{cases}$$

$X_0, X_1, \cdots$, $Y_n = X_n * sign(X_{n-1}), n \geq 1$.

. ($*$,)

: ξ $\xi * \mathbb{I}_A$, A

, $\mathbb{P}(A) = 1$.

(i) Y_n ; 20

X_n $sign(X_{n-1})$ $\mathbb{E}[Y_n] = 0$;

$$\mathbb{P}(sign^2(X_{n-1}) = 1) = 1 - \mathbb{P}(sign(X_{n-1}) = 0) = 1 - \mathbb{P}(X_{n-1} = 0) = 1,$$

$$\mathrm{Var}(Y_n) = \mathbb{E}[Y_n^2] = \mathbb{E}[X_n^2 * sign^2(X_{n-1})] = \mathbb{E}[X_n^2 * \mathbb{I}_{\{sign^2(X_{n-1})=1\}}] = \mathbb{E}[X_n^2] = 1;$$

(ii) $m, n \geq 1$ $m \neq n$, $\mathrm{Cov}(Y_m, Y_n)$; 10

$m > n$, $Y_m Y_n = X_m * sign(X_{m-1}) * X_n * sign(X_{n-1})$

,

$$\mathbb{E}[Y_m Y_n] = \mathbb{E}[X_m] * \mathbb{E}[sign(X_{m-1}) * X_n] * \mathbb{E}[sign(X_{n-1})] = 0;$$

(i) , $m, n \geq 1$ $m \neq n$, $\mathrm{Cov}(Y_m, Y_n) = 0$;

(iii) n, Y_n ? 10

$y \in \mathbb{R}$, Y_n

$$
\begin{aligned}
\mathbb{P}(Y_n \le y) &= \mathbb{P}(X_n * sign(X_{n-1}) \le y) \\
&= \mathbb{P}(X_n \le y, X_{n-1} > 0) + \mathbb{P}(-X_n \le y, X_{n-1} < 0) \\
&= \Phi(y)/2 + (1 - \Phi(-y))/2 \\
&= \Phi(y),
\end{aligned}
$$

$\Phi(\cdot)$, $Y_n \sim N(0,1)$;

(iv) (Y_1, Y_2) ? 10

: .

! , .

, $y_1, y_2 \in \mathbb{R}$,

$$
\begin{aligned}
&\mathbb{P}(Y_1 \le y_1, Y_2 \le y_2) \\
&\quad = \mathbb{P}(X_1 * sign(X_0) \le y_1, X_2 * sign(X_1) \le y_2) \\
&\quad = \mathbb{P}(X_1 \le y_1, X_2 \le y_2, X_0 > 0, X_1 > 0) \\
&\qquad\qquad + \mathbb{P}(X_1 \le y_1, X_2 \ge -y_2, X_0 > 0, X_1 < 0) \\
&\qquad\qquad + \mathbb{P}(X_1 \ge -y_1, X_2 \le y_2, X_0 < 0, X_1 > 0) \\
&\qquad\qquad + \mathbb{P}(X_1 \ge -y_1, X_2 \ge -y_2, X_0 < 0, X_1 < 0) \\
&\quad = \mathbb{P}(0 < X_1 \le y_1)\Phi(y_2)/2 + \mathbb{P}(X_1 \le \min\{0, y_1\})[1 - \Phi(-y_2)]/2 \\
&\qquad\qquad + \mathbb{P}(X_1 \ge \max\{0, -y_1\})\Phi(y_2)/2 + \mathbb{P}(-y_1 \le X_1 \le 0)[1 - \Phi(-y_2)]/2 \\
&\quad = \Phi(y_1)\Phi(y_2),
\end{aligned}
$$

$y_1 \ge 0, y_1 < 0$, $(Y_1, Y_2) \sim N(0,0;1,1;0)$.